Discrete Mathematics

Relations

Cartesian Product or Product Set

- If A and B are two nonempty sets, we define the product set or Cartesian product $A \times B$ as the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. Thus $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

Cartesian Product Example

- 1) If A = {1, 2, 3} and B = {a, b, c}, find A × B
- A × B = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c)}

- 2) A = {1, 2, 3} and B = {r,s}; then A × B = {(1,r), (1,s), (2,r), (2,s), (3,r),(3,s)}. Observe that the elements of A × B can be arranged in a convenient tabular array

$A \backslash B$	r	s
1	$(1, r)$	$(1, s)$
2	$(2, r)$	$(2, s)$
3	$(3, r)$	$(3, s)$

- If A and B are as in Example 2, then B × A = {(r, 1), (s, 1), (r, 2), (s, 2), (r, 3), (s, 3)}.

Using Matrices to Denote Cartesian Product

- For Cartesian Product of two sets, you can use a matrix to find the sets.
- Example: Assume A = {1, 2, 3} and B = {a, b, c}. The table below represents A × B.

	a	b	c
1	(1, a)	(1, b)	(1, c)
2	(2, a)	(2, b)	(2, c)
3	(3, a)	(3, b)	(3, c)

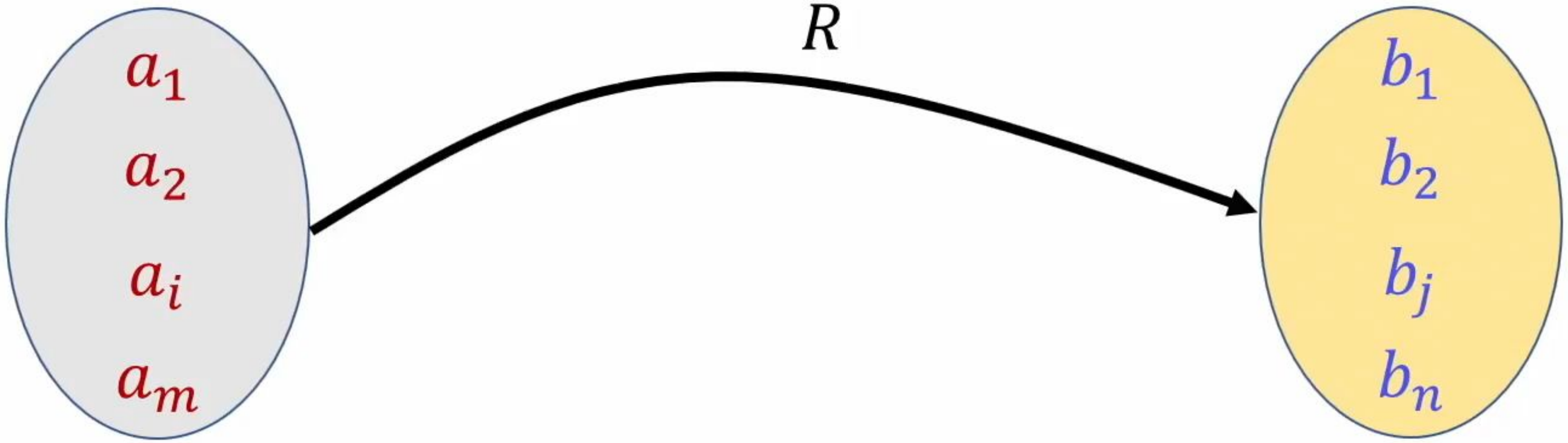
If A1, A2, …, Am are nonempty sets, then the Cartesian Product of them or Product Sets is the set of all ordered m-tuples (a1, a2, …, am), where ai $\in$ Ai, i = 1, 2m … m. Denoted A1 × A2 ×⋯× Am = {(a1, a2,..., am) | ai $\in$ Ai, i = 1, 2,..., m}

Theorem : Cardinality of Cartesian Product

The cardinality of the Cartesian Product equals the product of the cardinality of all of the sets:

$$| A_1 \times A_2 \times \ldots \times A_m | = | A_1 | \cdot | A_2 | \cdot \ldots \cdot | A_m |$$

Number of Binary Relations



$|A| = m$ $\qquad$ $|B| = n$

$|S| = n$

$|P(S)| = 2^n$

$|A \times B| = mn$

$|P(A \times B)| = 2^{mn}$

- How many relations possible from A to B ?
  - # of relations from A to B $= |P(A \text{ x } B)| = 2^{mn}$
  - Every relation (R) from A to B is a subset of $A \text{ x } B$

$A \times B = \{(a, b) : m\ a \in A,\ n\ b \in B\}$

$R \subseteq A \times B$

Subsets of the Cartesian Product

- Many of the results of operations on sets produce subsets of the Cartesian Product set
- Relational database
  - Each column in a database table can be considered a set
  - Each row is an m-tuple of the elements from each column or set
  - No two rows should be alike

Relations

- A relation, R, is a subset of a Cartesian Product that uses a definition to state whether an m-tuple is a member of the subset or not
- Terminology: ***Relation R from A to B***
- $R \subseteq A \times B$
- Denoted "x R y" where $x \in A$ and $y \in B$ and x has a relation with y
- If x does not have a relation with y, denoted

$$x \not{R}\, y$$

Relation Example

- A is the set of all students and B is the set of all courses
- A relation R may be defined as the course is required

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Relations Across Same Set

- Relations may be from one set to the same set, i.e., $A = B$
- Terminology: ***Relation R on A***
  $R \subseteq A \times A$

$S = \{1, 2\}$

$\mathcal{P}(S) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$ ✓

$S = \{1, 2, 3\}$

$\mathcal{P}(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\},$
$\{1, 2, 3\}\}$

$A, B \in \mathcal{P}(S)$ are related if
$A \subseteq B$.

<u>Congruence Modulo Relation</u>

$\mathbb{Z}$ = set of all integers.

$7 \in \mathbb{Z}$.

$x = 14 \quad y = 28 \quad y - x = 28 - 14 = 14$

$7 | 14 \quad x \equiv y \bmod 7$.

x congruent to y modulo 7.

$m \quad a, b \in \mathbb{Z}$

$a \equiv b \bmod m \quad$ if $\quad m \mid b - a$

Union, Intersection and Complement of Relations

- Suppose R and S are relations from A to B.

- For $a \in A$ and $b \in B$:
  - $a\ R \cup S\ b$ if and only if $a\ R\ b$ or $a\ S\ b$.
  - $a\ R \cap S\ b$ if and only if $a\ R\ b$ and $a\ S\ b$.
  - $a\ \overline{R}\ \ b$ if and only if $(a, b) \notin R$.

Union, Intersection and Complement of Relations

- Let $A = \{1,2,3,4\}$. $R = \{(1,2),(1,3),(2,4)\}$ and $S = \{(1,2),(2,3),(4,4)\}$ are two relations.

- $R \cup S = \{(1,2),(1,3),(2,4),(2,3),(4,4)\}$.

- $R \cap S = \{(1,2)\}$.

- $\bar{R} = A \times A \setminus R =$ the set of all elements that are in $A \times A$ but not in R

- The **domain of R:**
- denoted by **Dom(R)**
- is the set of elements in A that are related to some element in B
- Also called a subset of A, is the set of all first elements in the pairs that make up R
- The **range of R:**
- denoted by **Ran(R)**
- to be the set of elements in B that are second elements of pairs in R
- all elements in B that are paired with some element in A.
- **Example:**
- Let A = {1, 2, 3, 4, 5}. Define the following relation R (less than) on A:
- aRb if and only if $a < b$.
- Then
- R = {(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}.
- Dom(R) = A and Ran(R) = B

A={1,2,3,4,5}
R = A xA ={ (1,2),(1,3),(1,4).(1,5),

Relation on a Single Set Example skip

- A is the set of all courses
- A relation R may be defined as the course is a prerequisite
- CSCI 2150 R CSCI 3400
- R = {(CSCI 2150, CSCI 3400), (CSCI 1710, CSCI 2910), (CSCI 2800, CSCI 2910), …}

Example: Features of Digital Cameras skip

- Megapixels = {<2, 3 to 4, >5}
- battery life = {<200 shots, 200 to 400 shots, >400 shots}
- optical zoom = {none, 2X to 3X, 4X or better}
- storage capacity = {<32 MB, 32MB to 128MB, >128MB}
- price = Z+

Digital Camera Example (continued)

Possible relations might be:

- Priced below $X
- above a certain megapixels
- a combination of price below $X and optical zoom of 4X or better

Theorems of Relations

- Let R be a relation from A to B, and let A_1 and A_2 be subsets of A
  - If $A_1 \subseteq A_2$, then $R(A_1) \subseteq R(A_2)$
  - $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$
  - $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$
- Let R and S be relations from A to B. If $R(a) = S(a)$ for all a in A, then $R = S$.

R- relative set of x

If R is a relation from A to B and $x \in A$, we define $R(x)$, the ***R*-relative set of *x***, to be the set of all y in B with the property that x is R-related to y. Thus, in symbols,

$$R(x) = \{y \in B \mid x \; R \; y\}.$$

Similarly, if $A_1 \subseteq A$, then $R(A_1)$, the ***R*-relative set of A_1**, is the set of all y in B with the property that x is R-related to y for some x in A_1. That is,

$$R(A_1) = \{y \in B \mid x \; R \; y \text{ for some } x \text{ in } A_1\}.$$

Example:

Let $A = \{a, b, c, d\}$ and let $R = \{(a, a), (a, b), (b, c), (c, a), (d, c), (c, b)\}$. Then $R(a) = \{a, b\}$, $R(b) = \{c\}$, and if $A_1 = \{c, d\}$, then $R(A_1) = \{a, b, c\}$. ◆

Restriction of R to B

If R is a relation on a set A, and B is a subset of A, the restriction of R to B is R ∩ (B × B).
Example:

Let $A = \{a, b, c, d, e, f\}$ and $R = \{(a, a), (a, c), (b, c), (a, e), (b, e), (c, e)\}$. Let $B = \{a, b, c\}$. Then

$$B \times B = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$$

and the restriction of R to B is $\{(a, a), (a, c), (b, c)\}$. ◆

Exercise:

A) Find the domain, range, matrix, and, when $A = B$, the digraph of the relation R.

1. $A = \{1, 2, 3, 4, 8\} = B$; aRb if and only if $a = b$.
2. $A = \{1, 2, 3, 4, 6\} = B$; aRb if and only if a is a multiple of b.
3. $A = \{1, 2, 3, 4, 8\} = B$; aRb if and only if $a + b \leq 9$.

B) Let $A = \mathbb{R}$. Consider the following relation R on A: aRb if and only if $2a + 3b = 6$. Find Dom(R) and Ran(R).

Theorem:

Let R be a relation from A to B, and let A_1 and A_2 be subsets of A. Then:

(a) If $A_1 \subseteq A_2$, then $R(A_1) \subseteq R(A_2)$.

(b) $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$.

(c) $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$.

Matrix of a Relation

- We can represent a relation between two finite sets with a matrix
- $M_R = [m_{ij}]$, where

$$\text{Mij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ 0 \text{ if } (a_i, b_j) \notin R \end{cases}$$

Example of Using a Matrix to Denote a Relation

- Using the previous example where A = {1, 2, 3} and B = {a, b, c}. The matrix below represents the relation R = {(1, a), (1, c), (2, c), (3, a), (3, b)}.

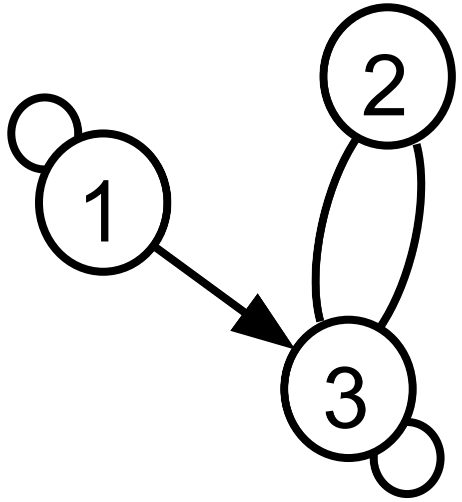
	a	b	c
1	1	0	1
2	0	0	1
3	1	1	0

Digraph of a Relation

- Let R be a relation on A
- We can represent R pictorially as follows
  - Each element of A is a circle called a vertex
  - If a_i is related to a_j, then draw an arrow from the vertex a_i to the vertex a_j
- In degree = number of arrows coming into a vertex
- Out degree = number of arrows coming out of a vertex

Representing a Relation

The following three representations depict the same relation on A = {1, 2, 3}.

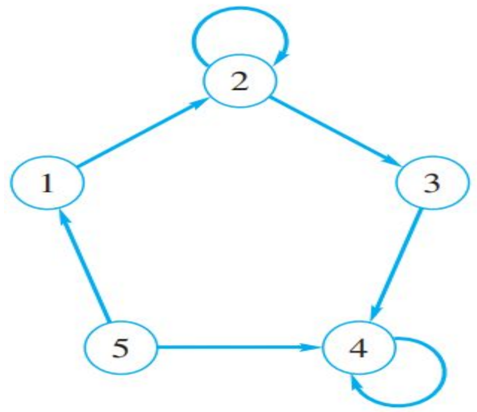


$$R = \{(1, 1), (1, 3), (2, 3), (3, 2), (3, 3)\}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

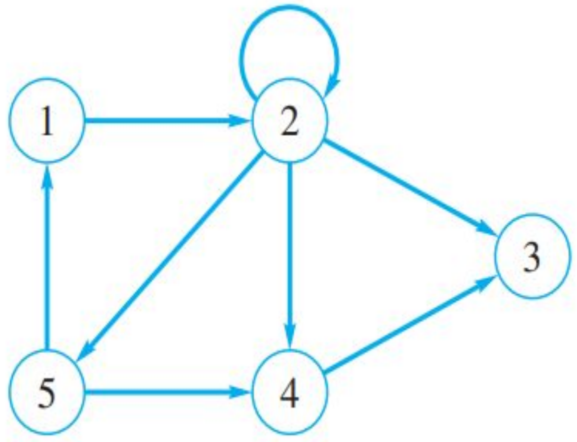
Exercise:

1. Find the relation determined by the digraph and give its matrix



Path in a relation and Diagraph

R is a relation on a set A. A path of length n in R from a to b is a finite sequence
π : a, x1, x2,..., xn−1, b, beginning with a and ending with b,
such that a R x1, x1 R x2, ..., xn−1 R b.
Note that a path of length n involves n + 1 elements of A, although they are not necessarily distinct.



Paths for diagraph :

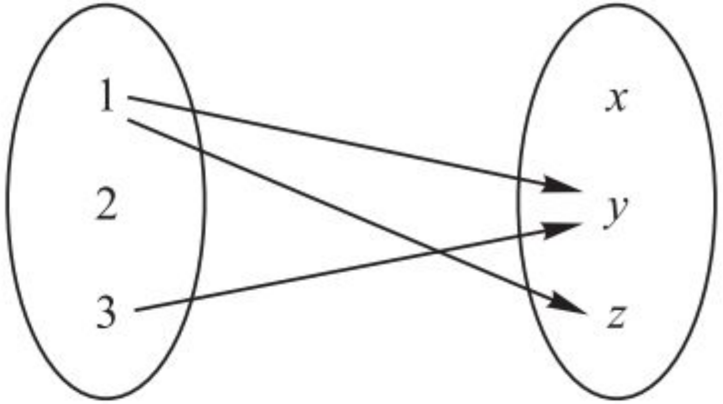
- π1 : 1, 2, 5, 4, 3 is a path of length 4 from vertex 1 to vertex 3,
- π2 : 1, 2, 5, 1 is a path of length 3 from vertex 1 to itself, And
- π3 : 2, 2 is a path of length 1 from vertex 2 to itself. A path that begins and ends at the same vertex is called a cycle.

- **Paths in a relation R can be used ---� to define new relations**
- If **n** is a fixed positive integer, we define a relation **R^n on A** as follows:
- **x R^n y** means **a path of length n from x to y in R**.
- We may also define a relation R^∞ on A, by letting **x R^∞ y mean that there is some path in R from x to y.**
- The relation **R^∞** is sometimes called the **connectivity relation for R**.

The relation R by the above two ways :

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

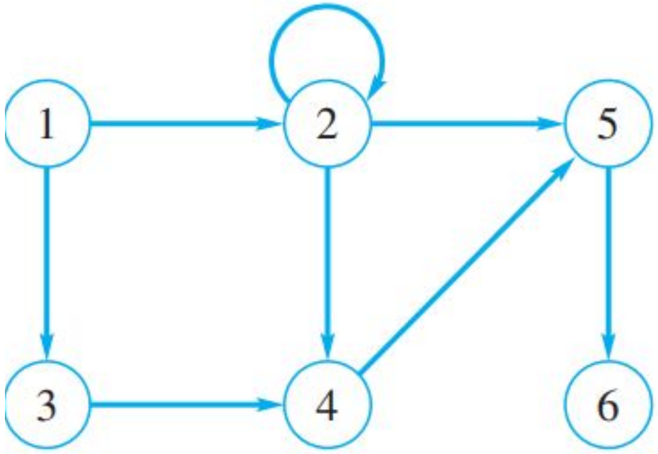
(i)



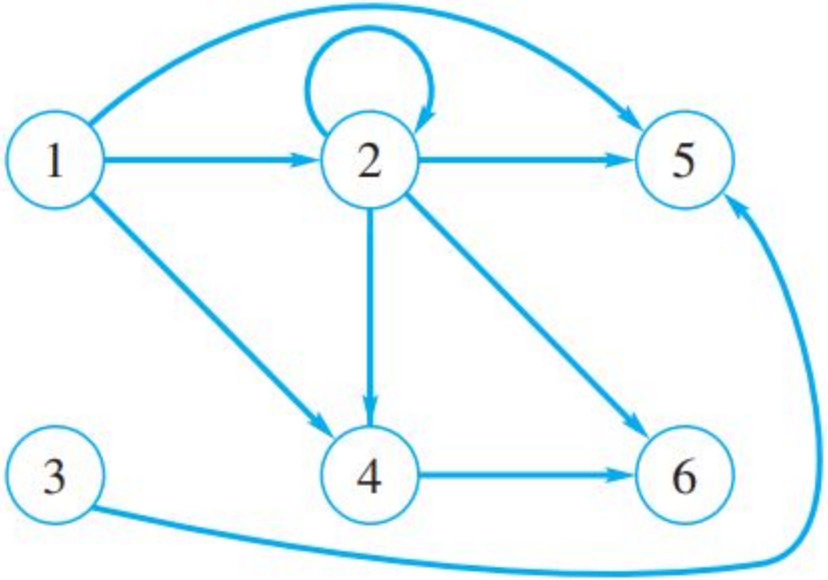
(ii)

$$R = \{(1, y),\ (1, z),\ (3, y)\}$$

Let A = {1, 2, 3, 4, 5, 6}. Let R be the relation whose digraph is shown in adjacent Figure.
If and only if they are $\mathbf{R^2}$– related
if and only if there is a path of length **two** connecting those vertices in Figure.
The ordered pairs relation $\mathbf{R^2}$ on A



$1\ R^2\ 2$	since	$1\ R\ 2$	and	$2\ R\ 2$
$1\ R^2\ 4$	since	$1\ R\ 2$	and	$2\ R\ 4$
$1\ R^2\ 5$	since	$1\ R\ 2$	and	$2\ R\ 5$
$2\ R^2\ 2$	since	$2\ R\ 2$	and	$2\ R\ 2$
$2\ R^2\ 4$	since	$2\ R\ 2$	and	$2\ R\ 4$
$2\ R^2\ 5$	since	$2\ R\ 2$	and	$2\ R\ 5$
$2\ R^2\ 6$	since	$2\ R\ 5$	and	$5\ R\ 6$
$3\ R^2\ 5$	since	$3\ R\ 4$	and	$4\ R\ 5$
$4\ R^2\ 6$	since	$4\ R\ 5$	and	$5\ R\ 6.$



we can construct the digraph of $\mathbf{R^n}$ for any n and try with the matrix of relation for the above problem.

COMPOSITION OF RELATIONS

Let A, B and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$.

Then R and S give rise to a relation from A to C denoted by $R \circ S$

And defined by: $a(R \circ S)c$ if for some $b \in B$ we have aRb and bSc.

That is ,

$$R \circ S = \{(a, c) \mid \text{there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$$

The relation $R \circ S$ is called the composition of R and S; it is sometimes denoted simply by RS

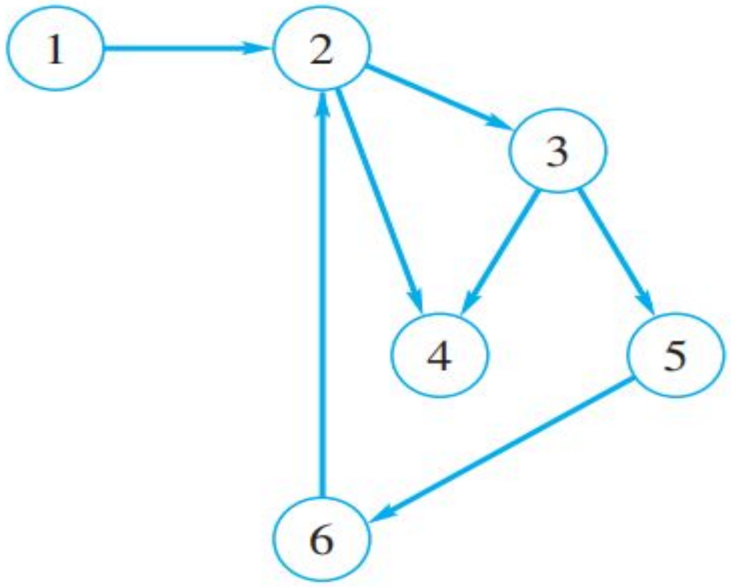
$R \circ R$, the composition of R with itself, is always defined.

Also, $R \circ R$ is sometimes denoted by $R^2 = R \circ R$.

Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$, and so on.

Thus R^n is defined for all positive n.

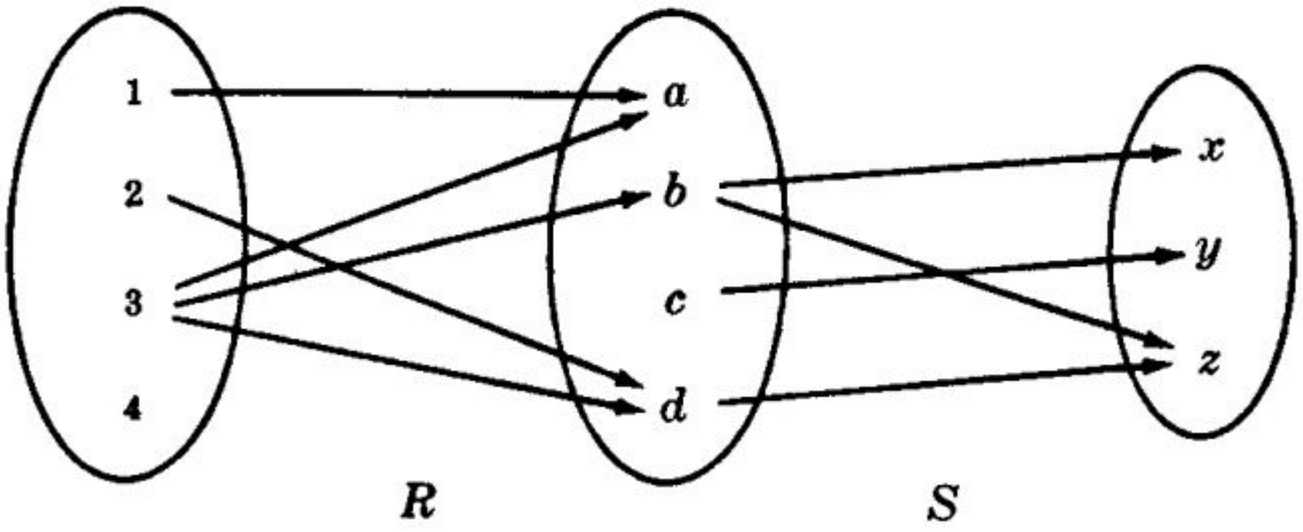
Let $\pi_1: a, x_1, x_2, \ldots, x_{n-1}, b$ be a path in a relation R of length n from a to b, and let $\pi_2: b, y_1, y_2, \ldots, y_{m-1}, c$ be a path in R of length m from b to c. Then the **composition of** π_1 **and** π_2 is the path $a, x_1, x_2, \ldots, b, y_1, y_2, \ldots, y_{m-1}, c$ of length $n + m$, which is denoted by $\pi_2 \circ \pi_1$. This is a path from a to c.



$$\pi_1: 1, 2, 3 \quad \text{and} \quad \pi_2: 3, 5, 6, 2, 4.$$

Then the composition of π_1 and π_2 is the path $\pi_2 \circ \pi_1: 1, 2, 3, 5, 6, 2, 4$ from 1 to 4 of length 6. ♦

Let A = {1, 2, 3, 4}, B = {a, b, c, d}, C = {x, y,z} and
Let R = {(1, a), (2, d), (3, a), (3, b), (3,d)} and S = {(b, x), (b, z), (c, y), (d, z)}



$R \circ S$ = {(2, z), (3, x), (3, z)}

Theorem: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then

$$(R \circ S) \circ T = R \circ (S \circ T)$$

Composition of Relations and Matrices

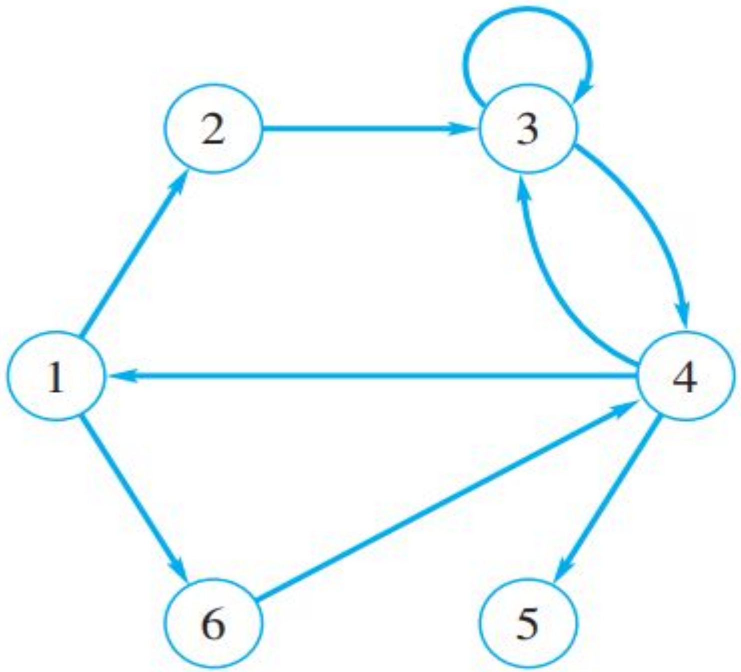
There is another way of finding R◦S. Let MR and MS denote respectively the matrix representations of the relations R and S. Then

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad M_S = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Multiplying $\mathbf{M_R}$ and $\mathbf{M_S}$ we obtain the matrix

The nonzero entries in this matrix tell us which elements are related by R◦S. Thus M = $\mathbf{M_R}\,\mathbf{M_S}$ and MR◦S have the same nonzero entries.

Solve:



1. List all paths of length 1.
2. (a) List all paths of length 2 starting from vertex 2.
   (b) List all paths of length 2.
3. (a) List all paths of length 3 starting from vertex 3.
   (b) List all paths of length 3.
4. Find a cycle starting at vertex 2.
5. Find a cycle starting at vertex 6.
6. Draw the digraph of R^2.
7. Find $\mathbf{M}_{R^2}$.
8. (a) Find R^∞.
   (b) Find $\mathbf{M}_{R^\infty}$.

Properties of Relations

A relation R on a set A is said to be

- reflexive: if $(a,a) \in R$ for all $a \in A$;
- irreflexive:if $(a,a) \notin R$ for all $a \in A$;
- symmetric: if $(a,b) \in R$, then $(b,a) \in R$, for all $a,b \in A$;
- antisymmetric: if $(a,b) \in R$ & $(b,a) \in R$, then $a=b$ for all $a,b \in A$
- asymmetric: if $(a,b) \in R$, then $(b,a) \notin R$, for all $a,b \in A$.
- transitive: if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$

Properties

1. Reflexive Relations

A relation R on a set A is reflexive if aRa for every a $\in$ A, that is, if (a, a) $\in$ R for every a $\in$ A.

Thus R is not reflexive if there exists a $\in$ A such that (a, a) $\notin$ R

Ex1: Consider the following five relations on the set A = {1, 2, 3, 4}:

R1 = {(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)}

R2 = {(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

R3 = {(1, 3), (2, 1)}

R4 = $\varnothing$, the empty relation

R5 = A × A, the universal relation

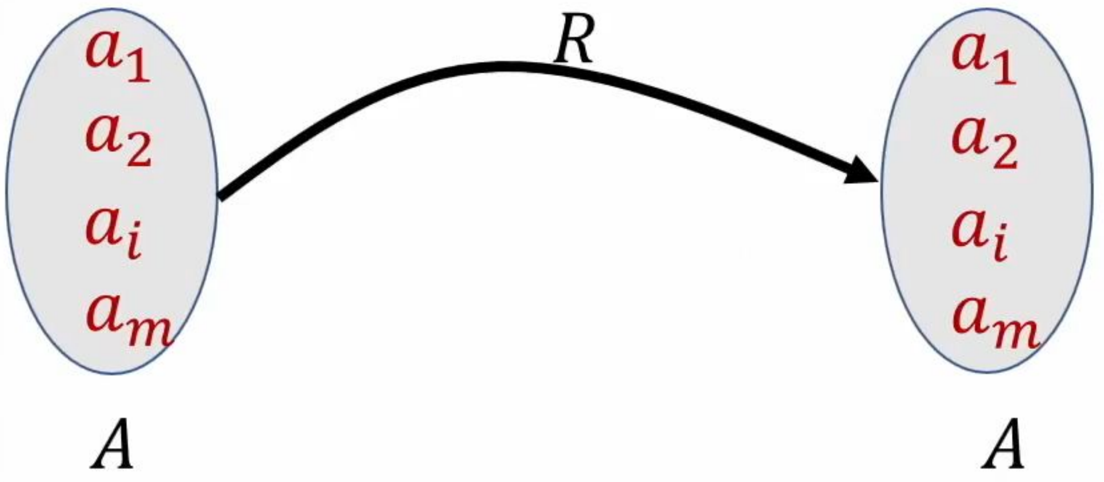
Determine which of the relations are reflexive.

Since A contains the four elements 1, 2, 3, and 4, a relation R on A is reflexive if it contains the four pairs (1, 1), (2, 2), (3, 3), and (4, 4).

Thus only R2 and the universal relation R5 = A × A are reflexive.

Note that **R1, R3, and R4 are not reflexive** since, for example, (2, 2) does not belong to any of them.

Reflexive Relation



- Relation R from A to A is reflexive if

  $\forall a: (a \in A \rightarrow (a, a) \in R)$ is true

- All diagonal entries of M_R will be 1
- Self loop at each node of the graph of R

$A \times A = \phi$

$R = \phi$

- Let $A = \{1, 2\}$. Which of the following are reflexive relations ?
  - $R_1 = \{(1,1), (2,2)\}$
  - $R_2 = \{(1,1), (2,2), (1,2)\}$
  - $R_3 = \{(1,1), (1,2), (2,1)\}$
  - $R_4 = \emptyset$

If $A = \phi$ then ϕ is a valid reflexive relation over A

- Can $\emptyset$ be a reflexive relation over any set A ?

- The matrix of a **reflexive relation** must have **all 1's on its main diagonal**, while
- the matrix of an **irreflexive** relation must have **all 0's on its main diagonal.**
- A reflexive relation has a cycle of length 1 at every vertex, while an irreflexive relation has no cycles of length 1.

- Reflexive: every element is related to itself.
- Irreflexive: no element is related to itself.
- Neither reflexive nor irreflexive: some elements are related to themselves but some aren't.

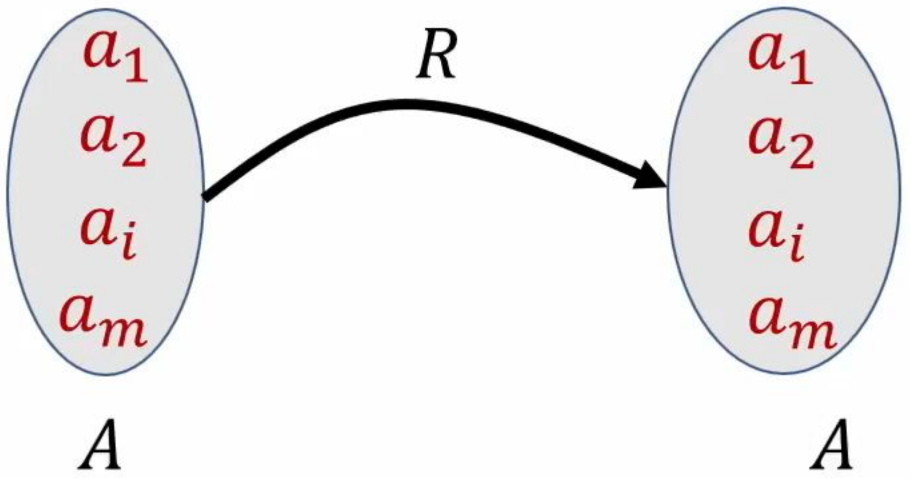
The formal definition states that if R is a relation on a set A then

- R is reflexive if xRx for all $x \in A$.
- R is irreflexive if $x \not R x$ for all $x \in A$.

Notice that irreflexive is not the negation of reflexive. The negation of reflexive would be:

- not reflexive: there is an $x \in A, x \not R x$

Irreflexive Relation



- ❑ Relation R from A to A is irreflexive if

$$\forall a: (a \in A \rightarrow (a, a) \notin R) \text{ is true}$$

- ❑ All diagonal entries of M_R will be 0
- ❑ No self loop at any node of the graph of R

❑ Let $A = \{1, 2\}$. Which of the following are irreflexive relations ?

A = ϕ

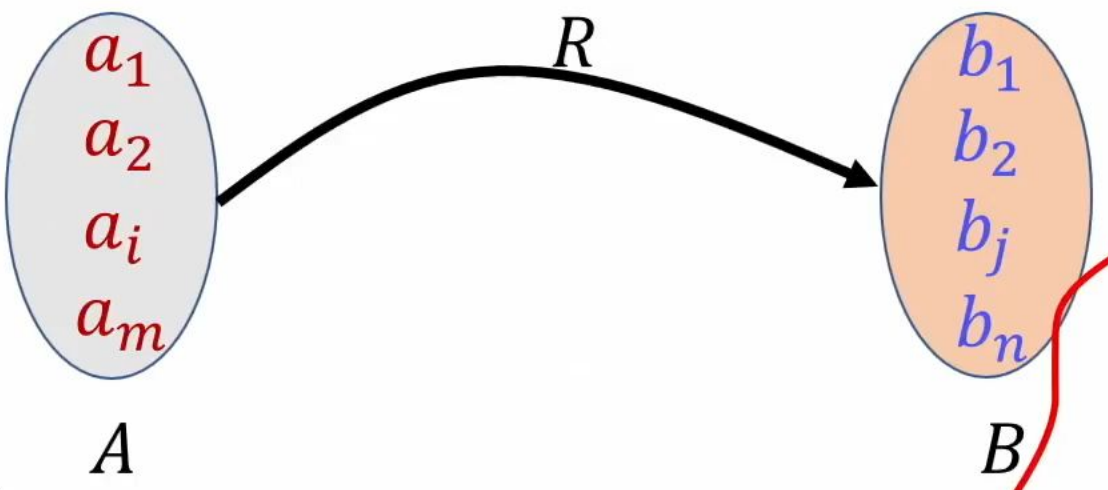
- ❖ $R_1 = \{(1,1), (2,2)\}$
- ❖ $R_2 = \{(1,1), (2,2), (1,2)\}$
- ❖ $R_3 = \{(1,1), (1,2), (2,1)\}$
- ❖ $R_4 = \emptyset$

R = ϕ

Reflexive Irreflexive

❑ Can a relation be both reflexive as well as irreflexive relation over any set A ?

Symmetric Relation



- Relation R from A to B is symmetric if

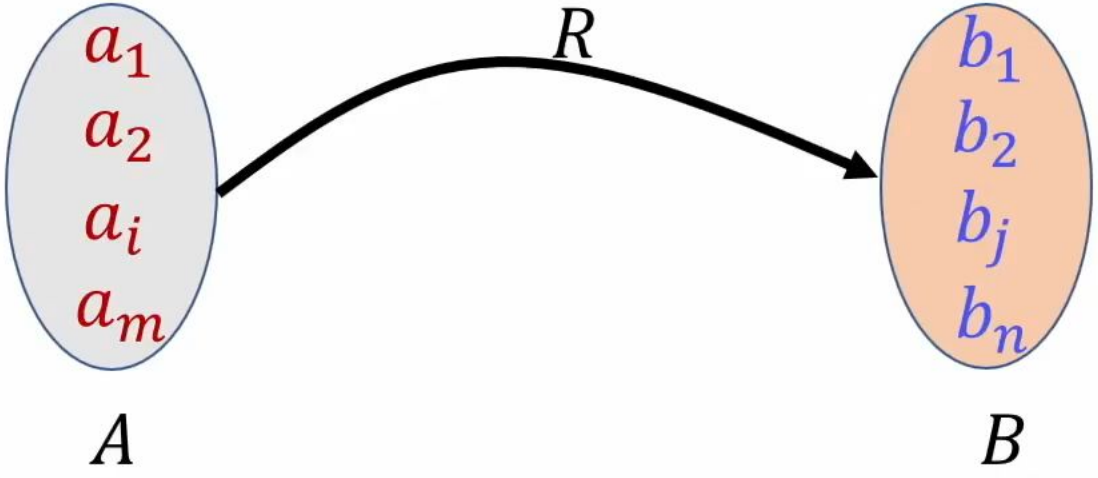
  $\forall a, b: [(a, b) \in R \rightarrow (b, a) \in R]$ is true
- M_R will be a symmetric matrix
- Edge (a_i, b_j) if and only if edge (b_j, a_i)

- Let $A = \{1, 2\}$. Which of the following are symmetric relations ?

  - ✓ $R_1 = \{(1, 1), (2, 2)\}$ a, b / b, a   a, b / b, a
  - ✓ $R_2 = \{(1, 2), (2, 1)\}$
  - ✓ $R_3 = \{(1, 1)\}$ a, b / b, a
  - ✓ $R_4 = \emptyset$ vacuously
  - ✗ $R_5 = \{(2, 1)\}$ a, b . but $(1,2) \notin R_5$

- Every reflexive relation is also a symmetric relation ?

Symmetric Relation



- ❑ Relation R from A to B is symmetric if $\forall a, b$: $[(a, b) \in R \rightarrow (b, a) \in R]$ is true
- ❑ M_R will be a symmetric matrix
- ❑ Edge (a_i, b_j) if and only if edge (b_j, a_i)

❑ Let $A = \{1, 2\}$. Which of the following are symmetric relations ?

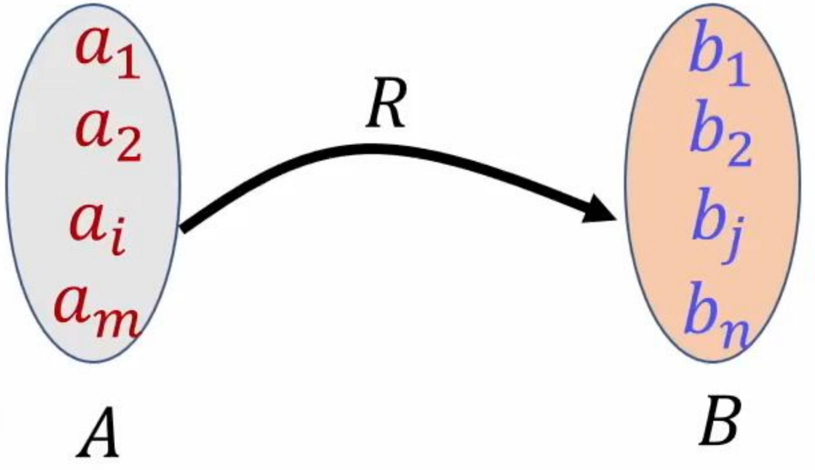
- ❖ $R_1 = \{(1,1), (2,2)\}$
- ❖ $R_2 = \{(1,2), (2, 1)\}$
- ❖ $R_3 = \{(1,1)\}$
- ❖ $R_4 = \emptyset$
- ❖ $R_5 = \{(2,1)\}$

❑ Every reflexive relation is also a symmetric ~~relation~~ ?

No

$R = \{(1,1), (2,2), (1,2)\}$

Asymmetric Relation



- Relation R from A to B is asymmetric if

$\forall a, b: [(a, b) \in R \rightarrow (b, a) \notin R]$ is true

- At most one of the entries among (i, j) or (j, i) can be 1 in M_R, for any i, j

1 1

diagonal entries will be 0

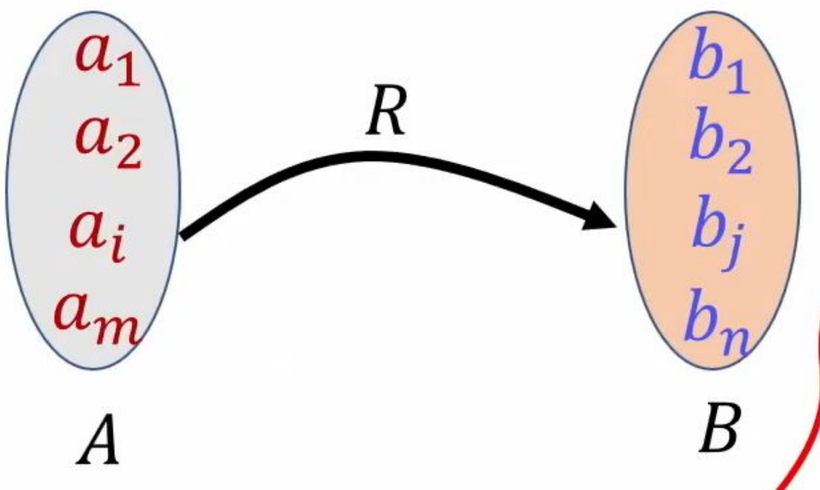
$(a_i, b_j) \in R$

$\wedge$

$(b_j, a_i) \in R$

$(a, a) \in R$

Asymmetric Relation



- Relation R from A to B is asymmetric if

  $\forall a, b: [(a, b) \in R \rightarrow (b, a) \notin R]$ is true

- At most one of the entries among (i, j) or (j, i) can be 1 in M_R, for any i, j
- Either edge (a_i, b_j) or (b_j, a_i) present for a_i, b_j

A = B

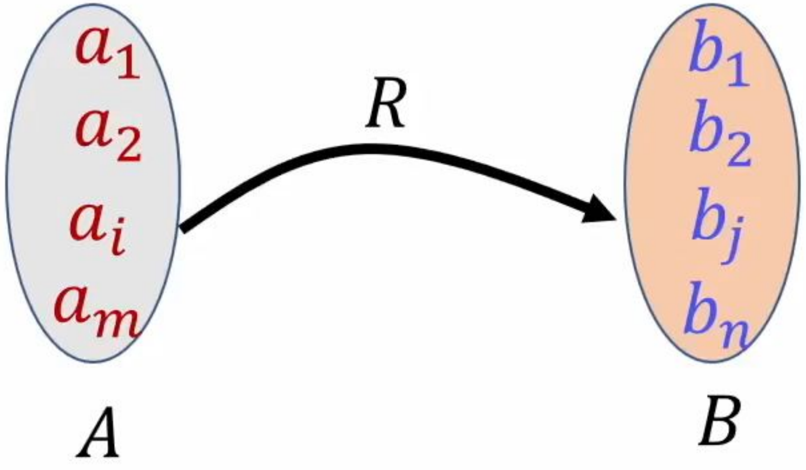
- Let $A = \{1, 2\}$. Which of the following are asymmetric relations ?

  - $R_1 = \{(1,1), (2,2)\}$
  - $R_2 = \{(1,2), (2, 1)\}$
  - $R_3 = \{(1,1)\}$

    (a, b)
    (b, a)
  - $R_4 = \emptyset$
  - $R_5 = \{(2,1)\}$

    (a, b)

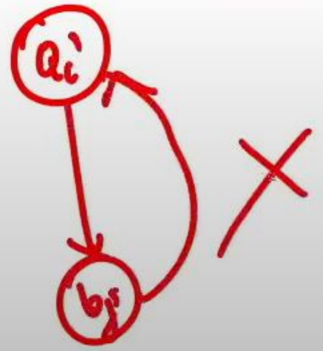
Antisymmetric Relation



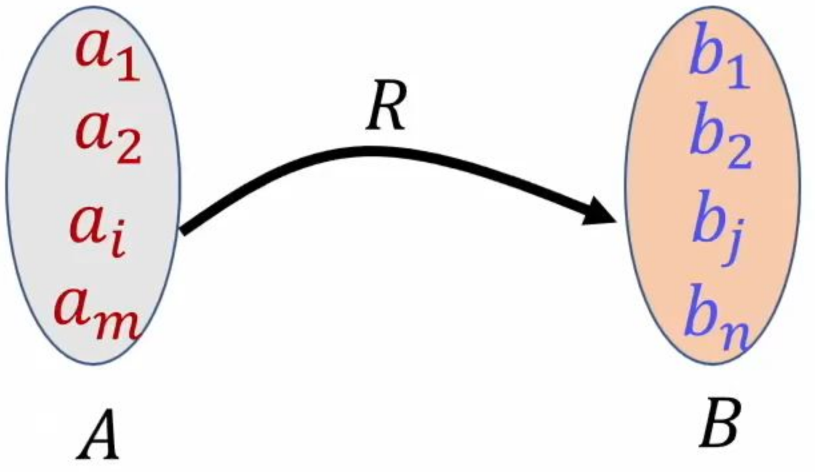
- Relation R from A to B is antisymmetric if

$\forall a, b: [(a, b) \in R \wedge (b, a) \in R \rightarrow (a = b)]$ is true

- (i, j) and (j, i) entries cannot be simultaneously 1 in M_R, for distinct i, j
- No edge (b_j, a_i) if edge (a_i, b_j) present for $a_i \neq b_j$



Antisymmetric Relation



- ❑ Relation R from A to B is antisymmetric if

  $\forall a, b: [(a, b) \in R \wedge (b, a) \in R \rightarrow (a = b)]$ is true
- ❑ (i, j) and (j, i) entries cannot be simultaneously 1 in M_R, for distinct i, j
- ❑ No edge (b_j, a_i) if edge (a_i, b_j) present for $a_i \neq b_j$

❑ Let $A = \{1, 2\}$. Which of the following are antisymmetric relations ?

- ❖ $R_1 = \{(1,1), (2,2)\}$
- ❖ $R_2 = \{(1,2), (2, 1)\}$

  (a, b) (b, a)
- ❖ $R_3 = \{(1,1)\}$
- ❖ $R_4 = \emptyset$
- ❖ $R_5 = \{(2,1)\}$

Symmetric vs Asymmetric vs Antisymmetric

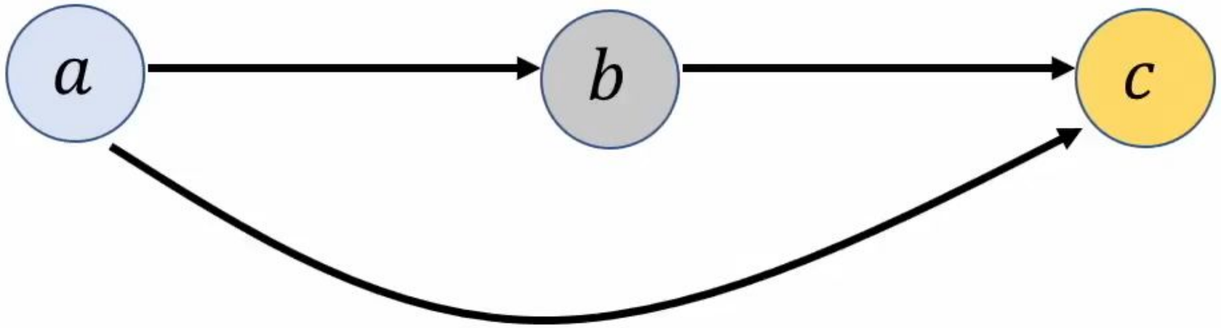
- ❑ Symmetric relation --- $\forall a, b: [(a, b) \in R \rightarrow (b, a) \in R]$
- ❑ Asymmetric relation --- $\forall a, b: [(a, b) \in R \rightarrow (b, a) \notin R]$
- ❑ Antisymmetric relation --- $\forall a, b: [(a, b) \in R \wedge (b, a) \in R \rightarrow (a = b)]$

❑ Absolutely no relationship:

- ❖ A relation can satisfy all the three properties
  - ➢ Ex: relation ∅ on the set $A = \{1, 2, 3\}$
- ❖ A relation may satisfy none of the three properties
  - ➢ Ex: relation $R = \{(1, 2), \underset{a,b}{(2, 3)}, \underset{b,a}{(3, 2)}\}$ on the set $\underline{A = \{1, 2, 3\}}$

Transitive Relation

- Transitive relation --- $\forall a, b, c : [(a, b) \in R \land (b, c) \in R \rightarrow (a, c) \in R]$



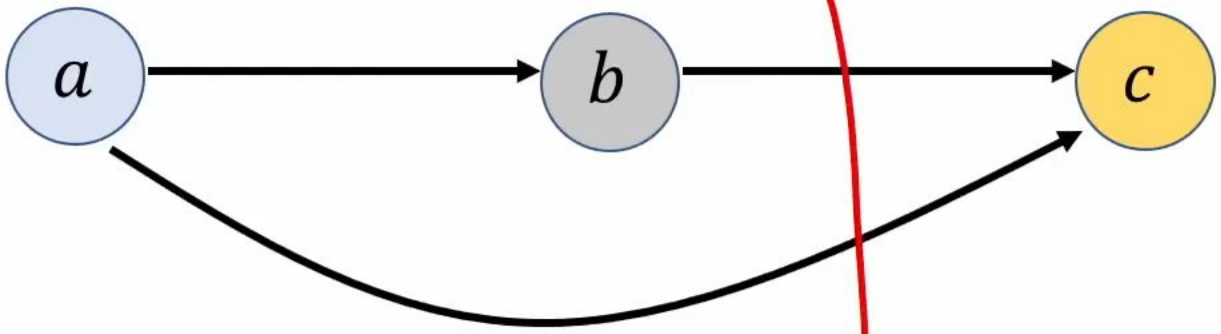
- Which of the following relations are transitive ?

A = {1,2)

- $R_1 = \{(1, 1), (2, 2)\}$ ✓✓
  - a, b / b, c / a, c
  - a, b / b, c / a, c
- $R_2 = \{(1, 2), (2, 1)\}$ ✗
  - (a, b) (b, c)
  - no (a, c)
- $R_3 = \{(1, 1)\}$
- $R_4 = \emptyset$
- $R_5 = \{(2, 1)\}$

Transitive Relation

- Transitive relation --- $\forall a, b, c : [(a, b) \in R \land (b, c) \in R \rightarrow (a, c) \in R]$



- Which of the following relations are transitive ?

$A = \{1, 2)$

❖ $R_1 = \{(1,1), (2,2)\}$ ❖ $R_2 = \{(1,2), (2,\ 1)\}$ ❖ $R_3 = \{(1,1)\}$

a, b
b, c
a, c

❖ $R_4 = \emptyset$ ❖ $R_5 = \{(2,1)\}$

,b

(a, b)

vacuously

Symmetric and Antisymmetric Relations

2. A relation R on a set A is symmetric

if whenever aRb then bRa, that is,

if whenever(a, b) $\in$ R then (b, a) $\in$ R.

Thus R is not symmetric or asymmetric if there exists a, b $\in$ A such that (a, b) $\in$ R but (b, a) $\notin$ R.

Ex1: Consider the following five relations on the set A = {1, 2, 3, 4}:

R1 = {(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)}

R2 = {(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

R3 = {(1, 3), (2, 1)}

R4 = $\varnothing$, the empty relation

R5 = A × A, the universal relation

R_1 is not symmetric since $(1, 2) \in R_1$ but $(2, 1) \notin R_1$. R_3 is not symmetric since $(1, 3) \in R_3$ but $(3, 1) \notin R_3$. The other relations are symmetric.

3. A relation R on a set A is antisymmetric

if whenever aRb and bRa then a = b,

that is, **if a = b and aRb then bRa**.

Thus R is not antisymmetric if there exist distinct elements a and b in A such that aRb and bRa.

Ex1: Consider the following five relations on the set A = {1, 2, 3, 4}:

R1 = {(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)}

R2 = {(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

R3 = {(1, 3), (2, 1)}

R4 = ∅, the empty relation

R5 = A × A, the universal relation

Exercise: Consider the following five relations:
Determine which relations are reflexive, symmetric, and antisymmetric.

(1) Relation $\leq$ (less than or equal) on the set $\mathbf{Z}$ of integers.

(2) Set inclusion $\subseteq$ on a collection C of sets.

(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.

(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.

(5) Relation $|$ of divisibility on the set $\mathbf{N}$ of positive integers. (Recall $x \mid y$ if there exists z such that $xz = y$.)

The relation (3) is not reflexive since no line is perpendicular to itself. Also (4) is not reflexive since no line is parallel to itself. The other relations are reflexive; that is, $x \leq x$ for every $x \in \mathbf{Z}$, $A \subseteq A$ for any set $A \in C$, and $n \mid n$ for every positive integer $n \in \mathbf{N}$.

The relation $\perp$ is symmetric since if line a is perpendicular to line b then b is perpendicular to a. Also, $\parallel$ is symmetric since if line a is parallel to line b then b is parallel to line a. The other relations are not symmetric. For example:

$$3 \leq 4 \text{ but } 4 \not\leq 3; \quad \{1, 2\} \subseteq \{1, 2, 3\} \text{ but } \{1, 2, 3\} \not\subseteq \{1, 2\}; \quad \text{and} \quad 2 \mid 6 \text{ but } 6 \nmid 2.$$

The relation $\leq$ is antisymmetric since whenever $a \leq b$ and $b \leq a$ then $a = b$. Set inclusion $\subseteq$ is antisymmetric since whenever $A \subseteq B$ and $B \subseteq A$ then $A = B$. Also, divisibility on $\mathbf{N}$ is antisymmetric since whenever $m \mid n$ and $n \mid m$ then $m = n$. (Note that divisibility on $\mathbf{Z}$ is not antisymmetric since $3 \mid -3$ and $-3 \mid 3$ but $3 \neq -3$.) The relations $\perp$ and $\parallel$ are not antisymmetric.

Remark: The properties of being symmetric and being antisymmetric are not negatives of each other. For example, the relation $R = \{(1, 3), (3, 1), (2, 3)\}$ is neither symmetric nor antisymmetric. On the other hand, the relation $R' = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric.

Symmetric: Transpose of a matrix and matrix are same. if mi j = 1, then m ji = 1. if m ji = 0, then mi j = 0 ,

Asymmetric: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

if mi j = 1, then m ji = 0

and mii = 0 for all i;

that is, the main diagonal of the matrix MR consists entirely of 0's.

Antisymmetric : if $i \neq j$, then mi j = 0 or m ji = 0

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \mathbf{M}_{R_1}$$

(a)

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \mathbf{M}_{R_2}$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{M}_{R_3}$$

(c)

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathbf{M}_{R_4}$$

(d)

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{M}_{R_5}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{M}_{R_6}$$

R1, R2…..Symmetric

R3 Antisymmetric, not asymmetric as 1’s in diagonal

R4….Not all 3 symmetric, asymmetric (4,1 1,4) antisymmetric

R5 : antisymmetric, not asymmetric

R6 : asymmetric and antisymmetric

What is an Equivalence Relation?

A relation R on a set A is called an equivalence relation if it is

1. **Reflexive Relation**: $(a, a) \in R \ \forall\ a \in A$, i.e. aRa for all $a \in A$.
2. **Symmetric Relation**: $\forall\ a, b \in A, (a, b) \in R \leftrightarrow (b, a) \in R$.
3. **Transitive Relation**: $\forall\ a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

where R is a subset of (A x A), i.e. the cartesian product of set A with itself.

Connectivity Relationship

- R : a relation over the set $A = \{a_1, a_2, \dots, a_i, \dots\}$

$R \subseteq A \times A$

- $R^* \overset{\text{def}}{=} R \cup R^2 \cup \cdots$
  - R^*: connectivity relationship
  - $(a_i, a_j) \in R^*$, if there exists some path from a_i to a_j in the directed graph of R

$(a_i, a_j) \in R^n$, iff there exists a path of length n from a_i to a_j in the directed graph of R

proved by Induction

either $(a_i, a_j) \in R^1$

or $(a_i, a_j) \in R^3$

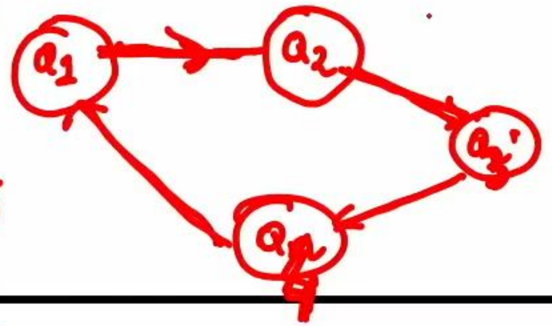
.

.

$(a_i, a_j) \in R^i$

$(a_i, a_j \in R^r$

Connectivity Relationship



$(a_1, a_2) \in R$

$(a_1, a_2) \in R^5$

- R : a relation over the set $A = \{a_1, a_2, \ldots, a_i, \ldots\}$

- $R^* \stackrel{\text{def}}{=} R \cup R^2 \cup \cdots$

$R^* = R \cup R^2 \cup R^3 \cup R^4 \mid R^5$

$(a_1, a_2) \in R^9$

$n = 4$

  - R^*: connectivity relationship
  - $(a_i, a_j) \in R^*$, if there exists some path from a_i to a_j in the directed graph of R

$(a_i, a_j) \in R^n$, iff there exists a path of length n from a_i to a_j in the directed graph of R

- What will be R^*, if $A = \{a_1, a_2, \ldots, a_n\}$? → R is defined over a finite set
  - $R^* = \underline{R \cup R^2 \cup \cdots \cup R^n} \mid \cup R^{n+1} \cup R^{n+2} \cup R^{n+3}$
    - Maximum path length can be n

maximum path length where path has distinct edges

Transitive Closure and Connectivity Relationship

- R : a relation over the set $A = \{a_1, a_2, \ldots, a_i, \ldots\}$
- $R^* \stackrel{\text{def}}{=} R \cup R^2 \cup \cdots$

$\forall a, b, c\, [(a,b) \in R^* \wedge (b,c) \in R^* \Rightarrow (a,c) \in R^*]$

- **Theorem**: Transitive closure of $R = R^*$
  - To prove the theorem we need to show the following

- R is present in R^* --- $R \subseteq R^*$
  - Follows from the definition of R^*

$R^* = R \cup R^2 \ldots\ldots$

- R^* is transitive : $(a,b) \in R^* \wedge (b,c) \in R^* \rightarrow (a,c) \in R^*$

arbitrary

Transitive Closure and Connectivity Relationship

- R : a relation over the set $A = \{a_1, a_2, \dots, a_i, \dots\}$
- $R^* \stackrel{\text{def}}{=} R \cup R^2 \cup \cdots$

$\forall a, b, c\, [(a,b) \in R^* \wedge (b,c) \in R^* \Rightarrow (a,c) \in R^*]$

- Theorem: Transitive closure of $R = R^*$
  - To prove the theorem we need to show the following

▪ R is present in R^* --- $R \subseteq R^*$

➢ Follows from the definition of R^*

▪ R^* is transitive : $(a, b) \in R^* \wedge (b, c) \in R^* \to (a, c) \in R^*$

➢ $(a, b) \in R^* \Rightarrow (a, b) \in R^j$

➢ $(b, c) \in R^* \Rightarrow (b, c) \in R^k$

$(a, c) \in R^{j+k} = R^k \circ R^j$

Transitive Closure and Connectivity Relationship

- R : a relation over the set $A = \{a_1, a_2, \ldots, a_i, \ldots\}$
- $R^* \stackrel{\text{def}}{=} R \cup R^2 \cup \cdots$

$\forall a, b, c\, [(a,b) \in R^* \wedge (b,c) \in R^* \Rightarrow (a,c) \in R^*]$

- **Theorem**: Transitive closure of $R = R^*$
  - To prove the theorem we need to show the following

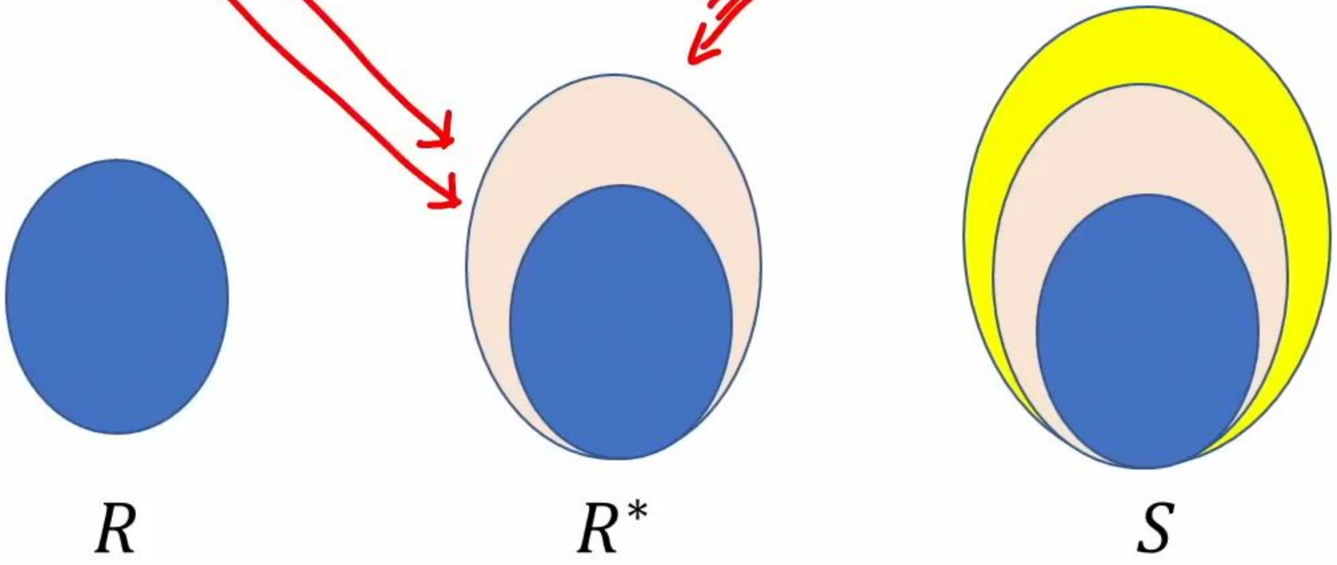
- R is present in R^* --- $R \subseteq R^*$
  - Follows from the definition of R^*

- R^* is transitive : $(a,b) \in R^* \wedge (b,c) \in R^* \rightarrow (a,c) \in R^*$
  - $(a,b) \in R^* \Rightarrow (a,b) \in R^j$
  - $(b,c) \in R^* \Rightarrow (b,c) \in R^k$

$$\left.\begin{array}{l}(a,b) \in R^* \Rightarrow (a,b) \in R^j \\ (b,c) \in R^* \Rightarrow (b,c) \in R^k\end{array}\right\} (a,c) \in R^{j+k} \Rightarrow (a,c) \in R^*$$

Transitive Closure and Connectivity Relationship

- **Theorem**: Transitive closure of $R = R^*$
  - R is present in R^* --- $R \subseteq R^*$ ✓
  - R^* is transitive : $(a, b) \in R^* \wedge (b, c) \in R^* \rightarrow (a, c) \in R^*$ ✓
  - R^* is the smallest transitive relation which includes R



If S is a transitive relation which includes R

Then R^* should be present in S

Operations on Relations:

Let R and S be relations from a set A to a set B. Then, if we remember that R and S are simply subsets of A × B, we can use set operations on R and S

1. complementary relation: the complement of R, $\overline{R}$ is referred to as the complementary relation

It is, of course, a relation from A to B that can be expressed simply in terms of R:

$$a\ \overline{R}\ b \quad \text{if and only if} \quad a \not R\ b.$$

2. We can also form the intersection R ∩ S and the union R ∪ S of the relations R and S. In relational terms, we see that a R ∩ S b means that aRb and aSb.

3. A different type of operation on a relation R from A to B is the formation of the inverse, usually written R^{-1}. The relation R^{-1} is a relation from B to A (reverse order from R) defined by b R^{-1} a if and only if aRb

Example1:

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Let

$$R = \{(1, a), (1, b), (2, b), (2, c), (3, b), (4, a)\}$$

and

$$S = \{(1, b), (2, c), (3, b), (4, b)\}.$$

Compute **(a)** $\overline{R}$; **(b)** $R \cap S$; **(c)** $R \cup S$; and **(d)** R^{-1}.

Solution

(a) We first find

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\}.$$

Then the complement of R in $A \times B$ is

$$\overline{R} = \{(1, c), (2, a), (3, a), (3, c), (4, b), (4, c)\}.$$

(b) We have $R \cap S = \{(1, b), (3, b), (2, c)\}$.

(c) We have

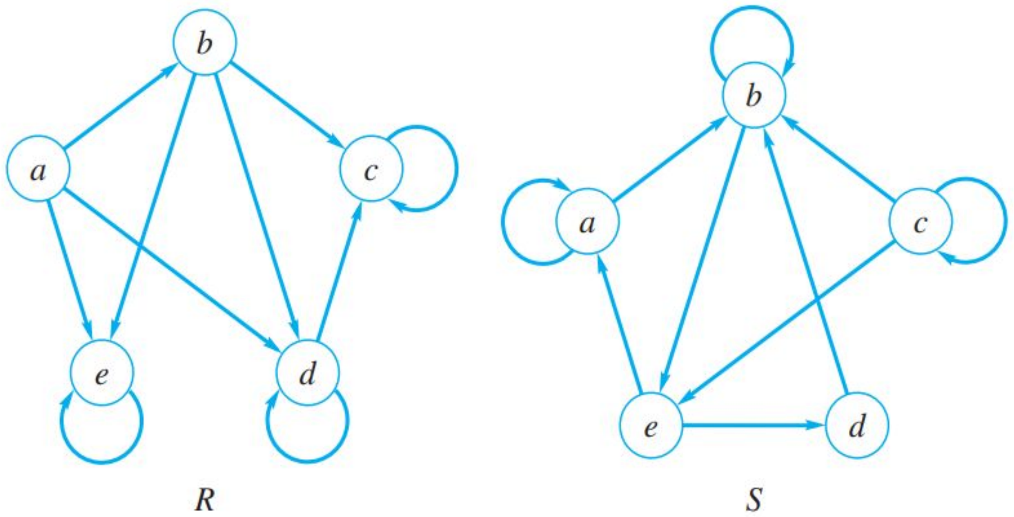
$$R \cup S = \{(1, a), (1, b), (2, b), (2, c), (3, b), (4, a), (4, b)\}.$$

(d) Since $(x, y) \in R^{-1}$ if and only if $(y, x) \in R$, we have

$$R^{-1} = \{(a, 1), (b, 1), (b, 2), (c, 2), (b, 3), (a, 4)\}.$$

Example 2:

Let A = {a, b, c, d, e} and let R and S be two relations on A whose corresponding digraphs are shown in Figure. Find $\overline{R}$, R^{-1} , $R \cap S$



$$\overline{R} = \{(a, a), (b, b), (a, c), (b, a), (c, b), (c, d), (c, e), (c, a), (d, b), (d, a), (d, e), (e, b), (e, a), (e, d), (e, c)\}$$

$$R^{-1} = \{(b, a), (e, b), (c, c), (c, d), (d, d), (d, b), (c, b), (d, a), (e, e), (e, a)\}$$

$$R \cap S = \{(a, b), (b, e), (c, c)\}.$$

Example 3: Let A = {1, 2, 3} and let R and S be relations on A. Suppose that the matrices of R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then we can verify that

$$\mathbf{M}_{\overline{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{M}_{R^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R\cap S} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{M}_{R\cup S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Recalling the operations on Boolean matrices

$$\mathbf{M}_{R\cap S} = \mathbf{M}_R \wedge \mathbf{M}_S$$
$$\mathbf{M}_{R\cup S} = \mathbf{M}_R \vee \mathbf{M}_S$$
$$\mathbf{M}_{R^{-1}} = (\mathbf{M}_R)^T.$$

Moreover, if $\mathbf{M}$ is a Boolean matrix, we define the **complement** $\overline{\mathbf{M}}$ of $\mathbf{M}$ as the matrix obtained from $\mathbf{M}$ by replacing every 1 in $\mathbf{M}$ by a 0 and every 0 by a 1. Thus, if

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

then

$$\overline{\mathbf{M}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

if R is a relation on a set A, then $\mathbf{M}_{\overline{R}} = \overline{\mathbf{M}}_R.$

a symmetric relation is a relation R such that

$\mathbf{M}_R = (\mathbf{M}_R)^T$ and since $(\mathbf{M}_R)^T = \mathbf{M}_{R^{-1}}$, we see that R is symmetric if and only if $R = R^{-1}$.

THEOREM 1 Suppose that R and S are relations from A to B.

(a) If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$.

(b) If $R \subseteq S$, then $\overline{S} \subseteq \overline{R}$.

(c) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ and $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

(d) $\overline{R \cap S} = \overline{R} \cup \overline{S}$ and $\overline{R \cup S} = \overline{R} \cap \overline{S}$.

THEOREM 2 Let R and S be relations on a set A.

(a) If R is reflexive, so is R^{-1}.

(b) If R and S are reflexive, then so are $R \cap S$ and $R \cup S$.

(c) R is reflexive if and only if $\overline{R}$ is irreflexive.

THEOREM 3 Let R be a relation on a set A. Then

(a) R is symmetric if and only if $R = R^{-1}$.

(b) R is antisymmetric if and only if $R \cap R^{-1} \subseteq \Delta$.

(c) R is asymmetric if and only if $R \cap R^{-1} = \varnothing$.

THEOREM 4 Let R and S be relations on A.

(a) If R is symmetric, so are R^{-1} and $\overline{R}$.

(b) If R and S are symmetric, so are $R \cap S$ and $R \cup S$.

THEOREM 5 Let R and S be relations on A.

(a) $(R \cap S)^2 \subseteq R^2 \cap S^2$.

(b) If R and S are transitive, so is $R \cap S$.

(c) If R and S are equivalence relations, so is $R \cap S$.