Complex Numbers

★ De Moivre's Theorem

$$\cos n\theta = {}^nC_0 \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + \ldots$$
$$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \ldots$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Q $\left(\dfrac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right)^n$

$\rightarrow$ $1 = \sin^2\alpha + \cos^2\alpha = \sin^2\alpha - i^2\cos^2\alpha$ $\quad (\because i^2 = -1)$

$1+\sin\alpha + i\cos\alpha = (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha) + (\sin\alpha + i\cos\alpha)$

$= (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha + 1)$

$\therefore \dfrac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha} = \sin\alpha + i\cos\alpha = \cos\left(\dfrac{\pi}{2}-\alpha\right) + i\sin\left(\dfrac{\pi}{2}-\alpha\right)$

$\therefore \left(\dfrac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right)^n = \cos n\left(\dfrac{\pi}{2}-\alpha\right) + i\sin n\left(\dfrac{\pi}{2}-\alpha\right)$

Q $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos\left(\dfrac{n\pi}{3}\right)$

$\rightarrow (1+i\sqrt{3})^n = 2\left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right) = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

$(1-i\sqrt{3})^n = 2\left(\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right) = 2\left(\cos\dfrac{\pi}{3} - i\sin\dfrac{\pi}{3}\right)$

$$\therefore \left(1+i\sqrt{3}\right)^n + \left(1-i\sqrt{3}\right)^n = 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + 2^n\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$$

$$= 2^n\left(2\cos\frac{n\pi}{3}\right)$$

$$= 2^{n+1}\cos\frac{n\pi}{3}$$

$\boxed{Q}$ α & β are roots of the eqn $x^2-2x+2=0$, P.T. $\alpha^n+\beta^n = 2 \cdot 2^{n/2}\cos\frac{n\pi}{4}$, $\alpha^8+\beta^8=32$

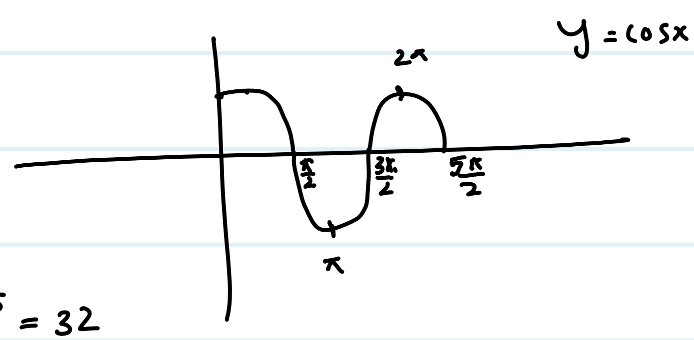
$\rightarrow$ $$x = \frac{-(-2) \pm \sqrt{4-4(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

$$\alpha = 1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\beta = 1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$\alpha^n + \beta^n = 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$

$$\alpha^2 + \beta^n = 2 \cdot 2^{n/2}\left(\cos\frac{n\pi}{4}\right)$$



Putting $n=8$, $\alpha^8+\beta^8 = 2 \cdot 2^4 \cos 2\pi = 2^5 = 32$

* Roots of a Complex Number :-

$$\boxed{(\cos\theta + i\sin\theta)^{1/n} = \cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)}$$

If $x = 1 \Rightarrow \cos 0^\circ + i\sin 0^\circ$ / If $x = -1 = \cos\pi + i\sin\pi$

Q] $(1-\omega)^6 = -27$

$\rightarrow$ $x^3 = 1 \Rightarrow x = 1^{1/3}$

$\therefore x = (\cos\theta + i\sin\theta)^{1/3} = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}$

Putting $k=0,1,2$ the cube roots of unity are:-

$\therefore x_0 = 1$

$\therefore x_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$

$\therefore x_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]^2 = \omega^2$

$\therefore 1 + \omega + \omega^2 = 1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$= 1 - \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$

$= 1 - 2\cos\frac{\pi}{3} = 1 - 2\left(\frac{1}{2}\right) = 0$

$1 + \omega^2 = -\omega$

Now, $(1-\omega)^6 = [(1-\omega)^2]^3 = (1 - 2\omega + \omega^2)^3 = (-3\omega)^3 = -27\omega^3 = -27$

$\boxed{Q}$ $x^6 + 1 = 0$

$x^6 = -1 = \cos\pi + i\sin\pi$

$x = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/6}$

$x = \cos(2k+1)\frac{\pi}{6} + i\sin(2k+1)\frac{\pi}{6}$

$k = 0, 1, 2, 3, 4, 5$

$k=0$, $x_0 = \frac{\sqrt{3}}{2} + \frac{i}{2}$

$k=1$, $x_1 = 0 + i = i$

$k=2$, $x_2 = -\frac{\sqrt{3}}{2} + \frac{i}{2}$

$k=3$, $x_3 = -\frac{\sqrt{3}}{2} - \frac{i}{2}$

$k=4$, $x_4 = 0 - i = -i$

$k=5$, $x_5 = \frac{\sqrt{3}}{2} - \frac{1}{2}$

$\therefore$ Roots : $\pm i$, $\frac{\sqrt{3} \pm i}{2}$, $\frac{-\sqrt{3} \pm i}{2}$

★ Hyperbolic Functions

- Circular functions of Complex Numbers

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

$$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

- Hyperbolic functions

$$\boxed{\cosh x = \frac{e^{x} + e^{-x}}{2}}$$

$$\boxed{\sinh x = \frac{e^{x} - e^{-x}}{2}}$$

- Relation b/w hyperbolic & circular functions :-

- $\sin ix = \sinh x$; $\sinh ix = i\sin x$
- $\cos ix = \cosh x$; $\cosh ix = \cos x$
- $\tan ix = i\tanh x$; $\tanh ix = i\tan x$
- $\sinh x = -i\sin ix$
- $\cosh x = \cos ix$
- $\tanh x = -i\tan ix = \frac{1}{i}\tan ix$

• Hyperbolic Identities

- $\sinh(-x) = -\sinh x$
- $\cosh(-x) = \cosh x$
- $e^{x} = \cosh x + \sinh x$
- $e^{-x} = \cosh x - \sinh x$
- $\cosh^2 x - \sinh^2 x = 1$

• Differentiation & Integration

$y = \sinh x$, $\frac{dy}{dx} = \cosh x$

$y = \cosh x$, $\frac{dy}{dx} = \sinh x$

$y = \tanh x$, $\frac{dy}{dx} = \text{sech}^2 x$

$\int \cosh x\,dx = \sinh x$

$\int \sinh x\,dx = \cosh x$

$\int \text{sech}^2\,dx = \tanh x$

We give below the complete list of formulae of Hyperbolic Functions.

(a) Square Relations

1. $\cos h^2 x - \sin h^2 x = 1$
2. $\sec h^2 x + \tan h^2 x = 1$
3. $\cot h^2 x - \text{cosec}\, h^2 x = 1$

(b) Addition Formulae

1. $\sin h(x \pm y) = \sin h\, x \cos h\, y \pm \cos h\, x \sin h\, y$
2. $\cos h(x \pm y) = \cos h\, x \cos h\, y \pm \sin h\, x \sin h\, y$
3. $\tan h(x \pm y) = \dfrac{\tan hx \pm \tan hy}{1 \pm \tan hx \tan hy}$

(c) Formulae for 2x and 3x

1. $\sin h\, 2x = 2 \sin h\, x \cos h\, x$
2. $\cos h\, 2x = \cos h^2 x + \sin h^2 x = 2\cos h^2 x - 1 = 2 \sin h^2 x + 1$
3. $\tan h\, 2x = \dfrac{2\tan hx}{1 + \tan h^2 x}$
4. $\sin h\, 3x = 3 \sin h\, x + 4 \sin h^3 x$
5. $\cos h\, 3x = 4 \cos h^3 x - 3 \cos h\, x$
6. $\tan h\, 3x = \dfrac{3\tan h\, x + \tan h^3 x}{1 + 3\tan h^2 x}$
7. $\sin h\, x = \dfrac{2\tan h(x/2)}{1 - \tan h^2(x/2)}$
8. $\cos h\, x = \dfrac{1 + \tan h^2(x/2)}{1 - \tan h^2(x/2)}$
9. $\tan h\, x = \dfrac{2\tan h(x/2)}{1 + \tan h^2(x/2)}$

(d) Product Formulae

1. $\sin h(x + y) + \sin h(x - y) = 2 \sin h\, x \cos h\, y$
2. $\sin h(x + y) - \sin h(x - y) = 2 \cos h\, x \sin h\, y$
3. $\cos h(x + y) + \cos h(x - y) = 2 \cos h\, x \cos h\, y$
4. $\cos h(x + y) - \cos h(x - y) = 2 \sin h\, x \sin h\, y$
5. $\sin hx + \sin hy = 2 \sin h\left(\dfrac{x+y}{2}\right) \cos h\left(\dfrac{x-y}{2}\right)$
6. $\sin hx - \sin hy = 2 \cos h\left(\dfrac{x+y}{2}\right) \sin h\left(\dfrac{x-y}{2}\right)$
7. $\cos hx + \cos hy = 2 \cos h\left(\dfrac{x+y}{2}\right) \cos h\left(\dfrac{x-y}{2}\right)$
8. $\cos hx - \cos hy = 2 \sin h\left(\dfrac{x+y}{2}\right) \sin h\left(\dfrac{x-y}{2}\right)$

• Periodic func^n

$\sinh x$: period $2\pi i$

$\cosh x$: period $2\pi i$

$\tanh x$: period $2\pi i$

Binomial Expansion :-

$$(x+y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{(n-1)} y^1 + {}^nC_2\, x^{(n-2)} y^2$$

$$(x-y)^n = {}^nC_0\, x^n y^0 - {}^nC_1\, x^{(n-1)} y^1 + {}^nC_2\, x^{(n-2)} y^2$$

$\boxed{Q}$ If $\tan h x = 1/2$, find the value of x & $\sin h\, 2x$

$\rightarrow$ $\tan x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{1}{2}$

Dividing by e^{-x},

$$\frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1}{2}$$

$\therefore\ e^{2x} = 3 \Rightarrow x = \dfrac{1}{2} \ln 3$

Now,

$$\sin h\, 2x = \frac{e^{2x} - e^{-2x}}{2} = \frac{3 - (1/3)}{2} = \frac{4}{3}$$

Q] P.T. $\left[\sin(\alpha+\theta) - e^{i\alpha}\sin\theta\right]^n = \sin^n\alpha\, e^{-in\theta}$

$\rightarrow \quad e^{i\alpha} = \cos\alpha \pm i\sin\alpha$

$$\text{L.H.S.} = \left[\sin\alpha\cos\theta + \cos\alpha\sin\theta - (\cos\alpha + i\sin\alpha)\sin\theta\right]^n$$

$$= \left[\sin\alpha\cos\theta - i\sin\alpha\sin\theta\right]^n = \sin^n\alpha \cdot (e^{-i\theta})^n$$

$$= \sin^n\alpha \cdot e^{-in\theta}$$

Q] P.T. $\cos^{-1}z = -i\log\left(z \pm \sqrt{z^2-1}\right)$, $\tan^{-1}z = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$

i]

$\rightarrow \quad \cos u = \dfrac{e^{iu}+e^{-iu}}{2} \Rightarrow z = \dfrac{e^{iu}+e^{-iu}}{2} \Rightarrow 2z = e^{iu} + \dfrac{1}{e^{iu}}$

$$\therefore \quad e^{2iu} - 2ze^{iu} + 1 = 0$$

$$e^{iu} = \frac{2z \pm \sqrt{4z^2-4}}{2} = z \pm \sqrt{z^2-1}$$

$$iu = \log\left(z \pm \sqrt{z^2-1}\right) \Rightarrow u = -i\log\left(z \pm \sqrt{z^2-1}\right)$$

$$\therefore \quad \cos^{-1}z = -i\log\left(z \pm \sqrt{z^2-1}\right)$$

ii]

$\tan^{-1}z = u \qquad \therefore \quad z = \tan u$

$$z = \frac{\sin u}{\cos u} = \frac{e^{iu}-e^{-iu}}{i(e^{iu}+e^{-iu})} = \frac{z}{i} = \frac{e^{iu}-e^{-iu}}{i^2(e^{iu}+e^{-iu})} = \frac{e^{-iu}-e^{iu}}{e^{iu}+e^{-iu}}$$

By componendo-dividendo,

$\therefore \; e^{-2iu} = \frac{i+z}{i-z}$

$\therefore \; -2iu = \log\left(\frac{i+z}{i-z}\right)$

$\therefore \; u = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right)$

Q] P.T. $16\cosh^5 x = \cosh 5x + 5\cosh 3x + 10\cosh x$

$\rightarrow$ L.H.S. $= 16\left(\frac{e^{x}+e^{-x}}{2}\right)^5$

$= \frac{16}{2^5}\left[e^{5x} + 5e^{4x}\cdot e^{-x} + 10e^{3x}\cdot e^{-2x} + 10e^{2x}\cdot e^{-3x} + 5e^{x}\cdot e^{-4x} + e^{-5x}\right]$

(By Binomial Theorem)

$= \frac{e^{5x}+e^{-5x}}{2} + 5\frac{(e^{3x}+e^{-3x})}{2} + 10\frac{(e^{x}+e^{-x})}{2}$

$= \cosh 5x + 5\cosh 3x + 10\cosh x$ = R.H.S

Q] If $\sin\alpha \cosh\beta = \frac{x}{2}$, $\cos\alpha \sinh\beta = \frac{y}{2}$ the P.T.

(i) $\operatorname{cosec}(\alpha - i\beta) + \operatorname{cosec}(\alpha + i\beta) = \frac{4x}{x^2+y^2}$

(ii) $\operatorname{cosec}(\alpha - i\beta) - \operatorname{cosec}(\alpha + i\beta) = \frac{4iy}{x^2+y^2}$

$\rightarrow$ $\operatorname{cosec}(\alpha+i\beta) = \frac{1}{\sin(\alpha+i\beta)} = \frac{1}{\sin\alpha\cosh\beta + \cos\alpha\sinh\beta}$

$$\operatorname{cosec}(\alpha+i\beta) = \frac{1}{(x/2) + i(y/2)} = \frac{2}{x+iy}$$

$\therefore$ $\operatorname{cosec}(\alpha - i\beta) = \frac{2}{x-iy}$

$\therefore$ $\operatorname{cosec}(\alpha-i\beta) + \operatorname{cosec}(\alpha+i\beta) = \frac{2}{x-iy} + \frac{2}{x+iy} = \frac{4x}{x^2+y^2}$

$\therefore$ $\operatorname{cosec}(\alpha-i\beta) - \operatorname{cosec}(\alpha+i\beta) = \frac{2}{x-iy} - \frac{2}{x+iy} = \frac{4iy}{x^2+y^2}$

Q] P.T. $\cosh^2 x = \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{1-\cosh^2 x}}}$

$\rightarrow$ RHS $= \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{-\sinh^2 x}}}$

$$= \cfrac{1}{1 - \cfrac{1}{1+\operatorname{cosech}^2 x}} = \cfrac{1}{1 - \cfrac{1}{\coth^2 x}} = \frac{1}{1-\tanh^2 x}$$

$$= \frac{1}{1 - \frac{\sinh^2 x}{\cosh^2 x}} = \frac{\cosh^2 x}{\cosh^2 x - \sinh^2 x} = \cosh^2 x$$

Q] If $\log \tan x = y$, P.T. $\sinh ny = \frac{1}{2}(\tan^n x - \cot^n x)$ &

$\cosh(n+1)y + \cosh(n-1)y = 2\cosh ny \operatorname{cosec} 2x$

→ $y = \log(\tan x) \Rightarrow e^y = \tan x$ & $e^{-y} = \cot x$

Now, $\sinh ny = \frac{e^{ny} - e^{-ny}}{2} = \frac{1}{2}(\tan^n x - \cot^n x)$

Again, $\cosh(n+1)y + \cosh(n-1)y = 2\cosh ny \operatorname{cosec} 2x$

$$= 2\cosh ny . \left(\frac{e^y + e^{-y}}{2}\right) = 2\cosh ny . \left(\frac{\tan x + \cot x}{2}\right)$$

$$= 2\cosh ny . \left(\frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}{2}\right)$$

$$= 2\cosh ny . \frac{1}{2\sin x \cos x} = 2\cosh ny . \operatorname{cosec} 2x$$

Q] $u = \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, P.T. i) $\cosh u = \sec\theta$, ii) $\sinh u = \tan\theta$, iii) $\tanh u = \sin\theta$, iv) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

→ $e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1 + \tan\theta/2}{1 - \tan\theta/2} = \frac{\cos(\theta/2) + \sin(\theta/2)}{\cos(\theta/2) - \sin(\theta/2)} . \frac{\cos(\theta/2) + \sin(\theta/2)}{\cos(\theta/2) + \sin(\theta/2)}$

$$= \frac{1 + 2\sin(\theta/2)\cos(\theta/2)}{\cos^2(\theta/2) - \sin^2(\theta/2)} = \frac{1 + \sin\theta}{\cos\theta} = \sec\theta + \tan\theta = e^u$$

$e^{-u} = \sec\theta - \tan\theta$

i) $e^{u} + e^{-u} = 2\sec\theta \implies \cosh u = \dfrac{e^{u}+e^{-u}}{2} = \sec\theta$

ii) $e^{u} - e^{-u} = 2\tan\theta \Rightarrow \sinh u = \dfrac{e^{u}-e^{-u}}{2} = \tan\theta$

iii) $\tanh u = \dfrac{\sinh u}{\cosh u} = \dfrac{\tan\theta}{\sec\theta} = \sin\theta$

(iv) $\tanh\left(\dfrac{u}{2}\right) = \dfrac{\sinh(u/2)}{\cosh(u/2)} = \dfrac{2}{2}\dfrac{\sinh(u/2)}{\cosh(u/2)} \cdot \dfrac{\cosh(u/2)}{\cosh(u/2)} = \dfrac{\sinh u}{1+\cosh u} = \dfrac{\tan\theta}{1+\sec\theta}$

$\therefore \tanh\left(\dfrac{u}{2}\right) = \dfrac{\frac{\sin\theta}{\cos\theta}}{1+\frac{1}{\cos\theta}} = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{2\sin(\theta/2)\cancel{\cos(\theta/2)}}{2\cancel{\cos^2}(\theta/2)} = \dfrac{\sin\theta/2}{\cos\theta/2} = \tan\dfrac{\theta}{2}$

* Separation of Real & Imaginary Parts

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin ix = i\sinh x \,, \quad \cos ix = \cosh x$$

i] $\sin z$

$$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$

$$\sin(x+iy) = \underbrace{\sin x \cosh y}_{\text{Real Part}} + i\,\underbrace{\cos x \sinh y}_{\text{Imaginary Part}}$$

ii] $\cos z$

$$\cos(x+iy) = \cos x \cos iy - \sin x \sin iy$$

$$\cos(x+iy) = \underbrace{\cos x \cosh y}_{\text{Real Part}} \underbrace{- i \sin x \cos hy}_{\text{Imaginary Part}}$$

iii] $\tan z$

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} \cdot \frac{\cos(x-iy)}{\cos(x-iy)}$$

$$\tan(x+iy) = \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i\sinh 2y}{\cos 2x + i\cosh 2y}$$

$\therefore$ Real Part $= \dfrac{\sin 2x}{\cos 2x + \cosh 2y}$, Imaginary Part $= \dfrac{\sinh 2y}{\cos 2x + \cosh 2y}$

iv] $\cot z$

Real Part $= \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ & Imaginary Part $= \dfrac{-\sinh 2y}{\cosh 2y - \cos 2x}$

(v) sec z

$$\sec z = \frac{1}{\cos(x+iy)}$$

As above
$$\sec z = \frac{1}{2\cos(x+iy)} \cdot \frac{2\cos(x-iy)}{\cos(x-iy)}$$
$$= \frac{2\cos x\cos(iy) + 2\sin x\sin(iy)}{\cos 2x + \cos 2iy}$$
$$= \frac{2\cos x\cos h\, y + i\, 2\sin x\sin h\, y}{\cos 2x + \cos h\, 2y}$$

$\therefore$ Real part $= \dfrac{2\cos x\cos hy}{\cos 2x + \cos h2y}$

and Imaginary part $= \dfrac{2\sin x\sin hy}{\cos 2x + \cos h2y}$

(vi) cosec z

Sol. : Left to you.

Real part $= \dfrac{2\sin x\cos hy}{-\cos 2x + \cos h2y}$

and Imaginary part $= \dfrac{-2\cos x\sin hy}{-\cos 2x + \cos h2y}$

(vii) sin hz

$$\sin h(x+iy) = \sin hx\cos h\, iy + \cos hx\sin h\, iy$$
$$= \sin hx\cos y + i\cos hx\sin y$$

$\therefore$ Real part $= \sin hx\cos y$

Imaginary part $= \cos hx\sin y$

(viii) cos hz

$$\cos h(x+iy) = \cos hx\cos h\, iy + \sin hx\sin h\, iy$$
$$= \cos hx\cos y + i\sin hx\sin y$$

$\therefore$ Real part $= \cos hx\cos y$

Imaginary part $= \sin hx\sin y$

(ix) tan hz

$$\tan h(x+iy) = \frac{\sin h(x+iy)}{\cos h(x+iy)}$$
$$= \frac{2\sin h(x+iy)}{2\cos h(x+iy)} \cdot \frac{\cos h(x-iy)}{\cos h(x-iy)}$$
$$= \frac{\sin h2x + \sin h2iy}{\cos h2x + \cos h2iy} = \frac{\sin h2x + i\sin 2y}{\cos h2x + \cos 2y}$$

$\therefore$ Real part $= \dfrac{\sin h2x}{\cos h2x + \cos 2y}$

Imaginary part $= \dfrac{\sin 2y}{\cos h2x + \cos 2y}$

Q] Separate into real & imaginary parts $\tan^{-1}(e^{i\theta})$

→ $\tan^{-1} e^{i\theta} = x+iy \Rightarrow e^{i\theta} = \tan(x+iy)$

$\therefore \cos\theta + i\sin\theta = \tan(x+iy)$

Similarly, $\cos\theta - i\sin\theta = \tan(x-iy)$

Now, $\tan 2x = \dfrac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$

$$\tan 2x = \frac{(\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)}{1-(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)} = \frac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)} = \frac{2\cos\theta}{0} = \infty$$

$\therefore 2x = \dfrac{\pi}{2} \Rightarrow x = \dfrac{\pi}{4}$

$$\tan 2iy = \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)} = \frac{(\cos\theta-i\sin\theta)+(\cos\theta-i\sin\theta)}{1+(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)} = \frac{2i\sin\theta}{1+(\cos^2\theta+\sin^2\theta)} = i\sin\theta$$

$\therefore i\tanh 2y = i\sin\theta \Rightarrow \tanh 2y = \sin\theta$

$2y = \tan^{-1}(\sin\theta) \quad \therefore y = \frac{1}{2}\tanh^{-1}(\sin\theta)$

Q] If $\sin(\alpha - i\beta) = x+iy$ P.T. $\dfrac{x^2}{\cosh^2\beta} + \dfrac{y^2}{\sinh^2\beta} = 1$ & $\dfrac{x^2}{\sin^2\alpha} - \dfrac{y^2}{\cos^2\alpha} = 1$

→ $\sin(\alpha - i\beta) = x+iy$

$\therefore \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta = x+iy$

Let, $\sin\alpha\cosh\beta = x$ & $\cos\alpha\sinh\beta = y$

$\therefore \dfrac{x^2}{\cosh^2\beta} + \dfrac{y^2}{\sinh^2\beta} = \sin^2\alpha + \cos^2\alpha = 1$ & $\dfrac{x^2}{\sin^2\alpha} - \dfrac{y^2}{\cos^2\alpha} = \cosh^2\beta - \sinh^2\beta = 1$

Q] $\cos(x+iy) = \cos\alpha + i\sin\alpha$, P.T. i] $\sin\alpha = \pm\sin^2 x = \pm\sinh^2 y$, ii] $\cos 2x + \cos 2y = 2$

$\rightarrow$ $\cos(x+iy) = \cos\alpha + i\sin\alpha$

$\cos x \cos(iy) - \sin x \sin(iy) = \cos\alpha + i\sin\alpha$

$\cos x \cosh y - i\sin x \sinh y = \cos\alpha + i\sin\alpha$

$\therefore \cos x \cosh y = \cos\alpha$ & $-\sin x \sinh y = \sin\alpha$

(i) $\sin^2\alpha + \cos^2\alpha = 1$

$\therefore \sin^2 x \sinh^2 y + \cos^2 x \cosh^2 y = 1$

$\therefore \sin^2 x \sinh^2 y + (1-\sin^2 x)(1+\sinh^2 y) = 1$

$\therefore \cancel{\sin^2 x \sinh^2 y} + 1 + \sinh^2 y - \sin^2 x - \cancel{\sin^2 x \sinh^2 y} = 1$

$\therefore 1 + \sinh^2 y - \sin^2 x = 1$

$\therefore \sinh^2 y = \sin^2 x$ ...... (i)

$\therefore \sinh y = \pm \sin x$

$\therefore \sin\alpha = -\sin x \sinh y = \pm \sin^2 x$

(ii) $\cos 2x + \cos 2y = 1 - 2\sin^2 x + 1 + 2\sinh^2 y$

$= 2 - 2\sin^2 x + 2\sin^2 x$

$= 2$

Q] Seperate into real and imaginary parts $\sqrt{i}^{\sqrt{i}}$

$\sqrt{i} = i^{1/2} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

Also $\sqrt{i} = \left(e^{i\pi/2}\right)^{1/2} = e^{i\pi/4}$

$$(\sqrt{i})^{\sqrt{i}} = \left(e^{i\pi/4}\right)^{\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)} = e^{\frac{i\pi}{4\sqrt{2}} - \frac{\pi}{4\sqrt{2}}} = e^{\frac{-\pi}{4\sqrt{2}}} . e^{\frac{i\pi}{4\sqrt{2}}}$$

$$= e^{\frac{-\pi}{4\sqrt{2}}}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$$

$\therefore$ Real part $= e^{-\frac{\pi}{4\sqrt{2}}} \cos\frac{\pi}{4\sqrt{2}}$

$\therefore$ Imaginary Part $= e^{\frac{-\pi}{4\sqrt{2}}} \sin\frac{\pi}{4\sqrt{2}}$

* Inverse Hyperbolic Functions

If x is real

(i) $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$

(ii) $\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$

(iii) $\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Q] P.T. $\tanh \log \sqrt{x} = \frac{x-1}{x+1}$, Hence deduce that $\tanh \log \sqrt{\frac{5}{3}} + \tan\log\sqrt{7} = 1$

→ Let $\tanh \log \sqrt{x} = a$

$$\log \sqrt{x} = \tanh^{-1}(a)$$

$$\frac{1}{2}\log x = \frac{1}{2}\log\left(\frac{1+a}{1-a}\right)$$

$$x = \frac{1+a}{1-a}$$

$$\frac{x-1}{x+1} = \frac{(1+a)-(1-a)}{(1+a)+(1-a)} = \frac{2a}{2} = a$$

$$\therefore \tanh \log \sqrt{x} = \frac{x-1}{x+1}$$

Putting $x = 5/3$ & $x = 7$

$$\tanh\left[\log\sqrt{5/3}\right] + \tanh(\log\sqrt{7}) = \frac{(5/3)-1}{(5/3)+1} + \frac{7-1}{7+1} = \frac{2}{8} + \frac{6}{8} = 1$$

Q i] P.T. $\cosh^{-1}\sqrt{1+x^2} = \sinh^{-1}x$

$\rightarrow$ $\cosh^{-1}\sqrt{1+x^2} = y$

$\therefore \sqrt{1+x^2} = \cosh y$

$\therefore 1+x^2 = \cosh^2 y$

$\therefore x^2 = \cosh^2 y - 1 = \sinh^2 y$

$\therefore x = \sinh y$

$\therefore y = \sinh^{-1}(x)$

$\therefore \cosh^{-1}\sqrt{1+x^2} = \sinh^{-1}x$

ii] P.T. $\tanh^{-1}x = \sinh^{-1}\dfrac{x}{\sqrt{1-x^2}}$

$\tanh^{-1}x = y \Rightarrow x = \tanh y$

$$\frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\sqrt{1-\tanh^2 y}} = \frac{\tanh y}{\sqrt{\frac{\cosh^2 y - \sinh^2 y}{\cosh^2 y}}} = \frac{\sinh y}{\cosh y} \times \frac{\cosh y}{1} = \sinh y$$

$\therefore y = \sinh^{-1}\dfrac{x}{\sqrt{1-x^2}}$ $\quad \therefore \tanh^{-1}(x) = \sinh^{-1}\dfrac{x}{\sqrt{1-x^2}}$

(iv) $\cot^{-1}\left(\frac{x}{a}\right) = y$. $\therefore \frac{x}{a} = \coth y \Rightarrow \therefore \tanh y = \frac{1}{\coth y} = \frac{1}{x/a} = \frac{a}{x}$

$y = \tanh^{-1}\left(\frac{a}{x}\right) = \frac{1}{2}\log\left(\frac{1+a/x}{1-a/x}\right) = \frac{1}{2}\log\left(\frac{x+a}{x-a}\right)$

$$\coth^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}\log\left(\frac{x+a}{x-a}\right)$$

* Logarithms of Complex Numbers :-

$z = x + iy$, $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\log z = \log(r(\cos\theta + i\sin\theta)) = \log(r\, e^{i\theta})$$

$$\log z = \log r + \log e^{i\theta} = \log r + i\theta$$

$$\therefore \log(x+iy) = \log r + i\theta$$

$$\therefore \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right) \quad \ldots\ldots\ldots\text{(i)} \rightarrow \text{Principle value}$$

or else,

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\left[2n\pi + \tan^{-1}\left(\frac{y}{x}\right)\right] \quad \ldots\ldots\ldots\text{(ii)} \rightarrow \text{General value}$$

$\boxed{Q}$ P.T. $\log_2(-3) = \dfrac{\log 3 + i\pi}{\log 2}$

→

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

Putting $x = -3$, $y = 0$

$$\log(-3) = \frac{1}{2}(9) + i\tan^{-1}\left(\frac{0}{-3}\right) = \frac{1}{2}\log 3^2 + i\pi = \log 3 + i\pi$$

$$\therefore \log_2(-3) = \frac{\log_e(-3)}{\log_e 2} = \frac{\log 3 + i\pi}{\log 2}$$

Q3] P.T. $\log(1+e^{2i\theta}) = \log(2\cos\theta) + i\theta$

$\rightarrow$ $\log(1+e^{2i\theta}) = \log(1+\cos 2\theta + i\sin 2\theta)$

$= \log(2\cos^2\theta + 2i\sin\theta\cos\theta)$

$= \log(2\cos\theta(\cos\theta + i\sin\theta))$

$= \log(2\cos\theta \,.\, e^{i\theta})$

$= \log(2\cos\theta) + \log(e^{i\theta})$

$= \log(2\cos\theta) + i\theta$

Q] Principle value of $(1+i)^{1-i}$

$\rightarrow$ $z = (1+i)^{1-i}$

$\log z = (1-i)\log(1+i)$

$\log z = (1-i)\left[\log\sqrt{1+1} + i\tan^{-1} 1\right]$

$= 1-i\left[\frac{1}{2}\log 2 + \frac{i\pi}{4}\right]$

$= \left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) = x + iy$ (assume)

$z = e^{x+iy} = e^x . e^{iy} = e^x(\cos y + i\sin y)$

$= e^{\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right)}\left[\cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)\right]$

$= \sqrt{2}\, e^{\pi/4}\left[\cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)\right]$ $\boxed{e^{\frac{1}{2}\log 2} = e^{\log\sqrt{2}} = \sqrt{2}}$