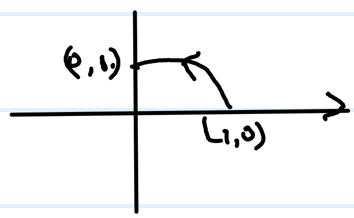
Q] Evaluate $\int_c \vec{F}.\vec{dr}$ where $\vec{F} = \cos y\,\hat{i} - x\sin y\,\hat{j}$

c: $y = \sqrt{1-x^2}$ from $(1,0)$ to $(0,1)$

$\rightarrow$



$$y = \sqrt{1-x^2}$$

$$x^2 + y^2 = 1$$

$$\int_c \vec{F}.\vec{dr} = \int_c \cos y\, dx - x\sin y\, dy$$

nullifies each other

$$\int_c d(x\cos y)$$

$$\left[x\cos y\right]_{1,0}^{0,1}$$

$$0 - 1$$

$$= -1$$

Q] Find work done in moving a particle once around the circle $x^2+y^2=a^2$, $z=0$ in the force field $\vec{F} = \sin y\, i + (x + x\cos y)\, j$

$\rightarrow$ $$\oint_C \vec{F}\cdot\vec{dr}$$

$$= \int_C \sin y\, dx + (x + x\cos y)\, dy$$

$$= \oint_C (\sin y\, dx + x\cos y\, dy) + \oint_C x\,dy$$

$$= \cancelto{0}{\oint_C d(x\sin y)} + \oint_C x\,dy$$

substitute $x = a\cos\theta$, $y = a\sin\theta$

$y = a\sin\theta$

$dy = a\cos\theta\, d\theta$

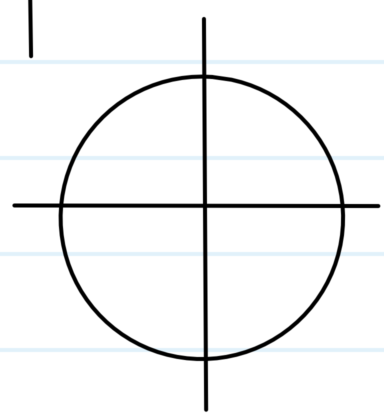
$$\int x\,dy = \int_0^{2\pi} a\cos\theta \cdot a\cos\theta\, d\theta$$

$$= a^2 \int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= a^2 \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\left[\theta + \sin 2\theta\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\cdot 2\pi = \pi a^2$$



0 to 2π

Q] P.T $\vec{F} = (y^2\cos x + z^3)i + (2y\sin x - 4)j + (3xz^2 + 2)K$

① Find ϕ.

② Find work done from moving an object from $(0,1,-1)$ to $\left(\frac{\pi}{2},-1,2\right)$

$\rightarrow$

① P.T : $\nabla \times \vec{F} = 0$

$\therefore$ $\vec{F}$ is conservative $\exists \phi$ such that $\vec{F} = \nabla\phi$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= (y^2\cos x + z^3)\,dx + (2y\sin x - 4)\,dy + (3xz^2+2)\,dz$$

$$= (y^2\cos x\,dx + 2y\sin x\,dy) + (z^3dx + 3xz^2dz) - 4dy + 2dz$$

$$d\phi = \vec{F}.\vec{dr} = F_1\,dx + F_2dy + F_3\,dz$$

$$= d(y^2\sin x) + d(xz^3) + d(-4y) + d(2z)$$

$$d\phi = d(y^2\sin x + xz^3 - 4y + 2z)$$

$$\phi = y^2\sin x + xz^3 - 4y + 2z + c$$

② $$\int_c d\phi = [\phi]_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)}$$

$$= 4\pi + 15$$

$\boxed{Q}$ $\int_A^B (y^2 dx + xy dy)$ along $x = t^2$ & $y = 2t$ from $A(1,-2)$ to $B(0,0)$

$\rightarrow$

$$\int_C \vec{F} \cdot \vec{dr} = \int 4t^2 (2t\,dt) + 2t^3 (2dt)$$

at A, $x = 1, y = -2$

$\Rightarrow t^2 = 1$ && $2t = -2$ $\Rightarrow t = -1$

at B, $x = y = 0$

$\Rightarrow t = 0$

$$\int_{-1}^{0} 8t^3 + 4t^3 \, dt$$

$$\left[3t^4\right]_{-1}^{0} = -3$$

* Green Theorem

$$\oint (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Q] Find $\oint (e^{-x} \sin y\, dx + e^{-x} \cos y\, dy)$ where C is a rectangle whose vertices are $(0,0)\ (\pi,0),\ \left(\pi, \frac{\pi}{2}\right), \left(0, \frac{\pi}{2}\right)$

→ $P = e^{-x} \sin y$ & $Q = e^{-x} \cos y$

$\frac{\partial P}{\partial y} = -e^{-x} \cos y$ && $\frac{\partial Q}{\partial x} = -e^{-x} \cos y$

$$\therefore \oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= -2 \int_0^{\pi} \int_0^{\frac{\pi}{2}} e^{-x} \cos y\, dx\, dy$$

$$= -2 \int_0^{\pi} e^{-x} [\sin y]_0^{\frac{\pi}{2}}\, dx$$

$$= -2 \int_0^{\pi} e^{-x}\, dx$$

$$= +2 \left[e^{-x}\right]_0^{\pi}$$

$$= 2 \left[e^{-\pi} - 1\right]$$

Q] Verify Green theorem for following integral in plane

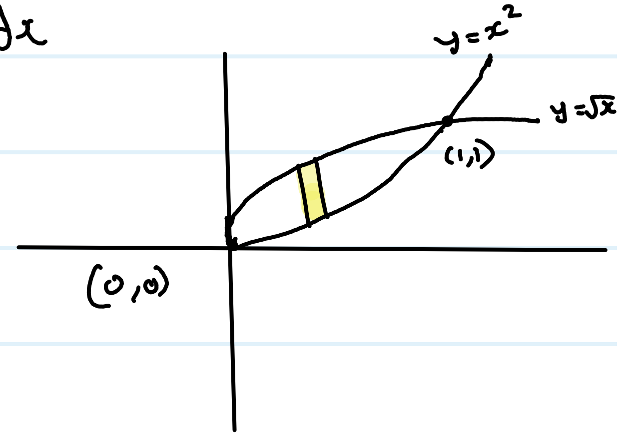
$$\iint_R [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$$

where c is boundary of region bounded by parabola $y = \sqrt{x}$ & $y = x^2$

$\rightarrow \quad P = 3x^2 - 8y^2 \quad , \quad Q = 4y - 6xy$

$$\frac{\partial P}{\partial y} = -16y \quad , \quad \frac{\partial Q}{\partial x} = -6y$$

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(-6y - (-16y)\right) dy\,dx$$



$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y \; dy\,dx$$

$$= \frac{10}{2} \int_0^1 \left[y^2\right]_{x^2}^{\sqrt{x}} dx$$

$$= 5 \int_0^1 (x - x^4)\, dx$$

$$= 5 \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 dx$$

$$= 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2}$$

* Surface Integral

$$\hat{n} = \frac{\text{grad } f}{|\text{grad} f|}$$

XY plane : $ds = \dfrac{dx\,dy}{|\hat{n}\,\hat{k}|}$

YZ plane : $ds = \dfrac{dy\,dz}{|\hat{n}\,\hat{i}|}$

XZ plane : $ds = \dfrac{dx\,dz}{|\hat{n}\,\hat{j}|}$

* Stoke's Theorem

$$\oint_c \vec{F}.\vec{dr} = \iint_s n \text{ curl } F.ds$$

$$\vec{F}.\vec{dr} = \iint_s (\nabla \times \vec{F}).\hat{n}.ds$$

plane

Area

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$\hat{n}$
- xy plane $\rightarrow \hat{k}$
- yz plane $\rightarrow \hat{i}$
- xz plane $\rightarrow \hat{j}$

$x^2+y^2=a^2$

$x = a\cos\theta$, $y = a\sin\theta$

$\theta = 0$ to 2π

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

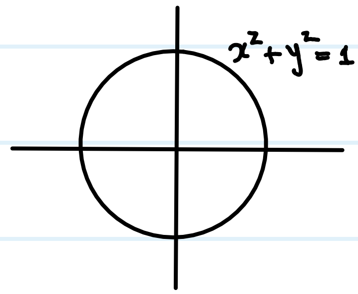
$x = a\cos\theta$, $y = b\sin\theta$

$\theta = 0$ to 2π

$$\int_0^{2\pi} \sin^2\theta \, d\theta = 4\int_0^{\frac{\pi}{2}} \sin^2\theta$$

$$\int_0^{2\pi} \cos^4\theta \, d\theta = 4\int_0^{\frac{\pi}{2}} \cos^4\theta$$

Q] Let $F = yi + zj + xk$ & S is part of sphere $x^2+y^2+z^2=1$ above x-y plane



$$\int F.dr = \iint_S \text{curl } F.\hat{n}\, ds$$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = i(-1) - j(1) + k(-1)$$

$$= -i - j - k$$

$$\hat{n} = k$$

$$\therefore \iint (-i-j-k)k\, dx dy$$

$$\therefore -\iint_R dx dy = \text{Area}$$

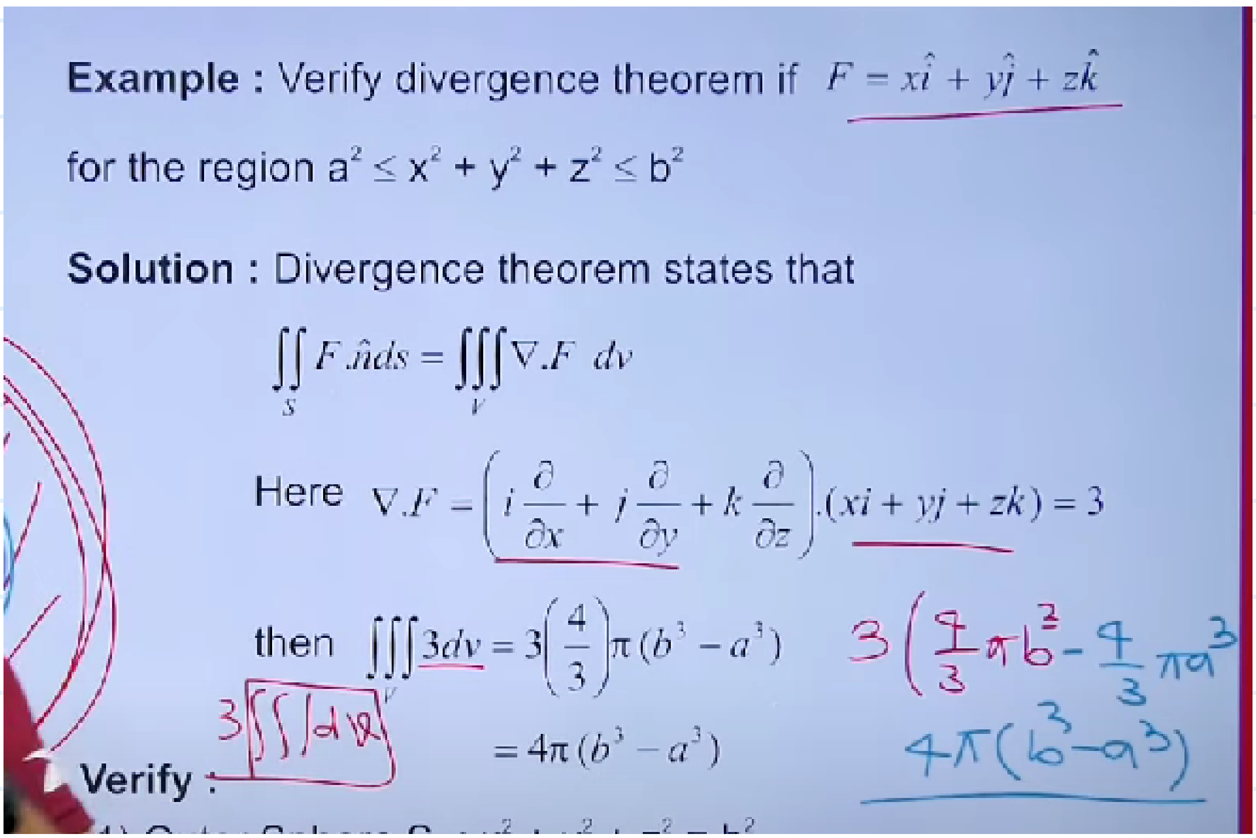
$$\therefore -\pi(1)^2 = -\pi$$

* Gauss Divergence Theorem

$$\iint_S F.n\,dS = \iiint \operatorname{div} F\,dV$$

Another form

$$\iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$



$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}\,dv$$

Q] $\int_A^B (2xy^3 - y^2\cos x)dx + (1 - 2y\sin x + 3x^2y^2)\,dy = \frac{\pi^2}{4}$ along arc

$2x = \pi y^2$ from $A(0,0)$ to $B(\pi/2, 1)$

$\rightarrow$ $F_1 = 2xy^3 - y^2\cos x$

$F_2 = 1 - 2y\sin x + 3x^2y^2$

$F_3 = 0$

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = i(0-0) - j(0-0) + k(-2y\cos x + 6xy^2 - (6xy^2 - 2y\cos x))$$

$$= 0$$

$\therefore$ $\nabla \times \vec{F} = 0$

$\therefore$ $\vec{F}$ is an irrotational

$d\phi = F_1\,dx + F_2\,dy + F_3 dz$

$d\phi = (2xy^3 - y^2\cos x)dx + (1 - 2y\sin x + 3x^2y^2)\,dy$

Integrating both sides

$$\phi = \left[\left(\cancel{2}y^3\frac{x^2}{\cancel{2}} - y^2\sin x\right) + \left(y - \cancel{2}\frac{y^2}{\cancel{2}}\sin x + \cancel{3}x^2\frac{y^3}{\cancel{3}}\right)\right]$$

$$= y^3x^2 - y^2\sin x + y - y^2\sin x + x^2y^3$$

$\Rightarrow$ Write repeated terms once only.

$$\phi = x^2y^3 - y^2\sin x + y$$

$$W \cdot D = \Big[\ \phi\ \Big]_{(0,0)}^{(\frac{\pi}{2},1)}$$

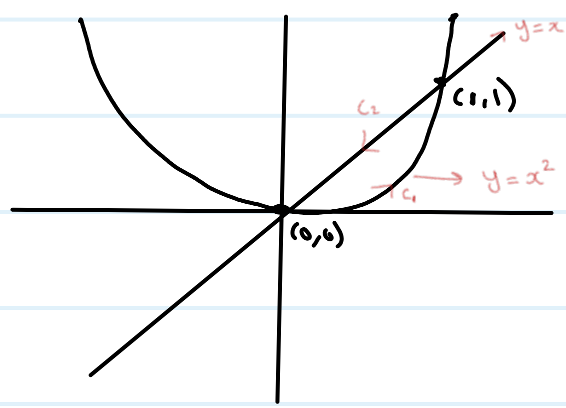
$$= \Big[x^2y^3 - y^2\sin x + y\Big]_{(0,0)}^{(\frac{\pi}{2},1)}$$

$$= \Big[\frac{\pi^2}{4}\cdot 1^3 - \cancel{1^3 \cdot \sin\frac{\pi}{2}} + \cancel{1} - 0\Big]$$

$$= \frac{\pi^2}{4}$$

Q] Verify Green's Theorem in the plane for $\int_c (xy + y^2)\,dx + x^2\,dy$ where c is closed curve of the region bounded by $y = x$ & $y = x^2$

→



$y = x$ & $y = x^2$

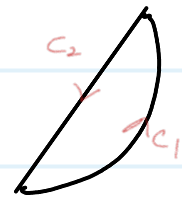
$\therefore\ x = x^2$

$\therefore\ x^2 - x = 0$

$\therefore\ x(x-1) = 0$

$\therefore\ x = 0,\ x = 1$

Along C_1



$y = x^2$

$dy = 2x\,dx$

$$\therefore \int P\,dx + Q\,dy = \int_{C_1} (xy + y^2)\,dx + x^2\,dy$$

$$= \int_0^1 \left(x.x^2 + (x^2)^2\right) dx + x^2.2x\,dx$$

$$= \int_0^1 (x^4 + 3x^3)\,dx$$

$$= \left[\frac{x^5}{5} + \frac{3.x^4}{4}\right]_0^1 = \frac{19}{20} \quad \text{—— (1)}$$

Along C_2,

$y = x$, $dy = dx$

$$\int P\,dx + Q\,dy = \int_{C_2} (xy + y^2)\,dx + x^2\,dy$$

$$= \int_1^0 (x.x + x^2)\,dx + x^2.\,dx$$

$$= \int_1^0 (3x^2)\,dx = \left[3.\frac{x^3}{3}\right]_1^0$$

$$= -1 \quad \text{—— (2)}$$

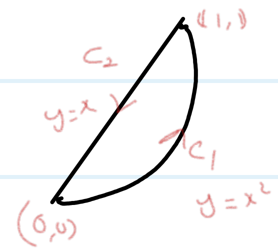
$$\int_C P\,dx+Q\,dy = \int_{C_1} P\,dx+Q\,dy + \int_{C_2} P\,dx+Q\,dy$$

$$= \frac{19}{20}-1$$

$$\int_C P\,dx+Q\,dy = \frac{-1}{20} \quad \text{— ③}$$

By using Green's Theorem

$$\int_C P\,dx+Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dx\,dy$$



$$= \int_0^1\int_{x^2}^{x} \left[2x-(x+2y)\right] dx\,dy$$

$$= \int_0^1\int_{x^2}^{x} \left[x-2y\right] dx\,dy$$

$P = xy+y^2 \qquad Q = x^2$

$\frac{\partial P}{\partial y} = x+2y \qquad \frac{\partial Q}{\partial x} = 2x$

$$= \int_0^1 \left[xy - \frac{\cancel{2}y^2}{\cancel{2}}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\cancel{x^2}-\cancel{x^2}-(x^3-x^4)\right] dx$$

$$= \left[\frac{-x^4}{4}+\frac{x^5}{5}\right]_0^1 = \frac{-1}{20} \quad \text{— ④}$$

From ③ & ④

Green Theorem verified

Q] Apply Stokes Theorem to calculate $\int_C 4y\,dx + 2z\,dy + 6y\,dz$ where C is the curve of intersection of $x^2+y^2+z^2 = 6z$ and $z = x+3$

→

$$\int_C 4y\,dx + 2z\,dy + 6y\,dz$$

According to Stoke's Theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = i(6-2) - j(0-0) + k(0-4)$$

$$= 4\hat{i} - 4\hat{k}$$

$\boxed{x = z-3}$

$Z = x+3$

$\phi = x+3-z$

$\nabla\phi = i + 0j - k$

$\nabla\phi = \hat{i} - \hat{k} \Rightarrow |\nabla\phi| = \sqrt{2}$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i}-\hat{k}}{\sqrt{2}}$$

$x^2+y^2+z^2 = 6z$

$x^2+y^2+z^2-6z = 0$

$x^2+y^2+z^2-2(z)(3)+9-9 = 0$

$x^2+y^2+(z-3)^2 = 3^2$

Radius of circle = 3

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$$

$$= \iint \left(4\hat{i} - 4\hat{k}\right) \cdot \frac{\hat{i} - \hat{k}}{\sqrt{2}} \cdot ds$$

$$= \frac{1}{\sqrt{2}} \iint \left[4(1) + (-4)(-1)\right] ds$$

$$= \frac{1}{\sqrt{2}} \iint 8\, ds$$

$$= \frac{8}{\sqrt{2}} \boxed{\iint ds} \rightarrow \text{Area}$$

$$= \frac{8}{\sqrt{2}} \times \pi (3)^2$$

$$= 4\sqrt{2} \times 9\pi$$

$$\vec{F} \cdot \vec{dr} = 36\pi\sqrt{2}$$

Q] Evaluate $\iint_S (\nabla \times \vec{F}) \vec{ds}$ where $\vec{F} = (x^3 - y^3)\hat{i} - xyz\hat{j} + y^3\hat{k}$ & S is surface $x^2 + 4y^2 + z^2 - 2x = 4$ above plane $x = 0$

$\rightarrow$ $\vec{F} = (x^3 - y^3)\hat{i} - xyz\hat{j} + y^3\hat{k}$

Using Stoke's Theorem

$$\int_C \vec{F} \cdot \vec{dr} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \cdot ds$$

$$\int_C (x^3 - y^3) dx - (xyz) dy + y^3 dz$$

$x^2 + 4y + z^2 - 2x = 4$

Put $x = 0$ $x = 0 \rightarrow$ Given, $dx = 0$

$4y^2 + z^2 = 4$

$\frac{y^2}{1} + \frac{z^2}{4} = 1$

$y = a\cos\theta = \cos\theta \Rightarrow dy = -\sin\theta\, d\theta$

$z = b\sin\theta = 2\sin\theta \Rightarrow dz = 2\cos\theta\, d\theta$

$$\int_c (x^3-y^3)\,dx + (-xyz)\,dy + y^3\,dz$$

$$\int (\quad)0 + (0) + y^3 dz$$

$$\int_c y^3 dz = \int_0^{2\pi} (\cos\theta)^3 \cdot 2\cos\theta\, d\theta = 2\int_0^{2\pi} \cos^4\theta\, d\theta = 2\cdot 4\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \cos^4\theta = \cancel{8}\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{3\pi}{2}$$

Q] Verify Gauss Divergence Theorem $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ & S the surface area of bounded by the planes $x=0$, $x=2$, $y=0$, $y=2$, $z=0$, $z=2$

→ $\nabla \vec{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$

Using Gauss Divergence Theorem

$$\nabla\cdot\vec{F} = 4z - 2y + y$$

$$\nabla\cdot\vec{F} = 4z - y$$

$$\int_{x=0}^{x=2}\int_{y=0}^{y=2}\int_{z=0}^{z=2} \nabla\cdot\vec{F}\,.dv = \int_0^2\int_0^2\int_0^2 (4z-y)\,dx\,dy\,dz$$

$$= \int_0^2\int_0^2 \left[\int_0^2 4z-y\, dz\right] dx\,dy$$

$$= \int_0^2\int_0^2 \left[\frac{4z^2}{2} - yz\right]_0^2 dx\,dy$$

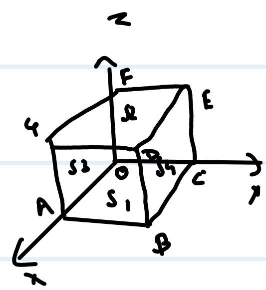
$$= \int_0^2 \int_0^2 8 - 2y \; dx\, dy$$

$$= \int_0^2 \left[8y - \cancel{2} \frac{y^2}{\cancel{2}} \right]_0^2 dx$$

$$= \int_0^2 12 \; dx$$

$$= \left[12x \right]_0^2$$

$$= 24 \quad \text{—} \quad (1)$$



$S_1 = OABC$, $S_2 = GDEF$, $S_3 = OAGF$, $S_4 = BCED$

$S_5 = OCEF$, $S_6 = ABDG$

$$S_1 = \int_0^2 \int_0^2 \vec{F} \cdot \hat{n} \, ds$$

$\hat{n} = -k$

$$\vec{F} \cdot \hat{n}_{z=0} = (yz)(-1) = -yz$$

$$-y(0) = 0$$

$S_1 = 0$, $S_3 = 0$