Fourier Series

* Dirichlet's Conditions :-

Consider a single valued function $f(x)$ in interval $(a, a+2L)$ which satisfies below conditions is known as Dirichlet's conditions.

① $f(x)$ is defined in interval $(a, a+2L)$ & $f(x) = f(x+2L)$

② $f(x)$ is continuous function OR has finite number of discontinuites in interval $(a, a+2L)$

③ $f(x)$ has no maxima or minima OR has finite numbers of maxima or minima

Fourier - Euler's formula

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \; ; \quad a < x < \alpha + 2\pi$$

OR

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \longrightarrow$$ [Fourier series in interval $(\alpha, \alpha+2L)$]

$$f(x) = \frac{1}{L} \int_a^{a+2L} f^2(x)\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \longrightarrow$$ [Parseval's Identity Use whenever you need square of summation series]

where,

$$a_0 = \frac{1}{L}\int_a^{a+2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier Coefficients

• Formulas

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

Where

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$\boxed{0, 2\pi}$

$$f(x) = \frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$\boxed{(-\pi, \pi) \text{ or } (-L, L)}$

Check for even or odd

[L or π both works]

Even	Odd
$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$	$a_0 = 0$
	$a_n = 0$
$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$	$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$
$b_n = 0$	

- Half Range Series

$0 \le x \le 2$

$\hookrightarrow L = 2$

Half range cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Half Range Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where, $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

- Complex form

$f(x)$ in interval $(a, a+2L)$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{L}}$$

where, $C_n = \frac{1}{2}(a_n - ib_n)$ & $C_0 = \frac{a_0}{2}$

$= \frac{1}{2L}\int_0^{a+2L} f(x)\, e^{-\frac{in\pi x}{L}}\,dx$

$C_{-n} = \frac{1}{2}(a_n + ib_n)$

Q] Fourier series in interval $(0, 2\pi)$ →

Obtain F.S. for $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 \leq x \leq 2\pi$ & $f(x+2\pi) = f(x)$

Deduce. (i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

(ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

(iv) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$

→ $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2\,dx$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = \frac{-1}{12\pi}(-\pi^3 - \pi^3)$$

$\therefore$ $\boxed{a_0 = \frac{\pi^2}{6}}$

Remember!!!

When period = $(0, 2\pi)$ or $(-\pi, \pi)$

$$\cos\left(\frac{n\pi x}{L}\right) = \cos(nx)$$

$$\sin\left(\frac{n\pi x}{L}\right) = \sin(nx)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot \cos nx \, dx$$

$$a_n = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cdot \cos nx \, dx$$

ILATE

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \int \cos nx - \int \frac{d}{dx}(\pi - x)^2 \int \cos nx \, dx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin(nx)}{n} - \int 2(\pi - x) \cdot (-1) \frac{\sin(nx)}{n}\right]_0^{2\pi}$$

ILATE

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin(nx)}{n} - \left[\int -2(\pi - x) \frac{\sin(nx)}{n}\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin(nx)}{n} - \left[-2(\pi - x) \int \frac{\sin(nx)}{n} + \int \frac{d}{dx}(-2(\pi - x)) \int \frac{\sin(nx)}{n}\right] dx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin(nx)}{n} - \left(-2(\pi - x) \frac{-\cos(nx)}{n^2}\right) + \int 2 \cdot \frac{-\cos(nx)}{n^2} dx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin(nx)}{n} - \left(-2(\pi - x) \frac{-\cos(nx)}{n^2}\right) + 2 \cdot \frac{-\sin(nx)}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \cancel{\frac{\sin(nx)}{n}}^{0} - 2(\pi - x) \frac{\cos(nx)}{n^2} + 2 \cdot \left(-\cancel{\frac{\sin(nx)}{n^3}}^{0}\right)\right]_0^{2\pi}$$

$$= \frac{-1}{2n^2\pi}\left[(\pi - x) \cos(nx)\right]_0^{2\pi} = \frac{-1}{2n^2\pi}(-\pi - \pi) \Rightarrow \boxed{a_n = \frac{1}{n^2}}$$

$$b_n = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cdot \sin(nx)\, dx$$

ILATE

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi-x)^2 \cdot \frac{-\cos nx}{n} - \cancelto{0}{(-2(\pi-x)) \cdot \frac{-\sin nx}{n^2}} + 2 \cdot \frac{\cos(nx)}{n^3} \Bigg]_0^{2\pi}$$

$$= \frac{1}{4\pi n}\left[-(\pi - x)^2 \cos(nx) + \frac{2\cos(nx)}{n^2} \right]_0^{2\pi}$$

$$= \frac{1}{4\pi n}\left[-\pi^2 .(1) + \frac{2}{n^2} - (-\pi^2) . 1 - \frac{2}{n^2} \right]$$

$$\boxed{b_n = 0}$$

$$\therefore \left(\frac{\pi - x}{2}\right)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} . \cos(nx) + 0$$

Putting $x = 0$,

$$\therefore \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$\therefore \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{— } (1)$$

Putting $x = \pi$,

$$\therefore 0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \quad \text{— } (2)$$

Adding ① & ②,

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{1^2} + \cancel{\frac{1}{2^2} - \frac{1}{2^2}} + \frac{1}{3^2} + \frac{1}{3^2} + \cancel{\frac{1}{4^2} - \frac{1}{4^2}} + \cdots$$

$$\frac{18\pi^2}{72} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2}$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Parseval's Identity

$$\frac{1}{L}\int_0^{a+2L} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\therefore \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^4 dx = \frac{\pi^2}{36 \times 2} + \sum_{n=1}^{\infty} \left(\left(\frac{1}{n^2}\right)^2 + 0^2\right)$$

$$\therefore \frac{1}{16\pi}\int_0^{2\pi} (\pi - x)^4\, dx = \frac{\pi^4}{72} + \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \frac{1}{16\pi}\left[\frac{(\pi - x)^5}{-5}\right]_0^{2\pi} = \frac{\pi^4}{72} + \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{40} - \frac{\pi^4}{72} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots$$

9] $f(x) = e^{-x} \quad (0, 2\pi)$

i) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$ ii) $\operatorname{cosech}(\pi)$

→

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx$$

$$a_0 = -\frac{1}{\pi}\left[e^{-x}\right]_0^{2\pi} = \frac{1-e^{2\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos(nx)\,dx$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$a = -1 \quad b = n$

$$\therefore a_n = \frac{1}{\pi}\left[\frac{e^{-x}}{n^2+1}\left(-\cos nx + \cancelto{0}{n\sin nx}\right)\right]_0^{2\pi}$$

$$\therefore a_n = \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}\cdot(-1) - 1(-1)\right] = \frac{1}{n^2+1}\cdot\left(\frac{1-e^{-2\pi}}{\pi}\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin(nx)\,dx$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{e^{-x}}{n^2+1}\left(\cancelto{0}{-\sin(nx)} - n\cos(nx)\right)\right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi(n^2+1)}\left(e^{-2\pi}\cdot(-n) - 1(-n)\right) = \frac{n}{n^2+1}\left(\frac{1-e^{-2\pi}}{\pi}\right)$$

$$\therefore f(x) = \left(\frac{1-e^{-2\pi}}{2\pi}\right) + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\left[\frac{\cos(nx)}{n^2+1} + \frac{n.\sin(nx)}{n^2+1}\right]$$

(i) Putting $x = \pi$

$$e^{-\pi} = \left(\frac{1-e^{-2\pi}}{2\pi}\right) + \left(\frac{1-e^{-2\pi}}{\pi}\right)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\therefore e^{-\pi} = \cancel{\left(\frac{1-e^{-2\pi}}{2\pi}\right)} + \cancel{\left(\frac{-1}{2}\left(\frac{1-e^{-2\pi}}{\pi}\right)\right)} + \left(\frac{1-e^{-2\pi}}{\pi}\right)\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\therefore \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1} = \frac{\pi.e^{-\pi}}{1-e^{-2\pi}} = \frac{\pi}{e^{\pi}-e^{-\pi}}$$

(ii) $$\therefore \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1} = \frac{\pi}{2\left(\frac{e^{\pi}-e^{-\pi}}{2}\right)} = \frac{\pi}{2}\cdot\operatorname{cosech}(\pi)$$

$$\therefore \operatorname{cosech}(\pi) = \frac{2}{\pi}\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{n^2+1}\right)$$

Q] $f(x) = \cos(px)$ in $(0, 2\pi)$ p is not an integer

Deduce that :- 1) $\pi\operatorname{cosec}(p\pi) = \frac{1}{p} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{p+n} + \frac{1}{p-n}\right]$

ii) $\pi\cot(2\pi p) = \frac{1}{2p} + p\sum_{n=1}^{\infty}\frac{1}{p^2-n^2}$

$$\rightarrow a_0 = \frac{1}{\pi}\int_0^{2\pi}\cos(px)\,dx$$

$$a_0 = \frac{1}{\pi}\left[\frac{\sin px}{p}\right]_0^{2\pi} = \frac{1}{p\pi}\left(\sin 2\pi p - \sin 0\right)$$

$$a_0 = \frac{\sin(2p\pi)}{p\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos(px)\,\cos nx \; dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} 2\cos px \;\cos nx \; dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos[(p+n)x] + \cos[(p-n)x] \; dx$$

$$a_n = \frac{1}{2\pi}\left[\frac{\sin[(p+n)x]}{p+n} + \frac{\sin[(p-n)x]}{p-n}\right]_0^{2\pi}$$

But, $\sin(p\pm n)2\pi = \sin 2p\pi \cdot \cos(2n\pi) = \sin(2p\pi)$

$$a_n = \frac{\sin(2p\pi)}{2\pi}\left[\frac{1}{p+n} + \frac{1}{p-n}\right]$$

$$a_n = \frac{\sin(2p\pi)}{2\pi}\left(\frac{2p}{p^2-n^2}\right) \quad //$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \cos(px)\cdot\sin(nx)\,dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} \sin((p+n)x) - \sin((p-n)x)\,dx$$

$$b_n = \frac{1}{2\pi}\left[\frac{\cos((p-n)x)}{p-n} - \frac{\cos((p+n)x)}{p+n}\right]_0^{2\pi}$$

But $\cos(p\pm n)2\pi = \cos 2p\pi$

$$b_n = \frac{1}{2\pi}\left[\frac{\cos(2p\pi)-1}{p-n} - \frac{\cos(2p\pi)-1}{p+n}\right]$$

$$b_n = \frac{\cos(2p\pi)-1}{2\pi}\left(\frac{2n}{p^2-n^2}\right) = \frac{n[\cos(2p\pi)-1]}{\pi(p^2-n^2)} \quad //$$

$$f(x) = \cos(px) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$\cos(px) = \frac{\sin(2p\pi)}{2p\pi} + \frac{p\sin(2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{\cos(nx)}{p^2-n^2} + \frac{\cos(2p\pi)-1}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{p^2-n^2}$$

$$\frac{1}{p^2-n^2} = \frac{1}{2p}\left(\frac{1}{p+n} + \frac{1}{p-n}\right)$$

i) Putting $x=\pi$

$$\cos(p\pi) = \frac{2\sin p\pi\cos p\pi}{2p\pi} + \frac{p\sin(2p\pi)}{\pi}\frac{1}{p}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{p+n} + \frac{1}{p-n}\right) + 0$$

$$\therefore \cancel{\cos p\pi} = \frac{2\sin p\pi\,\cancel{\cos p\pi}}{\cancel{2}p\pi} + \frac{\cancel{2}p\,\sin p\pi\,\cancel{\cos p\pi}}{\pi}\cdot\frac{1}{\cancel{2p}}\cdot\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{p+n} + \frac{1}{p-n}\right) + 0$$

$$\therefore \pi\,\mathrm{cosec}(p\pi) = \frac{1}{p} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{p+n} + \frac{1}{p-n}\right] \quad //$$

ii) Putting $x=2\pi$

$$\cos(2p\pi) = \frac{\sin(2p\pi)}{2p\pi} + \frac{p\sin(2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{1}{p^2-n^2}$$

$$\pi\cot(2p\pi) = \frac{1}{2p} + p\sum_{n=1}^{\infty}\frac{1}{p^2-n^2} \quad //$$