Pumping Lemma

Pumping Lemma

- Very useful in Proving certain sets are not regular

For eg-

- $a^n b^n$ is not regular
- We cannot generate with FSM
- Finite State Automaton has a finite amount of memory
- If we want to accept $a^n b^n$, it should remember ‘n’
- But FSA cannot remember infinite number of values

Pumping Lemma:Why ?

- z= string that belongs to L
- z=uvw
- We can pump the middle portion "v"
- We can also have "u" empty
- i.e. The pump can be in the beginning also
- We are putting more and more weight to the middle portion
- We are pumping the middle portion
- Thus called Pumping Lemma

Pumping Lemma:Why ?

- $z=uvw$ such that $uv^iw \in L$ for $i>0$

- $i=1$, $z=uvw$
- $i=2$, $z=uvvw$
- $i=3$, $z=uvvvw$

- v is not empty $|v|>=1$
- $|uv|<=n$

What?

- To Show L is not Regular

- Problem Definition gives L
- Problem Definition gives n
- We choose z depending on question such that z=uvw
- We show that $uv^iw \in L$
- Thus not regular

Pumping Lemma:How?

- Step 1-Assume language L is regular, Let n be the number of states in the corresponding FA
- Step 2- Choose a string z in language L such that $|z|>n$, Use pumping lemma to write the string $z=uvw$ with $|uv|<=n$, $|v|>=1$ such that $uv^iw \in L$ for $i>0$
- Step 3-Find a suitable integer "i" such that $uv^iw \notin L$. This contradicts our assumption. Hence L is not regular

$|uv| \leq n$

$u v^i w$

Pumping Lemma :Example 1

$\boxed{1 \leq |v| \leq n}$

L={a^{n^2} | n>=1}
Step 1-Suppose L is regular
Let n be the number of states in FA accepting L

Step 2-Choose a string z ∈ L such that |z|>n
z can be written in the form z=uvw with |uv|<=n and |v|>=1
such that uv^iw ∈ L for i>0
Consider a =aaaaa……..aaaaaaaa for n^2 no of times
String of a's aaaaaaaaaaaaa…….aaaaaaaaa

u v w

uv^2w uv^3w ....

$\boxed{1 \le v \le n^2}$

Pumping Lemma:Example 1

a^{n^2}

Step 3- So length of v alone is 1<=|v|<=n

For i=2,

$|uv^2w|=|uvw|+|v|$

$=n^2+|v|$

$<= n^2+ n$

$n=1 \quad a^1 \to a^1 \to |a| = 1 = n^2$

$n=2 \quad a^{2^2} \to |aaaa| = 4 = n^2$

$n=3 \quad a^{3^2} \to |a^9| = 9 = n^2$

we get a^P such that value of P is <= n^2+ n and > n^2

After L={a, a^4, a^9,………………. a_{n^2}, $a_{(n+1)^2}$,……………}

The string higher than a_{n^2} is $a_{(n+1)^2}$ i.e. a_{n^2+2n+1}

But n2 + n lies between these two values, Thus it is not regular

Pumping Lemma :Example 2

L={$a^{n!}$| n>=1}
Step 1-Suppose L is regular
Let n be the number of states in FA accepting L

Step 2-Choose a string z ∈ L such that |z|>n
z can be written in the form z=uvw with |uv|<=n and |v|>=1
such that uv^iw ∈ L for i>0

Consider a
n=1 String=a
n=2 String=aa
n=3 String=aaaaaa

$a^{1!} = |a^1| = 1$ ✓
$a^{2!} = |a^2| = |aa| = 2$ ✓
$a^{3!} = a^6 = |aaaaaa| = 6$ ✓

n=4 ✓
n
|string| = |w| = n!

Pumping Lemma :Example 2

uv^iw

$1 \le |v| \le n$

Step 3- So length of v alone is $1 <= |v| <= n$

$|uv^2w| = |uvw| + |v|$

$\cancel{= n! + |v|}$

$<= n! + n$

$n! + n$

$n!$

$n \times n! + n!$

we get a^P such that value of P is $<= n! + n$ and $> n!$

After L={a, a^2, a^6, a^{24} .... $a_{n!}$, $a_{(n+1)!}$ ....}

$n=1$ $n=2$ $n=3$ $n=4$

a^{n^2}

$n=n$ $n=(n+1)$

$(n+1)! = (n+1).n!$

$= n \times n! + n!$

But n! + n lies between these two values, Thus it is not regular

Pumping Lemma :Example 3

$L=\{a^n b^n \mid n>=1\}$

Step 1-Suppose L is regular

Let n be the ~~number of~~ states in FA accepting L

Step 2-Choose a string $z \in L$ such that $|z|>n$

z can be written in the form $z=uvw$ with $|uv|<=n$ and $|v|>=1$

such that $uv^iw \in L$ for $i>0$

q_0 q_1 q_2 q_3 $n=4$

$\underbrace{10101011101}$ z $|z|>n$ or $|z|>4$

Pumping Lemma :Example 3

$n=4$

q_0
q_1
q_2
q_3

$n < m$

$a^9 b^9$ $|uv| \le n$

$L=\{a^n b^n \mid n>=1\}$

Consider $a^m b^m$, $m>n$

aaaaa……..abbbbbbb……b

m m

aaaaaaaaabbbbbbbbb

uv w

uv w

Using Pumping Lemma, This can be written as

z=uvw , such that |uv|<=n and |v|>=1

$|uv| \le n$

≤ 4

As m>n and |uv|<=n

$n=6$

aaaaaaaaaa…………………aaabbbbbbbbbbbbbbbbbb

v will not fall into “b”

uv **w**

Pumping Lemma :Example 3

As m>n

aaaaaaaaaaa……………………..abbbbbbbbbbbbbbbbbbbb

Let v=a^p

u=a^q

w=$a^r b^m$

p+q+r=m

uv ⇒ aaaa - - - - -

w = aa - - - bbbbb

m

Let z=a^q (a^p) a^r b^m

$\underbrace{m}_{\text{symbols of a}} = p+q+r$

Pumping v part

Let z=a^q $(a^p)^i$ a^r b^m

So many strings, we can get where no of a's are not equal to no of bs

There is ~~a~~ Contradiction bbbb

Thus, Not Regular