Fourier Series

* Dirichlet's Conditions :-

Consider a single valued function $f(x)$ in interval $(a, a+2L)$ which satisfies below conditions is known as Dirichlet's conditions.

① $f(x)$ is defined in interval $(a, a+2L)$ & $f(x) = f(x+2L)$

② $f(x)$ is continuous function **OR** has finite number of discontinuites in interval $(a, a+2L)$

③ $f(x)$ has no maxima or minima **OR** has finite numbers of maxima or minima

Fourier - Euler's formula

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \; ; \quad a < x < \alpha + 2\pi$$

OR

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \longrightarrow \left[\text{Fourier series in interval } (\alpha, \alpha+2L) \right]$$

$$f(x) = \frac{1}{L} \int_a^{a+2L} f^2(x)\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \longrightarrow \left[\begin{array}{l} \text{Parseval's Identity} \\ \text{Use whenever you need square of} \\ \text{summation series} \end{array} \right]$$

where, $$a_0 = \frac{1}{L}\int_a^{a+2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier Coefficients

Q] Fourier series in interval $(0, 2\pi)$

Obtain F.S. for $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 \leq x \leq 2\pi$ & $f(x+2\pi) = f(x)$

Deduce. (i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$

(ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

(iv) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$

→ $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2\,dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = \frac{-1}{12\pi}(-\pi^3 - \pi^3)$$

$$\therefore \boxed{a_0 = \frac{\pi^2}{6}}$$

Remember!!!

When period $= (0, 2\pi)$ or $(-\pi, \pi)$

$$\cos\left(\frac{n\pi x}{L}\right) = \cos(nx)$$

$$\sin\left(\frac{n\pi x}{L}\right) = \sin(nx)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \, . \cos nx \, dx$$

$$a_n = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 . \cos nx \, dx$$

ILATE

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \int \cos nx - \int \frac{d}{dx}(\pi - x)^2 \int \cos nx \, dx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin(nx)}{n} - \int 2(\pi - x).(-1) \frac{\sin(nx)}{n}\right]_0^{2\pi}$$

ILATE

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin(nx)}{n} - \left[\int -2(\pi - x) \frac{\sin(nx)}{n}\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin(nx)}{n} - \left[-2(\pi - x)\int \frac{\sin(nx)}{n} + \int \frac{d}{dx}(-2(\pi - x)) \int \frac{\sin(nx)}{n}\right] dx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin(nx)}{n} - \left(-2(\pi - x) \frac{-\cos(nx)}{n^2}\right) + \int 2 . \frac{-\cos(nx)}{n^2} dx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin(nx)}{n} - \left(-2(\pi - x) \frac{-\cos(nx)}{n^2}\right) + 2 . \frac{-\sin(nx)}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\cancelto{0}{(\pi - x)^2 \frac{\sin(nx)}{n}} - 2(\pi - x)\frac{\cos(nx)}{n^2} + 2 . \cancelto{0}{\left(-\frac{\sin(nx)}{n^3}\right)}\right]_0^{2\pi}$$

$$= \frac{-1}{2n^2\pi}\left[(\pi - x)\cos(nx)\right]_0^{2\pi} = \frac{-1}{2n^2\pi}(-\pi - \pi) \Rightarrow \boxed{a_n = \frac{1}{n^2}}$$

$$b_n = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cdot \sin(nx)\, dx$$

ILATE

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi-x)^2 \cdot \frac{-\cos nx}{n} - \cancelto{0}{(-2(\pi-x)) \cdot \frac{-\sin nx}{n^2}} + 2 \cdot \frac{\cos(nx)}{n^3} \Bigg]_0^{2\pi}$$

$$= \frac{1}{4\pi n}\left[-(\pi-x)^2 \cos(nx) + \frac{2\cos(nx)}{n^2} \right]_0^{2\pi}$$

$$= \frac{1}{4\pi n}\left[-\pi^2 \cdot (1) + \frac{2}{n^2} - (-\pi^2) \cdot 1 - \frac{2}{n^2} \right]$$

$$\boxed{b_n = 0}$$

$$\therefore \left(\frac{\pi - x}{2}\right)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \cos(nx) + 0$$

Putting $x=0$,

$$\therefore \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$\therefore \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{— } (1)$$

Putting $x=\pi$,

$$\therefore 0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \quad \text{— } (2)$$

Adding ① & ②,

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{1^2} + \cancel{\frac{1}{2^2} - \frac{1}{2^2}} + \frac{1}{3^2} + \frac{1}{3^2} + \cancel{\frac{1}{4^2} - \frac{1}{4^2}} + \cdots$$

$$\frac{18\pi^2}{72} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2}$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Parseval's Identity

$$\frac{1}{L}\int_0^{a+2L} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\therefore \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^4 dx = \frac{\pi^2}{36\times 2} + \sum_{n=1}^{\infty} \left(\left(\frac{1}{n^2}\right)^2 + 0^2\right)$$

$$\therefore \frac{1}{16\pi}\int_0^{2\pi} (\pi - x)^4\,dx = \frac{\pi^4}{72} + \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \frac{1}{16\pi}\left[\frac{(\pi - x)^5}{-5}\right]_0^{2\pi} = \frac{\pi^4}{72} + \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{40} - \frac{\pi^4}{72} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots$$

9] $f(x) = e^{-x} \quad (0, 2\pi)$

i) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ ii) $\operatorname{cosech}(\pi)$

$\rightarrow$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} e^{-x}\, dx$$

$$a_0 = -\frac{1}{\pi} \left[e^{-x}\right]_0^{2\pi} = \frac{1-e^{2\pi}}{\pi}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos(nx)\, dx$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2} \left[a \cos bx + b \sin bx\right]$$

$a = -1 \quad b = n$

$$\therefore\ a_n = \frac{1}{\pi} \left[\frac{e^{-x}}{n^2+1} \left(-\cos nx + \cancelto{0}{n \sin nx}\right) \right]_0^{2\pi}$$

$$\therefore\ a_n = \frac{1}{\pi(n^2+1)} \left[e^{-2\pi} \cdot (-1) - 1(-1) \right] = \frac{1}{n^2+1} \cdot \left(\frac{1-e^{-2\pi}}{\pi} \right)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \sin(nx)\, dx$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2} \left[a \sin bx - b \cos bx\right]$$

$$b_n = \frac{1}{\pi} \left[\frac{e^{-x}}{n^2+1} \left(-\cancelto{0}{\sin(nx)} - n \cos(nx)\right) \right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi(n^2+1)} \left(e^{-2\pi} \cdot (-n) - 1(-n) \right) = \frac{n}{n^2+1} \left(\frac{1-e^{-2\pi}}{\pi} \right)$$

$$\therefore\ f(x) = \left(\frac{1-e^{-2\pi}}{2\pi}\right) + \frac{1-e^{-2\pi}}{\pi} \sum_{n=1}^{\infty} \left[\frac{\cos(nx)}{n^2+1} + \frac{n.\sin(nx)}{n^2+1}\right]$$

(i) Putting $x = \pi$

$$e^{-\pi} = \left(\frac{1-e^{-2\pi}}{2\pi}\right) + \left(\frac{1-e^{-2\pi}}{\pi}\right) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$\therefore\ e^{-\pi} = \cancel{\left(\frac{1-e^{-2\pi}}{2\pi}\right)} + \cancel{\left(\frac{-1}{2}\left(\frac{1-e^{-2\pi}}{\pi}\right)\right)} + \left(\frac{1-e^{-2\pi}}{\pi}\right) \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$\therefore\ \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{\pi.\, e^{-\pi}}{1-e^{-2\pi}} = \frac{\pi}{e^{\pi}-e^{-\pi}}$$

(ii) $$\therefore\ \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{\pi}{2\left(e^{\pi}-\frac{e^{-\pi}}{2}\right)} = \frac{\pi}{2} \cdot \operatorname{cosech}(\pi)$$

$$\therefore\ \operatorname{cosec} h(\pi) = \frac{2}{\pi} \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{n^2+1}\right)$$