

# Pumping Lemma

# Pumping Lemma

- Very useful in Proving certain sets are not regular

For eg-

- $a^n b^n$  is not regular
- We cannot generate with FSM
- Finite State Automaton has a finite amount of memory
- If we want to accept  $a^n b^n$ , it should remember 'n'
- But FSA cannot remember infinite number of values

# Pumping Lemma: Why ?

- $z$  = string that belongs to  $L$
- $z = uvw$
- We can pump the middle portion “ $v$ ”
- We can also have “ $u$ ” empty
- i.e. The pump can be in the beginning also
- We are putting more and more weight to the middle portion
- We are pumping the middle portion
- Thus called Pumping Lemma

# Pumping Lemma: Why ?

- $z=uvw$  such that  $uv^i w \in L$  for  $i>0$

- $i=1, z=uvw$

- $i=2, z=uvvw$

- $i=3, z=uvv vw$

- $v$  is not empty  $|v| \geq 1$

- $|uv| \leq n$

# What?

- To Show L is not Regular
- Problem Definition gives L
- Problem Definition gives n
- We choose z depending on question such that  $z=uvw$
- We show that  $uv^i w \in L$
- Thus not regular ✓

# Pumping Lemma:How?

- Step 1-Assume language  $L$  is regular, Let  $n$  be the number of states in the corresponding FA
- Step 2- Choose a string  $z$  in language  $L$  such that  $|z| > n$ , Use pumping lemma to write the string  $z = uvw$  with  $|uv| \leq n$ ,  $|v| \geq 1$  such that  $uv^i w \in L$  for  $i > 0$
- Step 3-Find a suitable integer “ $i$ ” such that  $uv^i w \notin L$ . This contradicts our assumption. Hence  $L$  is not regular

$$|uv| \leq n$$

$$uv^i w$$

# Pumping Lemma : Example 1

$$1 \leq |v| \leq n$$

$$L = \{a^{n^2} \mid n \geq 1\}$$

Step 1-Suppose L is regular

Let n be the number of states in FA accepting L

Step 2-Choose a string  $z \in L$  such that  $|z| > n$

$z$  can be written in the form  $z = uvw$  with  $|uv| \leq n$  and  $|v| \geq 1$

such that  $uv^i w \in L$  for  $i > 0$

Consider a = aaaaaa.....aaaaaaaaa for  $n^2$  no of times

String of  $a^{n^2}$  aaaaaaaaaaaa.....aaaaaaaaaa

$$u \quad v \quad w$$

$uv^2w$   $uv^3w$  - - -

$$1 \leq |v| \leq n$$

## Pumping Lemma: Example 1

Step 3- So length of  $v$  alone is  $1 \leq |v| \leq n$

For  $i=2$ ,

$$|uv^2w| = |uvw| + |v|$$

$$= n^2 + |v|$$

$$\leq n^2 + n$$

$n=1$   $a^1 \rightarrow a^1 \rightarrow |a| = 1 = n^2$

$n=2$   $a^{2^2} \rightarrow |aaaa| = 4 = n^2$

$n=3$   $a^{3^2} \rightarrow |a^9| = 9 = n^2$

we get  $a^P$  such that value of  $P$  is  $\leq n^2 + n$  and  $> n^2$

After  $L = \{a, a^4, a^9, \dots, a_{n^2}, a_{(n+1)^2}, \dots\}$

The string higher than  $a$  is  $a$  i.e.  $a$

$n^2$   $(n+1)^2$   $n^2 + 2n + 1$

But  $n^2 + n$  lies between these two values, Thus it is not regular



# Pumping Lemma : Example 2

$$L = \{a^{n!} \mid n \geq 1\}$$

Step 1- Suppose L is regular

Let n be the number of states in FA accepting L

Step 2- Choose a string  $z \in L$  such that  $|z| > n$

z can be written in the form  $z = uvw$  with  $|uv| \leq n$  and  $|v| \geq 1$  such that  $uv^i w \in L$  for  $i > 0$

Consider a

$n=1$  String = a

$n=2$  String = aa

$n=3$  String = aaaaaa

$$\begin{aligned} &= a^{1!} = |a^1| = 1 \quad \checkmark \\ &= a^{2!} = |a^2| = |aa| = 2 \quad \checkmark \\ &= a^{3!} = a^6 = |aaaaaa| = 6 \quad \checkmark \end{aligned}$$

$|string| = |w| = n!$

$n=4$

$n$

# Pumping Lemma :Example 2

$uv^i w$

$$1 \leq |v| \leq n$$

Step 3- So length of v alone is  $1 \leq |v| \leq n$

$$|uv^2w| = |uvw| + |v|$$

$$= n! + |v|$$

$$\leq n! + n$$

$$n! + n$$

$$n!$$

$$n \times n! + n!$$

we get  $a^P$  such that value of P is  $\leq n! + n$  and  $> n!$

After  $L = \{a, a^2, a^6, a^{24}, \dots, a_{n!}, a_{(n+1)!}, \dots\}$

$a_{n!}^2$

$$(n+1)! = (n+1) \cdot n!$$

$$= n \times n! + n!$$

$$a_{n!}, a_{(n+1)!}$$

But  $n! + n$  lies between these two values, Thus it is not regular

# Pumping Lemma :Example 3

$$L = \{a^n b^n \mid n \geq 1\}$$

Step 1-Suppose L is regular

Let  $n$  be the ~~number of states~~ in FA accepting L

Step 2-Choose a string  $z \in L$  such that  $|z| > n$

$z$  can be written in the form  $z = uvw$  with  $|uv| \leq n$  and  $|v| \geq 1$   
such that  $uv^i w \in L$  for  $i > 0$

$q_0$   $q_1$   $q_2$   $q_3$   $n=4$

$\rightarrow$   $|0|0|0|1|1|0|$   $z$   $|z| > n$  or  $|z| > 4$

# Pumping Lemma :Example 3

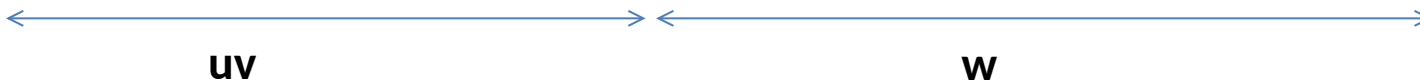
$$L = \{a^n b^n \mid n \geq 1\}$$

Consider  $a^m b^m$ ,  $m > n$   
~~aaaaa.....abbbbbbb.....b~~

Using Pumping Lemma, This can be written as  
 $z = uvw$ , such that  $|uv| \leq n$  and  $|v| \geq 1$

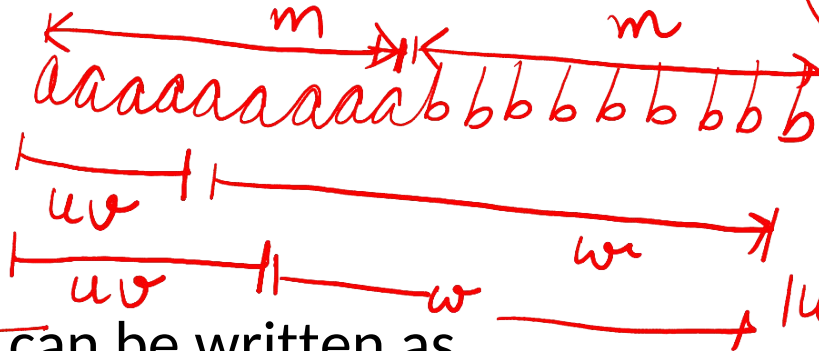
~~As  $m > n$  and  $|uv| \leq n$~~   
 aaaaaaaaaa.....aaabbbbbbbbbbbbbbbbbbb

      
 $v$  will not fall into "b"

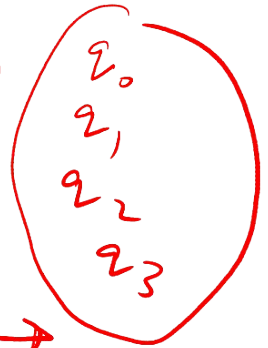


$$n < m$$

$$a^9 b^9 \quad |uv| \leq n$$



$$n = 4$$



$$|uv| \leq n$$

$$< 4$$

$$n = 6$$

# Pumping Lemma : Example 3

As  $m > n$

aaaaaaaaaaaa.....abbbbbbbbbbbbbbbbbbbbbbb

Let  $v = a^p$

$u = a^q$

$w = a^r b^m$

$p + q + r = m$

Let  $z = a^q (a^p)^i a^r b^m$

Pumping  $v$  part

Let  $z = a^q (a^p)^i a^r b^m$

So many strings, we can get where no of a's are not equal to no of bs

There is a Contradiction

Thus, Not Regular

$uv \Rightarrow aaaaa.....$

$w = aa.....bbbbbb$

$\longleftarrow m \longrightarrow$

$m = p + q + r$

symbols of a  
b b b b