

Laplace Transform

$$\mathcal{L}\{f(t)\} = f(s) = \int_0^\infty e^{-st} f(t) dt$$

(i) Let $f(t) = 1$,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_0^\infty = \left(\frac{0-1}{-s} \right) = \frac{1}{s} \quad (s > 0)$$

(ii) Let $f(t) = t^n$,

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} \cdot t^n dt$$

Let $st = u \Rightarrow sdt = du$

$$\begin{aligned} &= \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \left(\frac{du}{s}\right) = \frac{1}{s^{n+1}} \int_0^\infty u^n e^{-u} du = \frac{1}{s^{n+1}} \int_0^\infty u^{(n+1)-1} e^u du = \\ &= \frac{\Gamma(n+1)}{s^{n+1}} \quad s > 0 \text{ & } n+1 > 0 \quad (\text{Gamma Functions}) \end{aligned}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^q}\right) = \frac{t^q}{\Gamma_q}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\Gamma_n}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

1.	$L(1) = \frac{1}{s}$	2.	$L(e^{at}) = \frac{1}{s-a}$, $L(e^{-at}) = \frac{1}{s+a}$, $L(c^{at}) = \frac{1}{s-a\log c}$
3.	$L(t^n) = \frac{ n+1 }{s^{n+1}} = \frac{n!}{s^{n+1}}$ if $n \in N$	4.	$L(\cos at) = \frac{s}{s^2 + a^2}$
5.	$L(\sin at) = \frac{a}{s^2 + a^2}$	6.	$L(\cosh at) = \frac{s}{s^2 - a^2}$
7.	$L(\sinh at) = \frac{a}{s^2 - a^2}$		

- Change of scale property

$$L[f(t)] = \Phi(s), \text{ then } L[f(at)] = \frac{1}{a} \Phi\left(\frac{s}{a}\right)$$

- Error function (Error function integral or Probability integral)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du$$

- Complementary Error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

Q] $f(t) = \begin{cases} t & \text{when } 0 < t < 4 \\ 5 & \text{when } t > 4 \end{cases}$

$$\rightarrow L[f(t)] = \int_0^\infty e^{-st} f(t) dt = \int_0^4 e^{-st} (t) dt + \int_4^\infty e^{-st} (5) dt$$

$$= \left[t \left(\frac{e^{-st}}{-s} \right) - (1) \left(\frac{e^{-st}}{s^2} \right) \right]_0^4 + 5 \left(\frac{e^{-st}}{-s} \right)_4^\infty$$

$$= \left[4 \left(\frac{e^{-4s}}{-s} \right) - 1 \left(\frac{e^{-4s} - 1}{s^2} \right) \right] + \frac{5}{s} [0 - e^{-4s}]$$

$$L[f(t)] = \frac{1}{s^2} + \left(\frac{1}{s} - \frac{1}{s^2} \right) e^{-4s}$$

Q] $f(t) = \begin{cases} \cos t & \text{when } 0 < t < \pi \\ \sin t & \text{when } t > \pi \end{cases}$

→

$$L[f(t)] = \int_0^\pi e^{-st} \cos(t) dt + \int_\pi^\infty e^{-st} \sin(t) dt$$

$$\left[\begin{aligned} \int e^{ax} \sin(bx) dx &= \frac{e^{ax}}{a^2 + b^2} (\sin(bx) - b \cos(bx)) \\ \int e^{ax} \cos(bx) dx &= \frac{e^{ax}}{a^2 + b^2} (\cos(bx) + b \sin(bx)) \end{aligned} \right]$$

$$= \left[\frac{e^{-st}}{s^2 + 1} (-s \cos(t) + \sin(t)) \right]_0^\pi + \left[\frac{e^{-st}}{s^2 + 1} (-s \sin(t) - \cos(t)) \right]_0^\infty$$

$$= \left[\frac{e^{-\pi s}}{s^2 + 1} (-s(-1)) - \frac{1}{s^2 + 1} (-s) \right] + \left[0 - \frac{e^{-\pi s}}{s^2 + 1} (0 - (-1)) \right]$$

$$= \frac{s \cdot e^{-\pi s}}{s^2 + 1} + \frac{s}{s^2 + 1} - \frac{e^{-\pi s}}{s^2 + 1}$$

$$= \frac{1}{s^2 + 1} [s + (s-1)e^{-\pi s}]$$

Q] (i) $L \{ 3t^4 - 2t^3 + 4e^{-3t} - 2 \sin 5t + 3 \cos 2t \}$

(ii) $L \{ \sin^3 t \}$ (iii) $L \{ \cos^3 t \}$ (iv) $L \{ (t^2 + 4)^2 \}$ (v) $L \{ \sin(\omega t + \alpha) \}$ ω & α being const

→

(i) Using the linearity property

$$L \{ 3t^4 - 2t^3 + 4e^{-3t} - 2 \sin 5t + 3 \cos 2t \}$$

$$= 3L[t^4] - 2L[t^3] + 4[e^{-3t}] - 2L[\sin 5t] + 3L[\cos 2t]$$

$$= 3 \frac{4!}{s^5} - 2 \frac{3!}{s^4} + \frac{4}{s+3} - \frac{2 \cdot 5}{s^2 + 5^2} + \frac{3 \cdot 5}{s^2 + 2^2}$$

$$= \frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2 + 25} + \frac{3s}{s^2 + 4}$$

(iii) $L \{ \sin^3 t \} = L \left[\frac{3}{4} \sin t - \frac{1}{4} \sin 3t \right]$

$$= \frac{3}{4} L\{\sin t\} - \frac{1}{4} L\{\sin 3t\}$$

$$= \frac{3}{4} \frac{1}{s^2+1} - \frac{1}{4} \frac{1}{s^2+9} = \frac{3}{4} \left[\frac{1}{s^2+1} - \frac{1}{s^2+9} \right] = \frac{6}{(s^2+1)(s^2+9)}$$

$$\begin{aligned} \text{(iii)} \quad L\{\cos^3 t\} &= L\left[\frac{3}{4} \cos t + \frac{1}{4} \cos 3t\right] = \frac{3}{4} L\{\cos t\} + \frac{1}{4} L\{\cos 3t\} \\ &= \frac{3}{4} \frac{s}{s^2+1} + \frac{1}{4} \frac{s}{s^2+9} = \frac{3}{4} \left[\frac{s}{s^2+1} + \frac{s}{s^2+9} \right] = \frac{s(s^2+7)}{(s^2+1)(s^2+9)} \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad L\{(t^2+1)^2\} &= L\{t^4 + 2t^2 + 1\} = L[t^4] + 2L[t^2] + L[1] \\ &= \frac{4!}{s^5} + 2 \cdot \frac{2!}{s^3} + \frac{1}{s} = \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s} \end{aligned}$$

$$\text{(V)} \quad \sin(\omega t + \alpha) = \sin \omega t \cos \alpha + \cos \omega t \sin \alpha$$

$$\begin{aligned} L\{\sin(\omega t + \alpha)\} &= L[\sin \omega t \cos \alpha] + L[\cos \omega t \sin \alpha] \\ &= \cos \alpha L[\sin \omega t] + \sin \alpha L[\cos \omega t] \\ &= \cos \alpha \left[\frac{\omega}{s^2 + \omega^2} \right] + \sin \alpha \left[\frac{s}{s^2 + \omega^2} \right] = \frac{\omega \cos \alpha + s \sin \alpha}{s^2 + \omega^2} \end{aligned}$$

Q] Evaluate $L[\sin 2t \cdot \sin 3t]$

$$\rightarrow L[\sin 2t \cdot \sin 3t] = \frac{1}{2} L[\cos t - \cos 5t]$$

$$\left\{ \sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)] \right\}$$

$$= \frac{1}{2} \left[\frac{s}{s^2+1} - \frac{s}{s^2+25} \right]$$

Q] Evaluate $L[\cos t \cos 2t \cos 3t]$

$$\rightarrow \cos A \cos B = \frac{1}{2} \{ \cos(A+B) + \cos(A-B) \}$$

$$L \left[\cos t \cdot \frac{1}{2} (\cos 5t + \cos t) \right] = \frac{1}{2} L [\cos t \cos 5t + \cos^2 t]$$

$$\frac{1}{2} L \left[\frac{1}{2} (\cos 6t + \cos 4t) + \frac{1}{2} (1 + \cos 2t) \right]$$

$$= \frac{1}{4} \left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4} + \frac{1}{s} \right]$$

Q] P.T $L[\sin^5 t] = \frac{5!}{(s^2+1)(s^2+9)(s^2+25)}$

$$\rightarrow \text{Let } x = \cos t + i \sin t \Rightarrow \frac{1}{x} = \cos t - i \sin t$$

$$\therefore \sin t = \frac{1}{2i} \left(x - \frac{1}{x} \right)$$

Also $x^n = \cos nt + i \sin nt$ & $\frac{1}{x^n} = \cos nt - i \sin nt$

$$\therefore x^n - \frac{1}{x^n} = 2i \sin nt$$

$$\therefore \sin^5 t = \left(\frac{1}{2i}\right)^5 \left(x - \frac{1}{x}\right)^5$$

$$= \frac{1}{32i} \left(x^5 - 5x^4 \cdot \frac{1}{x} + 10x^3 \cdot \frac{1}{x^2} - 10x^2 \cdot \frac{1}{x^3} + 5x \cdot \frac{1}{x^4} - \frac{1}{x^5} \right)$$

$$= \frac{1}{32i} \left[\left(x^5 - \frac{1}{x^5} \right) - 5 \left(x^3 - \frac{1}{x^3} \right) + 10 \left(x - \frac{1}{x} \right) \right]$$

$$= \frac{1}{32i} \left[2i \sin st - 5(2i \sin 3t) + 10(2i \sin t) \right]$$

$$= \frac{1}{16} \left[\sin st - 5 \sin 3t + 10 \sin t \right]$$

$$\mathcal{L}\{\sin^5 t\} = \frac{1}{16} \left[\frac{s}{s^2+25} - 5 \cdot \frac{3}{s^2+9} + 10 \cdot \frac{1}{s^2+1} \right]$$

$$= \frac{5}{16} \left[\frac{1}{s^2+25} - \frac{3}{s^2+9} + \frac{2}{s^2+1} \right]$$

$$= \frac{5!}{(s^2+1)(s^2+9)(s^2+1)}$$

Q] If $f(t) = (\sin 2t - \cos 2t)^2$ then find $\mathcal{L}[f(t)]$, Hence find $\mathcal{L}[f(zt)]$

$$\rightarrow f(t) = (\sin 2t - \cos 2t)^2$$

$$= \sin^2 2t - 2 \sin 2t \cos 2t + \cos^2 2t$$

$$= 1 - \sin 4t$$

$$\mathcal{L}(f(t)) = \mathcal{L}[1 - \sin 4t] = \frac{1}{s} - \frac{4}{s^2 + 16}$$

$$= \frac{s^2 + 16 - 4s}{s(s^2 + 16)}$$

Now using change of scale property

$$\text{If } L[f(t)] = \Phi(s) \text{ then } L[f(at)] = \frac{1}{a} \Phi\left(\frac{s}{a}\right)$$

$$\therefore L[f(2t)] = \frac{1}{2} \left[\frac{\left(\frac{s}{2}\right)^2 - 4\left(\frac{s}{2}\right) + 16}{\left(\frac{s}{2}\right)\left[\left(\frac{s}{2}\right)^2 + 16\right]} \right] = \frac{s^2 - 8s + 64}{s(s^2 + 64)}$$

Q] If $L(f(t)) = \log\left(\frac{s+3}{s+1}\right)$. find $L[f(2t)]$

→ Using change of scale property

$$\text{If } L(f(t)) = \Phi(s) \text{ then } L(f(at)) = \frac{1}{a} \Phi\left(\frac{s}{a}\right)$$

$$\text{Now } L[f(t)] = \log\left(\frac{s+3}{s+1}\right)$$

$$\therefore L[f(2t)] = \frac{1}{2} \log\left(\frac{\frac{s}{2} + 3}{\frac{s}{2} + 1}\right) = \frac{1}{2} \log\left(\frac{s+6}{s+2}\right)$$

Q] If $L[\operatorname{erf} \sqrt{t}] = \frac{1}{s\sqrt{s+1}}$ find $L[\operatorname{erf} 3\sqrt{t}]$

→

Using change of scale property

$$L[\operatorname{erf} \sqrt{t}] = \frac{1}{s\sqrt{s+1}} = \Phi(s)$$

$$\mathcal{L}[\operatorname{erf} 3\sqrt{t}] = \mathcal{L}[\operatorname{erf} \sqrt{9t}]$$

$$= \frac{1}{9} \Phi\left(\frac{s}{9}\right) = \frac{1}{9} \cdot \frac{1}{\sqrt{\frac{s}{9} + 1}} = \frac{3}{s\sqrt{s+9}}$$

Q] Find Laplace transform of $\sin \sqrt{t}$. Hence find $\mathcal{L}[\sin 2\sqrt{t}]$

$$\rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin \sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \dots$$

$$\therefore \mathcal{L}[\sin \sqrt{t}] = \mathcal{L}[t^{1/2}] - \frac{1}{3!} \mathcal{L}[t^{3/2}] + \frac{1}{5!} \mathcal{L}[t^{5/2}] - \frac{1}{7!} \mathcal{L}[t^{7/2}] + \dots$$

Now $\mathcal{L}[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$. $\Gamma n = (n-1)\Gamma n-1$ & $\frac{1}{2} = \sqrt{\pi}$

$$\mathcal{L}[\sin \sqrt{t}] = \frac{\sqrt{\frac{1}{2}}}{s^{3/2}} - \frac{1}{3!} \frac{\sqrt{\frac{5}{2}}}{s^{5/2}} + \frac{1}{5!} \frac{\sqrt{\frac{7}{2}}}{s^{7/2}} - \frac{1}{7!} \frac{\sqrt{\frac{9}{2}}}{s^{9/2}} + \dots$$

$$= \frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{s^{3/2}} - \frac{1}{3!} \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\frac{1}{2}}}{s^{5/2}} + \frac{1}{5!} \frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\frac{1}{2}}}{s^{7/2}} \dots$$

$$= \frac{\sqrt{\frac{1}{2}}}{2s^{3/2}} \left[1 - \left(\frac{1}{2^2 \cdot s} \right) + \frac{1}{2!} \left(\frac{1}{2^2 \cdot s} \right)^2 \dots \right]$$

$$\mathcal{L}[\sin \sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s} \left[e^{-x} = 1 - x + \frac{x^2}{2!} \dots \right]$$

Using change of scale property

$$\mathcal{L}[\sin 2\sqrt{t}] = \mathcal{L}[\sin \sqrt{4t}] = \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2 \left(\frac{s}{4}\right)^{3/2}} e^{-1/4\left(\frac{s}{4}\right)} = \frac{\sqrt{\pi}}{s^{3/2}} e^{-1/3}$$

* First Shifting Theorem.—

$$\mathcal{L}[f(t)] = \Phi(s), \quad \mathcal{L}[e^{-at} f(t)] = \Phi(s+a), \quad \mathcal{L}[e^{at} f(t)] = \Phi(s-a)$$

$$\mathcal{L}[e^{-bt} \sin at] = \frac{a}{(s+b)^2 + a^2}$$

Q] If $\mathcal{L}[f(t)] = \frac{s}{s^2+s+4}$. find $\mathcal{L}[e^{-3t} f(2t)]$

→

By change of scale property

$$\mathcal{L}[f(2t)] = \frac{1}{2} \frac{\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)^2 + \left(\frac{s}{2}\right) + 4} = \frac{s}{s^2 + 2s + 16} = \Phi(s)$$

Now, using first shifting property

$$\mathcal{L}[e^{-3t} f(2t)] = \Phi(s+3)$$

$$= \frac{s+3}{(s+3)^2 + 2(s+3) + 16} = \frac{s+3}{s^2 + 8s + 31}$$

Q] find $\mathcal{L}[\cosh 2t \cos 2t]$

→ = $\mathcal{L}\left[\frac{1}{2} (e^{2t} + e^{-2t}) \cos 2t\right]$

$$= \frac{1}{2} \left[L(e^{2t} \cos 2t) + L(e^{-2t} \cos 2t) \right]$$

$$L|\cos 2t| = \frac{s}{s^2 + 4}$$

By shifting theorem,

$$L|\cosh 2t \cos 2t| = \frac{1}{2} \left[\frac{s-2}{(s-2)^2 + 4} + \frac{s+2}{(s+2)^2 + 4} \right] = \frac{s^3}{s^4 + 64}$$

Q] Find $L|(t^2 \sinh t)^2|$

$$\rightarrow (t^2 \sinh t)^2 = t^4 \left(\frac{e^t - e^{-t}}{2} \right)^2 = \frac{t^4}{4} [e^{2t} - 2 + e^{-2t}]$$

$$\therefore L|(t^2 \sinh t)^2| = L \left[\frac{t^4}{4} (e^{2t} - 2 + e^{-2t}) \right]$$

$$= \frac{1}{4} \left[L(e^{2t} + e^{-2t}) - 2L(t^4) + L(e^{-2t} + e^{2t}) \right]$$

$$L(t^4) = \frac{4!}{s^5}$$

$$\therefore L|(t^2 \sinh t)^2| = \frac{1}{4} \left[\frac{4!}{(s-2)^5} - \frac{2 \cdot 4!}{s^5} + \frac{4!}{(s+2)^5} \right]$$

$$= 6 \left[\frac{1}{(s-2)^5} - \frac{2}{s^5} + \frac{1}{(s+2)^5} \right]$$

$$\text{Q] P.T. } L \left[\sinh \left(\frac{t}{2} \right) \sin \left(\frac{\sqrt{3}t}{2} \right) \right] = \frac{\sqrt{3}}{2} \cdot \frac{s}{s^4 + s^2 + 1}$$

$$\rightarrow \sinh \left(\frac{t}{2} \right) \sin \left(\frac{\sqrt{3}t}{2} \right) = \left(\frac{e^{t/2} - e^{-t/2}}{2} \right) \sin \left(\frac{\sqrt{3}t}{2} \right)$$

$$\text{Now, } L \left[\sin \left(\frac{\sqrt{3}t}{2} \right) \right] = \frac{\sqrt{3}/2}{s^2 + \frac{3}{4}}$$

By First shifting theorem,

$$L \left| e^{t/2} \sin \left(\frac{\sqrt{3}t}{2} \right) \right| = \frac{\sqrt{3}/2}{\left(s - \frac{1}{2} \right)^2 + \frac{3}{4}} = \frac{\sqrt{3}/2}{s^2 - s + 1}$$

$$L \left| e^{-t/2} \sin \left(\frac{\sqrt{3}t}{2} \right) \right| = \frac{\sqrt{3}/2}{\left(s + \frac{1}{2} \right)^2 - \frac{3}{4}} = \frac{\sqrt{3}/2}{s^2 + s - 1}$$

$$L \left(\sinh \left(\frac{t}{2} \right) \sin \left(\frac{\sqrt{3}t}{2} \right) \right) = \frac{1}{2} \left[\frac{\sqrt{3}/2}{(s^2 + 1) - s} - \frac{\sqrt{3}/2}{(s^2 + 1) + s} \right]$$

$$= \frac{\sqrt{3}}{2} \left[\frac{s}{s^4 + s^2 + 1} \right]$$

* Second Shifting Theorem :-

$$L[f(t)] = \Phi(s) \quad \text{and} \quad g(t) = f(t-a) \quad \text{when } t > a \quad \text{and} \quad g(t) = 0 \quad \text{when } t < a$$

$$\text{then } L[g(t)] = e^{-as} \Phi(s)$$

Q] (i) $L\{f(t)\}$ where $f(t) = \cos(t - \alpha)$, $t > \alpha$ & $f(t) = 0$, $t < \alpha$

(ii) $L\{f(t)\}$ where $f(t) = e^{t-k}$, $t > k$ & $f(t) = 0$, $t < k$

$$\rightarrow (i) L(\cos t) = \frac{s}{s^2 + 1}$$

Hence by second shifting theorem

$$L(\cos(t - \alpha)) = e^{-\alpha s} \cdot \frac{s}{s^2 + 1}$$

$$(ii) L(e^t) = \frac{1}{s-1}$$

Hence by second shifting theorem

$$L(e^{t-k}) = e^{-ks} \cdot \frac{1}{s-1}$$

* Effect of Multiplication by t :-

Let $f(t)$ be a function & if $L\{f(t)\} = f(s)$ then $L\{t f(t)\} = -\frac{d}{ds} f(s)$

$$\& L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

Q] Find $L\{t e^{-t} \cosh 2t\}$

$$\rightarrow e^{-t} \cosh 2t = e^{-t} \left[\frac{e^{2t} + e^{-2t}}{2} \right] = \frac{e^t + e^{-3t}}{2}$$

$$L(e^{-t} \cosh 2t) = \frac{1}{2} \left[L(e^t) + L(e^{-3t}) \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-1} + \frac{1}{s+3} \right]$$

$$\therefore L\{t e^{-t} \cosh 2t\} = - \frac{d}{ds} \cdot \frac{1}{2} \left[\frac{1}{s-1} + \frac{1}{s+3} \right]$$

$$= \frac{1}{2} \left[\frac{1}{(s-1)^2} + \frac{1}{(s+3)^2} \right]$$

Q] Find $L[(1+te^{-t})^3]$

$$\rightarrow L[(1+te^{-t})^3] = L[1 + 3te^{-t} + 3t^2e^{-2t} + t^3e^{-3t}]$$

$$= L[1] + 3L[te^{-t}] + 3L[t^2e^{-2t}] + L[t^3e^{-3t}]$$

$$= \frac{1}{s} - 3 \cdot \frac{d}{ds} [L(e^{-t})] + 3 \frac{d^2}{ds^2} [L(e^{-2t})] - \frac{d^3}{ds^3} [L(e^{-3t})]$$

$$= \frac{1}{s} - 3 \frac{d}{ds} \left[\frac{1}{s+1} \right] + 3 \frac{d^2}{ds^2} \left[\frac{1}{s+2} \right] - \frac{d^3}{ds^3} \left[\frac{1}{s+3} \right]$$

$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^2} + \frac{6}{(s+3)^4}$$

Q] Find $L[t e^{-4t} \sin 3t]$

$$L[\sin 3t] = \frac{3}{s^2 + 9}$$

$$L[t \sin 3t] = -\frac{d}{ds} \cdot \left[\frac{3}{s^2 + 9} \right]$$

$$= \frac{6s}{(s^2 + 9)^2}$$

$$L[e^{-4t} t \sin 3t] = \frac{6(s+4)}{(s+4)^2 + 9} \quad [\text{Using First shifting}]$$

$$= \frac{6(s+4)}{(s^2 + 8s + 25)^2}$$

Q] Find $L[t^5 \cosh t]$

$$\rightarrow L[t^5 \cosh t] = L \left[t^5 \left(\frac{e^t + e^{-t}}{2} \right) \right]$$

$$= \frac{1}{2} L \left[e^t t^5 + e^{-t} t^5 \right]$$

But $L[t^5] = \frac{5!}{s^6}$ & using first shifting property

$$L[t^5 \cosh t] = \frac{1}{2} \left[\frac{5!}{(s-1)^6} + \frac{5!}{(s+1)^6} \right] = 60 \left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6} \right]$$

Q] Find $L[t\sqrt{1+\sin t}]$

$$\begin{aligned} \rightarrow \sqrt{1+\sin t} &= \sqrt{\sin^2\left(\frac{t}{2}\right) + \cos^2\left(\frac{t}{2}\right) + 2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)} \\ &= \sqrt{\left(\sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)\right)^2} \\ &= \sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right) \end{aligned}$$

$$L[\sqrt{1+\sin t}] = L\left[\sin\left(\frac{t}{2}\right)\right] + L\left[\cos\left(\frac{t}{2}\right)\right]$$

$$= \frac{1/2}{s^2 + \frac{1}{4}} + \frac{s}{s^2 + \frac{1}{4}}$$

$$= \frac{2}{4s^2 + 1} + \frac{4s}{4s^2 + 1} = \frac{2(2s+1)}{(4s^2+1)}$$

Now using multiplication by t property

$$L[t\sqrt{1+\sin t}] = -\frac{d}{ds} \left[\frac{2(2s+1)}{(4s^2+1)} \right]$$

$$= -2 \left[\frac{(4s^2+1) \cdot 2 - (2s+1) \cdot 8s}{(4s^2+1)^2} \right]$$

$$L[t\sqrt{1+\sin t}] = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

Q] Find $L[te^{3t} \operatorname{erf}\sqrt{t}]$

$$\rightarrow L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$$

Using Multiplication by t

$$L[t\operatorname{erf}\sqrt{t}] = -\frac{d}{ds} \left[\frac{1}{s\sqrt{s+1}} \right]$$

$$= - \left[\frac{-1}{s^2(s+1)} \cdot \frac{d}{ds} (s\sqrt{s+1}) \right]$$

$$= \frac{1}{s^2(s+1)} \left[s \cdot \frac{1}{2\sqrt{s+1}} + \sqrt{s+1} \right]$$

$$= \frac{1}{s^2(s+1)} \left[\frac{s+2(s+1)}{2\sqrt{s+1}} \right]$$

$$L[t\operatorname{erf}\sqrt{t}] = \frac{3s+2}{2s^2(s+1)^{3/2}}$$

Now using First shifting theorem,

$$L[e^{3t} t\operatorname{erf}\sqrt{t}] = \frac{3(s-3)+2}{2(s-3)^2(s-3+1)^{3/2}} = \frac{3s-7}{2(s-3)^2(s-2)^{3/2}}$$

Q] $L\left(t\left(\frac{\sin t}{e^t}\right)^2\right)$

$$f(t) = t \left(\frac{\sin t}{e^t} \right)^2 = t e^{-2t} \sin^2 t = t e^{-2t} \left(\frac{1 - \cos 2t}{2} \right) = \frac{1}{2} t e^{-2t} (1 - \cos 2t)$$

$$\mathcal{L}(1 - \cos 2t) = \mathcal{L}(1) - \mathcal{L}(\cos 2t)$$

$$= \frac{1}{s} - \frac{s}{s^2 + 4}$$

$$\mathcal{L}(t(1 - \cos 2t)) = -\frac{d}{ds} \left[\frac{1}{s} - \frac{s}{s^2 + 4} \right]$$

$$= - \left[\frac{-1}{s^2} - \frac{(s^2 + 4)(1 - s(2s))}{(s^2 + 4)^2} \right]$$

$$= - \left[\frac{-1}{s^2} - \frac{4 - s^2}{(s^2 + 4)^2} \right] = \frac{1}{s^2} + \frac{4 - s^2}{(s^2 + 4)^2}$$

Now using first shifting theorem,

$$\mathcal{L}[e^{-2t} + \sin^2 t] = \frac{1}{(s+2)^2} + \frac{4 - (s+2)^2}{[(s+2)^2 + 4]^2} = \frac{1}{(s+2)^2} - \frac{s^2 + 4s}{(s^2 + 4s + 8)^2}$$

* Effect of division by t

$$\text{If } L[f(t)] = \phi(s) , \text{ then } L\left(\frac{f(t)}{t}\right) = \int_s^\infty \phi(s) ds$$

$$\text{Q] } L\left(\frac{e^{-at} - e^{-bt}}{t}\right)$$

$$\rightarrow L(e^{-at} - e^{-bt}) = \frac{1}{s+a} - \frac{1}{s-b}$$

Now effect by division by t

$$L = \left[\frac{1}{t} (e^{-at} - e^{-bt}) \right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s-b} \right) ds$$

$$= \left[\log(s+a) - \log(s-b) \right]_s^\infty$$

$$= \log\left(\frac{s+a}{s-b}\right) \Big|_s^\infty$$

$$= \log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)_s^\infty$$

$$= \log(1) - \log\left(\frac{1+a/s}{1+b/s}\right)$$

$$L\left(\frac{e^{-at} - e^{-bt}}{t}\right) = \log\left(\frac{s+b}{s+a}\right)$$

$$\text{Q] } L \left[\frac{\sin^2 2t}{t} \right]$$

$$\rightarrow L[\sin^2 2t] = L \left[\frac{1 - \cos 4t}{2} \right] = \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2 + 16} \right]$$

$$L \left[\frac{\sin^2 2t}{t} \right] = \int_s^\infty \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2 + 16} \right] ds$$

$$= \frac{1}{2} \int_s^\infty \frac{1}{s} ds - \frac{1}{2 \cdot 2} \int_s^\infty \frac{2s}{s^2 + 16} ds$$

$$= \frac{1}{2} \left[\log s - \frac{1}{2} \log(s^2 + 16) \right]_s^\infty$$

$$= \frac{1}{2} \left[\log \left(\frac{s}{\sqrt{s^2 + 16}} \right) \right]_s^\infty$$

$$= \frac{1}{2} \left[\log(1) - \log \left(\frac{1}{\sqrt{1 + \frac{16}{s^2}}} \right) \right]$$

$$L \left(\frac{\sin^2 2t}{t} \right) = \frac{1}{2} \log \left(\frac{\sqrt{s^2 + 16}}{s} \right)$$

$$\text{Q] } L \left[\frac{e^{-2t} \sin 2t \cosh t}{t} \right]$$

$$\rightarrow e^{-2t} \sin 2t \cosh t = e^{-2t} \sin 2t \left(\frac{e^t + e^{-t}}{2} \right) = \frac{1}{2} [e^{-t} \sin 2t + e^{-3t} \sin 2t]$$

$$\mathcal{L}[\sin 2t] = \frac{2}{s^2 + 4}$$

By shifting theorem,

$$\mathcal{L}(e^{-t} \sin 2t) = \frac{2}{(s+1)^2 + 4} \quad \& \quad \mathcal{L}[e^{-3t} \sin 2t] = \frac{2}{(s+3)^2 + 4}$$

$$\mathcal{L}[e^{-2t} \sin 2t \cosh t] = \frac{1}{2} \left[\frac{2}{(s+1)^2 + 4} + \frac{2}{(s+3)^2 + 4} \right]$$

$$= \frac{1}{(s+1)^2 + 2^2} + \frac{1}{(s+3)^2 + 2^2}$$

By effect of division by t

$$\mathcal{L}\left[\frac{e^{-2t} \sin 2t \cosh t}{t}\right] = \int_s^\infty \frac{1}{(s+1)^2 + 2^2} + \frac{1}{(s+3)^2 + 2^2} ds$$

$$= \left[\frac{1}{2} \tan^{-1} \left(\frac{s+1}{2} \right) + \frac{1}{2} \tan^{-1} \left(\frac{s+3}{2} \right) \right]_s^\infty$$

$$= \left[\frac{\pi}{2} - \frac{1}{2} \tan^{-1} \left(\frac{s+1}{2} \right) \right] + \left[\frac{\pi}{2} - \frac{1}{2} \tan^{-1} \left(\frac{s+3}{2} \right) \right]$$

Q] $L\left[\frac{\sin at}{t}\right]$ Also does $L\left[\frac{\cos at}{t}\right]$ exist?

$$\rightarrow L[\sin at] = \frac{a}{s^2 + a^2}$$

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2 + a^2} ds = \left[\tan^{-1}\left(\frac{s}{a}\right) \right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

Now,

$$L[\cos at] = \frac{s}{s^2 + a^2}$$

$$L\left[\frac{\cos at}{t}\right] = \int_s^\infty \frac{s}{s^2 + a^2} ds = \frac{1}{2} \int_s^\infty \frac{2s}{s^2 + a^2} ds = \left[\frac{1}{2} \log(s^2 + a^2) \right]_s^\infty$$

Since $\log(s^2 + a^2)$ is infinite when $s \rightarrow \infty$,

$L\left[\frac{\cos at}{t}\right]$ does not exist.

* Laplace Transforms of derivatives

$$L(f'(t)) = -f(0) + sL(f(t))$$

$$L(f''(t)) = s^2 L(f(t)) - sf(0) - f'(0)$$

Q] Given $f(t) = t+1$, $0 \leq t \leq 2$ & $f(t) = 3$, $t > 2$. Find $Lf(t)$, $Lf'(t)$, $Lf''(t)$

$$\rightarrow L[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

$$= \int_0^2 e^{-st} (t+1) dt + \int_2^\infty e^{-st} (3) dt$$

$$= \left[(t+1) \left(\frac{e^{-st}}{-s} \right) - (1) \left(\frac{e^{-st}}{s^2} \right) \right]_0^2 + 3 \left(\frac{e^{-st}}{-s} \right)_2^\infty$$

$$= 3 \frac{e^{-2s}}{-s} - \frac{e^{-2s}}{s^2} - (1) \left(-\frac{1}{s} \right) + \frac{1}{s^2} + 3 \left(0 - \frac{e^{-2s}}{-s} \right)$$

$$L[f(t)] = \frac{1}{s} + \frac{1}{s^2} - \frac{e^{-2s}}{s^2}$$

$$\text{Now, } L[f'(t)] = -f(0) + sL[f(t)]$$

But by data $f(0) = 1$

$$\therefore L[f'(t)] = -1 + s \left[\frac{1}{s} + \frac{1}{s^2} (1 - e^{-2s}) \right] = \frac{1}{s} (1 - e^{-2s})$$

$$L[f''(t)] = s^2 L[f(t)] - s[f(0)] - f'(0)$$

$$= s^2 \left[\frac{1}{s} + \frac{1}{s^2} (1 - e^{-2s}) \right] - s - 1$$

$$= s + (1 - e^{-2s}) - s - 1 = -e^{-2s}$$

Q] $L \left[\frac{d}{dt} \left(\frac{\sin 3t}{t} \right) \right]$

$$L[\sin 3t] = \frac{3}{s^2 + 9}$$

$$L \left[\frac{\sin 3t}{t} \right] = \int_s^\infty \frac{3}{s^2 + 9} ds = \tan^{-1} \left(\frac{s}{3} \right) \Big|_s^\infty = \frac{\pi}{2} - \tan^{-1} \left(\frac{s}{3} \right) = \cot^{-1} \left(\frac{s}{3} \right)$$

$$\mathcal{L}[f'(t)] = -f(0) - s \mathcal{L}[f(t)]$$

$$f(0) = \lim_{t \rightarrow 0} \frac{\sin 3t}{t} = 3 \lim_{t \rightarrow 0} \frac{\sin 3t}{3t} = 3 \cdot 1 = 3$$

$$\therefore \mathcal{L}[f'(t)] = -3 - s \cot^{-1}\left(\frac{s}{3}\right)$$

* Laplace Transforms of Integrals

$$\mathcal{L}[f(t)] = \phi(s), \quad \mathcal{L}_0^t \int f(u) du = \frac{1}{s} \phi(s)$$

Q] $\mathcal{L} \left[\int_0^t \sin 2u du \right]$

→

$$\mathcal{L}[\sin 2t] = \frac{2}{s^2 + 4} = \phi(s)$$

$$\mathcal{L} \left[\int_0^t \sin 2u du \right] = \frac{1}{s} \phi(s) = \frac{2}{s(s^2 + 4)}$$

$$\frac{1}{2} [1 + \cos 4u]$$

Q] $\mathcal{L} \left[\int_0^t u e^{-3u} \cos^2 2u du \right]$

$$\rightarrow \cos^2 2u = \frac{1 + \cos 4u}{2}$$

$$\mathcal{L}[\cos^2 u] = \frac{1}{2} \mathcal{L}[1 + \cos 4u] = \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2 + 16} \right]$$

$$\mathcal{L}[u \cos^2 u] = -\frac{d}{ds} \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2 + 16} \right] \quad [\text{Multiplication by } u]$$

$$= -\frac{1}{2} \left[-\frac{1}{s^2} + \frac{(s^2 + 16)(1) - s(2s)}{(s^2 + 16)^2} \right]$$

$$= \frac{1}{2} \left[\frac{1}{s^2} + \frac{s^2 - 16}{(s^2 + 16)^2} \right]$$

$$\therefore \mathcal{L}[e^{-3u} u \cos^2 u] = \frac{1}{2} \left[\frac{1}{(s+3)^2} + \frac{(s+3)^2 - 16}{[(s+3)^2 + 16]^2} \right] \quad [\text{First shifting theorem}]$$

$$= \frac{1}{2} \left[\frac{1}{(s+3)^2} + \frac{s^2 + 6s - 7}{(s^2 + 6s + 25)^2} \right] = \Phi(s)$$

$$\mathcal{L}\left[\int_0^t e^{-3u} u \cos^2 u du\right] = \frac{1}{s} \Phi(s) = \frac{1}{2s} \left[\frac{1}{(s+3)^2} + \frac{s^2 + 6s - 7}{(s^2 + 6s + 25)^2} \right]$$

$$\textcircled{1} \quad \mathcal{L}\left[t \int_0^t e^{-4u} \sin 3u du\right]$$

$$\rightarrow \mathcal{L}[\sin 3u] = \frac{3}{s^2 + 9}$$

$$\mathcal{L}[e^{-4u} \sin 3u] = \frac{3}{(s+4)^2 + 9} \quad [\text{By first shifting theorem}]$$

$$\mathcal{L} \left[\int_0^t e^{-4u} \sin 3u \, du \right] = \frac{1}{s} \cdot \frac{3}{(s+4)^2 + 9} \quad [\text{Laplace Transform of } \int f(u) \, du]$$

$$\mathcal{L} \left[t \int_0^t e^{-4u} \sin 3u \, du \right] = (-1) \frac{d}{ds} \left[\frac{3}{s^3 + 8s^2 + 25s} \right]$$

$$\mathcal{L} \left[t \int_0^t e^{-4u} \sin 3u \, du \right] = \frac{3(3s^2 + 16s + 25)}{(s^3 + 8s^2 + 25s)^2}$$

Q) $\mathcal{L} \left[e^{-t} \int_0^t e^u \cosh u \, du \right] \quad \frac{s-1}{s^2(s-2)}$

$$\rightarrow \mathcal{L} [\cosh u] = \frac{s}{s^2 - 1}$$

$$\mathcal{L} [e^u \cosh u] = \frac{s-1}{(s-1)^2 - 1} = \frac{s-1}{s^2 - 2s + 1 - 1} = \frac{s-1}{s(s-2)} \quad [\text{First shifting theorem}]$$

$$\mathcal{L} \left[e^{-t} \int_0^t e^u \cosh u \, du \right] = \frac{1}{s} \cdot \frac{s-1}{s(s-2)} = \frac{s-1}{s^2(s-2)} \quad [\text{Laplace of } \int f(u) \, du]$$

$$\mathcal{L} \left[e^{-t} \int_0^t e^u \cosh u \, du \right] = \frac{(s+1)-1}{(s+1)^2 [(s+1)-2]} = \frac{s}{(s+1)^2 (s-1)} \quad [\text{First shifting theorem}]$$

Q) $\mathcal{L} [\operatorname{erf} \sqrt{t}]$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \, du$$

$$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} \, du$$

put $u^2 = v$

$$\therefore u = \sqrt{v} \Rightarrow du = \frac{1}{2\sqrt{v}} dv$$

u	0	\sqrt{t}
v	0	t

$$\therefore \operatorname{erf} \sqrt{t} = \frac{2}{\sqrt{\pi}} \int_0^t e^{-v} \frac{1}{2\sqrt{v}} dv = \frac{1}{\sqrt{\pi}} \int_0^t e^{-v} \frac{v^{-1/2}}{2} dv$$

$$\therefore L[v^{-1/2}] = \frac{1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$L[e^{-v} v^{-1/2}] = \frac{\sqrt{\pi}}{\sqrt{s+1}}$$

$$L \left[\int_0^t e^{-v} v^{-1/2} dv \right] = \frac{\sqrt{\pi}}{s\sqrt{s+1}}$$

$$L[\operatorname{erf} \sqrt{t}] = \frac{1}{\sqrt{\pi}} L \left[\int_0^t e^{-v} v^{-1/2} dv \right] = \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{s\sqrt{s+1}}$$

$$\therefore L[\operatorname{erf} \sqrt{t}] = \frac{1}{s\sqrt{s+1}}$$

Q] Evaluate $\int_0^\infty e^{-2t} \sin^3 t dt$

$$\rightarrow L[\sin^3 t] = L\left[\frac{3}{4} \sin t - \frac{1}{4} \sin 3t\right]$$

$$= \frac{3}{4} \cdot \frac{1}{s^2+1} \cdot \frac{-1}{4} \cdot \frac{3}{s^2+9} = \frac{3}{4} \left[\frac{1}{s^2+1} - \frac{1}{s^2+9} \right]$$

This means that $\int_0^\infty e^{-st} \sin^3 t dt = \frac{3}{4} \left[\frac{1}{s^2+1} - \frac{1}{s^2+9} \right]$

Now put $s=2$

$$\therefore \int_0^\infty e^{-2t} \sin^3 t dt = \frac{3}{4} \left[\frac{1}{5} - \frac{1}{13} \right] = \frac{3}{4} \left[\frac{8}{65} \right] = \frac{6}{65}$$

Q] $\int_0^\infty e^{-2t} \sin(t+\alpha) \cos(t-\alpha) dt = \frac{1}{4}$, find α

$$\rightarrow \sin(t+\alpha) \cos(t-\alpha) = \frac{1}{2} [\sin(2t) + \sin(2\alpha)]$$

$$L[\sin(t+\alpha) \cos(t-\alpha)] = \frac{1}{2} \left[\left[\frac{2}{s^2+4} \right] + \sin 2\alpha \left(\frac{1}{s} \right) \right]$$

$$= \frac{1}{s^2+4} + \frac{1}{2} \sin 2\alpha \left(\frac{1}{s} \right)$$

Put $s=2$

$$\int_0^\infty e^{-2t} \sin(t+\alpha) \cos(t-\alpha) dt = \frac{1}{8} + \frac{1}{4} \sin 2\alpha$$

$$\frac{1}{4} = \frac{1}{8} + \frac{1}{4} \sin 2\alpha$$

$$\sin 2\alpha = \frac{1}{2} \Rightarrow 2\alpha = \frac{\pi}{6} \Rightarrow \alpha = \frac{\pi}{12}$$

Inverse Laplace Transforms

1. Table of Inverse Laplace Transforms:

$L(1) = \frac{1}{s}$	$L^{-1}\left(\frac{1}{s}\right) = 1$
$L(e^{-at}) = \frac{1}{s+a}$	$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$
$L(e^{at}) = \frac{1}{s-a}$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
$L(t^{n-1}) = \frac{ n }{s^n}$	$L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{ n }$
$L(t^{n-1}) = \frac{(n-1)!}{s^n}$	$L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$
$L(\sin at) = \frac{a}{s^2 + a^2}$	$L^{-1}\left(\frac{1}{s^2 + a^2}\right) = \frac{1}{a} \sin at$
$L(\cos at) = \frac{s}{s^2 + a^2}$	$L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$
$L(\sinh at) = \frac{a}{s^2 - a^2}$	$L^{-1}\left(\frac{1}{s^2 - a^2}\right) = \frac{1}{a} \sinh at$
$L(\cosh at) = \frac{s}{s^2 - a^2}$	$L^{-1}\left(\frac{s}{s^2 - a^2}\right) = \cosh at$

(i) distinct linear factor	$\frac{px+q}{(x-a)(x-b)}$	Express : $\frac{px+q}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$
(ii) distinct linear factor	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	Express : $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} +$
(iii) repetitive linear factor	$\frac{px+q}{(x-a)^2}$	Express : $\frac{px+q}{(x-a)^2} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
(iv) repetitive linear factor	$\frac{px^2+qx+r}{(x-a)^3}$	Express : $\frac{px^2+qx+r}{(x-a)^3} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$
(v) repetitive linear factor	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	Express : $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
(vi) Linear & quadratic factor	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	Express: $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$

Q] Find $L^{-1} \left[\frac{1-\sqrt{s}}{s^2} \right]^2$

$$\rightarrow L^{-1} \left[\frac{1-\sqrt{s}}{s^2} \right]^2 = L^{-1} \left[\frac{1-2\sqrt{s}+s}{s^4} \right] = L^{-1} \left(\frac{1}{s^4} \right) - 2L^{-1} \left(\frac{1}{s^{7/2}} \right) + L^{-1} \left(\frac{1}{s^3} \right)$$

But $L^{-1} \left[\frac{1}{s^n} \right] = \frac{t^{n-1}}{\Gamma n}$ or $\frac{t^{n-1}}{(n-1)!}$

$$\therefore L^{-1} \left[\frac{1-\sqrt{s}}{s^2} \right]^2 = \frac{t^3}{3!} - 2 \cdot \frac{t^{5/2}}{\Gamma \frac{7}{2}} + \frac{t^2}{2!}$$

$$= \frac{t^3}{6} - \frac{16 t^{5/2}}{15\sqrt{\pi}} + \frac{t^2}{2}$$

Q] If $L(f(t)) = \frac{s+2}{s^2+2}$. Find $L[f'(t)]$

→

$$L[f'(t)] = -f(0) + sL[f(t)]$$

$$f(t) = L^{-1} \left[\frac{s+2}{s^2+2} \right] = L^{-1} \left[\frac{s}{s^2+2} \right] + 2L^{-1} \left[\frac{1}{s^2+2} \right]$$

$$= \cos \sqrt{2}t + 2 \cdot \frac{1}{\sqrt{2}} \sin \sqrt{2}t$$

$$f(t) = \cos \sqrt{2}t + \sqrt{2} \sin \sqrt{2}t$$

$$f(0) = \cos(0) + \sqrt{2} \sin(0) = 1$$

$$\mathcal{L}[f'(t)] = -f(0) + s\mathcal{L}[f(t)]$$

$$= -1 + s \left[\frac{s+2}{s^2+2} \right]$$

$$\mathcal{L}[f'(t)] = \frac{2(s-1)}{s^2+2}$$

Q] i) $\frac{2s+3}{s^2+9}$

$$\rightarrow \mathcal{L}^{-1}\left[\frac{2s+3}{s^2+9}\right] = 2\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= 2\cos 3t + 3 \cdot \frac{1}{3} \sin 3t$$

$$\mathcal{L}^{-1}\left[\frac{2s+3}{s^2+9}\right] = 2\cos 3t + \sin 3t$$

$$(ii) \mathcal{L}^{-1}\left[\frac{1}{4s+5}\right] = \frac{1}{4} \mathcal{L}^{-1}\left[\frac{1}{s+\frac{5}{4}}\right] = \frac{1}{4} e^{-\frac{5}{4}t}$$

$$(iii) \mathcal{L}^{-1}\left[\frac{4s+15}{16s^2-2s}\right] = \frac{4}{16} \mathcal{L}^{-1}\left[\frac{s}{s^2-\frac{2s}{16}}\right] + \frac{15}{16} \mathcal{L}^{-1}\left[\frac{1}{s^2-\frac{2s}{16}}\right]$$

$$= \frac{4}{16} \mathcal{L}^{-1}\left[\frac{s}{s^2-(\frac{s}{4})^2}\right] + \frac{15}{16} \mathcal{L}^{-1}\left[\frac{1}{s^2-(\frac{s}{4})^2}\right]$$

$$= \frac{4}{16} \cosh \left[\frac{5}{4} t \right] + \frac{3}{16} \frac{4}{5} \sinh \left(\frac{5}{4} t \right)$$

$$\mathcal{L}^{-1} \left[\frac{4s+15}{16s^2-25} \right] = \frac{1}{4} \cosh \left(\frac{5}{4} t \right) + \frac{3}{4} \sinh \left(\frac{5}{4} t \right)$$

Q] $\mathcal{L}^{-1} \left[\frac{1}{(s-3)^3} \right]$

$$= e^{3t} \mathcal{L}^{-1} \left[\frac{1}{s^3} \right] = e^{3t} \cdot \frac{t^2}{2!} = \frac{e^{3t} t^2}{2}$$

Q] Find $\mathcal{L}^{-1} \left[\frac{s}{s^2+2s+2} \right]$

$$\rightarrow \mathcal{L}^{-1} \left[\frac{s+1-1}{(s+1)^2+1} \right] = \mathcal{L}^{-1} \left[\frac{s+1}{(s+1)^2+1} \right] - \mathcal{L}^{-1} \left[\frac{1}{(s+1)^2+1} \right]$$

$$= e^{-t} \mathcal{L}^{-1} \left[\frac{s}{s^2+1} \right] - e^{-t} \left[\frac{1}{s^2+1} \right]$$

$$= e^{-t} \cos t - e^{-t} \sin t$$

$$= e^{-t} (\cos t - \sin t)$$

Q] Inverse Laplace Transform of

$$\left[\frac{1}{(s+1)^2} + \frac{s-2}{s^2-4s+5} + \frac{s-2}{s^2+4s+3} \right]$$

$$\rightarrow L^{-1}\left(\frac{1}{(s+1)^2}\right) + L^{-1}\left(\frac{s-2}{s^2-4s+5}\right) + L^{-1}\left(\frac{s-2}{s^2+4s+3}\right)$$

$$= e^{-t} L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left[\frac{s-2}{(s-2)^2+1}\right] + L^{-1}\left[\frac{s-2}{(s-2)^2-1}\right]$$

$$= e^{-t} \cdot \frac{t}{1} + e^{2t} \cos t + e^{2t} \cosh t$$

$$= te^{-t} + e^{2t} \cos t + e^{2t} \cosh t$$

Q] Find $L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^2}\right]$

Let $\frac{5s^2-15s-11}{(s+1)(s-2)^2} = \frac{a}{s+1} + \frac{b}{s-2} + \frac{c}{(s-2)^2}$

$$\therefore 5s^2-15s-11 = a(s-2)^2 + b(s+1)(s-2) + c(s+1)$$

$$\text{Put } s=2, 20-30-11 = c(3) \Rightarrow -21 = 3c \Rightarrow c = -7$$

$$\text{Put } s=-1, 5+15-11 = 9a \Rightarrow 9a = 9 \Rightarrow a = 1$$

$$\text{Put } s=0, -11 = 4a - 2b + c \Rightarrow b = 4$$

$$\therefore \frac{5s^2 - 15s - 11}{(s+1)(s-2)^2} = \frac{1}{s+1} + \frac{4}{s-2} - \frac{7}{(s-2)^2}$$

$$\begin{aligned} L^{-1} \left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^2} \right] &= L^{-1} \left[\frac{1}{s+1} \right] + 4 L^{-1} \left[\frac{1}{s-2} \right] - 7 L^{-1} \left[\frac{1}{(s-2)^2} \right] \\ &= e^{-t} + 4e^{2t} - 7e^{2t} \cdot t \end{aligned}$$

Q] $L^{-1} \left[\frac{s+29}{(s+4)(s^2+9)} \right]$

→ Let $\frac{s+29}{(s+4)(s^2+9)} = \frac{a}{s+4} + \frac{bs+c}{s^2+9}$

$$s+29 = a(s^2+9) + (bs+c)(s+4)$$

$$\therefore s+29 = (a+b)s^2 + (4b+c)s + (9a+4c)$$

Comparing the coefficients of similar powers of s

$$a+b = 0, \quad 4b+c = 1, \quad 9a+4c = 29$$

$$4a+4b = 0$$

$$\text{(1)} \quad \underline{c + 4b = 1}$$

$$4a-c = -1$$

~~$$16a - 4c = -4$$~~

~~$$\underline{9a + 4c = 29}$$~~

$$25a = 25 \Rightarrow \boxed{a = 1}$$

$$\therefore b = -1 \quad \text{also} \quad c = 1 - 4b = 1 + 4 \Rightarrow c = 5$$

$$\therefore \frac{s+29}{(s+4)(s^2+9)} = \frac{1}{s+4} + \frac{(-s+5)}{s^2+9}$$

$$\therefore L^{-1} \left[\frac{s+29}{(s+4)(s^2+9)} \right] = L^{-1} \left[\frac{1}{s+4} \right] - L^{-1} \left[\frac{s}{s^2+9} \right] + 5 L^{-1} \left[\frac{1}{s^2+9} \right]$$

$$= e^{-4t} - \cos 3t + \frac{5}{3} \sin 3t$$

Q] $L^{-1} \left[\frac{s^2}{(s^2+a^2)(s^2+b^2)} \right]$

$$\rightarrow s^2 = x$$

$$\text{Let } \frac{x}{(x+a^2)(x+b^2)} = \frac{A}{x+a^2} + \frac{B}{x+b^2}$$

$$\dots x = A(x+b^2) + B(x+a^2)$$

$$\text{Put } x = -a^2, -a^2 = A(-a^2 + b^2) \Rightarrow A = \frac{a^2}{a^2 - b^2}$$

$$\text{Put } x = -b^2, -b^2 = B(-b^2 + a^2) \Rightarrow B = \frac{-b^2}{a^2 - b^2}$$

$$\therefore \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a^2 - b^2} \left[\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2} \right]$$

$$= \frac{1}{a^2 - b^2} \left[a^2 \cdot \frac{1}{a} \sin at - b^2 \cdot \frac{1}{b} \sin bt \right]$$

$$\therefore L^{-1} \left[\frac{s^2}{(s^2+a^2)(s^2+b^2)} \right] = \frac{a \sin at - b \sin bt}{a^2 - b^2}$$

Q] Find $L^{-1} \left[\frac{s^2 + 2s + 3}{(s^2+2s+5)(s^2+2s+2)} \right]$

$$\rightarrow = L^{-1} \left[\frac{(s+1)^2 + 2}{[(s+1)^2 + 4][(s+1)^2 + 1]} \right] = e^{-t} L^{-1} \left[\frac{s^2 + 2}{(s^2+4)(s^2+1)} \right]$$

Assume, $s^2 = x$

$$\text{Let } \frac{x+2}{(x+4)(x+1)} = \frac{A}{(x+4)} + \frac{B}{(x+1)}$$

$$x+2 = A(x+1) + B(x+4)$$

$$\text{Put } x = -1, \quad 1 = B(3) \Rightarrow B = \frac{1}{3}$$

$$\text{Put } x = -4, \quad -2 = -3A \Rightarrow A = \frac{2}{3}$$

$$\begin{aligned} L^{-1} \left[\frac{s^2 + 2s + 3}{(s^2+2s+5)(s^2+2s+2)} \right] &= e^{-t} L^{-1} \left[\frac{\frac{2}{3}}{s^2+4} + \frac{\frac{1}{3}}{s^2+1} \right] \\ &= e^{-t} \left\{ \frac{\frac{2}{3}}{\frac{1}{2}} \sin 2t + \frac{\frac{1}{3}}{1} \sin t \right\} \\ &= \frac{e^{-t}}{3} (\sin 2t + \sin t) \end{aligned}$$

$$\text{Q] } L^{-1} \left[\frac{s}{(s^2+a^2)(s^2+b^2)} \right]$$

→ First we consider,

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2} \left[\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2} \right]$$

$$\frac{s}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2} \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \right]$$

$$L^{-1} \left[\frac{s}{(s^2+a^2)(s^2+b^2)} \right] = \frac{i}{b^2-a^2} \left[\cos at - \cos bt \right]$$

* Inverse by Convolution theorem:

$$(f_1(t))^* f_2(t) = \int_0^t f_1(t-u) f_2(u) du$$

$$L[f_1(t)] = \phi_1(s) \quad \& \quad L[f_2(t)] = \phi_2(s), \quad L^{-1}[\phi_1(s) \phi_2(s)] = \int_0^t f_1(u) f_2(t-u) du$$

Procedure of Applying Convolution Theorem:

To find $L^{-1}[\phi_1(s) \phi_2(s)]$

1. Find $L^{-1}[\phi_1(s)] = f_1(u)$, say putting u in place to t .
2. Find $L^{-1}[\phi_2(s)] = f_2(t-u)$, say putting $(t-u)$ in place to t .
3. Find $L^{-1}[\phi_1(s) \phi_2(s)] = \int_0^t f_1(u) f_2(t-u) du$

Q] Find inverse Laplace transform using convolution theorem

$$\textcircled{1} \quad \frac{s}{(s^2 + a^2)^2}$$

$$\rightarrow \text{Let } \phi_1(s) = \frac{s}{s^2 + a^2}, \quad \phi_2(s) = \frac{1}{s^2 + a^2}$$

$$\therefore f_1(t) = L^{-1}|\phi_1(s)| = \cos at, \quad f_2(t) = L^{-1}|\phi_2(s)| = \frac{\sin at}{a}$$

$$\therefore L^{-1}|\phi(s)| = L^{-1}[\phi_1(s) \cdot \phi_2(s)] \\ = \int_0^t f_1(u) f_2(t-u) du$$

$$= \int_0^t \cos au \cdot \frac{\sin a(t-u)}{a} du$$

$$= \frac{1}{a} \int_0^t \cos au \sin(a(t-au)) du$$

$$\cos A \sin B = \frac{1}{2} (\sin(A+B) - \sin(A-B))$$

$$= \frac{1}{2a} \int_0^t [\sin at - \sin(au - at + au)] du$$

$$= \frac{1}{2a} \int_0^t [\sin at - \sin(2au - at)] du$$

$$= \frac{1}{2a} \left[\sin at(u) + \frac{\cos(2au - at)}{2a} \right]_0^t$$

$$= \frac{1}{2a} \left[t \sin at + \frac{\cos at}{2a} - 0 - \frac{\cos at}{2a} \right]$$

$$= t \sin at$$

Q] $\phi(s) = \frac{1}{(s-2)(s+2)^2}$

$$\rightarrow \phi(s) = \frac{1}{(s-2)(s+2)^2} = \frac{1}{(s+2)^2} \cdot \frac{1}{s-2}$$

$$= \phi_1(s) \phi_2(s)$$

$$f_1(t) = L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}$$

$$f_2(t) = L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$$

By convolution theorem,

$$\begin{aligned} L^{-1}[\phi(s)] &= L^{-1}[\phi_1(s) \phi_2(s)] \\ &= \int_0^t f_1(u) f_2(t-u) du \\ &= \int_0^t ue^{-2u} \cdot e^{2(t-u)} du \\ &= \int_0^t ue^{2t-4u} du \\ &= \left[u \left(\frac{e^{2t-4u}}{-4} \right) - (1) \left(\frac{e^{2t-4u}}{16} \right) \right]_0^t \end{aligned}$$

$$= \left[t \left(\frac{e^{-2t}}{-4} \right) - \left(\frac{e^{-2t}}{16} \right) \right] - \left[0 - \frac{e^{2t}}{16} \right]$$

$$= \left(\frac{e^{2t} - e^{-2t}}{16} \right) - \frac{te^{-2t}}{4} = \frac{1}{16} \left[e^{2t} - 4te^{-2t} - e^{-2t} \right]$$

Q] $\phi(s) = \frac{s}{(s+1)^2}$

$$\rightarrow \phi(s) = \frac{s}{(s^2+1)^2} = \frac{s}{s^2+1} \cdot \frac{1}{s^2+1}$$

$$= \phi_1(s) \phi_2(s)$$

$$f_1(t) = L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$$

$$f_2(t) = L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

By convolution theorem,

$$\begin{aligned} L^{-1}[\phi(s)] &= L^{-1}[\phi_1(s) \cdot \phi_2(s)] \\ &= \int_0^t f_1(u) f_2(t-u) du \\ &= \int_0^t \cos u \sin(t-u) du \\ &= \frac{1}{2} \int_0^t (\sin t + \sin(t-2u)) du \end{aligned}$$

$$= \frac{1}{2} \left[u \sin t - \frac{\cos(t-2u)}{-2} \right]_0^t$$

$$= \frac{1}{2} [(t \sin t + \cos t) - (0 + \cos t)]$$

$$= \frac{t \sin t}{2}$$

Q] $\phi(s) = \frac{1}{(s-a)(s-b)}$

$$= \frac{1}{s-a} \cdot \frac{1}{s-b}$$

$$= \phi_1(s) \cdot \phi_2(s)$$

$$f_1(t) = L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$f_2(t) = L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s-b}\right] = e^{bt}$$

By Convolution theorem,

$$\begin{aligned} L^{-1}[\phi(s)] &= L^{-1}[\phi_1(s) \cdot \phi_2(s)] \\ &= \int_0^t f_1(u) \cdot f_2(t-u) du \\ &= \int_0^t e^{au} \cdot e^{b(t-u)} du \\ &= \int_0^t e^{bt + (a-b)u} du = \left[\frac{e^{bt + (a-b)u}}{(a-b)} \right]_0^t \\ &= \frac{1}{a-b} [e^{at} - e^{bt}] \end{aligned}$$

* Use of differentiation of $\Phi(s)$

$$\mathcal{L}^{-1} \Phi(s) = -\frac{1}{t} \mathcal{L}^{-1} [\Phi'(s)]$$

Q) $\mathcal{L}^{-1} \left[\log \left(\frac{s+a}{s+b} \right) \right]$

$$\rightarrow \mathcal{L}^{-1} \left[\log \left(\frac{s+a}{s+b} \right) \right] = -\frac{1}{t} \mathcal{L}^{-1} \left[\frac{d}{ds} \left(\log \left(\frac{s+a}{s+b} \right) \right) \right]$$

$$= -\frac{1}{t} \mathcal{L}^{-1} \left[\frac{d}{ds} \left[\log(s+a) - \log(s+b) \right] \right]$$

$$= -\frac{1}{t} \mathcal{L}^{-1} \left[\frac{1}{s+a} - \frac{1}{s+b} \right]$$

$$= -\frac{1}{t} [e^{-at} - e^{-bt}]$$

Q) $\mathcal{L}^{-1} \left[\log \left(\frac{s^2+a^2}{\sqrt{s+b}} \right) \right]$

$$\rightarrow = -\frac{1}{t} \mathcal{L}^{-1} \left[\frac{d}{ds} \left(\log \left(\frac{s^2+a^2}{\sqrt{s+b}} \right) \right) \right]$$

$$= -\frac{1}{t} \mathcal{L}^{-1} \left[\frac{d}{ds} \left[\log(s^2+a^2) - \frac{1}{2} \log(s+b) \right] \right]$$

$$= -\frac{1}{t} \mathcal{L}^{-1} \left[\frac{2s}{s^2+a^2} - \frac{1}{2} \cdot \frac{1}{s+b} \right]$$

$$= \frac{-1}{t} \left[2 \cos at - \frac{1}{2} e^{-bt} \right]$$

$$= \frac{1}{t} \left[\frac{1}{2} e^{-bt} - 2 \cos at \right]$$

Q] $L^{-1} \left[\tan^{-1} \left(\frac{1}{s} \right) \right]$

$$\rightarrow = -\frac{1}{t} L^{-1} \left[\frac{d}{ds} \left(\tan^{-1} \left(\frac{1}{s} \right) \right) \right]$$

$$= -\frac{1}{t} L^{-1} \left[\frac{1}{1 + (\frac{1}{s})^2} \cdot \left(-\frac{1}{s^2} \right) \right]$$

$$= \frac{1}{t} L^{-1} \left[\frac{1}{1 + s^2} \right]$$

$$= \frac{1}{t} \sin t$$

Q] $L^{-1} [\cot^{-1}(s)]$

$$\rightarrow = -\frac{1}{t} L^{-1} \left[\frac{d}{ds} \cot^{-1}(s) \right]$$

$$= -\frac{1}{t} \left[\frac{-1}{1+s^2} \right] = \frac{1}{t} \sin t$$

$$\text{Q] } L^{-1} \left[\tan^{-1} \left(\frac{s+a}{b} \right) \right]$$

$$\rightarrow = -\frac{1}{t} L^{-1} \left\{ \frac{d}{ds} \left[\tan^{-1} \left(\frac{s+a}{b} \right) \right] \right\}$$

$$= -\frac{1}{t} L^{-1} \left[\frac{1}{1 + \left(\frac{s+a}{b} \right)^2} \cdot \frac{1}{b} \right]$$

$$= -\frac{1}{t} L^{-1} \left[\frac{b}{(s+a)^2 + b^2} \right] = -\frac{1}{t} e^{-at} L^{-1} \left[\frac{b}{s^2 + b^2} \right]$$

$$= -\frac{1}{t} e^{-at} \sin bt$$

* Periodic Functions

If $f(t)$ is a periodic function of period a

$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t) dt$$

Q] $f(t) = \begin{cases} \frac{t}{a}, & 0 < t \leq a \\ \frac{1}{a}(2a-t), & a < t < 2a \end{cases}$ If $f(t) = f(t+2a)$

→ As $f(t)$ is periodic function with period $2a$,

$$L(f(t)) = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2as}} \left\{ \int_0^a \frac{t}{a} e^{-st} dt + \int_a^{2a} \frac{1}{a}(2a-t) e^{-st} dt \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \frac{1}{a} \left[t \left(\frac{e^{-st}}{-s} \right) - 1 \left(\frac{e^{-st}}{s^2} \right) \right] \Big|_0^a + \frac{1}{a} \left[(2a-t) \left(\frac{e^{-st}}{-s} \right) - (-1) \left(\frac{e^{-st}}{-s^2} \right) \right] \Big|_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \frac{1}{a} \left[a \left(\frac{e^{-as}}{-s} \right) - \left(\frac{e^{-as}}{s^2} \right) - 0 + \frac{1}{s^2} \right] + \frac{1}{a} \left[0 + \frac{e^{-2as}}{s^2} - (a) \left(\frac{e^{-as}}{-s} \right) - \left(\frac{e^{-as}}{s^2} \right) \right] \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \frac{1}{as^2} \left[1 - 2e^{-as} + e^{-2as} \right] \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \frac{1}{as^2} (1-e^{-as})^2 \right\}$$

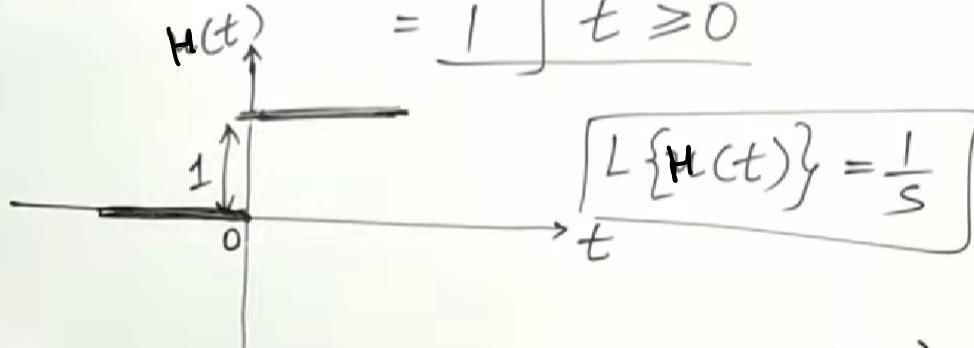
$$= \frac{1}{as^2} \left(\frac{1-e^{-as}}{1+e^{-as}} \right) = \frac{1}{as^2} \left(\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}} \right)$$

$$= \frac{1}{as^2} \tanh\left(\frac{as}{2}\right)$$

* Heaviside Functions

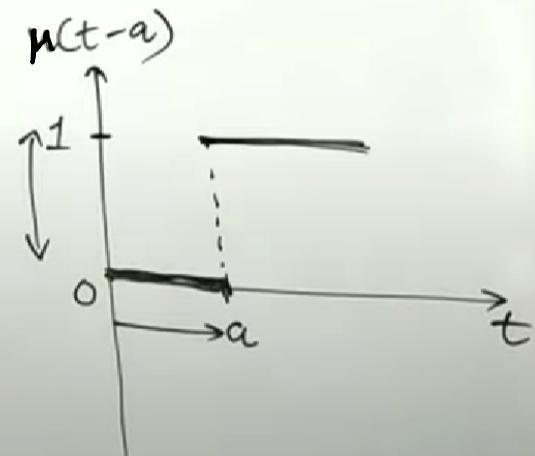
Unit step function (Heaviside function):

① No shift $\mu(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$



② with shift $\mu(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

$L\{\mu(t-a)\} = \frac{e^{-as}}{s}$



$$\text{Q} \quad L[(1+2t-t^2+t^3) H(t-1)]$$

$$\rightarrow L[f(t)H(t-a)] = e^{-as} L[f(t+a)]$$

$$f(t) = 1 + 2t - t^2 + t^3 \quad \text{for } a=1$$

$$\therefore L[(1+2t-t^2+t^3) H(t-1)] = e^{-is} L[f(t+1)]$$

$$\text{Now, } f(t+1) = 1 + 2(t+1) - (t+1)^2 + (t+1)^3$$

$$= 1 + 2t+2 - (t^2+2t+1) + (t^3+3t^2+3t+1)$$

$$f(t+1) = 3 + 3t + 2t^2 + t^3$$

$$\therefore L[(1+2t-t^2+t^3) H(t-1)] = e^{-s} L[3 + 3t + 2t^2 + t^3]$$

$$= e^{-s} \left[\frac{3}{s} + \frac{3}{s^2} + \frac{2 \cdot 2!}{s^3} + \frac{3!}{s^4} \right]$$

$$L[(1+2t-t^2+t^3) H(t-1)] = e^{-s} \left[\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4} \right]$$

$$\text{Q} \quad L[e^{-t} \sin t H(t-\pi)]$$

$$\rightarrow L[f(t)H(t-\pi)] = e^{-as} L[f(t+\pi)]$$

$$a=\pi, \quad f(t) = e^{-t} \sin t$$

$$L[e^{-t} \sin t H(t-\pi)] = e^{-\pi s} L[f(t+\pi)]$$

$$\begin{aligned} f(t+\pi) &= e^{-t-\pi} \sin(t+\pi) \\ &= -e^{-\pi} [e^{-t} \sin t] \end{aligned}$$

$$L[e^{-t} \sin t H(t-\pi)] = -e^{-\pi(s+1)} \frac{1}{(s+1)^2 + 1} = \frac{-e^{-\pi(s+1)}}{s^2 + 2s + 2}$$

* Differential Equations in Laplace

$$\mathcal{L}[y'] = s \mathcal{L}[y] - y(0)$$

$$\mathcal{L}[y''] = s^2 \mathcal{L}[y] - sy(0) - y'(0)$$

$$\mathcal{L}[y'''] = s^3 \mathcal{L}[y] - s^2 y(0) - sy'(0) - y''(0)$$

