Q.2]

$$Z = \frac{x^2+y^2}{x+y}$$

$$\frac{\partial z}{\partial x} = \frac{(x+y)2x-(x^2+y^2)}{(x+y)^2} = \frac{x^2-y^2+2xy}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)2y-(x^2+y^2)}{(x+y)^2} = \frac{-x^2+y^2+2xy}{(x+y)^2}$$

$$LHS = \left[\frac{x^2+2xy-y^2+x^2-2xy-y^2}{(x+y)^2}\right]^2$$

$$= \left[2\cdot\frac{(x^2-y^2)}{(x+y)^2}\right]^2 = \left[2\frac{(x-y)}{(x+y)}\right]^2 = \frac{4(x-y)^2}{(x+y)^2}$$

$$RHS = 4\left[1-\frac{x^2+2xy-y^2}{(x+y)^2}-\frac{-x^2+2xy+y^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right] = \frac{4(x-y)^2}{(x+y)^2}$$

$\therefore LHS = RHS$

Q3] $u = \log(\tan x + \tan y)$

$$\sin 2x \cdot \frac{\partial u}{\partial x} + \sin 2y \cdot \frac{\partial u}{\partial y} = 2$$

$\rightarrow$

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y} \cdot \sec^2 x$$

$$\sin 2x \frac{\partial u}{\partial x} = \frac{2\sin x \cancel{\cos x}}{\tan x + \tan y} \times \frac{1}{\cos^2 x} = \frac{2\tan x}{\tan x + \tan y}$$

Similarly,

$$\sin 2y \frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y}$$

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = \frac{2\tan x}{\tan x + \tan y} + \frac{2\tan y}{\tan x + \tan y} = 2$$

Q4] $u = (1 - 2xy + y^2)^{-1/2}$

P.T. $\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right) = y^2 u^3$

$$\rightarrow \frac{\partial u}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2y) = y\left[(1 - 2xy + y^2)^{-1/2}\right]^3 = yu^3$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2x + 2y) = (x-y)\left[(1 - 2xy + y^2)^{-1/2}\right]^3 = (x-y)u^3$$

$$\therefore LHS = x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xyu^3 - y(x-y)u^3 = \cancel{xyu^3} - \cancel{xyu^3} + y^2u^3 = y^2u^3$$

5] $u = \log(x^3+y^3+z^3-3xyz)$ P.T. $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

→ LHS : $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) \text{———} \; (1)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz), \quad \frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}, \quad \frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = 3\left(\frac{\cancel{x^2+y^2+z^2-xy-yz-zx}}{\cancel{x^3+y^3+z^3-3xyz}}\right) \times \frac{\cancel{(x+y+z)}}{(x+y+z)}$$

$$\left[\because (x^2+y^2+z^2-xy-yz-zx)(x+y+z) = x^3+y^3+z^3-3xyz\right]$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

Hence from (1),

$$\text{LHS} = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\frac{3}{(x+y+z)}$$

$$= 3\left[\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}\right]$$

$$= \frac{-9}{(x+y+z)^2} = \text{R.H.S.}$$

Q6] $z = x^y + y^x$, P.T. $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

$\rightarrow$ $\frac{\partial z}{\partial y} = x^y \log x + x y^{x-1}$

Differentiating this partially wrt x we get,

$$\frac{\partial^2 z}{\partial x \partial y} = y x^{y-1} \log x + x^y \cdot \frac{1}{x} + 1 \cdot y^{x-1} + x y^{x-1} \log y$$

$$= y x^{y-1} \log x + x^{y-1} + y^{x-1} + x y^{x-1} \log y \quad \text{——} ①$$

$\frac{\partial z}{\partial x} = y x^{y-1} + y^x \log y$

Differentiating this partially wrt y we get,

$$\frac{\partial^2 z}{\partial y \partial x} = x^{y-1} + y x^{y-1} \log x + \frac{y^x}{y} + x y^{y-1} \log y$$

$$= y x^{y-1} \log x + x^{y-1} + x y^{y-1} \log y + y^{x-1} \quad \text{——} ②$$

from ① & ② : $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

Q7] $u = x^3 y + e^{xy^2}$ P.T. $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

$\rightarrow$

$\frac{\partial u}{\partial x} = 3x^2 y \quad e^{xy^2} \cdot y^2$

$\frac{\partial^2 u}{\partial y \partial x} = 3x^2 + e^{xy^2} 2y + y^2 e^{xy^2} 2xy$ —— ①

$\frac{\partial u}{\partial y} = x^3 + e^{xy^2} 2xy$

$\frac{\partial^2 u}{\partial x \partial y} = 3x^2 + e^{xy^2} \cdot 2y + 2xy e^{xy^2} y^2$ —— ②

From ① & ②,

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Q 8] $e^{x^2+y^2+z^2}$ P.T. $\frac{\partial^3 u}{\partial x \partial y \partial z} = 8xyzu.$

$\rightarrow$ $\frac{\partial u}{\partial z} = e^{x^2+y^2+z^2} . 2z$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)$$

$$= 2z . e^{x^2+y^2+z^2} . 2y$$

$$= 4yz . e^{x^2+y^2+z^2}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = 4yz . e^{x^2+y^2+z^2} . 2x$$

$$= 8xyz . e^{x^2+y^2+z^2}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = 8xyzu$$

Q11] $u = f(r)$, $r^2 = x^2+y^2+z^2$, P.T. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$.

$\rightarrow$ $u = f(r)$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(r) = \frac{d}{dr} f(r) . \frac{\partial r}{\partial x}$$

$$= f'(r) . \frac{\partial r}{\partial x} \quad \text{——— (1)}$$

$r^2 = x^2+y^2+z^2$

Differentiating r^2 partially wrt x,

$2r \frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$

$$\therefore \frac{\partial u}{\partial x} = f'(r) . \frac{x}{r}$$

Differentiating $\frac{\partial u}{\partial x}$ partially wrt x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r).\frac{x}{r}\right]$$

$$= f''(r)\frac{\partial r}{\partial x}.\frac{x}{r} + \frac{f'(r)}{r} + xf'(r)\left(\frac{-1}{r^2}\right).\frac{\partial r}{\partial x}$$

$$= f''(r)\frac{x}{r}.\frac{x}{r} + \frac{f'(r)}{r} - \frac{x}{r^2}f'(r).\frac{x}{r}$$

$$= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r} - \frac{x^2}{r^3}f'(r) \quad \text{——— (2)}$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - \frac{y^2}{r^3}f'(r)$ — (3) | $\frac{\partial^2 u}{\partial z^2} = f''(r)\frac{z^2}{r^2} + \frac{f'(r)}{r} - \frac{z^2}{r^3}f'(r)$ — (4)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f''(r)}{r^2}(x^2+y^2+z^2) + \frac{3f'(r)}{r} - \frac{(x^2+y^2+z^2)}{r^3}f'(r)$$

$$= \frac{f''(r)}{r^2}.r^2 + \frac{3f'(r)}{r} - \frac{r^2}{r^3}f'(r)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2f'(r)}{r}$$

Q9] $\theta = t^n e^{-r^2/4t}$, find n which will make $\frac{\partial \theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right)$.

$\rightarrow$ $$\frac{\partial \theta}{\partial t} = nt^{n-1}.e^{-r^2/4t} + t^n e^{-r^2/4t}.\left(\frac{r^2}{4t^2}\right)$$

$$= \frac{n}{t}\theta + \frac{r^2}{4t^2}\theta = \left(\frac{n}{t} + \frac{r^2}{4t^2}\right)\theta \quad \text{——— (1)}$$

Also, $\frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t} \cdot \left(\frac{-2r}{4t}\right) = \frac{-r\theta}{2t}$

$$\therefore r^2 \frac{\partial \theta}{\partial r} = -\frac{r^3\theta}{2t}$$

Also, $\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{-r^3\theta}{2t}\right) = \frac{-1}{2t}\frac{\partial}{\partial r}(r^3\theta)$

$$= \frac{-1}{2t}\left[r^3 \frac{\partial \theta}{\partial r} + 3r^2\theta\right]$$

$$= \frac{-1}{2t}\left[r^3 \frac{r\theta}{2t} + 3r^2\theta\right]$$

$$= \frac{-1}{2t}\left[\frac{r^4\theta}{2t} + 3r^2\theta\right]$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2t}\left[-\frac{r^2\theta}{2t} + 3\theta\right] \quad \text{—— (2)}$$

Equating (1) & (2), $\frac{n}{t} = -\frac{3}{2t}$

$$\therefore \boxed{n = -\frac{3}{2}}$$

Q 12 $\rightarrow$ $x = \cos\theta - r\sin\theta$, $y = \sin\theta + r\cos\theta$, P.T. $\frac{\partial r}{\partial x} = \frac{x}{r}$

$$x^2+y^2 = (\cos\theta - r\sin\theta)^2 + (\sin\theta + r\cos\theta)^2$$

$$= \cos^2\theta - 2r\cos\theta\sin\theta + r^2\sin^2\theta + \sin^2\theta + 2r\sin\theta\cos\theta + r^2\cos^2\theta$$

$$= \cos^2\theta + \sin^2\theta + r^2(\cos^2\theta + \sin^2\theta)$$

$$= 1 + r^2$$

$$\therefore r^2 = x^2 + y^2 - 1$$

Differentiating partially wrt x,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

Home Assignment

Q1 $U = e^{xyz}$

$$\frac{\partial u}{\partial z} = xy e^{xyz}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}(xy \cdot e^{xyz})$$

$$\therefore \frac{\partial^2 u}{\partial y \partial z} = xy \frac{\partial}{\partial y}(e^{xyz}) + e^{xyz} \cdot \frac{\partial}{\partial y}(xy)$$

$$\therefore \frac{\partial^2 u}{\partial y \partial z} = (xy)(xz) e^{xyz} + e^{xyz} \cdot x$$

$$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz}(x^2 yz + x)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = (x^2 yz + x) yz \cdot e^{xyz} + e^{xyz} \cdot (2xyz + 1)$$

$$\therefore \frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}\left[x^2 y^2 z^2 + 3xyz + 1\right]$$

$$\therefore \frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}\left[1 + 3xyz + x^2 y^2 z^2\right]$$

Q2 $z = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ P.T. $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial z} = \frac{x^2-y^2}{x^2+y^2}$

$\rightarrow$

$$\frac{\partial z}{\partial y} = x^2\left(\frac{x^2}{x^2+y^2}\times\frac{1}{x}\right) - \left(y^2\cdot\frac{y^2}{x^2+y^2}\cdot\frac{-x}{y^2}\right) + 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial y} = \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial y} = \frac{x\,\cancel{(x^2+y)}}{\cancel{x^2+y^2}} - 2y\tan^{-1}\left(\frac{x}{y}\right) \quad \text{---} (1)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 1 - 2y\cdot\frac{y^2}{x^2+y^2}\times\frac{1}{y}$$

$$\frac{\partial^2 z}{\partial x\partial y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

$$\frac{\partial z}{\partial x} = \left(2x\tan^{-1}\left(\frac{y}{x}\right) - x^2\cdot\frac{x^2}{x^2+y^2}\times\frac{-y}{x^2}\right) - \left[y^2\,\frac{y^2}{x^2+y^2}\times\frac{1}{y}\right]$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right)\cdot\frac{-x^2y}{x^2+y^2} - \frac{y^2}{x^2+y^2}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{y\,\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 2x\cdot\frac{x^2}{x^2+y^2}\cdot\frac{1}{x} - 1$$

$$\frac{\partial^2 z}{\partial y\partial x} = \frac{2x^2}{x^2+y^2} - 1 = \frac{2x^2-x^2-y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

$$\boxed{\therefore \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x} = \frac{x^2-y^2}{x^2+y^2}}$$

Hence Proved

2022

Q1 vi) $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) + y^2 \sin^{-1}\left(\frac{x}{y}\right)$ P.T. $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2u$

$$\frac{\partial u}{\partial x} = \left[2x \tan^{-1}\left(\frac{y}{x}\right) + \cancel{x^2} \cdot \frac{x^2}{x^2+y^2} \cdot \frac{-y}{\cancel{x^2}}\right] + \cancel{y^2} \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{-x}{\cancel{y^2}}$$

$$\frac{\partial u}{\partial x} = \left[2x \tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2+y^2} - \frac{xy}{\sqrt{y^2-x^2}}\right]$$

Q3] $u = f\left(\frac{x^2}{y}\right)$ P.T. $x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = 0$

And $x^2\frac{\partial^2 u}{\partial x^2} + 3xy\frac{\partial^2 u}{\partial x \partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} = 0$

$\longrightarrow$

a) $$\frac{\partial u}{\partial x} = f'\left(\frac{x^2}{y}\right)\cdot\frac{2x}{y}$$

$$\frac{\partial u}{\partial y} = f'\left(\frac{x^2}{y}\right)\cdot\frac{-x^2}{y^2}$$

$$\therefore x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = x\cdot f'\left(\frac{x^2}{y}\right)\cdot\frac{2x}{y} - 2\cancel{y}f'\left(\frac{x^2}{y}\right)\cdot\frac{-x^2}{y^2} = 0$$

b) $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = f''\left(\frac{x^2}{y}\right)\frac{2x}{y}\times\frac{2x}{y} = f''\left(\frac{x^2}{y}\right)\frac{4x^2}{y^2}$$

$$\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -f''\left(\frac{x^2}{y}\right)\frac{x^2}{y^2}\cdot\frac{2x}{y} = -f''\left(\frac{x^2}{y}\right)\left(\frac{2x^3}{y^3}\right)$$

$$\therefore \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial y}\right) = f''\left(\frac{x^2}{y}\right)\frac{x^4}{y^4}$$

$$\therefore f''\left(\frac{x^2}{y}\right)\left[\frac{4x^4}{y^2} - 3xy\left(\frac{2x^3}{y^3}\right) + 2y^2\frac{x^4}{y^4}\right]$$

$$\therefore f''\left(\frac{x^2}{y}\right)\left[\frac{6x^4 - 6x^4 + 2x^4}{y^2}\right] = 0$$

Q4 $u = f(e^{y-z}, e^{z-x}, e^{x-y})$ P.T. $U_x + U_y + U_z = 0$

$\longrightarrow$

$y - z = t_1$, $z - x = t_2$, $x - y = t_3$

$u = f(e^{t_1}, e^{t_2}, e^{t_3})$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\frac{\partial t_2}{\partial x} + \frac{\partial u}{\partial t_3}\frac{\partial t_3}{\partial x}$$

$$\frac{\partial u}{\partial x} = 0 + -1\cancel{e^{t_2}} + \cancel{e^{t_3}}$$

$$\frac{\partial u}{\partial y} = \cancel{e^{t_1}} + 0 - \cancel{e^{t_3}}$$

$$\frac{\partial u}{\partial z} = -\cancel{e^{t_1}} + \cancel{e^{t_2}} + 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 0$$

Q1] $u = x^2 y^3$, $x = \log t$, $y = e^t$, find $\frac{du}{dt}$

$\rightarrow$ $$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= 2xy^3 \cdot \frac{1}{t} + 3y^2 x^2 \cdot e^t$$

Substituting x & y,

$$\frac{du}{dt} = 2(\log t) e^{3t} \cdot \frac{1}{t} + 3(\log t)^2 e^{2t} \cdot e^t$$

$$\frac{du}{dt} = \frac{2}{t} (\log t) e^{3t} + 3(\log t)^2 e^{3t}$$

Q2] $u = xy + yz + zx$, $x = \frac{1}{t}$, $y = e^t$, $z = e^{-t}$, find $\frac{du}{dt}$

$\rightarrow$ $$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

$$= (y+z)\left(-\frac{1}{t^2}\right) + (x+z)e^t - (y+z)e^{-t}$$

Substituting x, y & z,

$$\frac{du}{dt} = -\frac{1}{t^2}(e^t + e^{-t}) + \left(\frac{1}{t} + e^{-t}\right)e^t - \left(e^t + \frac{1}{t}\right)e^{-t}$$

$$\frac{du}{dt} = -\frac{1}{t^2}(e^t + e^{-t}) + \frac{1}{t}(e^t - e^{-t})$$

Q3 $z = x^2y + y^2x$, $x = at^2$, $y = 2at$ find $\frac{dz}{dt}$.

$\rightarrow$ $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{dz}{dt} = (2xy + y^2)\,2at + (x^2 + 2yx)\,2a$$

Substituting x & y.

$$\frac{dz}{dt} = (2(at^2)(2at) + (2at)^2)\,2at + ((at^2)^2 + 2(2at)(at^2))\,2a$$

$$= (4a^2t^3 + 4a^2t^2)\,2at + (a^2t^4 + 4a^2t^3)\,2a$$

$$= 8a^3t^4 + 8a^3t^3 + 2a^3t^4 + 8a^3t^3$$

$$\frac{dz}{dt} = 10a^3t^4 + 16a^3t^3$$

Q4 $z = e^{xy}$, $x = t\cos t$, $y = t\sin t$, find $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$

$\rightarrow$ $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{dz}{dt} = ye^{xy}.[\cos t - t\sin t] + xe^{xy}.[\sin t + t\cos t]$$

At $t = \frac{\pi}{2}$, $x = 0$, $y = \frac{\pi}{2}$

$$\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} = e^0\left[\frac{\pi}{2}\left(0 - \frac{\pi}{2}\right) + 0\right] = \frac{-\pi^2}{4}$$

Q5 $z = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$ find $\frac{dz}{dt}$

$\rightarrow$ $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}} \cdot 3 + \frac{(-1)}{\sqrt{1-(x-y)^2}} \, 12t^2$$

$$= \frac{3 - 12t^2}{\sqrt{1-(x-y)^2}}$$

Substituting x & y,

$$\frac{dz}{dt} = \frac{3 - 12t^2}{\sqrt{1-(3t-4t^3)^2}}$$

$\boxed{Q6}$ $x^2 = au + bv$, $y^2 = au - bv$ & $z = f(x, y)$

P.T. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$

$\rightarrow$ $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{a}{2x} + \frac{\partial z}{\partial y} \cdot \frac{a}{2y}$$

$$u \cdot \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{au}{2x} + \frac{\partial z}{\partial y} \cdot \frac{au}{2y} \quad \ldots\ldots\ldots \text{(i)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{b}{2x} + \frac{\partial z}{\partial y}\left(\frac{-b}{2y}\right)$$

$$v \cdot \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{bv}{2x} - \frac{\partial z}{\partial y} \cdot \frac{bv}{2y} \quad \ldots\ldots\ldots \text{(ii)}$$

Adding eqn (i) & (ii)

$$u\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{au}{2x} + \frac{\partial z}{\partial y}\cdot\frac{au}{2y} + \frac{\partial z}{\partial x}\cdot\frac{bv}{2x} - \frac{\partial z}{\partial y}\cdot\frac{bv}{2y}$$

$$= \frac{\partial z}{\partial x}\left(\frac{au+bv}{2x}\right) + \frac{\partial z}{\partial y}\left(\frac{au-bv}{2}\right)$$

$$= \frac{\partial z}{\partial x}\left(\frac{x^2}{2x}\right) + \frac{\partial z}{\partial y}\left(\frac{y^2}{2y}\right)$$

$$u\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v} = \frac{1}{2}\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

Q7 $z = f(u,v)$ & $u = \log(x^2+y^2)$, $v = \frac{y}{x}$ P.T. $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1+v^2)\frac{\partial z}{\partial v}$

→

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2}\frac{\partial z}{\partial u} + \frac{1}{x}\frac{\partial z}{\partial v}$$

$$x\frac{\partial z}{\partial y} = \frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \ldots\ldots (1)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2}\frac{\partial z}{\partial u} + \frac{-y}{x^2}\frac{\partial z}{\partial v}$$

$$y\frac{\partial z}{\partial x} = \frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} - \frac{y^2}{x^2}\frac{\partial z}{\partial v} \quad \ldots\ldots\ldots (2)$$

(1) − (2)

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v} + \frac{y^2}{x^2}\frac{\partial z}{\partial v} = (1+v^2)\frac{\partial z}{\partial v}$$

$\boxed{Q8}$ $u = f(x^2-y^2,\ y^2-z^2,\ z^2-x^2)$ P.T. $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$

$\rightarrow$ Let $t_1 = x^2-y^2$, $t_2 = y^2-z^2$, $t_3 = z^2-x^2$

$$\frac{\partial t_1}{\partial x} = 2x, \quad \frac{\partial t_2}{\partial x} = 0, \quad \frac{\partial t_3}{\partial x} = -2x$$

$$\frac{\partial t_1}{\partial y} = -2y, \quad \frac{\partial t_2}{\partial y} = 2y, \quad \frac{\partial t_3}{\partial y} = 0$$

$$\frac{\partial t_1}{\partial z} = 2y \qquad \frac{\partial t_2}{\partial z} = 0, \quad \frac{\partial t_3}{\partial z} = -2y$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial x} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial x} = \frac{\partial u}{\partial t_1}(2x) + 0 + \frac{\partial u}{\partial t_3}(-2x)$$

$$\therefore \frac{1}{x}\frac{\partial u}{\partial x} = \cancel{2\frac{\partial u}{\partial t_1}} - \cancel{2\frac{\partial u}{\partial t_3}} \quad \text{— (1)}$$

$$\therefore \frac{1}{y}\frac{\partial u}{\partial y} = \cancel{-2\frac{\partial u}{\partial t_1}} + \cancel{2\frac{\partial u}{\partial t_2}} \quad \text{— (2)}$$

$$\therefore \frac{1}{z}\frac{\partial u}{\partial z} = \cancel{-2\frac{\partial u}{\partial t_2}} + \cancel{2\frac{\partial u}{\partial t_3}} \quad \text{— (3)}$$

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$$

$\boxed{Q9}$ $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ & ψ is func of x, y, z

P.T. $x\frac{\partial \psi}{\partial x} + y\frac{\partial \psi}{\partial y} + z\frac{\partial \psi}{\partial z} = u\frac{\partial \psi}{\partial u} + v\frac{\partial \psi}{\partial v} + w\frac{\partial \psi}{\partial w}$

$$\rightarrow \frac{\partial \psi}{\partial u} = \frac{\partial \psi}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial \psi}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial \psi}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= \frac{\partial \psi}{\partial x}(0) + \frac{\partial \psi}{\partial y}\cdot\frac{\sqrt{w}}{2\sqrt{u}} + \frac{\partial \psi}{\partial z}\cdot\frac{\sqrt{v}}{2\sqrt{u}}$$

$$u\frac{\partial \psi}{\partial u} = \frac{u\sqrt{w}}{2\sqrt{u}}\frac{\partial \psi}{\partial y} + \frac{u\sqrt{v}}{2\sqrt{u}}\frac{\partial \psi}{\partial z} = \frac{\sqrt{uw}}{2}\frac{\partial \psi}{\partial y} + \frac{\sqrt{uv}}{2}\frac{\partial \psi}{\partial z}$$

$$u\frac{\partial \psi}{\partial u} = \frac{1}{2}\left(y\frac{\partial \psi}{\partial y} + z\frac{\partial \psi}{\partial z}\right) \quad \ldots\ldots\ldots (1)$$

$0 + y + z$

$$v\frac{\partial \psi}{\partial v} = \frac{1}{2}\left(x\frac{\partial \psi}{\partial x} + z\frac{\partial \psi}{\partial z}\right) \quad \ldots\ldots\ldots (2)$$

$x + 0 + z$

$x + y + 0$

$$w\frac{\partial \psi}{\partial w} = \frac{1}{2}\left(x\frac{\partial \psi}{\partial x} + y\frac{\partial \psi}{\partial y}\right) \quad \ldots\ldots\ldots (3)$$

$$u\frac{\partial \psi}{\partial u} + v\frac{\partial \psi}{\partial v} + w\frac{\partial \psi}{\partial w} = x\frac{\partial \psi}{\partial x} + y\frac{\partial \psi}{\partial y} + z\frac{\partial \psi}{\partial z}$$

Q10] $x = e^u \operatorname{cosec} v$, $y = e^u \cot v$ & z is $f(x, y)$

P.T. $$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$\rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^u \operatorname{cosec} v + \frac{\partial z}{\partial y}e^u \cot v$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^u \operatorname{cosec} v \cot v) + \frac{\partial z}{\partial y}(-e^u \operatorname{cosec} v)$$

$$R.H.S. = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= e^{-2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 e^{2u}\operatorname{cosec}^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cot^2 v + 2\,\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\, e^{2u}\operatorname{cosec} v\cot v + \right.$$

$$(-\sin^2 v)\left(\frac{\partial z}{\partial x}\right)^2 (e^{2u}\operatorname{cosec}^2 v\cot^2 v) + (-\sin^2 v)\left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\operatorname{cosec}^4 v +$$

$$\left.(-\sin^2 v)\, 2.\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\, e^{2u}\operatorname{cosec}^3 v\cot v\right]$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 (\operatorname{cosec}^2 v - \cot^2 v) + \left(\frac{\partial z}{\partial y}\right)^2 (\cot^2 v - \operatorname{cosec}^2 v)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

$$= L.H.S.$$

19. If $u = x\log(x+r) - r,\ r^2 = x^2 + y^2$, prove that $i)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{x+r}$ $\quad ii)\ \frac{\partial^3 u}{\partial x^3} = -\left(\frac{x}{r^3}\right)$

$\rightarrow$

$$r^2 = x^2 + y^2$$

$$\cancel{2}r \cdot \frac{\partial r}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\cancel{2}r \cdot \frac{\partial r}{\partial y} = \cancel{2}y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial s}{\partial y} = \frac{\partial r}{\partial y}$$

Let $s = x + r$

$$\frac{\partial s}{\partial x} = 1 + \frac{\partial r}{\partial x} = 1 + \frac{x}{r} = \frac{x+r}{r} = \frac{s}{r}$$

$$\left(\frac{\frac{\partial s}{\partial x}}{s} = \frac{\frac{s}{r}}{s} = \frac{1}{r}\right)$$

$$z = x\log s - r$$

$$\frac{\partial z}{\partial x} = \log s + \frac{1}{s} \cdot \frac{\partial s}{\partial x} \cdot x - \frac{\partial r}{\partial x}$$

$$\frac{\partial z}{\partial x} = \log s + \frac{x}{r} - \frac{x}{r}$$

$$\frac{\partial z}{\partial x} = \log s$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{s} \cdot \frac{\partial s}{\partial x} = \frac{1}{r}$$

$$\frac{\partial^3 z}{\partial x^3} = -\frac{1}{r^2} \cdot \frac{\partial r}{\partial x} = \frac{1}{r^2} \cdot \frac{x}{r}$$

$$\boxed{\frac{\partial^3 z}{\partial x^3} = -\frac{x}{r^3}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{s} \cdot \frac{\partial s}{\partial y} \cdot x - \frac{\partial r}{\partial y} = \frac{\partial r}{\partial y} \cdot \frac{x}{s} - \frac{\partial r}{\partial y} = \frac{\partial r}{\partial y}\left(\frac{x}{s} - 1\right)$$

$$\frac{\partial z}{\partial y} = \frac{\partial r}{\partial y} \cdot \frac{x-s}{s} = \frac{y}{r} \cdot -\frac{r}{s}$$

$$\frac{\partial z}{\partial y} = \frac{-y}{s}$$

$$\left(\frac{\partial z}{\partial y}\right)^2 = \frac{s - y \cdot \frac{\partial s}{\partial y}}{s^2} = \frac{-s - y\frac{\partial r}{\partial y}}{s^2}$$

$$\left(\frac{\partial z}{\partial y}\right)^2 = -\frac{s - \frac{y^2}{r}}{s^2} = -\frac{(x+r)r - y^2}{rs^2} = -\frac{xr + r^2 - y^2}{rs^2} = -\frac{xr + x^2 \cancel{+y^2} \cancel{-y^2}}{rs^2}$$

$$= \frac{-xr + x^2}{r(x+r)(x+r)} = -\frac{x\cancel{(x+r)}}{r(x+r)\cancel{(x+r)}} = \frac{-x}{rs}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{r} - \frac{x}{rs} = \frac{s-x}{rs}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{s}$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{x+r}}$$

2022

Q2B(ii) $u = f(r)$, $r^2 = x^2+y^2+z^2$ PT $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$

$\rightarrow$ $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$u = f(r)$

$$\frac{\partial u}{\partial x} = f'(r) \frac{\partial r}{\partial x} = \frac{f'(r) \cdot x}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{r\left[f'(r) \cdot 1 + x f''(r) \left(\partial r/\partial x\right)\right] - x f'(r) \cdot \partial r/\partial x}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2}\left[r f'(r) + x^2 f''(r) - \frac{x^2}{r} f'(r)\right] \text{ ——— } (1)$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[r f'(r) + y^2 f''(r) - \frac{y^2}{r} f'(r)\right]$ ——— (2)

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}\left[r f'(r) + z^2 f''(r) - \frac{z^2}{r} f'(r)\right] \text{ ——— } (3)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}\left[3r f'(r) + (x^2+y^2+z^2) f''(r) - \frac{(x^2+y^2+z^2)}{r} f'(r)\right]$$

$$= \frac{1}{r^2}\left[3r f'(r) + r^2 f''(r) - r f'(r)\right]$$

$$= \frac{1}{r^2}\left[r^2 f''(r) + 2r f'(r)\right]$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r} f'(r) + f''(r)$$

Hence Proved

Sahiya

Q] $z = f(u,v)$, $u = e^x$, $v = e^y$ P.T. $\frac{\partial^2 z}{\partial x \partial y} = uv \frac{\partial^2 z}{\partial u \partial v}$

$\rightarrow$

$$\frac{\partial z}{\partial y} = \cancel{\frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y}}^{0} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = e^y \cdot \frac{\partial z}{\partial v}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = e^y \cdot \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= e^y \; f''(u,v) \left[\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial x}}^{0}\right]$$

$$= e^y \cdot f''(u,v)\, e^x = e^{x+y} f''(u+v) \qquad = \text{L.H.S}$$

$$\text{R.H.S} = uv \frac{\partial^2 z}{\partial u \partial v} = uv \; f''(u,v) = e^x e^y f''(u,v) = e^{x+y} f''(u,v)$$

$$\boxed{\frac{\partial^2 z}{\partial u \partial v} = f''(u,v)}$$

$\therefore$ LHS = RHS

Q3] A]

$$u = e^{x^2} + e^{y^2} + e^{z^2}$$

$$\frac{\partial u}{\partial x} = 2x e^{x^2} \quad \frac{\partial u}{\partial y} = 2y e^{y^2} \quad \frac{\partial u}{\partial z} = 2z e^{z^2}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = 8xyz\, e^{x^2+y^2+z^2}$$

$$= 8xyzu$$

Q $u = \log(x^2+y^2) + \frac{x^2+y^2}{x+y} - 2\log(x+y)$. Find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

→

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}\cdot 2x + \frac{(x+y)2x - (x^2+y^2)\cdot 1}{(x+y)^2} - 2\frac{1}{x+y}\cdot 1$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} + \frac{2x^2+2xy - x^2 - y^2}{(x+y)^2} - \frac{2}{x+y}$$

$$x\frac{\partial u}{\partial x} = \frac{2x^2}{x^2+y^2} + \frac{x(x^2+2xy-y^2)}{(x+y)^2} - \frac{2x}{x+y}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2} + \frac{(x+y)2y - (x^2+y^2)\cdot 1}{(x+y)^2} - \frac{2}{x+y}$$

$$y\frac{\partial u}{\partial y} = \frac{2y^2}{x^2+y^2} + \frac{y(y^2+2xy-x^2)}{(x+y)^2} - \frac{2y}{x+y}$$

$x^3+2x^2y-xy^2$

$\frac{x^3+y^3+x^2y+xy^2}{}$

$\frac{y^3+2xy^2-x^2y}{(x+y)^3}$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} + \frac{x^3+y^3+x^2y+xy^2}{(x+y)^2} - \frac{2\cancel{(x+y)}}{\cancel{x+y}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\cancel{(x+y)}(x^2+y^2)}{(x+y)\cancel{(x+y)}} = \frac{x^2+y^2}{x+y}$$

Q $u = f(x)$, $v = f(x,y)$, $w = f(x,y,z)$, P.T. $J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}$

→

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} f'(x) & 0 & 0 \\ f'(x,y) & f'(x,y) & 0 \\ f'(x,y,z) & f'(x,y,z) & f'(x,y,z) \end{vmatrix} = u_x v_y w_z$$

Q] $x = e^v \sec u$, $y = e^v \tan u$, P.T. $JJ' = 1$

$$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} e^v \sec u \tan u & e^v \sec u \\ e^v \sec^2 u & e^v \tan u \end{vmatrix}$$

$$\therefore J = e^{2v} \sec u \tan^2 u - e^{2v} \sec^3 u$$

$$= e^{2v} \sec u (\tan^2 u - \sec^2 u)$$

$$= -e^{2v} \sec u$$

$$J = -x e^v$$

$$\frac{y}{x} = \frac{e^v \tan u}{e^v \sec u} = \sin u$$

$$u = \sin^{-1} \frac{y}{x} \quad \text{—} ①$$

As $\sec^2 u - \tan^2 u = 1$

$$\left(\frac{x}{e^v}\right)^2 - \left(\frac{y}{e^v}\right)^2 = 1$$

$$\therefore x^2 - y^2 = e^{2v}$$

$$v = \frac{1}{2} \log (x^2 - y^2) \quad \text{—} ②$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{-y}{x\sqrt{x^2-y^2}} & \dfrac{1}{\sqrt{x^2-y^2}} \\ \dfrac{x}{x^2-y^2} & \dfrac{-y}{x^2-y^2} \end{vmatrix} = \frac{y^2}{x(x^2-y^2)^{3/2}} - \frac{x}{(x^2-y^2)^{3/2}} = \frac{y^2 - x^2}{x(x^2-y^2)^{3/2}} = \frac{-1}{x\sqrt{x^2-y^2}} = \frac{-1}{x} e^{-v}$$

$$J' = \frac{-1}{x} e^{-v}$$

$$\therefore JJ' = 1$$