

Pumping Lemma

Pumping Lemma

- Very useful in Proving certain sets are not regular

For eg-

- $a^n b^n$ is not regular
- We cannot generate with FSM
- Finite State Automaton has a finite amount of memory
- If we want to accept $a^n b^n$, it should remember ‘n’
- But FSA cannot remember infinite number of values

Pumping Lemma:Why ?

- z = string that belongs to L
- $z=uvw$
- We can pump the middle portion “ v ”
- We can also have “ u ” empty
- i.e. The pump can be in the beginning also
- We are putting more and more weight to the middle portion
- We are pumping the middle portion
- Thus called Pumping Lemma

Pumping Lemma:Why ?

- $z=uvw$ such that $uv^iw \in L$ for $i>0$
- $i=1$, $z=uvw$
- $i=2$, $z=uvvw$
- $i=3$, $z=uvvvw$
- v is not empty $|v|>=1$
- $|uv|<=n$

What?

- To Show L is not Regular
- Problem Definition gives L
- Problem Definition gives n
- We choose z depending on question such that $z=uvw$
- We show that $uv^iw \in L$
- Thus not regular \checkmark

Pumping Lemma:How?

- Step 1-Assume language L is regular, Let n be the number of states in the corresponding FA
- Step 2- Choose a string z in language L such that $|z| > n$, Use pumping lemma to write the string $z = uvw$ with $|uv| \leq n$, $|v| \geq 1$ such that $uv^i w \in L$ for $i > 0$
- Step 3-Find a suitable integer “i” such that $uv^i w \notin L$. This contradicts our assumption. Hence L is not regular

$$|uv| < n$$

$\textcircled{u} \textcircled{v}^i \textcircled{w}$

Pumping Lemma :Example 1

→ $L = \{a^{n^2} \mid n \geq 1\}$

Step 1-Suppose L is regular

Let n be the number of states in FA accepting L

$$1 \leq |v| \leq n$$

Step 2-Choose a string $z \in L$ such that $|z| > n$

z can be written in the form $z = uvw$ with $|uv| \leq n$ and $|v| \geq 1$

such that $uv^i w \in L$ for $i > 0$

Consider a $=aaaaa.....aaaaaaa$ for n^2 no of times

String of a^{n^2} aaaaaaaaaaaaaa.....aaaaaaa

u v w

uv^2w uv^3w

$$1 \leq v \leq n$$

Pumping Lemma: Example 1

Step 3- So length of v alone is $1 \leq |v| \leq n$

For $i=2$,

$$\begin{aligned} |uv^2w| &= |uvw| + |v| \\ &= n^2 + |v| \\ &\leq n^2 + n \end{aligned}$$

$$\begin{aligned} n=1 & \quad a^1 \rightarrow a^1 \rightarrow a = 1 = n^2 \\ n=2 & \quad a^{2^2} \rightarrow [aaaa] = 4 = n^2 \\ n=3 & \quad a^{3^2} \rightarrow [a^9] = 9 = n^2 \end{aligned}$$

we get a^P such that value of P is $\leq n^2 + n$ and $> n^2$

After $L = \{a, a^4, a^9, \dots\}$ $a_{n^2}, a_{(n+1)^2}, \dots\}$

The string higher than a is a $a_{(n+1)^2}$ i.e. a_{n^2+2n+1}

But $n^2 + n$ lies between these two values, Thus it is not regular

Pumping Lemma :Example 2

$$L = \{a^n \mid n \geq 1\}$$

Step 1-Suppose L is regular

Let n be the number of states in FA accepting L

Step 2-Choose a string $z \in L$ such that $|z| > n$

z can be written in the form $z = uvw$ with $|uv| \leq n$ and $|v| \geq 1$
such that $uv^i w \in L$ for $i > 0$

Consider a ↑

$n=1$ String=a

$n=2$ String=aa

$n=3$ String=aaaaaa

$$\begin{aligned} &= a^1! &= |a^1| &= 1 &\checkmark \\ &= a^{2!} &= |a^2| = |aa| &= 2 &\checkmark \\ &= a^{3!} &= a^6 = |\text{aaaaaa}| &= 6 &\checkmark \end{aligned}$$

$n=9$

n

Pumping Lemma :Example 2

$$uv^iw$$
$$1 \leq |v| \leq n$$

Step 3- So length of v alone is $1 \leq |v| \leq n$

$$\begin{aligned} |uv^2w| &= |uvw| + |v| \\ &= n! + |v| \\ &\leq n! + n \\ &\quad \cancel{\text{---}} \end{aligned}$$

$$\begin{aligned} &n! + n \\ &\cancel{\text{---}} \end{aligned}$$
$$\cancel{n!} + n!$$
$$n \times n! + n!$$

we get a^P such that value of P is $\leq n! + n$ and $> n!$

After $L = \{a, a^2, a^6, a^{24}, \dots\}$

$$\begin{aligned} a^n &= a^{n^2} \\ n=1 & n=2 & n=3 & n=4 \\ (n+1)! &= (n+1) \cdot n! \\ &= n \times n! + n! \end{aligned}$$

$$\begin{aligned} &a_{n!} & a_{(n+1)!} \\ &\cancel{n=n} & \cancel{n=(n+1)} \end{aligned}$$

But $n! + n$ lies between these two values, Thus it is not regular

Pumping Lemma :Example 3

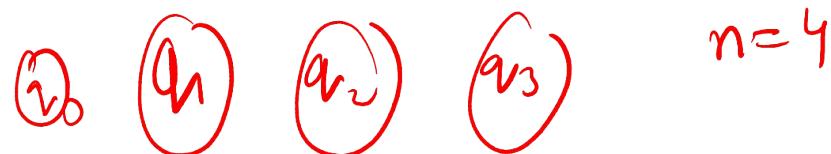
$$L = \{a^n b^n \mid n \geq 1\}$$

Step 1-Suppose L is regular

Let n be the ~~number of~~ states in FA accepting L

Step 2-Choose a string $z \in L$ such that $|z| > n$

z can be written in the form $z = uvw$ with $|uv| \leq n$ and $|v| \geq 1$
such that $uv^i w \in L$ for $i > 0$



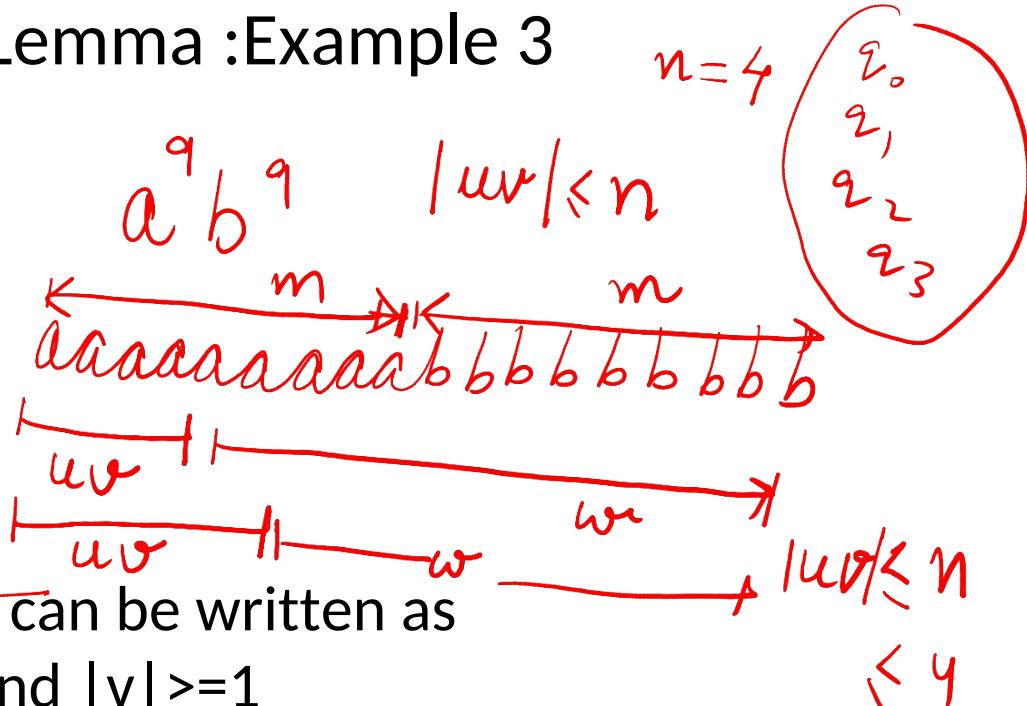
$\overbrace{|b|01010110|}^z$ $|z| > n$ or $|z| > 4$

Pumping Lemma :Example 3

$$L = \{a^n b^n \mid n \geq 1\}$$

Consider $a^m b^m$, $m > n$
~~aaaaaa.....abbbbbbb.....b~~

$$n < m$$



Using Pumping Lemma, This can be written as
 $z = uvw$, such that $|uv| \leq n$ and $|v| \geq 1$

As $m > n$ and $|uv| \leq n$

~~aaaaaaaaaaa.....aaabb...bbb...bbb...bbb...bbb~~

$$n=6$$

v will not fall into "b"



Pumping Lemma :Example 3

As $m > n$

aaaaaaaaaaaa.....abbbbbbbbbb

Let $v = a^p$

~~u = a^q~~

~~w = a^rb^m~~

~~p+q+r=m~~

~~=~~

Let $z = a^q (a^p) a^r b^m$

Pumping v part

Let $z = a^q (a^p)^i a^r b^m$

So many strings, we can get where no of a's are not equal to no of bs

There is ~~a~~ Contradiction

Thus, Not Regular



$uv \Rightarrow aaaa \dots$

$w = aa \dots bbbb$

$\underbrace{m}_{\text{symbols of } a} = p + q + r$ \overbrace{m}

~~0~~
~~Contradiction~~
~~Thus, Not Regular~~