Quantum Mechanics – Exercise Set 1: de Broglie equations

- Data:
  Mass of electron $m_e = 9.1 \times 10^{-31}$ kg
  Mass of proton $m_p = 1.67 \times 10^{-27}$ kg
  Speed of light c in vacuum $= 3 \times 10^8$ m/s
  Planck's constant $h = 6.63 \times 10^{-34}$ J-s
  Reduced Planck's constant $\hbar = h/2\pi = 1.055 \times 10^{-34}$ J-s
  Elementary charge $q = 1.6 \times 10^{-19}$ C

- Class-work:

1) Calculate de'Broglie wavelength of a subatomic particle of mass $200m_e$ moving with a speed of 0.98c.
2) Calculate de'Broglie wavelengths of (i) cricket ball of mass 450 gm thrown at a speed of 150 km/hr (ii) electron in a cathode ray tube (CRT) fired at a speed of 10^6 m/s. Comment on your results.
3) What is de'Broglie wavelength of a neutron having energy 1 MeV. Use $m_n = m_p$.
4) Calculate de'Broglie wavelength of a proton accelerated through 100 kV potential difference.
5) Through what potential difference must a deuteron be accelerated in order to have de'Broglie wavelength of 15 fm? Given $m_d = 2m_p$ and $q_d = q$.
6) Find kinetic energy of an electron whose de'Broglie wavelength is the same as that of a photon possessing energy 100 keV.
7) An electron and a proton have the same de'Broglie wavelengths. Compare their kinetic energies. Given $m_p = 1800\ m_e$.
8) An electron and a muon (μ^-) are accelerated through the same potential difference. How do their de'Broglie wavelengths compare? Given $m_\mu = 207\ m_e$ and $q_\mu = q_e = q$
9) Estimate the minimum energy possessed by an electron in an atom using uncertainty principle.
10) What is the uncertainty in the determination of position of a particle if its momentum is measured to be 2×10^{-24} kg-m/s with an uncertainty of 0.05%?
11) Calculate uncertainty in the determination of momentum of an electron confined to a quantum well of size 1 nm. What is the percentage uncertainty in the momentum if its mean speed is 10^6 m/s?
12) Determine uncertainty in the measurement of momentum of a marble of mass 10 gm confined to a box of dimensions 50 cm. What is the percentage uncertainty in the momentum if it is moving with a speed of 20 cm/s. Is it significant as compared to the result of preceding example? What can you say about the measurement?
13) Calculate the percentage uncertainty in the measurement of momentum of a neutron having energy 20 MeV confined to a region of width equal to (i) 3 nuclei (ii) 3 atoms. Comment on the results.
14) The position of a proton is determined within an accuracy of 1 Å. Determine uncertainty in the measurement of its position 1 μs later.
15) The lifetime of an excited state of nucleus is usually 1 ps. Estimate uncertainty in energy of a γ-ray emitted by a nucleus.
16) Calculate the width of a spectral line if the transition giving rise to this spectral line has occurred during 0.01 μs seconds. What fraction it is of the frequencies of light in the visible spectrum?
17) Calculate the energy, momentum and de' Broglie wavelength of an electron trapped in a one-dimensional quantum well of size 10 Å in its ground state.

18) Calculate the difference in energies of the first two allowed states for (i) electron confined to a strip of 10 nm and (ii) marble of mass 10 gm confined to a box of size 10 cm. What can you say about the results?

19) The wave function of a particle is given by $\varphi(x) = \sqrt{\frac{\pi}{2}}\, x;\ 0 \leq x \leq 1$. Find the probability that the particle can be found between x = 0.45 to x = 0.55.

20) Determine the normalization constant for a particle whose wave function is given by $\varphi(x) = Ae^{-2x}$ in the interval x = 0 to L when L = 0.2.

- Answers to Class-Work:
(1) 1.24×10^{-14} m, (2) 3.54×10^{-35} m, this value is beyond any measurement. Hence, wave nature is not revealed at macroscopic level, 7.286×10^{-10} m, this value is verifiable experimentally. Hence, wave nature is significant at microscopic level, (3) 2.868×10^{-14} m, (4) 9.07×10^{-14} m, (5) 1.83×10^{6} volt, (6) 9768 eV, (7) 1800:1, (8) 14.387:1, (9) 3.4 eV (10) ~ 0.1 micron, (11) 11.6%, (12) 1.055×10^{-30} %. Uncertainty in momentum is infinitesimally small and hence insignificant as compared to earlier example. Momentum is measured with perfect accuracy, (13) (i) 34%, (ii) 0.00034%. In first case, the position of neutron is almost accurate which leads to larger uncertainty in its momentum. In the second case, the position uncertainty is large, which effectively reduced the momentum uncertainty, (14) 0.83 mm (15) 0.00066 eV (16) 100 MHz, part of a million, (17) 0.377 eV, $\pm 3.315 \times 10^{-15}$ kg-m/s, 20 Å, (18) Electron: 0.011 eV; energies are discrete, Marble: ~ 10^{-45} eV; difference in energies are so small that energy can be regarded as continuous, (19) 3.9% (20) ± 2.69.

Numericals on Quantum Mechanics

Q.1. Mass = 200 me

Velocity $v = 0.98c$

Formula for De-broglie's Wavelength $\lambda = \frac{h}{mv}$

$$= \frac{6.64 \times 10^{-34}}{200 \times 9.1 \times 10^{-31} \times 0.98 \times 3 \times 10^{8}}$$

$$= 1.24 \times 10^{-14} \text{ m.}$$

Q.2: (i) De-Broglie's Wavelength for a cricket ball of mass 450 gm thrown at a speed of 150 km/hr.

$$\lambda_b = \frac{h}{mv} = \frac{6.64 \times 10^{-34}}{0.45 \times 41.66} = 3.54 \times 10^{-35} \text{ m.}$$

(ii) An electron in a CRT fired at a speed of 10^6 m/s.

$$\lambda_e = \frac{h}{mv} = \frac{6.64 \times 10^{-34}}{9.1 \times 10^{-31} \times 10^{6}} = 7.296 \times 10^{-10} \text{ m.}$$

From the results, it is quite evident the ~~the~~ de-broglie's wavelength associated with the cricket ball is too small to measure, hence it is not applicable for macroscopic bodies.

While, the de broglie's wavelength for an electron has already been measured & the results are verified.

Q.3. Energy of the neutron = 1 MeV = 10^6 eV.

$$E = 10^6 \text{ eV} = 1.6 \times 10^{-13} \text{ J}$$

$$m_n = 1.67 \times 10^{-27} \text{ Kg.}$$

$\lambda = \frac{h}{\sqrt{2mE}}$ ⇒ As momentum $p = \sqrt{2mE}$ for charged particle with energy E.

$\therefore \quad \lambda = 2.86 \times 10^{-14}$ m.

Q.4:- Accelerating voltage of a proton $V = 100$ kV.
Mass of a proton $= 1.67 \times 10^{-27}$ kg.
De broglies wavelength for an electron/proton accelerating through a voltage V is given as

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV}}$$

$$= \frac{6.64 \times 10^{-34}}{(2 \times 1.67 \times 10^{-27} \times 1.6 \times 10^{-19} \times 10^{5})}$$

$$= 9.08 \times 10^{-14} \text{ m.}$$

Q.5. Given $\lambda = 15$ fm $= 15 \times 10^{-15}$ m.

$m_d = 2m_p = 2 \times 1.67 \times 10^{-27}$ kg

$q_d = q$

As De-broglies wavelength is $\lambda = \frac{h}{\sqrt{2 e_d m_d V}}$

$$\lambda^2 = \frac{h^2}{2meV} \Rightarrow V = \frac{h^2}{2 m_d e_d \lambda^2}$$

$$V = 1833247 \text{ V} = 1.83 \times 10^{6} \text{ V}$$

Q.6. Energy possessed by a photon $= 100$ keV

Energy of a photon $= \frac{hc}{\lambda}$

$$\therefore \lambda = \frac{hc}{E_{ph.}} = \frac{h}{P_e}$$

$\therefore \lambda$ (wavelength of the photon is) $= \frac{6.64 \times 10^{-34} \times 3 \times 10^{8}}{E \; 100 \text{ keV}}$

$\lambda = 1.24 \times 10^{-11}$ m.

Kinetic energy of an electron in terms of momentum is given as

$$K.E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

As $p = h/\lambda$

$$\therefore K.E = \frac{h^2}{2m\lambda^2} = 1.567 \times 10^{-15} \text{ J}$$

$$\text{or } KE = \frac{1.567 \times 10^{-15}}{1.6 \times 10^{-19}} = 9782 \text{ eV}.$$

Q.7: Given : $\lambda_e = \lambda_p$; $m_p = 1800\, m_e$

Kinetic energy in terms of momentum is given as

$$KE = \frac{p^2}{2m} \quad \left[\lambda_e = \frac{h}{P_e} ; \lambda_p = \frac{h}{P_p} \right]$$

For an electron $(KE)_{el.} = \frac{p_e^2}{2m_e} = \frac{h^2}{2m_e\lambda_e^2}$

Similarly for a proton $(KE)_{pr.} = \frac{p_p^2}{2m_p} = \frac{h^2}{2m_p\lambda_p^2}$

If we take the ratio $\frac{(KE)_{el}}{(KE)_{pr}} = \frac{h^2/2m_e\lambda_e^2}{h^2/2m_p\lambda_p^2}$

As $\lambda_e = \lambda_p$ i.e. $\frac{(KE)_{el.}}{(KE)_{pr.}} = \frac{m_p}{m_e} = \frac{1800\, m_e}{m_e}$

$$\therefore (KE)_{el.} = 1800\, (KE)_{pr.}$$

Q.8. Accelerating voltage for an electron and muon are same.

$V_e = V_\mu$

$m_\mu = 207\, m_e$

$q_\mu = q_e = q$

De-broglie's wavelength for such cases λ is given as

$$\lambda = \frac{h}{\sqrt{2meV}}$$

For electrons $\lambda_e = \frac{h}{\sqrt{2 m_e e_e V_e}}$

For muons $\lambda_m = \frac{h}{\sqrt{2 m_\mu e_\mu V_\mu}}$

We take the ratio of λ_e to λ_μ

$$\therefore \frac{\lambda_e}{\lambda_m} = \frac{h / \sqrt{2 m_e e_e V_e}}{h / \sqrt{2 m_\mu e_\mu V_\mu}}$$

As V_e & V_μ also e_e & e_μ are same

$$\therefore \frac{\lambda_e}{\lambda_\mu} = \sqrt{\frac{m_\mu}{m_e}} = \sqrt{\frac{207 m_e}{m_e}}$$

$$\therefore \lambda_e = \sqrt{207}\ \lambda_\mu$$

$$\lambda_e = 14.38\ \lambda_\mu$$

Q.9. Minimum energy possessed by an electron Uncertainty Principle is given as

$$\Delta x . \Delta p \simeq \frac{\hbar}{2}$$

The radius of the ~~nucleus~~ atom is of the order of 10^{-10} m.

Uncertainty in the position of the e^- will be within the dimension of the size of the ~~nucleus~~ atom.

$$\therefore \Delta x = 2 \times 10^{-10} \text{ m.}$$

From the uncertainty principle, the minimum uncertainty in the momentum is given as

$$\Delta p = \frac{\hbar}{\Delta x} = \frac{1.05 \times 10^{-34}}{2\times 10^{-10}\ m} = \cancel{51.21 \times 10^{-24}\ kgm/s} \Rightarrow 5.2\times 10^{-25}\ kgm/s$$

It is assumed to be the momentum of an electron.
The minimum energy of the electron in the atom is given as

$$E_{min} = p_{min}\, c$$
$$= 5.2 \times 10^{-25} \times 3 \times 10^{8}$$
$$= 1.575 \times 10^{-16}\ J$$
$$= 984.375\ eV$$

Q.10 Momentum $p = 2\times 10^{-24}$ kg-m/s.
Uncertainty in momentum is $= 0.05\%$.
$\therefore$ $\Delta p = 0.05\%$ of p i.e $\Delta p = 10^{-27}$ kg m/s.

From HUP $\Delta x . \Delta p = \hbar$

$$\therefore \Delta x = \frac{\hbar}{\Delta p} = \frac{1.05 \times 10^{-34}}{10^{-27}} = 1.05 \times 10^{-7}\ m$$

$\therefore$ $\Delta x \simeq 0.1\ \mu m$.

Q.11. Electron confined in the quantum well $= 1$ nm.
$\Delta x = 10^{-9}$ m.
From the uncertainty principle $\Delta x\, \Delta p \simeq \hbar$

$$\Delta p \simeq \frac{\hbar}{\Delta x} = \frac{1.05\times 10^{-34}}{10^{-9}} = 1.05 \times 10^{-25}\ kgm/s$$

Mean speed $\Delta v = 10^{6}$ m/s.
Considering this value if the uncertainty in momentum is calculated. $\Delta p = m \Delta v$.

$$m_e = 9.1 \times 10^{-31}\ kg.$$
$$\Delta p = 9.1 \times 10^{-31} \times 10^{6} = 9.1 \times 10^{-25}\ kg\ m/s.$$

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Percentage uncertainty in momentum is given as

$$\% \text{ Uncertainty} = \frac{1.05 \times 10^{-25}}{9.1 \times 10^{-25}} = 11.5\%.$$

Q.12. Given : Mass of the marble = 10 gm. = 0.01 kg

Dimension of the box (Uncertainty in position)

i.e $\Delta x = 50 \text{ cm} = 0.5 \text{ m}$

Mean speed $v = 20 \text{ cm/s} = 0.2 \text{ m/s}$.

Uncertainty in momentum $\Delta p = \frac{\hbar}{\Delta x} = \frac{1.05 \times 10^{-34}}{0.5}$

$$\Delta p = 2.1 \times 10^{-34} \text{ kg m s}^{-1}.$$

From the given mass & velocity if we calculate the uncertainty in momentum, we get

$$\Delta p = m v = 0.01 \times 0.2 = 0.002 \text{ kg m/s}.$$

$$\% \text{ Uncertainty in momentum} = \frac{2.1 \times 10^{-34}}{0.002} \times 100$$

$$= 1.05 \times 10^{-29} \%$$

This value is infinitesimally small to be measured. It is not significant. It can't be measured accurately.

Q.13. Energy of the neutron = 20 MeV

$$E = 20 \times 10^{6} \text{ eV} = 20 \times 10^{6} \times 1.6 \times 10^{-19}$$

$$= 3.2 \times 10^{-12} \text{ J}.$$

Momentum of the neutron is : $p = \sqrt{2mE}$

$$p = \sqrt{2 \times 1.67 \times 10^{-27} \times 3.2 \times 10^{-12}}$$

$$p = 1.034 \times 10^{-19} \text{ kg m/s}$$

(i) Uncertainty for 3 nuclei

$$\Delta x = 3 \times 10^{-15} \text{ m}$$

$$\therefore \Delta p = \frac{\hbar}{\Delta x} = \frac{1.05 \times 10^{-34}}{3 \times 10^{-15}} = 3.5 \times 10^{-20} \text{ kg m/s}.$$

Uncertainty in momentum is $\frac{\Delta p}{p}$

$\therefore$ Uncertainity $= \dfrac{3.5\times10^{-20}}{1.034\times10^{-19}} = 0.339 \simeq 0.34$

% Uncertainity $= 34\%$.

(ii) Uncertainity for 3 atoms.

$\Delta x = 3\times10^{-10}$ m

$\Delta p = \dfrac{\hbar}{\Delta x} = \dfrac{1.05\times10^{-34}}{3\times10^{-10}} = 3.5\times10^{-25}$ kg m/s.

As p is already calculated $\therefore p = 1.034\times10^{-19}$ kg m/s

$\therefore$ Uncertainity in momentum $= \dfrac{\Delta p}{p} = \dfrac{3.5\times10^{-25}}{1.034\times10^{-19}}$

$= 3.385\times10^{-6}$

% Uncertainity $= 3.385\times10^{-4}$ %

$= 0.000338$ %

Q.14. $\Delta x = 1A^\circ = 10^{-10}$ m.

time delay $= 1\mu s$.

$\Delta p = \dfrac{\hbar}{\Delta x} = 1.055\times10^{-24}$ kg m/s.

As $p = mv \Rightarrow \Delta p = m\Delta v$

$\Delta v = \dfrac{\Delta p}{m} = \dfrac{1.055\times10^{-24}}{1.67\times10^{-27}}$

$= 631$ m/s

As $\Delta v = \dfrac{\Delta x}{\Delta t} \Rightarrow \Delta x = \Delta v\, \Delta t$

$\Delta x = 631\times10^{-6}$ m

$\Delta x = 0.631$ mm

Q.15: $\Delta t = 1\,ps = 10^{-12}\,s$

Uncertainty in energy & time is given by

$$\Delta E . \Delta t \simeq \hbar$$

$$\Delta E = \frac{\hbar}{\Delta t} = \frac{1.05 \times 10^{-34}}{10^{-12}} = 1.05 \times 10^{-22}\,J$$

$$\Delta E = 0.00066\,eV.$$

Q.16. $\Delta E . \Delta t \simeq \hbar$

$$\Delta t = 0.01\,\mu s = 10^{-8}\,s.$$

$$\therefore \Delta E = \frac{\hbar}{\Delta t} = 1.05 \times 10^{-26}\,J$$

Now to calculate the frequency / width of the spectral lines

$$\Delta E = \hbar \Delta \nu \Rightarrow \Delta \nu = \frac{\hbar}{\Delta E}$$

$$\therefore \Delta \nu = \frac{1.05 \times 10^{-26}}{1.05 \times 10^{-34}} = 10^{8}\,Hz = 100\,MHz$$

λ range of visible spectrum $= 4000\,A^\circ - 7000\,A^\circ$.

ν is of the range of $7.5 \times 10^{14} \sim 4.3 \times 10^{14}$ Hz

$\therefore$ It is almost ppm of the range of visible light frequencies.

Q.17. Width of the quantum well $= 10\,A^\circ = L = 10^{-9}\,m$

$$E_n = \frac{n^2 h^2}{8 m L^2}$$

For ground state energy $n = 1$

$$\therefore E = \frac{h^2}{8 m L^2}$$

Substitute the values of the known constant.

(i) $$E = \frac{(6.63 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (10^{-9})^2} = 6.023 \times 10^{-20}\,J$$

$$E = 0.375\,eV.$$

(ii) To calculate the momentum $= p^2 = 2mE$

$\therefore p = \sqrt{2mE}$

(Substitute E value in Joules)

$\therefore p = (2 \times 9.1 \times 10^{-31} \times 6.03 \times 10^{-20})^{1/2}$

$p = 3.312 \times 10^{-25}$ kg m/s

(iii) De Broglie's wavelength $\lambda = \frac{h}{p}$

$\therefore \lambda = \frac{6.64 \times 10^{-34}}{3.312 \times 10^{-25}} = 2 \times 10^{-9}$ m $= 20 A^{o}$.

Q.18 – (i) For an electron confined in a strip of 10 nm.

$\therefore L = 10 nm = 10 \times 10^{-9} m = 10^{-8}$ m.

$E_n = \frac{n^2 h^2}{8mL^2}$

First two allowed energy states i.e $n = 1$ & $n = 2$

$\therefore \Delta E = E_2 - E_1 = \frac{4h^2}{8mL^2} - \frac{h^2}{8mL^2} = \frac{3h^2}{8mL^2}$

$\therefore \Delta E = \frac{3 \times (6.63 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (10^{-8})^2} = 1.8 \times 10^{-21}$ J

$\Delta E = 0.011$ eV. (Energy is discrete)

(ii) For a marble of mass = 10 gm

$m = 0.01$ kg

$L = 10$ cm $= 0.1$ m.

As. $\Delta E = \frac{3h^2}{8mL^2} = \frac{3 \times (6.63 \times 10^{-34})^2}{8 \times 0.01 \times (0.1)^2} = 1.65 \times 10^{-63}$ J

$\Delta E = 10^{-44}$ eV (Difference in the energies is very small so energy can be regarded as continuous.

Q.19. $\phi(x) = \sqrt{\frac{\pi}{2}}\, x \quad 0 \le x \le 1$

limits $x = 0.45$ to $x = 0.55$.

$$\text{Probability} = \int_{L_1}^{L_2} |\psi(x)|^2 dx$$

$$P = \therefore \int_{0.45}^{0.55} \left(\sqrt{\frac{\pi}{2}}\, x\right)^2 dx = \frac{\pi}{2}\int_{0.45}^{0.55} x^2 dx$$

$$= \frac{\pi}{6}\left[x^3\right]_{0.45}^{0.55} = 0.039$$

Probability (%) = 3.9 %

Q.20 :- $\phi(x) = Ae^{-2x}$

Given interval $x = 0$ to L & $L = 0.2$

To determine the normalization constant the probability has to be equal to unity.

$$\therefore P = \int_0^L |\phi(x)|^2 dx = 1$$

$$\therefore \int_0^L \left[Ae^{-2x}\right]^2 dx = 1$$

$$A^2 \int_0^L e^{-4x} dx = 1 \Rightarrow A^2 \left(\frac{e^{-4x}}{-4}\right)\Bigg|_0^L$$

$$\Rightarrow \frac{-A^2}{4}\left[e^{-4L} - e^0\right] = 1$$

As $L = 0.2$

$$\therefore \frac{-A^2}{4}\left[e^{-0.8} - 1\right] = 1$$

$$\Rightarrow A^2 = 7.29 \Rightarrow A = \sqrt{7.29} = \pm 2.69.$$