$\boxed{Q1}$ $x\,dx + y\,dy = \dfrac{a(x\,dy - y\,dx)}{x^2+y^2}$

$\rightarrow$

$$x\,dx + y\,dy = \frac{ax}{x^2+y^2}dy - \frac{ay}{x^2+y^2}dx$$

$$\therefore \left(x + \frac{ay}{x^2+y^2}\right)dx + \left(y - \frac{ax}{x^2+y^2}\right)dy = 0$$

$$\therefore M = x + \frac{ay}{x^2+y^2}, \quad N = y - \frac{ax}{x^2+y^2}$$

$$\therefore \frac{\partial M}{\partial y} = a\frac{((x^2+y^2)1 - 2y^2)}{x^2+y^2} = \frac{a(x^2-y^2)}{x^2+y^2}$$

$$\therefore \frac{\partial N}{\partial x} = -\frac{a((x^2+y^2)1 - 2x^2)}{x^2+y^2} = -\frac{a(y^2-x^2)}{x^2+y^2} = \frac{a(x^2-y^2)}{x^2+y^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{a(x^2-y^2)}{x^2+y^2}$$

$\therefore$ The eqn is exact.

$$\therefore \int M\,dx + \int N \text{ (Except } x \text{ terms) } dy = c$$

$$\therefore \int\left(x + \frac{ay}{x^2+y^2}\right)dx + \int y\,dy = c$$

$$\therefore \frac{x^2}{2} + ay\left[\frac{1}{y}\tan^{-1}\frac{x}{y}\right] + \frac{y^2}{2} = c$$

$$\therefore \frac{x^2}{2} + \frac{y^2}{2} + a\tan^{-1}\left(\frac{x}{y}\right) = c$$

$$\therefore x^2 + y^2 + 2a\tan^{-1}\left(\frac{x}{y}\right) = c'$$

* Integration factor

- If $M\,dx + N\,dy = 0$

  but $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial y}$

  then $IF = \frac{1}{Mx + Ny}$ Multiply

then derative nikalo vo exact hoga

then solve it by exact waala method

- I.F. for an eqn of the type

  $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$

I.F. is $\frac{1}{Mx - Ny}$ when $Mx - Ny \neq 0$

Then I.F. $(M\,dx + N\,dy) = 0$ will be exact D.E.

- Rule: In the eqn $M\,dx + N\,dy = 0$

  If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ or $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ be a func^n of x or y only say f(x) or f(y)

Then $e^{\int f(x)dx}$ or $e^{\int f(y)dy}$ is an integrating factor of $M\,dx + N\,dy = 0$ resp.

What to use when: Denominator mein less term waala eqn use karo.

- I.F. for the eqn of type

$$x^a y^b (my\,dx + nx\,dy) + x^{a'} y^{b'} (m'y\,dx + n'x\,dy) = 0$$

and I.F. is $x^h y^k$

where $\frac{a+h+1}{m} = \frac{b+k+1}{n}$, $\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$

Rare!!

Q $(x^2+y^2+1)\,dx - 2xy\,dy = 0$

$M = x^2+y^2+1$ & $N = -2xy$

$\therefore \frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = -2y$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{4y}{2xy} = -\frac{2}{x} = f(x)$$

$$\therefore \text{I.F.} = e^{\int -(2/x)\,dx} = e^{-2\log x} = e^{\log(1/x^2)} = \frac{1}{x^2}$$

Multiplying the eqn by I.F. we get, $\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx + \left(-\frac{2y}{x}\right)dy = 0$

this is exact.

$\therefore \int M\,dx = \int \left[1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right] dx = x - \frac{y^2}{x} - \frac{1}{x}$ & $\int N\,dy = \int (\text{terms in } N \text{ free from } x)\,dy = 0$

$\therefore$ The solution is $x - \frac{y^2}{x} - \frac{1}{x} = c$

i.e. $x^2 - y^2 - 1 = cx$

Q] $y(xy+e^x)\,dx - e^x dy = 0$

$\rightarrow M = y(xy+e^x)$ & $N = -e^x$

$$\frac{\partial M}{\partial y} = 2xy + e^x \quad , \quad \frac{\partial N}{\partial x} = -e^x$$

$$\therefore \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-2\cancel{(xy+e^x)}}{y\cancel{(xy+e^x)}} = \frac{-2}{y} = f(y)$$

$$\therefore \quad I.F. = e^{\int \frac{-2}{y} dy} = e^{-2\log y} = e^{\log 1/y^2} = \frac{1}{y^2}$$

$$\therefore \text{The eqn: } \left(x + \frac{e^x}{y}\right) dx - \frac{e^x}{y^2}\, dy = 0$$

$$\int M\,dx = \int \left(x + \frac{e^x}{y}\right) = \frac{x^2}{2} + \frac{e^x}{y}$$

$$\int N\,dx = 0$$

$$\therefore \quad \text{The solu}^n : \frac{x^2}{2} + \frac{e^x}{y} = c$$

Q] $\dfrac{dy}{dx} = \dfrac{x^2y^3 + 2y}{2x - 2x^3y^2}$

$$\therefore (x^2y^3 + 2y)\,dx + (2x - 2x^3y^2)\,dy = 0$$

$$\therefore y(x^2y^2 + 2)\,dx + x(2 - 2x^2y^2)\,dy = 0$$

$$\frac{\partial M}{\partial y} = 3x^2y^2 + 2 \quad , \quad \frac{\partial N}{\partial x} = 2 - 6x^2y^2$$

$$Mx - Ny = x^3y^3 + \cancel{2xy} - \cancel{2xy} + 2x^3y^3$$

$$= 3x^3y^3 \neq 0$$

$$\therefore \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

$$\therefore \text{The eqn: } \left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \left(\frac{2}{3x^2y^3} - \frac{2}{3y}\right)dy = 0$$

$$\therefore \int M = \frac{1}{3}\log x - \frac{1}{3x^2y^2}$$

$$\int N = -\frac{2}{3y} = -\frac{2}{3}\log y$$

$$\therefore \text{The solution is } \frac{1}{3}\log x - \frac{1}{3x^2y^2} - \frac{2}{3}\log y = c$$

$$\therefore \frac{1}{3}\log\left(\frac{x}{y^2}\right) - \frac{1}{3x^2y^2} = c$$

$\boxed{Q}$ $$\left[2x\sinh\left(\frac{y}{x}\right) + 3y\cosh\left(\frac{y}{x}\right)\right]dx - 3x\cosh\left(\frac{y}{x}\right)dy = 0$$

$$M = 2x\sinh\left(\frac{y}{x}\right) + 3y\cosh\left(\frac{y}{x}\right)] \quad \& \quad N = -3x\cosh\left(\frac{y}{x}\right)$$

$$\therefore \frac{\partial M}{\partial y} = 2x\cos\left(\frac{y}{x}\right)\cdot\frac{1}{x} + 3\cosh\left(\frac{y}{x}\right) + 3y\sinh\left(\frac{y}{x}\right)\cdot\frac{1}{x} = 5\cosh\left(\frac{y}{x}\right) + \frac{3y}{x}\sinh\left(\frac{y}{x}\right)$$

$$\therefore \frac{\partial N}{\partial x} = -3\cdot\cosh\left(\frac{y}{x}\right) - 3x\sinh\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) = -3\cosh\left(\frac{y}{x}\right) + \frac{3y}{x}\sin\left(\frac{y}{x}\right)$$

$$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{8\cosh\left(\frac{y}{x}\right)}{-3x\cosh\left(\frac{y}{x}\right)} = \frac{-8}{3x} = f(x)$$

$$I.F. = e^{\int -(8/3x)\,dx} = e^{-8/3\log x} = x^{-8/3}$$

The eqn: $\left[2x^{-5/3}\cdot\sinh\left(\frac{y}{x}\right) + 3x^{-8/3}y\cosh\left(\frac{y}{x}\right)\right]dx - 3x^{-5/3}\cdot\cosh\left(\frac{y}{x}\right)dy = 0$

$\because \int M\,dx$ is a complex integral

$$\therefore \int N\,dy = \int\left[-3x^{-5/3}\cosh\left(\frac{y}{x}\right)\right]dx = -3x^{-2/3}\sinh\left(\frac{y}{x}\right)$$

$\therefore$ The solu^n is $x^{-2/3}\cdot\sinh\left(\frac{y}{x}\right) = -\frac{c'}{3} = c$

* Linear Differential Eqn :-

① $\dfrac{dy}{dx} + Py = Q$

② Find $\int P\,dx$ & then I.F. $= e^{\int P\,dx}$

③ Multiply the eqn by IF it becomes exact and hence can be solved by mere integration

④ $y \cdot e^{\int P\,dx} = \int \left[\left[e^{\int P\,dx} \right] Q \right] dx + c$

Q $\dfrac{dy}{dx} + \left(\dfrac{1-2x}{x^2} \right) y = 1$

→ $\int P\,dx = \int \left(\dfrac{1-2x}{x^2} \right) dx = \int \dfrac{1}{x^2}\,dx - 2\int \dfrac{dx}{x} = \dfrac{-1}{x} - 2\log x$

$\therefore e^{\int P\,dx} = e^{\left(-\frac{1}{x} - 2\log x\right)} = e^{-\frac{1}{x}} \cdot e^{-2\log x} = \dfrac{e^{-\frac{1}{x}}}{x^2}$

$\therefore y\dfrac{e^{-1/x}}{x^2} = \int \dfrac{e^{-1/x}}{x^2} \cdot 1\,dx + C$ $\left[\because \int e^{f(x)} \cdot f'(x)\,dx = e^{f(x)} + c \right.$

$y\dfrac{e^{-1/x}}{x^2} = e^{\frac{-1}{x}} + C$

$y = x^2 + c e^{1/x} \cdot x^2$

$\boxed{Q}$ $(1+x+xy^2)\,dy + (y+y^3)\,dx = 0$

$$-(1+x+xy^2)\,dy = (y+y^3)\,dx$$

$$\frac{dx}{dy} = \frac{-(1+x(1+y^2))}{y(1+y^2)}$$

$$\frac{dx}{dy} = \frac{-1}{y(1+y^2)} - \frac{x\cancel{(1+y^2)}}{y\cancel{(1+y^2)}}$$

$$\frac{dx}{dy} + \frac{x}{y} = \frac{-1}{y(1+y^2)}$$

$$\int P\,dy = \int \frac{1}{y}\,dy = \log y$$

$$\therefore e^{\int P\,dy} = e^{\log y} = y$$

$$\therefore xy = \int \left[y \cdot \frac{-1}{y(1+y^2)}\right] dy + c$$

$$\therefore xy = -\tan^{-1} y + c$$

$$\therefore xy + \tan^{-1} y = c$$

Q] $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

Dividing by $\cos^2 y$

$\therefore \sec^2 y \frac{dy}{dx} + x\, 2\tan y = x^3$

If $\tan y = t$, $\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore \frac{dt}{dx} + 2xt = x^3$

$\int P dx = 2x\, dx = x^2$

$I.F. = e^{\int P dx} = e^{x^2}$

$\therefore t e^{x^2} = \int e^{x^2} . x^3\, dx + c$

$x^2 = v$

$\therefore 2x\,dx = dv$

$\therefore x\,dx = \frac{dv}{2}$

$= \int e^v . v . \frac{dv}{2} = \frac{1}{2}\left[v.e^v - \int e^v . dt \right]$

ILATE

$= \frac{1}{2}\left[ve^v - e^v \right] = \frac{1}{2} e^v (v-1) = \frac{1}{2} e^{x^2}(x^2-1)$

The solu^n is $ve^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + c$

$\therefore \tan y . e^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + c$

$\therefore \tan y = \frac{1}{2}(x^2-1) + ce^{-x^2}$

* Complementry function

Case - I : All roots are distinct as $\alpha_2, \alpha_2, \ldots\ldots \alpha_n$ then CF is :-

$$y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + \ldots\ldots c_n e^{\alpha_n x}$$

Q] $\dfrac{\partial^3 y}{\partial x^3} + 6\dfrac{\partial^2 y}{\partial x^3} + 11\dfrac{\partial y}{\partial x} + 6y = 0$

$\rightarrow (D^3 + 6D^2 + 11D + 6)y = 0$

$m^3 + 6m^2 + 11m + 6 = 0$

$m = -1, -2, -3$

$\therefore\ y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

Case - 2 : If roots are equal like $\alpha, \alpha, \alpha_3, \alpha_4, \ldots \alpha_n$ then :-

$$\therefore\ y = (c_1 + c_2 x)e^{\alpha x} + c_3 e^{\alpha_3 x} + c_4 e^{\alpha_4 x} + \ldots\ldots c_n e^{\alpha_n x}$$

Q] $(D^4 + 2D^3 - 3D^2 - 4D + 4)y = 0$

$\rightarrow\ m^4 + 2m^3 - 3m^2 - 4m + 4 = 0$

$m = 1, 1, -2, -2$

$\therefore\ y = (c_1 + c_2 x)e^{x} + (c_3 + c_4 x)e^{-2x}$

Case - 3: If roots are complex number like $\alpha \pm i\beta$ then:—

$$y = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$$

Q] $(D^2+1)y = 0$

→ $m^2 + 1 = 0$

$m = \sqrt{-1} = \pm i = 0 \pm i$

$\therefore y = e^{0x}(c_1 \cos x + c_2 \sin x)$

Q] $(D^2 - 2D + 5)^2 y = 0$

→ $(m^2 - 2m + 5)^2 = 0$

$m = 1+2i, \; 1-2i$

But as the quadratic eqn is squarred

$\therefore m = 1+2i, 1+2i, 1-2i, 1-2i$

$\therefore y = e^{x}[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$

Case - 4: If roots are form of $\alpha \pm \sqrt{\beta}$ then:—

$$y = e^{\alpha x}[c_1 \cosh \sqrt{\beta}\, x + c_2 \sinh \sqrt{\beta}\, x]$$

Q] $(D^2 - 4D + 1)y = 0$

$m^2 - 4m + 1 = 0$; $m = 2 \pm \sqrt{3}$

$\therefore y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$; $y = e^{2x}[c_1 \cosh \sqrt{3}x + c_2 \sinh \sqrt{3}x]$

* Particular Integral

① e^{ax}

- Let $(D^n + a_1 D^{n-1} + \dots a_n)y = e^{ax}$ is a DE with constant coefficient

Working Rule:

Step 1: $f(D) = D^n + a_1 D^{n-1} + \dots a_n$

then P.I. is $\dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax}$ where $f(a) \neq 0$

Step 2: If $f(a) = 0$, then $f(D) = (D-a)^r \phi(D)$

So, P.I. $\dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax} = \dfrac{x e^{ax}}{f'(a)}$ where $f'(a) \neq 0$

Case I: A D.E of this type $[f(D)]y = 1$, then $[F(D)]y = e^{0x}$

Q) Find P.I. of $(D^2 - 3D + 2)y = 1$

→ $m^2 - 3m + 2 = 0$

$m = 1, 2$

$\therefore$ C.F : $y = c_1 e^{x} + c_2 e^{2x}$

& P.I : $y = \dfrac{1}{D^3 - 3D + 2} = \dfrac{e^{0x}}{D^2 - 3D + 2} = \dfrac{e^{0x}}{0 - 3(0) + 2} = \dfrac{e^{0x}}{2} = \dfrac{1}{2}$

Case – II : A D.E. of this type $[f(D)]y = e^{ax}$, when $f(a) \neq 0$

Q) find P.I. of $(D^2 + 4D + 4)y = e^{2x}$

$\rightarrow$ $m^2 + 4m + 4 = 0$

$m = -2, -2$

C.F. : $y = (C_1 + C_2 x)e^{-2x}$

P.I. : $y = \dfrac{e^{2x}}{D^4 + 4D + 4} = \dfrac{e^{2x}}{(2)^2 + 4(2) + 4} = \dfrac{e^{2x}}{16}$

$\therefore$ $y =$ P.I + C.F

Case – III : A D.E of this type $[f(D)]y = e^{ax}$, when $f(a) = 0$

Q) Solve for P.I. $(D^4 + D^3 + D^2 - D - 2)y = e^x$

$\rightarrow$ $m^4 + m^3 + m^2 - m - 2 = 0$

$m = 1, -1, \dfrac{-1 - \sqrt{7}i}{2}, \dfrac{-1 + \sqrt{7}i}{2}$

CF : $y = c_1 e^x + c_2 e^{-x} + e^{-x}\left[C_3 \cos \frac{\sqrt{7}}{2}x + C_4 \sin \frac{\sqrt{7}}{2}x\right]$

P.I : $y = \dfrac{e^x}{D^4 + D^3 + D^2 - D - 2} = \dfrac{xe^x}{4D^3 + 3D^2 + 2D - 1} = \dfrac{xe^x}{4 + 3 + 2 - 1} = \dfrac{xe^x}{8}$

$y =$ CF + PI

Case – IV : $[f(D)]y = \alpha e^{ax}$ then $PI = \alpha \dfrac{1}{f(D)} e^{\alpha x}$

Q Find P.I. of $(D^2 - 3D + 2)y = 3e^{3x}$

→ $m^2 - 3m + 2 = 0$

$\therefore m = 1, 2$

CF : $y = c_1 e^{x} + c_2 e^{2x}$

PI : $y = \dfrac{3e^{3x}}{D^2 - 3D + 2} = \dfrac{3}{2} e^{3x}$

$y = CF + PI$

Case V : $[f(D)]y = e^{ax} + e^{bx}$ then P.I.

$$\frac{1}{f(D)}(e^{ax} + e^{bx}) = \frac{1}{f(D)} e^{ax} + \frac{1}{f(D)} e^{bx}$$

Q Find P.I. for DE $(D^2 - 6D + 8)y = (e^{2x} + 1)^2$

→ $m^2 - 6m + 8 = 0$

$m = 2, 4$

CF : $y = c_1 e^{2x} + c_2 e^{4x}$

PI : $y = \dfrac{e^{4x} + e^{2x} + 1}{D^2 - 6D + 8} = \dfrac{e^{4x}}{D^2 - 6D + 8} + \dfrac{2e^{2x}}{D^2 - 6D + 8} + \dfrac{e^{0x}}{D^2 - 6D + 8}$

$y = \dfrac{xe^{4x}}{2} - \dfrac{2xe^{2x}}{2} + \dfrac{e^{0x}}{8}$

② $\sin ax / \cos ax$

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} \dots a_n)y = \sin ax / \cos ax$$

when $[f(D)]y = \sin ax / \cos ax$, then P.I. is

$$\frac{1}{f(D^2)} \sin ax / \cos ax$$

$$= \frac{1}{f(-a^2)} \sin ax / \cos ax$$

Case-I: When $f(-a^2) \neq 0$

Q] $(D^2+1)y = \cos 2x$

$$m^2+1=0$$

$$m = \pm i$$

CF : $y = c_1 \cos x + c_2 \sin x$

PI : $y = \dfrac{\cos 2x}{D^2+1} = \dfrac{\cos 2x}{(-(2)^2)+1} = \dfrac{\cos 2x}{-4+1} = \dfrac{\cos 2x}{-3}$

Case-II: When $f(-a^2) = 0$

Q] $(D^2+a^2)y = \sin ax$

$m^2 + a^2 = 0 \implies m = \pm ai$

CF : $y = c_1 \cos ax + c_2 \sin ax$

Integrate

P.I : $y = \dfrac{\sin ax}{D^2+a^2} = \dfrac{x \sin ax}{2D} = -\dfrac{x}{2} \dfrac{\cos ax}{a}$

Q] $(D^3 + 3D^2 + 3D + 1)y = \cos x$

$m^3 + 3m^2 + 3m + 1 = 0$

$m = -1, -1, -1$

CF: $y = (c_1 + c_2 x + c_3 x^2)e^{-x}$

PI: $y = \dfrac{\cos x}{D^2 \cdot D + 3D^2 + 3D + 1} = \dfrac{\cos x}{-D - 3 + 3D + 1} = \dfrac{(2D+2)}{(2D+2)} \dfrac{\cos x}{2D - 2} = \dfrac{(2D+2)\cos x}{4D^2 - 4}$

$y = \dfrac{2D(\cos x) + 2\cos x}{-4 - 4} = \dfrac{-2\sin x + 2\cos x}{-8}$

③ x^m

Let $[f(D)]y = x^m$, $m \in Z^+$ is a DE with constt coefficient then $\dfrac{1}{f(D)} x^m$ is P.I

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots$

Working Rule:

① $\dfrac{1}{f(D)} x^m = \dfrac{1}{[1 + \phi(D)]^n} x^m$

② Expand $[1 + \phi(D)]^n$ then we get PI

Q] $(D^4 - a^4)y = x^4$

$m^4 - a^4 = 0$

$m = \pm ai, \pm a$

CF: $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$

PI: $y = \dfrac{x^4}{D^4 - a^4}$

$$y = \frac{-1}{a^4}\left(\frac{x^4}{1 - \frac{D^4}{a^4}}\right) = \frac{-1}{a^4}\left(1 - \frac{D^4}{a^4}\right) x^4$$

$$y = \frac{-1}{a^4}\left(1 + \frac{D^4}{a^4} + \cancel{\left(\frac{D^4}{a^4}\right)^2} + \dots\right) x^4$$

$$y = \frac{-1}{a^4}\left(x^4 + \frac{D^4(x^4)}{a^4}\right) = \frac{-1}{a^4}\left(x^4 + \frac{24}{a^4}\right)$$

Q] $(D^3 - D^2 - 6D)y = x^2 + 1$

→ $m^3 - m^2 - 6m = 0$

$m(m-3)(m+2) = 0$

$m = 0, -2, 3$

Cf: $y = c_1 e^{0x} + c_2 e^{-2x} + c_3 e^{3x}$

PI: $y = \dfrac{x^2+1}{D^3 - D^2 - 6D} = -\dfrac{1}{6D} \dfrac{(x^2+1)}{\left(1 - \frac{D^3 - D^2}{6D}\right)}$

$$y = \frac{-1}{6D}\left(1 - \frac{D^2 - D}{6}\right)^{-1} (x^2+1)$$

$$y = -\frac{1}{6D}\left(1 + \frac{D^2 - D}{6} + \left(\frac{D^2 + D}{6}\right)^2 + \dots\right)(x^2+1)$$

$$y = -\frac{1}{6D}\left(1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36}\right)(x^2+1)$$

$$y = \frac{-1}{6D}\left(1 - \frac{D}{6} + \frac{7D^2}{36}\right)(x^2+1)$$

$$y = \frac{-1}{6D}\left[x^2+1 - \frac{1}{6}D(x^2+1) + \frac{7D^2}{36}(x^2+1)\right]$$

$$y = \frac{-1}{6D}\left(x^2+1 - \frac{1}{6}(2x) + \frac{7\times 2}{36}\right)$$

$$y = \frac{-1}{6D}\left(x^2 + \frac{x}{3} + \frac{7}{18}\right)$$

$$y = -\frac{1}{6}\left(\frac{x^3}{3} + \frac{x^2}{6} + \frac{7x}{18}\right)$$

$y = cf + PI$.

④ $e^{ax}V$

Q] $(D^2 - 2D + 1)y = x^2 e^{3x}$

$m^2 - 2m + 1 = 0$

$m = 1, 1$

$cf: \; y = (c_1 + c_2 x)e^x$

PI: $y = \dfrac{x^2 e^{3x}}{D^2-2D+1} = e^{3x}\left[\dfrac{x^2}{(D+3)^2-2(D+3)+1}\right]$

$y = e^{3x}\left(\dfrac{x^2}{D^2+6D+9-2D-6+1}\right) = e^{3x}\left(\dfrac{x^2}{D^2+4D+4}\right) = \dfrac{e^{3x}}{4}\left[1+\dfrac{D^2+D}{4}\right]^{-1}$

$y = \dfrac{e^{3x}}{4}\left[1-\dfrac{D^2+4D}{4}+\left(\dfrac{D^2+4D}{4}\right)^2-\dots\right)x^2$

$y = \dfrac{e^{3x}}{4}\left[1-\dfrac{D^2}{4}-D+D^2\right]x^2$

$y = \dfrac{e^{3x}}{4}\left[x^2-2x+\dfrac{3}{2}\right]$

$y = cf + PI$

5 xV

Q] $(D^2-2D+1)y = x\sin x$

$\rightarrow$ $m^2-2m+1=0$

$m = 1, 1$

cf : $y = (c_1+c_2x)e^x$

PI: $y = \dfrac{x\sin x}{D^2-2D+1}$

formula

$\boxed{x-\dfrac{f'(D)}{f(D)}\,\dfrac{V(x)}{f(D)}}$

$y = \left(x - \dfrac{2D-2}{D^2-2D+1}\right)\left(\dfrac{\sin x}{D^2-2D+1}\right)$

$$y = \left(x - \frac{2\cancel{(D-1)}}{(D-1)^{\cancel{2}}}\right) \frac{\sin x}{\cancel{-1} \quad 2D + \cancel{1}}$$

$$y = \frac{-1}{2}\left(x - \frac{2}{D-1}\right) - \cos x$$

$$y = \frac{1}{2}\left(x\cos x - \frac{2\,(D+1)\cos x}{(D+1)(D-1)}\right)$$

$$y = \frac{1}{2}\left[x\cos x - \frac{\cancel{2}(D+1)\cos x}{\cancel{2}}\right]$$

$$y = \frac{1}{2}\left[x\cos x - \sin x + \cos x\right]$$

* Method of Variation of Parameters

Q] $(D^2 - 2D + 1)y = e^x$

→ $m^2 - 2m + 1 = 0$

$m = 1, 1$

Cf: $y = (c_1 + c_2 x)e^x = c_1 e^x + x c_2 e^x$

$u = e^x$, $v = xe^x$

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$

PI: $y = u\,f(x) + v\,g(x)$

$$f(x) = -\int \frac{vR}{W}\,dx = -\int \frac{xe^x \cdot e^x}{\cancel{e^{2x}}} = -\frac{x^2}{2}$$

$$g(x) = \int \frac{uR}{W}\,dx = \int \frac{\cancel{e^x \cdot e^x}}{\cancel{e^{2x}}} = x$$

$$y = e^x\left(\frac{-x^2}{2}\right) + xe^x\,(x)$$

$$y = \frac{-x^2}{2}e^x + x^2 e^x$$

$$y = \frac{x^2 e^x}{2}$$

$y = cf + PI$