Space Forces

9.1 Introduction

We have so far discussed and worked with a coplanar system of forces, wherein all the forces in a system would lie in a single plane i.e. it was a two dimensional force system. In this chapter we deal with a system of forces lying in different planes forming a three dimensional force system. Such a system is also referred as a *Space Force System* and a vector approach is required to deal with these problems. The chapter begins with the study of basic operations with forces using a vector approach. Later on we will learn to find the resultant of a space force system and finally we shall deal with equilibrium problems in space forces.

9.2 Basic Operations Using Vector Approach

Since the forces have three dimensions, we employ a vector approach, which simplifies the working. Here we will learn to represent a force vectorially, to find vectorially moment of a force about a point and about a line, to find magnitude and direction of a force given in vector form and other basic operations useful in solution of a space force problem.

9.2.1 *Force in vector form.*

Fig. 9.1 shows a force of magnitude F in space passing through A (x_1, y_1, z_1) and B (x_2, y_2, z_2). The force in vector form is

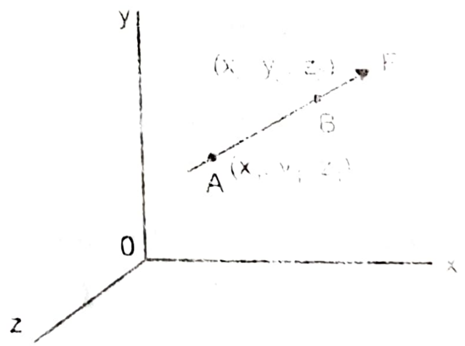

Fig. 9.1

$$\overline{F} = F.\hat{e}_{AB}$$

$$\overline{F} = F\left(\frac{(x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}\right)$$

$$\overline{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k} \qquad \text{........... Force in vector form}$$

Note: **i, j** *and* **k** *printed in bold type denote unit vectors along the x, y and z axis respectively.*

Ex. 9.1 A force of magnitude 650 N passes from P (0, 3, 0) to Q (5, 0, 4). Put this force in vector form.

Solution:

$$\bar{F} = F.\hat{e}_{PQ}$$

$$= 650\left(\frac{5\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{5^2 + 3^2 + 4^2}}\right)$$

$$\bar{F} = 459.6\,\hat{i} - 257.8\,\hat{j} + 367.7\,\hat{k} \text{ N}$$ .... **Ans.**

9.2.2. *Magnitude and direction of force*

Fig. 9.2 shows a force $\bar{F} = F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}$ making angles θ_x, θ_y and θ_z with the x, y and z axis respectively.

Here F_x is the component of force in the x direction. Similarly F_y and F_z are the force components in the y and z direction.

The magnitude of the force is

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

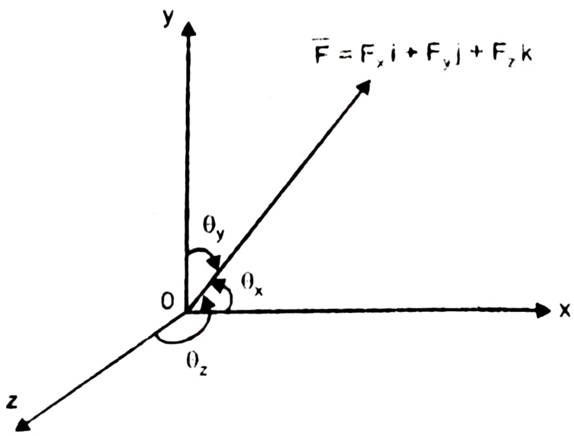

Fig. 9.2

Also $F_x = F \cos \theta_x$

$F_y = F \cos \theta_y$

$F_z = F \cos \theta_z$

Here θ_x, θ_y and θ_z are known as the *force directions*, the value of which lies between 0 and 180. There is an important identity which relates them.

$$\cos^2 \theta_x + \cos^2 \theta_y + \cos^2 \theta_z = 1$$

Ex. 9.2 Determine the magnitude and the directions of the force $\bar{F} = 345\,\hat{i} + 150\,\hat{j} - 290\,\hat{k}$ N

Solution: Magnitude of the force

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

$$F = \sqrt{345^2 + 150^2 + 290^2}$$

$$F = 475 \text{ N}$$ .... **Ans.**

Direction of the force

$$F_x = F\cos\theta_x$$
$$345 = 475\cos\theta_x$$
$$\theta_x = 43.42°$$
.... **Ans.**

$$F_y = F\cos\theta_y$$
$$150 = 475\cos\theta_y$$
$$\theta_y = 71.59°$$
.... **Ans.**

$$F_z = F\cos\theta_z$$
$$-290 = 475\cos\theta_z$$
$$\theta_z = 127.62°$$
.... **Ans.**

Ex.9.3 The direction of a force is given by $\theta_x = 66°$ and $\theta_y = 140°$. If $F_z = -4$ N determine i) θ_z ii) the magnitude of force iii) the other components.

Solution:

Using $\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$

$$\cos^2 66 + \cos^2 140 + \cos^2\theta_z = 1$$
$$\cos^2\theta_z = 0.2477$$
$$\therefore \quad \cos\theta_z = \pm 0.4977$$
$$\therefore \quad \theta_z = 60.14° \quad \text{or} \quad \theta_z = 119.85°$$

Since $F_z = -4$ N it implies that the force component is directed towards the negative direction of the z axis.

$$\therefore \quad \theta_z = 119.85°$$
.... **Ans.**

using
$$F_z = F\cos\theta_z$$
$$-4 = F\cos 119.85$$
$$\therefore \quad F = 8.036 \text{ N}$$
.... **Ans.**

using
$$F_y = F\cos\theta_y$$
$$= 8.036\cos 140$$
$$\therefore \quad F_y = 6.156 \text{ N}$$
.... **Ans.**

using
$$F_x = F\cos\theta_x$$
$$= 8.036\cos 66$$
$$\therefore \quad F_x = 3.269 \text{ N}$$
.... **Ans.**

9.2.3. *Moment of a force about a point*

This is a very important operation while dealing with forces. For coplanar forces, moment about a point was the product of the force and the $\perp$ distance. Here if the force is in space the moment calculation requires a vector approach.

Fig. 9.3 shows a force F in space passing through points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) on its line of action. Let C (x_3, y_3, z_3) be the moment centre i.e. the point about which we have to find the moment. The procedure of finding the moment of the force about the point is as follows.

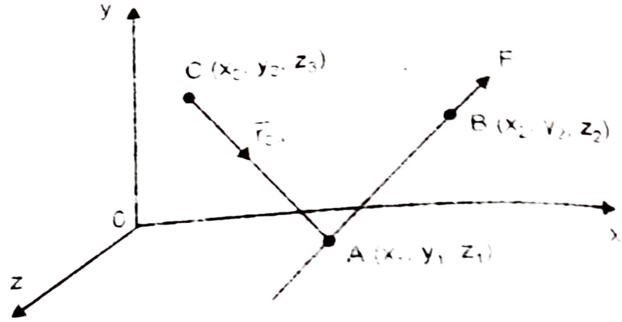

Fig. 9.3

Step 1: Put the force in vector form i.e.

$$\bar{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

Step 2: Find the position vector extending from the moment centre to any point on the force i.e. $\bar{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$

Step 3: Perform the cross product of the position vector and the force vector to get the moment vector i.e.

$$\overline{M}^F_{point} = \bar{r} \times \bar{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

Ex. 9.4 A force of magnitude 50 kN is acting at point A (2, 3, 4) m towards point B (6, –2, –3) m. Find the moment of the given force about a point D (–1, 1, 2) m.

Solution: The force in vector form is

$$\bar{F} = F.e_{AB}$$

$$= 50\left(\frac{4\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}}{\sqrt{4^2 + 5^2 + 7^2}}\right)$$

$$= 21.08\,\mathbf{i} - 26.35\,\mathbf{j} - 36.89\,\mathbf{k} \quad \text{kN}$$

$$\overline{M}^F_D = \bar{r}_{DA} \times \bar{F} \qquad \text{Here } \bar{r}_{DA} = 3\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k} \text{ m}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 2 \\ 21.08 & -26.35 & -36.89 \end{vmatrix}$$

$$\therefore \quad \overline{M}^F_D = -21.08\,\mathbf{i} + 152.8\,\mathbf{j} - 121.2\,\mathbf{k} \quad \text{kNm}$$

.... **Ans.**

9.2.4. *Moment of force about a line*

Moment of force about a line or axis implies finding the projection of the moment vector on the given axis. In other words, it is the component of the moment vector along the given axis.

Fig. 9.4 shows a force F in space passing through points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) on its line of action. To find the moment of the force about line CD, we follow the following steps.

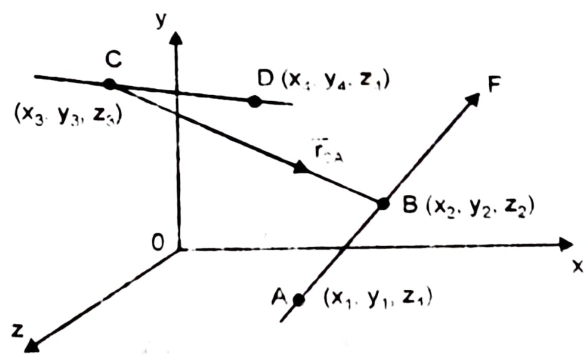

Fig. 9.4

Step 1: Put the force in vector form i.e. $\overline{F}$

Step 2: Find moment of the force about any point on the line i.e. $\overline{M}^F_{point}$

Step 3: Find the unit vector of the line about which we have to find moment i.e $\hat{e}_{line}$

Step 4: Perform the dot product of the moment vector and the unit vector of the line. This gives the magnitude of the moment about the line, i.e.

$$M^F_{line} = \overline{M}^F_{point} \cdot \hat{e}_{line}$$

Step 5: Finally to get the moment of the force about the line in vector form, multiply the magnitude with the unit vector of the line, i.e.

$$\overline{M}^F_{line} = M^F_{point} (\hat{e}_{line})$$

Ex. 9.5 A force of 10 kN acts at a point P (2, 3, 5) m and has its line of action passing through Q (10, -3, 4) m. Calculate moment of this force about an axis passing through ST where S is a point (1, – 10, 3) m and T is (5, -10, 8) m.

Solution: Putting the force in vector form

$$\overline{F} = F.\hat{e}_{PQ}$$

$$= 10\left(\frac{8\hat{i} - 6\hat{j} - \hat{k}}{\sqrt{8^2 + 6^2 + 1}}\right)$$

$$\overline{F} = 7.96\,\hat{i} - 5.97\,\hat{j} - 0.995\,\hat{k} \quad \text{kN}$$

Finding the moment of the force about any point on the axis, let us take point s as the moment centre.

$$\overline{M}_S^F = \bar{r}_{SP} \times \bar{F} \quad \text{where, } \bar{r}_{SP} = \mathbf{i} + 13\,\mathbf{j} + 2\,\mathbf{k} \text{ m}$$

$$\overline{M}_S^F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 13 & 2 \\ 7.96 & -5.97 & -0.995 \end{vmatrix}$$

$$= 0.995\,\mathbf{i} - 16.92\,\mathbf{j} - 109.45\,\mathbf{k} \text{ kNm}$$

Finding the Unit Vector of the axis

$\bar{e}_{ST}$

$$\hat{e}_{ST} = \left(\frac{4\,\mathbf{i} + 5\,\mathbf{k}}{\sqrt{4^2 + 5^2}}\right)$$

$$= 0.625\,\mathbf{i} + 0.78\,\mathbf{k}$$

Now $M^F{}_{ST} = \overline{M}_{ST}^F \cdot \hat{e}_{ST}$

$$= (-0.995\,\mathbf{i} - 16.92\,\mathbf{j} - 109.45\,\mathbf{k}) \cdot (0.625\,\mathbf{i} + 0.78\,\mathbf{k})$$

$$= -86 \text{ kNm}$$

Now $\overline{M}_{ST}^F = M^F{}_{ST}\,(\hat{e}_{ST})$

$$= -86\,[0.625\,\mathbf{i} + 0.78\,\mathbf{k}]$$

$\therefore \quad \overline{M}_{ST}^F = -53.75\,\mathbf{i} - 67.08\,\mathbf{k}$ kNm ....**Ans.**

Ex. 9.6 A force $\bar{F} = -120\,\mathbf{i} + 30\,\mathbf{j} + 40\,\mathbf{k}$ N acts at a point C (4, – 3, – 4) m. Find its moment about a line MP lying in the y–z plane and making an angle of 60° with the positive y axis. The point M has co-ordinates (0, 2, 3) m.

Solution: Given $\bar{F} = -120\,\mathbf{i} + 30\,\mathbf{j} + 40\,\mathbf{k}$ N

The line MP passes through M (0, 2, 3) m and lies in the y-z plane making 60° with the y axis. The angles it makes with positive x, y and z axis are therefore 90°, 60° and 30° respectively.

The unit vector of the line MP is

$$\bar{e}_{MP} = \cos\theta_x\,\mathbf{i} + \cos\theta_y\,\mathbf{j} + \cos\theta_z\,\mathbf{k}$$

$$= \cos 90\,\mathbf{i} + \cos 60\,\mathbf{j} + \cos 30\,\mathbf{k}$$

$$= 0.5\,\mathbf{j} + 0.866\,\mathbf{k}$$

Finding moment of the force about point M (0, 2, 3) m on the line MP.

$$\overline{M}_M^F = \bar{r}_{MC} \times \bar{F} \quad \text{where, } \bar{r}_{MC} = 4\,\mathbf{i} - 5\,\mathbf{j} - 7\,\mathbf{k} \text{ m}$$

$$= (4\,\mathbf{i} - 5\,\mathbf{j} - 7\,\mathbf{k}) \times (-120\,\mathbf{i} + 30\,\mathbf{j} + 40\,\mathbf{k})$$

$$= 10\,\mathbf{i} + 680\,\mathbf{j} - 480\,\mathbf{k} \text{ Nm}$$

Now $M^F_{MP} = \overline{M}^F_M . \hat{e}_{MP}$

$= (10\,\hat{i} + 680\,\hat{j} - 480\,\hat{k}).(0.5\,\hat{j} + 0.866\,\hat{k})$

$= -75.68$ Nm

Now $\overline{M}^F_{MP} = M^F_{MP}\,(\hat{e}_{MP})$

$= -75.68 \times (0.5\,\hat{j} + 0.866\,\hat{k})$

$\overline{M}^F_{MP} = -37.84\,\hat{j} - 65.54\,\hat{k}$ Nm ....**Ans.**

9.2.5. *Vector Components of force*

Vector component of a force along any axis represents the component of the force along that axis. If F is the given force then F' is said to be its vector component. Fig. 9.5 shows a force F passing through points A and B. To find the vector component of F along the axis CD, the following steps are required to be followed.

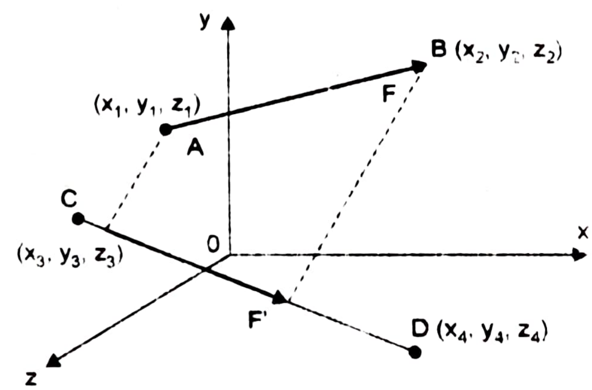

Fig. 9.5

Step 1: Put the force in vector form i.e. $\overline{F}$

Step 2: Find unit vector of the line along which the vector component is required i.e. $\hat{e}_{Line}$.

Step 3: Perform the dot product of the force vector and the unit vector of the line to get the magnitude of the force component, i.e.

$$F' = \overline{F} . \hat{e}_{Line}$$

Step 4: To get the vector component, multiply the magnitude with the unit vector of the line, i.e

$$\overline{F}' = F'\,(\hat{e}_{Line})$$

Ex. 9.7 A force $\overline{F} = 4\,\hat{i} - 3\,\hat{j} + 8\,\hat{k}$ N acts at a point A (2, -1, 3) m. Find the vector component of $\overline{F}$ along the line AB. The co-ordinates of point B are (3, 2, 3) m.

Solution : The force is already in the vector form

i.e $\overline{F} = 4\,\hat{i} - 3\,\hat{j} + 8\,\hat{k}$ N

Unit vector of line AB

$$\hat{e}_{AB} = \frac{\mathbf{i} + 3\,\mathbf{j}}{\sqrt{1^2 + 3^2}}$$

$$= 0.316\,\mathbf{i} + 0.943\,\mathbf{j}$$

The scalar component $\quad F' = F \,.\, \hat{e}_{AB}$

$$= (4\,\mathbf{i} - 3\,\mathbf{j} + 8\,\mathbf{k}).(0.316\,\mathbf{i} + 0.948\,\mathbf{j})$$

$$= -\,1.58 \text{ N}$$

The vector component of the force,

$$\overline{F}' = F' \, (\hat{e}_{AB})$$

$$= -\,1.58\,(0.316\,\mathbf{i} + 0.948\,\mathbf{j})$$

$$\overline{F}' = -\,0.5\,\mathbf{i} - 1.5\,\mathbf{j} \quad \text{N}$$

……… **Ans.**

9.3 Resultant of Concurrent Space Force System

Resultant of a concurrent space force system is a single force $\overline{R}$, which acts through the point of concurrence. Fig. 9.6 (a) shows a concurrent system at point P. The resultant of the system is shown in Fig. 9.6 (b) and calculated as,

$$\overline{R} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3 + \overline{F}_4$$

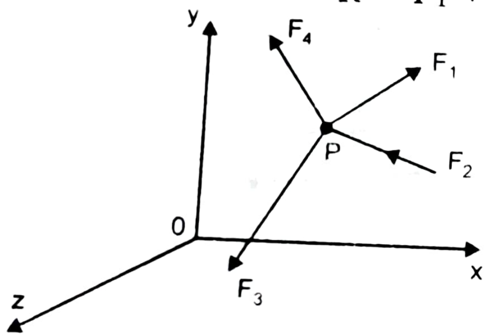

Fig. 9.6 (a)

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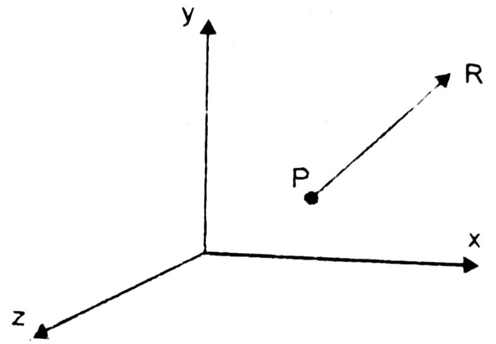
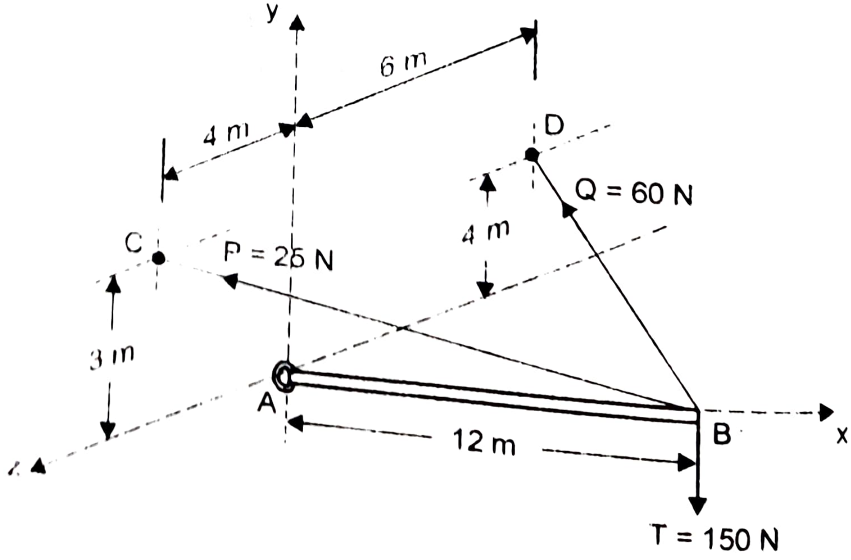


Fig. 9.6 (b)

Ex. 9.8 Three forces P, Q and T act at point B. Find the resultant of these forces.



Solution: The co-ordinates are found out, B (12, 0, 0), C (0, 3, 4) and D (0, 4, -6).

The given system is a concurrent space force system of three forces. Putting the forces in vector form.

$$\overline{P} = P.\hat{e}_{BC}$$

$$= 25\left(\frac{-12\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k}}{\sqrt{12^2 + 3^2 + 4^2}}\right)$$

$$= -23.07\,\mathbf{i} + 5.77\,\mathbf{j} + 7.69\,\mathbf{k} \text{ N}$$

$$\overline{Q} = Q\,.\,\hat{e}_{BD}$$

$$= 60\left(\frac{-12\,\mathbf{i} + 4\,\mathbf{j} + 6\,\mathbf{k}}{\sqrt{12^2 + 4^2 + 6^2}}\right)$$

$$= -51.42\,\mathbf{i} + 17.14\,\mathbf{j} - 25.71\,\mathbf{k} \text{ N}$$

$\overline{T} = -150\,\mathbf{j}$ N .... since it is parallel to the y axis and is directed in the – ve direction.

Now the resultant force $\overline{R} = \overline{P} + \overline{Q} + \overline{T}$

$$\overline{R} = (-23.07\,\mathbf{i} + 5.77\,\mathbf{j} + 7.69\,\mathbf{k}) + (-51.42\,\mathbf{i} + 17.14\,\mathbf{j} - 25.71\,\mathbf{k}) + (-150\,\mathbf{j})$$

$\overline{R} = -74.49\,\mathbf{i} - 127.09\,\mathbf{j} - 18.02\,\mathbf{k}$ N .... **Ans.**

Ex. 9.9 The lines of actions of three forces concurrent at origin O pass respectively through point A (– 1, 2, 4), B (3, 0 – 3), C (2, – 2, 4). Force $F_1 = 40$ N passes through A, $F_2 = 10$ N passes through B, $F_3 = 30$ N passes through C. Find magnitude and direction of their resultant.

Solution: The given system is a concurrent space force system of three forces.

Putting the forces in vector form.

$$\overline{F}_1 = F_1.\hat{e}_{OA}$$

$$= 40\left(\frac{-\mathbf{i} + 2\,\mathbf{j} + 4\,\mathbf{k}}{\sqrt{1^2 + 2^2 + 4^2}}\right)$$

$$= -8.73\,\mathbf{i} + 17.45\,\mathbf{j} + 34.91\,\mathbf{k} \text{ N}$$

$$\overline{F}_2 = F_2.\hat{e}_{OB}$$

$$= 10\left(\frac{3\,\mathbf{i} - 3\,\mathbf{k}}{\sqrt{3^2 + 3^2}}\right)$$

$$= 7.07\,\mathbf{i} - 7.07\,\mathbf{k} \text{ N}$$

$$\overline{F}_3 = F_3.\hat{e}_{OC}$$

$$= 30\left(\frac{2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}}{\sqrt{2^2 + 2^2 + 4^2}}\right)$$

$$\overline{F}_3 = 12.25\,\mathbf{i} - 12.25\,\mathbf{j} + 24.5\,\mathbf{k} \quad \text{N}$$

The resultant force $\overline{R} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3$

$$\overline{R} = (-8.73\,\mathbf{i} + 17.45\,\mathbf{j} + 34.91\,\mathbf{k}) + (7.07\,\mathbf{i} - 7.07\,\mathbf{k}) + (12.25\,\mathbf{i} - 12.25\,\mathbf{j} + 24.5\,\mathbf{k})$$

$\therefore \quad \overline{R} = 10.59\,\mathbf{i} + 5.2\,\mathbf{j} + 52.34\,\mathbf{k}$ N .... **Ans.**

Magnitude and direction of the resultant

$$R = \sqrt{10.59^2 + 5.2^2 + 52.34^2}$$

$\therefore \quad R = 53.65$ N .... **Ans.**

$$R_x = R \cos\theta_x$$

$$10.59 = 53.65 \cos\theta_x$$

$\therefore \quad \theta_x = 78.61°$ .... **Ans.**

$$R_y = R \cos\theta_y$$

$$5.2 = 53.65 \cos\theta_y$$

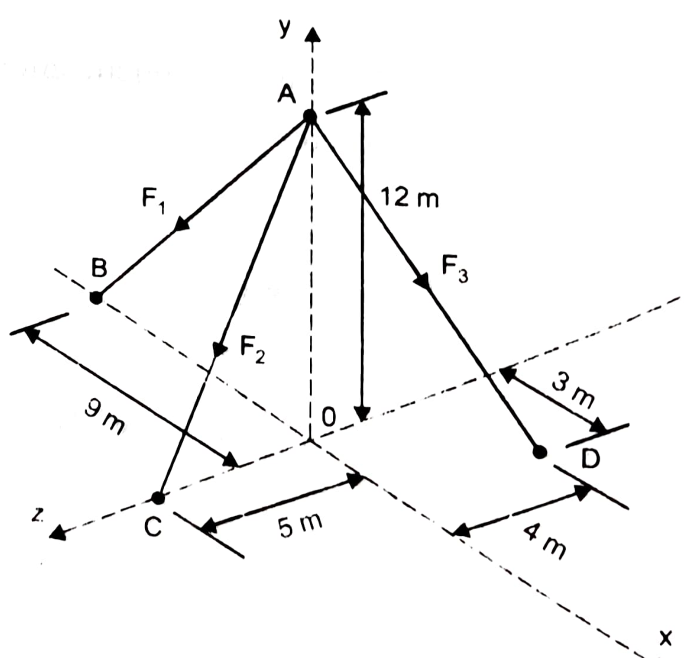
$\therefore \quad \theta_y = 84.44°$ .... **Ans.**

$$R_z = R \cos\theta_z$$

$$52.34 = 53.65 \cos\theta_z$$

$\therefore \quad \theta_z = 12.68°$ .... **Ans.**

Ex. 9.10 The resultant of the three concurrent space forces at A is $\overline{R} = -788\,\mathbf{j}$ N. Find the magnitude of F_1, F_2 and F_3 force.



Solution: This is a concurrent space force system of three forces. To put the forces in vector form, we need the coordinates of the points through which the forces pass.

From the figure the coordinates are, A (0, 12, 0) m, B (– 9, 0, 0) m, C (0, 0, 5) m and D (3, 0, – 4) m.

Putting the forces in vector form.

$$\bar{F}_1 = F_1.\hat{e}_{AB}$$

$$= F_1\left(\frac{-9\,\mathbf{i} - 12\,\mathbf{j}}{\sqrt{9^2 + 12^2}}\right)$$

$$= F_1(-0.6\,\mathbf{i} - 0.8\,\mathbf{j})\ \text{N}$$

$$\bar{F}_2 = F_2.\hat{e}_{AC}$$

$$= F_2\left(\frac{-12\,\mathbf{j} + 5\,\mathbf{k}}{\sqrt{12^2 + 5^2}}\right)$$

$$= F_2(-0.923\,\mathbf{j} + 0.385\,\mathbf{k})\ \text{N}$$

$$\bar{F}_3 = F_3.\hat{e}_{AD}$$

$$= F_3\left(\frac{3\,\mathbf{i} - 12\,\mathbf{j} - 4\,\mathbf{k}}{\sqrt{3^2 + 12^2 + 4^2}}\right)$$

$$= F_3(0.231\,\mathbf{i} - 0.923\,\mathbf{j} - 0.308\,\mathbf{k})\ \text{N}$$

The resultant of the forces at A is $\bar{R} = -788\,\mathbf{j}$ N

$$\bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3$$

$$0\,\mathbf{i} - 788\,\mathbf{j} + 0\,\mathbf{j} = F_1(-0.6\,\mathbf{i} - 0.8\,\mathbf{j}) + F_2(-0.923\,\mathbf{j} + 0.385\,\mathbf{k}) + F_3(0.231\,\mathbf{i} - 0.923\,\mathbf{j} - 0.308\,\mathbf{k})$$

$$0\,\mathbf{i} - 788\,\mathbf{j} + 0\,\mathbf{j} = (-0.6\,F_1 + 0.231\,F_3)\,\mathbf{i} + (-0.8\,F_1 - 0.923\,F_2 - 0.923\,F_3)\,\mathbf{j} + (0.385\,F_2 - 0.308\,F_3)\,\mathbf{k}$$

Equating the coefficients

$$-0.6\,F_1 - 0.231\,F_3 = 0 \qquad \text{.................(1)}$$

$$-0.8\,F_1 - 0.923\,F_2 - 0.923\,F_3 = -788 \qquad \text{.................(2)}$$

$$0.385\,F_2 - 0.308\,F_3 = 0 \qquad \text{.................(3)}$$

Solving equations (1), (2) and (3) we get,

$F_1 = 154$ N, $F_2 = 320$ N, $F_3 = 400$ N .... **Ans.**

9.4 Resultant of Parallel Space Force System

The resultant of a parallel space force system is a single force $\overline{R}$ which acts parallel to the force system. The location of the resultant can be found out using Varignon's theorem.

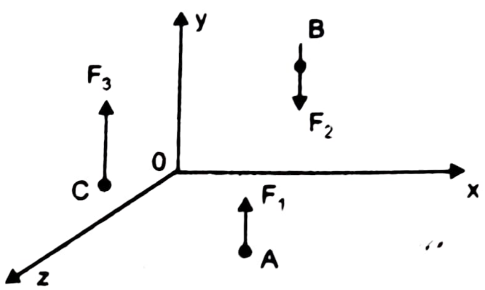


Fig. 9.7 (a)

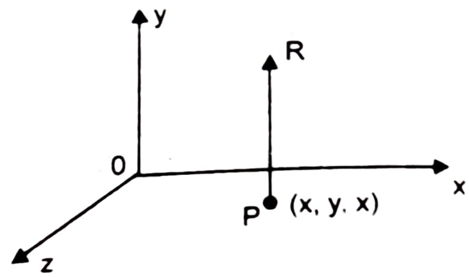


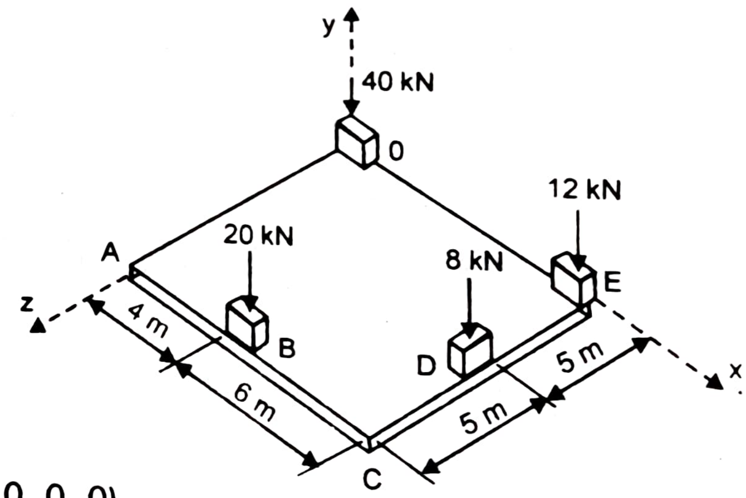
Fig. 9.7 (b)

Figure 9.7 (a) shows a parallel system of three forces F_1, F_2 and F_3. The resultant of the system is shown in figure 9.7 (b) and is calculated as,

$$\overline{R} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3$$

The resultant acts at P. The coordinates (x, y, z) of the point P can be calculated by using Varignon's theorem, the moments for which can be taken about any convenient point like point O. The equation of Varignon's theorem for space forces is $\Sigma \overline{M}_O^F = \overline{M}_O^R$

Ex.9.11 A square foundation mat supports the four columns as shown. Determine the magnitude and point of application of the resultant of the four loads.



Solution: The given system is a parallel force system of four forces. The co-ordinates through which the forces act are,

O (0, 0, 0), B (4, 0, 10), D (10, 0, 5), E (10, 0, 0)

Putting the forces in vector form

Let $F_1 = 20$ kN

$\therefore$ $\overline{F}_1 = -20\,\mathbf{j}$ kN since it is parallel to y axis and directed downwards.

Similarly

Let $F_2 = 8$ kN

$\therefore$ $\overline{F}_2 = -8\,\mathbf{j}$ kN

Let $F_3 = 12$ kN

$\therefore$ $\overline{F}_3 = -12\,\mathbf{j}$ kN

Let $\quad F_4 = 40$ kN

$\therefore \quad \overline{F}_4 = -40\,\mathbf{j}$ kN

The resultant $\quad \overline{R} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3 + \overline{F}_4$

$$\overline{R} = (-20\,\mathbf{j}) + (-8\,\mathbf{j}) + (-12\,\mathbf{j}) + (-40\,\mathbf{j})$$

$$\therefore \quad \overline{R} = -80\,\mathbf{j} \text{ kN}$$

.... **Ans.**

Point of application of the resultant:

Let the resultant act at a point P (x, 0, z) in the plane of the foundation mat. To use Varignon's theorem, we need to find the moments of all the forces and also of the resultant about point O.

$$\overline{M}_O^{F_1} = \overline{r}_{OB} \times \overline{F}_1 \qquad \text{where } \overline{r}_{OB} = 4\,\mathbf{i} + 10\,\mathbf{k} \text{ m}$$

$$= (4\,\mathbf{i} + 10\,\mathbf{k}) \times (-20\,\mathbf{j})$$

$$= 200\,\mathbf{i} - 80\,\mathbf{k} \text{ kNm}$$

$$\overline{M}_O^{F_2} = \overline{r}_{OD} \times \overline{F}_2 \quad \text{where } \overline{r}_{OD} = 10\,\mathbf{i} + 5\,\mathbf{k}$$

$$= (10\,\mathbf{i} + 5\,\mathbf{k}) \times (-8\,\mathbf{j})$$

$$= 40\,\mathbf{i} - 80\,\mathbf{k} \text{ kNm}$$

$$\overline{M}_O^{F_3} = \overline{r}_{OE} \times \overline{F}_3 \qquad \text{where } \overline{r}_{OE} = 10\,\mathbf{i} \text{ m}$$

$$= (10\,\mathbf{i}) \times (-12\,\mathbf{j})$$

$$= -120\,\mathbf{k} \text{ kNm}$$

$\overline{M}_O^{F_4} = 0$ ----------- since F_4 passes through O

$$\overline{M}_O^{R} = \overline{r}_{OP} \times \overline{R} \qquad \text{where } \overline{r}_{OP} = x\,\mathbf{i} + z\,\mathbf{k}$$

$$= (x\,\mathbf{i} + z\,\mathbf{k}) \times (-80\,\mathbf{j})$$

$$= (80\,z)\,\mathbf{i} + (-80\,x)\,\mathbf{k} \text{ kNm}$$

Using Varignon's theorem

$$\Sigma\,\overline{M}_O^{F} = \Sigma\,\overline{M}_O^{R}$$

$$\overline{M}_O^{F_1} + \overline{M}_O^{F_2} + \overline{M}_O^{F_3} + \overline{M}_O^{F_4} = \overline{M}_O^{R}$$

$$(200\,\mathbf{i} - 80\,\mathbf{k}) + (40\,\mathbf{i} - 80\,\mathbf{k}) + (-120\,\mathbf{k}) = (80\,z)\,\mathbf{i} + (-80x)\,\mathbf{k}$$

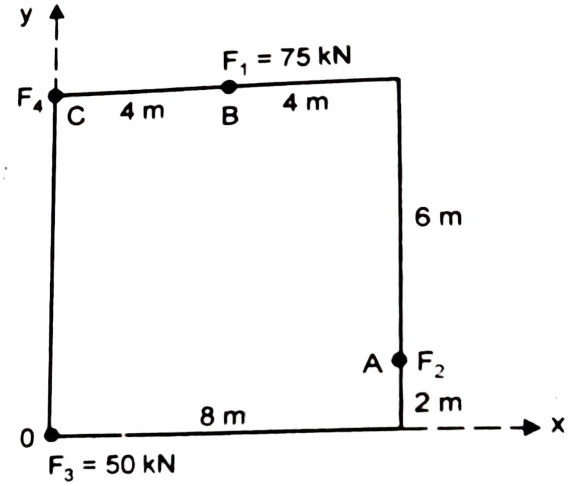
$$240\,\mathbf{i} - 280\,\mathbf{k} = (80\,z)\,\mathbf{i} + (-80\,x)\,\mathbf{k}$$

equating the coefficients

$$240 = 80\,z$$
$$z = 3\text{ m}$$
$$-280 = -80\,x$$
$$x = 3.5\text{ m}$$

$\therefore$ The resultant $\bar{R} = -80\,\mathbf{j}$ kN passes through point P (3.5, 0, 3) m .... **Ans.**

Ex. 9.12 A square foundation is acted upon by four column loads. The resultant of the loads acts at the centre of the foundation. Find the magnitude of forces F_2 and F_4. All the forces point in the –ve z direction.



Solution: The given system is a parallel force system of four forces. The co-ordinates through which the forces act are,
O (0, 0, 0), A (8, 2, 0), B (4, 8, 0), E (0, 8, 0)

Putting the forces in vector form

$$\bar{F}_1 = -75\,\mathbf{k} \text{ kN}$$
$$\bar{F}_2 = -F_2\,\mathbf{k} \text{ kN}$$
$$\bar{F}_3 = -50\,\mathbf{k} \text{ kN}$$
$$\bar{F}_4 = -F_4\,\mathbf{k} \text{ kN}$$

Let $\bar{R}$ be the resultant of the four forces.
It is given $\bar{R}$ acts at the centre G.

$\therefore$ G = (4, 4, 0)

Let us use Varignon's theorem to find the unknown forces F_2 and F_4. For this we need to find the moments of all the forces about any convenient point. Let us take G as the moment centre.

$$\bar{M}_G^{F_1} = \bar{r}_{GB} \times \bar{F}_1 \quad \text{where } \bar{r}_{GB} = 4\,\mathbf{j}\text{ m}$$
$$= (4\,\mathbf{j}) \times (-75\,\mathbf{k})$$
$$= -300\,\mathbf{i} \text{ kNm}$$

$$\bar{M}_G^{F_2} = \bar{r}_{GA} \times \bar{F}_2 \quad \text{where } \bar{r}_{GA} = 4\,\mathbf{i} - 2\,\mathbf{j}\text{ m}$$
$$= (4\,\mathbf{i} - 2\,\mathbf{j}) \times (-F_2\,\mathbf{k})$$
$$= 4\,F_2\,\mathbf{j} + 2\,F_2\,\mathbf{i} \text{ kNm}$$

$$\overline{M}_G^{F_3} = \bar{r}_{GO} \times \overline{F}_3 \quad \text{where } \bar{r}_{GO} = -4\,\mathbf{i} - 4\,\mathbf{j}\ \text{m}$$

$$= (-4\,\mathbf{i} - 4\,\mathbf{j}) \times (-50\,\mathbf{k})$$

$$= -200\,\mathbf{j} + 200\,\mathbf{i}\ \ \text{kNm}$$

$$\overline{M}_G^{F_4} = \bar{r}_{GC} \times \overline{F}_4 \quad \text{where } \bar{r}_{GC} = -4\,\mathbf{i} + 4\,\mathbf{j}\ \text{m}$$

$$= (-4\,\mathbf{i} + 4\,\mathbf{j}) \times (-F_4\,\mathbf{k})$$

$$= -4\,F_4\,\mathbf{j} - 4\,F_4\,\mathbf{i}\ \ \text{kNm}$$

Since resultant $\overline{R}$ passes through G, $\overline{M}_G^{R} = 0$

Using Varignon's theorem

$$\Sigma\, \overline{M}_G^{F} = \overline{M}_G^{R}$$

$$(-300\,\mathbf{i}) + (4\,F_2\,\mathbf{j} + 2\,F_2\,\mathbf{i}) + (-200\,\mathbf{j} + 200\,\mathbf{i}) + (-4\,F_4\,\mathbf{j} - 4\,F_4\,\mathbf{i}) = 0$$

$$(2\,F_2 - 4\,F_4 - 100)\,\mathbf{i} + (4\,F_2 - 4\,F_4 - 200)\,\mathbf{j} = 0$$

i.e. $2\,F_2 - 4\,F_4 - 100 = 0$ .... (1)

$4\,F_2 - 4\,F_4 - 200 = 0$ .... (2)

Solving equations (1) and (2) we get

$F_2 = 50$ kN and $F_4 = 0$ .... **Ans.**

9.5 Resultant of General Space Force system

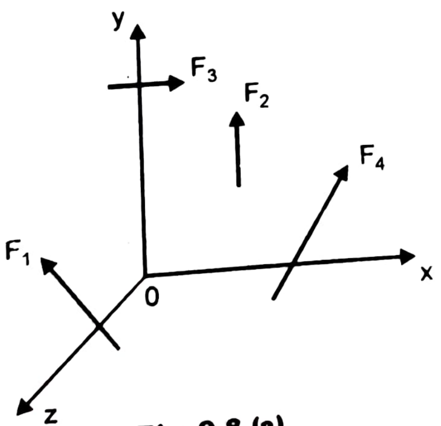


Fig. 9.8 (a)

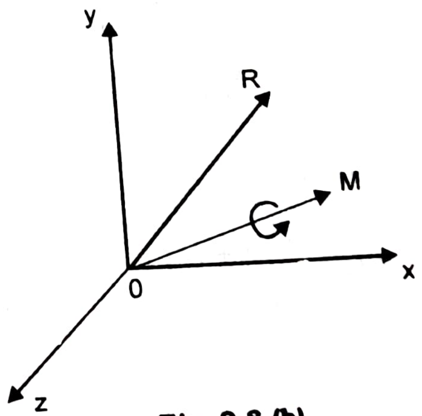
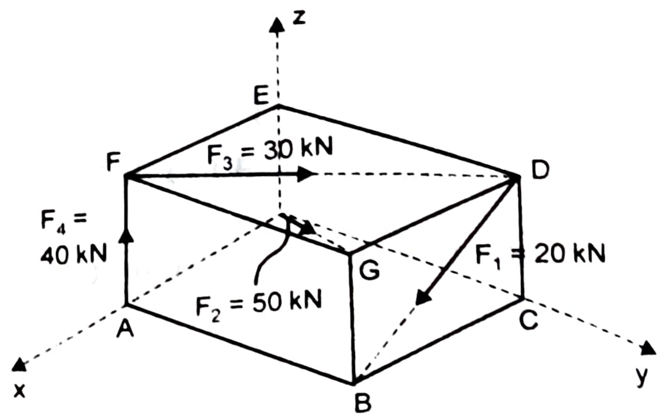


Fig. 9.8 (b)

A general space force system is neither a concurrent nor a parallel system. The resultant of such a system is a single force R and a moment M at any desired point. Since the resultant contains one force and one moment, it is also known as a *Force Couple System*. Fig.9.8 (a) shows a general system of four forces F_1, F_2, F_3 and F_4. If it is desired to have the resultant at point O, then as per figure 9.8 (b) the single force $\overline{R} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3 + \overline{F}_4$

and the single moment $\overline{M} = \overline{M}_O^{F_1} + \overline{M}_O^{F_2} + \overline{M}_O^{F_3} + \overline{M}_O^{F_4}$

Ex. 9.13 Determine the resultant force and couple moment about the origin of the force system shown in figure.
L (OA) = 4 m, L (OC) = 5 m, L(OE) = 3 m



Solution: The given system is a general system of four forces. The co-ordinates of the various points through which the force passes are, A (4, 0, 0), B (4, 5, 0), C (0, 5, 0), D (0, 5, 3), F (4, 0, 3), G (4, 5, 3) and O (0, 0, 0).

Putting the forces in vector form

$$\overline{F}_1 = F_1.\hat{e}_{DB}$$

$$= 20\left(\frac{4\mathbf{i} - 3\mathbf{k}}{\sqrt{4^2 + 3^2}}\right)$$

$$= 16\,\mathbf{i} - 12\,\mathbf{k} \text{ kN}$$

$$\overline{F}_2 = F_2.\hat{e}_{OG}$$

$$= 50\left(\frac{4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}}{\sqrt{4^2 + 5^2 + 3^2}}\right)$$

$$= 28.28\,\mathbf{i} + 35.35\,\mathbf{j} + 21.21\,\mathbf{k} \text{ kN}$$

$$\overline{F}_3 = F_3.\hat{e}_{FD}$$

$$= 30\left(\frac{-4\mathbf{i} + 5\mathbf{j}}{\sqrt{4^2 + 5^2}}\right)$$

$$= -18.74\,\mathbf{i} + 23.42\,\mathbf{j} \text{ kN}$$

$\overline{F}_4 = 40\,\mathbf{k}$ kN since the force is parallel to z axis.

The resultant force $\overline{R} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3 + \overline{F}_4$

$$\overline{R} = (16\,\mathbf{i} - 12\,\mathbf{k}) + (28.28\,\mathbf{i} + 35.35\,\mathbf{j} + 21.21\,\mathbf{k}) + (-18.74\,\mathbf{i} + 23.42\,\mathbf{j}) + (40\,\mathbf{k})$$

$$\therefore\ \overline{R} = 25.54\,\mathbf{i} + 58.77\,\mathbf{j} + 49.21\,\mathbf{k} \text{ kN}$$

.........**Ans.**

Taking moment of all forces about the specified point, which is the origin.

$\overline{M}_O^{F_1} = \bar{r}_{OD} \times \overline{F}_1$ where $\bar{r}_{OD} = 5\,\mathbf{j} + 3\,\mathbf{k}$ m

$= (5\,\mathbf{j} + 3\,\mathbf{k}) \times (16\,\mathbf{i} - 12\,\mathbf{k})$

$= -60\,\mathbf{i} + 48\,\mathbf{j} - 80\,\mathbf{k}$ kNm

$\overline{M}_O^{F_2} = 0$ since F_2 passes through O

$\overline{M}_O^{F_3} = \bar{r}_{OD} \times \overline{F}_3$ where $\bar{r}_{OD} = 5\,\mathbf{j} + 3\,\mathbf{k}$ m

$= (5\,\mathbf{i} + 3\,\mathbf{k}) \times (-18.74\,\mathbf{i} + 23.42\,\mathbf{j})$

$= -70.26\,\mathbf{i} - 56.22\,\mathbf{j} + 93.7\,\mathbf{k}$ kNm

$\overline{M}_O^{F_4} = \bar{r}_{OA} \times \overline{F}_4$ where $\bar{r}_{OA} = 4\,\mathbf{i}$ m

$= (4\,\mathbf{i}) \times (40\,\mathbf{k})$

$= -160\,\mathbf{j}$ kNm

The resultant moment at the origin is

$\overline{M} = \overline{M}_O^{F_1} + \overline{M}_O^{F_2} + \overline{M}_O^{F_3} + \overline{M}_O^{F_4}$

$= (-60\,\mathbf{i} + 48\,\mathbf{j} - 80\,\mathbf{k}) + 0 + (-70.26\,\mathbf{i} - 56.22\,\mathbf{j} + 93.7\,\mathbf{k}) + (-160\,\mathbf{j})$

$\therefore\ \overline{M} = -130.26\,\mathbf{i} - 168.2\,\mathbf{j} + 13.7\,\mathbf{k}$ kNm .... **Ans.**

The resultant force and couple moment at the origin is

$\overline{R} = 25.54\,\mathbf{i} + 58.77\,\mathbf{j} + 49.21\,\mathbf{k}$ kN

$\overline{M} = -130.26\,\mathbf{i} - 168.2\,\mathbf{j} + 13.7\,\mathbf{k}$ kNm .... **Ans.**

Ex. 9.14 Determine the resultant and resultant couple moment at a point A (3, 1, 2) m of the following force system

$\overline{F}_1 = 5\,\mathbf{i} + 8\,\mathbf{k}$ N acting at a point B (8, 3, 1) m

$\overline{F}_2 = 3\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k}$ N acting at O (0, 0, 0) m

$\overline{M}_1 = 12\,\mathbf{i} - 20\,\mathbf{j} + 9\,\mathbf{k}$ Nm

Solution: The given system is a general space force system of two forces and one couple. The forces and moments are already in the vector form.

$\overline{F}_1 = 5\,\mathbf{i} + 8\,\mathbf{k}$ N

$\overline{F}_2 = 3\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k}$ N

$\therefore$ The resultant force $\bar{R} = \bar{F}_1 + \bar{F}_2$

$$\bar{R} = (5\,\mathbf{i} + 8\,\mathbf{k}) + (3\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k})$$

$$\bar{R} = 8\,\mathbf{i} + 2\,\mathbf{j} + 4\,\mathbf{k} \text{ N}$$

.... **Ans.**

To find the resultant moment at point A

Taking moment of all forces about point A

$$\overline{M}_A^{F_1} = \bar{r}_{AB} \times \bar{F}_1 \qquad \text{where } \bar{r}_{AB} = 5\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k} \text{ m}$$

$$= (5\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}) \times (5\,\mathbf{i} + 8\,\mathbf{k})$$

$$= 16\,\mathbf{i} - 45\,\mathbf{j} - 10\,\mathbf{k} \text{ Nm}$$

$$\overline{M}_A^{F_2} = \bar{r}_{AO} \times \bar{F}_2 \qquad \text{where } \bar{r}_{AO} = -3\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k} \text{ m}$$

$$= (-3\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}) \times (3\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k})$$

$$= 8\,\mathbf{i} - 18\,\mathbf{j} - 3\,\mathbf{k} \text{ Nm}$$

$\therefore$ The resultant moment $\overline{M} = \overline{M}_A^{F_1} + \overline{M}_A^{F_2} + \overline{M}_1$

$$= (16\,\mathbf{i} - 45\,\mathbf{j} - 10\,\mathbf{k}) + (8\,\mathbf{i} - 18\,\mathbf{j} - 3\,\mathbf{k}) + (12\,\mathbf{i} - 20\,\mathbf{j} + 9\,\mathbf{k})$$

$$= 36\,\mathbf{i} - 83\,\mathbf{j} - 4\,\mathbf{k} \text{ Nm}$$

.........**Ans.**

The resultant force and couple moment at the point A is

$$\bar{R} = 8\,\mathbf{i} + 2\,\mathbf{j} + 4\,\mathbf{k} \text{ N}$$

$$\overline{M} = 36\,\mathbf{i} - 83\,\mathbf{j} - 4\,\mathbf{k} \text{ Nm}$$

.........**Ans.**