Fourier Series

* Dirichlet's Conditions :-

Consider a single valued function $f(x)$ in interval $(a, a+2L)$ which satisfies below conditions is known as Dirichlet's conditions.

① $f(x)$ is defined in interval $(a, a+2L)$ & $f(x) = f(x+2L)$

② $f(x)$ is continuous function OR has finite number of discontinuites in interval $(a, a+2L)$

③ $f(x)$ has no maxima or minima OR has finite numbers of maxima or minima

Fourier - Euler's formula

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \; ; \quad a < x < \alpha + 2\pi$$

OR

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \longrightarrow$$ [Fourier series in interval $(\alpha, \alpha+2L)$]

$$f(x) = \frac{1}{L} \int_a^{a+2L} f^2(x)\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \longrightarrow$$ [Parseval's Identity Use whenever you need square of summation series]

where,

$$\left.\begin{aligned} a_0 &= \frac{1}{L}\int_a^{a+2L} f(x)\,dx \\ a_n &= \frac{1}{L}\int_a^{a+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n &= \frac{1}{L}\int_a^{a+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned}\right\} \text{Fourier Coefficients}$$

- Formulas

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

Where

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$\boxed{0, 2\pi}$

$$f(x) = \frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$\boxed{(-\pi, \pi) \text{ or } (-L, L)}$

Check for even or odd

[L or π both works]

Even	Odd
$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$	$a_0 = 0$
	$a_n = 0$
$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$	$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$
$b_n = 0$	

- Half Range Series

$0 \leq x \leq 2$

$\hookrightarrow L = 2$

Half range cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_0 = \frac{2}{L} \int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\,dx$$

Half Range Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where, $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\,dx$

- Complex form

$f(x)$ in interval $(a, a+2L)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

where, $c_n = \frac{1}{2}(a_n - ib_n)$ & $c_0 = \frac{a_0}{2}$

$= \frac{1}{2L} \int_a^{a+2L} f(x) e^{-\frac{in\pi x}{L}}\,dx$

$c_{-n} = \frac{1}{2}(a_n + ib_n)$

Q. Fourier series in interval $(0, 2\pi)$

Obtain F.S. for $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 \le x \le 2\pi$ & $f(x+2\pi) = f(x)$

Deduce. (i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$

(ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

(iv) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$

→ $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2\,dx$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = \frac{-1}{12\pi}(-\pi^3 - \pi^3)$$

$\therefore \boxed{a_0 = \frac{\pi^2}{6}}$

Remember!!!

When period $= (0, 2\pi)$ or $(-\pi, \pi)$

$$\cos\left(\frac{n\pi x}{L}\right) = \cos(nx)$$

$$\sin\left(\frac{n\pi x}{L}\right) = \sin(nx)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot \cos nx \, dx$$

$$a_n = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cdot \cos nx \, dx$$

ILATE

$$= \frac{1}{4\pi}\left[(\pi-x)^2 \int \cos nx - \int \frac{d}{dx}(\pi-x)^2 \int \cos nx \, dx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2 \frac{\sin(nx)}{n} - \int 2(\pi-x)\cdot(-1) \frac{\sin(nx)}{n}\right]_0^{2\pi}$$

ILATE

$$= \frac{1}{4\pi}\left[(\pi-x)^2 \frac{\sin(nx)}{n} - \left[\int -2(\pi-x) \frac{\sin(nx)}{n}\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2 \frac{\sin(nx)}{n} - \left[-2(\pi-x)\int \frac{\sin(nx)}{n} + \int \frac{d}{dx}(-2(\pi-x)) \int \frac{\sin(nx)}{n}\right] dx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2 \frac{\sin(nx)}{n} - \left(-2(\pi-x) \frac{-\cos(nx)}{n^2}\right) + \int 2 \cdot \frac{-\cos(nx)}{n^2} dx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2 \frac{\sin(nx)}{n} - \left(-2(\pi-x) \frac{-\cos(nx)}{n^2}\right) + 2 \cdot \frac{-\sin(nx)}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2 \cancel{\frac{\sin(nx)}{n}}^{\,0} - 2(\pi-x)\frac{\cos(nx)}{n^2} + 2 \cdot \left(-\cancel{\frac{\sin(nx)}{n^3}}^{\,0}\right)\right]_0^{2\pi}$$

$$= \frac{-1}{2n^2\pi}\left[(\pi-x)\cos(nx)\right]_0^{2\pi} = \frac{-1}{2n^2\pi}(-\pi-\pi) \Rightarrow \boxed{a_n = \frac{1}{n^2}}$$

$$b_n = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cdot \sin(nx)\, dx$$

ILATE

$$= \frac{1}{4\pi}\int_0^{2\pi} \left[(\pi-x)^2 \cdot \frac{-\cos nx}{n} - \cancelto{0}{(-2(\pi-x))\cdot\frac{-\sin nx}{n^2}} + 2\cdot\frac{\cos(nx)}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi n}\left[-(\pi-x)^2 \cos(nx) + \frac{2\cos(nx)}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi n}\left[-\pi^2.(1) + \frac{2}{n^2} - (-\pi^2).1 - \frac{2}{n^2}\right]$$

$$\boxed{b_n = 0}$$

$$\therefore \left(\frac{\pi - x}{2}\right)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}.\cos(nx) + 0$$

Putting $x=0$,

$$\therefore \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$\therefore \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{— } (1)$$

Putting $x=\pi$,

$$\therefore 0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \quad \text{— } (2)$$

Adding ① & ② ,

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{1^2} + \cancel{\frac{1}{2^2} - \frac{1}{2^2}} + \frac{1}{3^2} + \frac{1}{3^2} + \cancel{\frac{1}{4^2} - \frac{1}{4^2}} + \cdots$$

$$\frac{18\pi^2}{72} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2}$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Parseval's Identity

$$\frac{1}{L}\int_0^{a+2L} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\therefore \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^4 dx = \frac{\pi^2}{36 \times 2} + \sum_{n=1}^{\infty} \left(\left(\frac{1}{n^2}\right)^2 + 0^2 \right)$$

$$\therefore \frac{1}{16\pi}\int_0^{2\pi} (\pi - x)^4\,dx = \frac{\pi^4}{72} + \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \frac{1}{16\pi}\left[\frac{(\pi - x)^5}{-5}\right]_0^{2\pi} = \frac{\pi^4}{72} + \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{40} - \frac{\pi^4}{72} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots$$

9] $f(x) = e^{-x} \quad (0, 2\pi)$

i) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$ ii) $\operatorname{cosech}(\pi)$

→

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx$$

$$a_0 = -\frac{1}{\pi}\left[e^{-x}\right]_0^{2\pi} = \frac{1-e^{2\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos(nx)\,dx$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$a = -1 \quad b = n$

$$\therefore a_n = \frac{1}{\pi}\left[\frac{e^{-x}}{n^2+1}\left(-\cos nx + \cancelto{0}{n\sin nx}\right)\right]_0^{2\pi}$$

$$\therefore a_n = \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}\cdot(-1) - 1(-1)\right] = \frac{1}{n^2+1}\cdot\left(\frac{1-e^{-2\pi}}{\pi}\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin(nx)\,dx$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{e^{-x}}{n^2+1}\left(\cancelto{0}{-\sin(nx)} - n\cos(nx)\right)\right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi(n^2+1)}\left(e^{-2\pi}\cdot(-n) - 1(-n)\right) = \frac{n}{n^2+1}\left(\frac{1-e^{-2\pi}}{\pi}\right)$$

$$\therefore f(x) = \left(\frac{1-e^{-2\pi}}{2\pi}\right) + \frac{1-e^{-2\pi}}{\pi} \sum_{n=1}^{\infty} \left[\frac{\cos(nx)}{n^2+1} + \frac{n.\sin(nx)}{n^2+1}\right]$$

(i) Putting $x = \pi$

$$e^{-\pi} = \left(\frac{1-e^{-2\pi}}{2\pi}\right) + \left(\frac{1-e^{-2\pi}}{\pi}\right) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$\therefore e^{-\pi} = \cancel{\left(\frac{1-e^{-2\pi}}{2\pi}\right)} + \cancel{\left(\frac{-1}{2}\left(\frac{1-e^{-2\pi}}{\pi}\right)\right)} + \left(\frac{1-e^{-2\pi}}{\pi}\right) \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$\therefore \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{\pi . e^{-\pi}}{1-e^{-2\pi}} = \frac{\pi}{e^{\pi}-e^{-\pi}}$$

(ii) $$\therefore \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{\pi}{2\left(\frac{e^{\pi}-e^{-\pi}}{2}\right)} = \frac{\pi}{2} . \operatorname{cosech}(\pi)$$

$$\therefore \operatorname{cosec} h(\pi) = \frac{2}{\pi} \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{n^2+1}\right)$$

Q] $f(x) = \cos(px)$ in $(0, 2\pi)$ p is not an integer

Deduce that :- 1) $\pi \operatorname{cosec}(p\pi) = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{p+n} + \frac{1}{p-n}\right]$

ii) $\pi \cot(2\pi p) = \frac{1}{2p} + p \sum_{n=1}^{\infty} \frac{1}{p^2-n^2}$

$$\rightarrow a_0 = \frac{1}{\pi} \int_0^{2\pi} \cos(px)\,dx$$

$$a_0 = \frac{1}{\pi}\left[\frac{\sin px}{p}\right]_0^{2\pi} = \frac{1}{p\pi}(\sin 2\pi p - \sin 0)$$

$$a_0 = \frac{\sin(2p\pi)}{p\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos(px)\cos nx \; dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} 2\cos px \cos nx \; dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos[(p+n)x] + \cos[(p-n)x] \; dx$$

$$a_n = \frac{1}{2\pi}\left[\frac{\sin[(p+n)x]}{p+n} + \frac{\sin[(p-n)x]}{p-n}\right]_0^{2\pi}$$

But, $\sin(p \pm n)2\pi = \sin 2p\pi \cdot \cos(2n\pi) = \sin(2p\pi)$

$$a_n = \frac{\sin(2p\pi)}{2\pi}\left[\frac{1}{p+n} + \frac{1}{p-n}\right]$$

$$a_n = \frac{\sin(2p\pi)}{2\pi}\left(\frac{2p}{p^2-n^2}\right) \quad //$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \cos(px) \cdot \sin(nx) \, dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} \sin((p+n)x) - \sin((p-n)x) \; dx$$

$$b_n = \frac{1}{2\pi}\left[\frac{\cos((p-n)x)}{p-n} - \frac{\cos((p+n)x)}{p+n}\right]_0^{2\pi}$$

But $\cos(p\pm n)2\pi = \cos 2p\pi$

$$b_n = \frac{1}{2\pi}\left[\frac{\cos(2p\pi)-1}{p-n} - \frac{\cos(2p\pi)-1}{p+n}\right]$$

$$b_n = \frac{\cos(2p\pi)-1}{2\pi}\left(\frac{2n}{p^2-n^2}\right) = \frac{n[\cos(2p\pi)-1]}{\pi(p^2-n^2)} \quad //$$

$$f(x) = \cos(px) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$\cos(px) = \frac{\sin(2p\pi)}{2p\pi} + \frac{p\sin(2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{\cos(nx)}{p^2-n^2} + \frac{\cos(2p\pi)-1}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{p^2-n^2}$$

$$\frac{1}{p^2-n^2} = \frac{1}{2p}\left(\frac{1}{p+n} + \frac{1}{p-n}\right)$$

i) Putting $x=\pi$

$$\cos(p\pi) = \frac{2\sin p\pi\cos p\pi}{2p\pi} + \frac{p\sin(2p\pi)}{\pi}\frac{1}{p}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{p+n} + \frac{1}{p-n}\right) + 0$$

$$\therefore \cancel{\cos p\pi} = \frac{2\sin p\pi\cancel{\cos p\pi}}{2p\pi} + \frac{\cancel{2p}\sin p\pi\cancel{\cos p\pi}}{\pi}\cdot\frac{1}{\cancel{2p}}\cdot\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{p+n} + \frac{1}{p-n}\right) + 0$$

$$\therefore \pi\,\mathrm{cosec}(p\pi) = \frac{1}{p} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{p+n} + \frac{1}{p-n}\right] \quad //$$

ii) Putting $x=2\pi$

$$\cos(2p\pi) = \frac{\sin(2p\pi)}{2p\pi} + \frac{p\sin(2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{1}{p^2-n^2}$$

$$\pi\cot(2p\pi) = \frac{1}{2p} + p\sum_{n=1}^{\infty}\frac{1}{p^2-n^2} \quad //$$

* Complex form of Fourier Series

① Fourier Series from c to $(c+2\ell)$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \& \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \left[\frac{e^{\frac{in\pi x}{\ell}} + e^{-\frac{in\pi x}{\ell}}}{2}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{e^{\frac{in\pi x}{\ell}} - e^{-\frac{in\pi x}{\ell}}}{2i}\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2}\right) e^{\frac{in\pi x}{\ell}} + \sum_{n=1}^{\infty} \left(\frac{a_n + ib_n}{2}\right) e^{-\frac{in\pi x}{\ell}}$$

$$= a_0 + \sum_{n=1}^{\infty} C_n e^{\frac{in\pi x}{\ell}} + \sum_{n=1}^{\infty} C_{-n} e^{-\frac{in\pi x}{\ell}}$$

$$= a_0 + \sum_{-\infty}^{\infty} C_n e^{\frac{in\pi x}{\ell}}$$

$$C_n = \frac{1}{2\ell} \int_{c}^{c+2\ell} f(x)\, e^{-\frac{in\pi x}{\ell}}$$

Q] Obtain complex form of fourier Series for

$f(x) = e^{ax}$ in $(-\pi, \pi)$ where a is not an integer

$\rightarrow$ $f(x) = a_0 + \sum^{\infty} C_n e^{\frac{in\pi x}{l}}$

$$C_n = \frac{1}{2l}\int_{c}^{c+2l} f(x)\, e^{-\frac{in\pi x}{l}}$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\cdot e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(a-in)}\, dx = \frac{1}{2\pi}\left[\frac{e^{x(a-in)}}{a-in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a-in)}\left[e^{\pi(a-in)} - e^{-\pi(a-in)}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}\cdot e^{-in\pi} - e^{-a\pi}\cdot e^{in\pi}\right]$$

As, $e^{\pm in\pi} = \cos n\pi \pm i\sin n\pi$

$= (-1)^n \pm 0$

$e^{\pm in\pi} = (-1)^n$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}(-1)^n - e^{-a\pi}(-1)^n\right]$$

$$= \frac{(-1)^n}{\pi(a-in)}\left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]$$

$$= \frac{(-1)^n}{\pi(a-in)}\left[\sinh a\pi\right]$$

$$= \frac{(-1)^n \sinh a\pi}{\pi(a-in)}$$

$$= \frac{(-1)^n \sinh a\pi}{\pi(a-in)} \times \frac{(a+in)}{(a+in)}$$

$$c_n = \frac{(-1)^n . \sinh a\pi . (a+in)}{\pi(a^2+n^2)}$$

$\therefore$ Complex form of fourier Series is,

$$e^{ax} = \sum_{-\infty}^{\infty} \frac{(-1)^n \sinh a\pi}{\pi(a^2+n^2)} . (a+in) e^{inx}$$

$$\int U.V\,dx = U' V_I - U'' V_{II} + U''' V_{III}$$

$$e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$$

$$e^{ax} \cos bx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$$

$$|x| \leq a \Rightarrow -a \leq x \leq a$$

$$|x| \geq a \Rightarrow x \geq a \text{ or } x \leq -a$$

* Fourier Transform $(-\infty < x < \infty)$

$$\boxed{F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{i\lambda u} . du}$$

Inverse Fourier Transform

$$\boxed{F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\lambda) . e^{-i\lambda x} . d\lambda}$$

Even function

$$F(\lambda) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(u). \cos\lambda u \, du$$

Inverse

$$F(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(\lambda) \cos\lambda x \, d\lambda$$

Odd Function

$$F(\lambda) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(u). \sin\lambda u \, du$$

Inverse

$$F(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(\lambda). \sin\lambda x . d\lambda$$

Fourier Cosine Transform $(0 < x < \infty)$

$$F(\lambda) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(u) . \cos\lambda u . du \longrightarrow$$

Inverse :

$$F(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(\lambda). \cos\lambda x . d\lambda$$

Fourier Sine Transform

$$F(\lambda) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(u) . \sin\lambda u \, du \longrightarrow$$

Inverse :

$$F(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(\lambda). \cos\lambda x . d\lambda$$

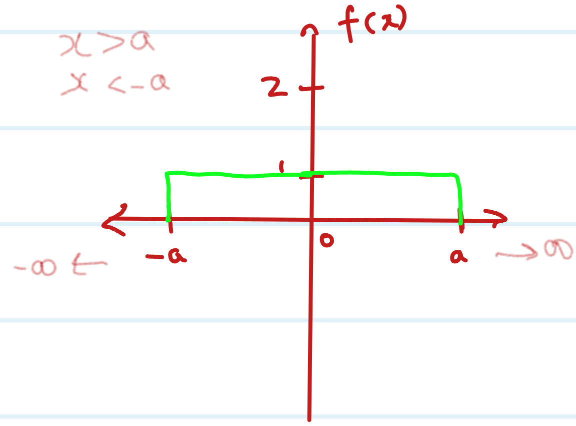
Q] Find Fourier Transform for

$-a \leq x \leq a$

$x > a$

$x < -a$

$$F(x) = 1 \qquad |x| \leq a$$
$$\quad = 0 \qquad |x| \geq a$$



F(x) is even function

$$F(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(u).\cos\lambda u.\,du$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a f(u).\cos\lambda u\,du + \cancelto{0}{\int_a^\infty f(u)\cos\lambda u.\,du} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a 1.\cos\lambda u.\,du \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin\lambda u}{\lambda} \right]_0^a$$

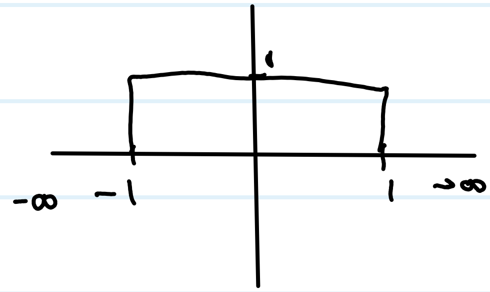
$$F(\lambda) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin\lambda a}{\lambda}$$

Q] Find Fourier Integral representation of function.

$$F(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

Deduce the value of $\int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda$

→ $f(x)$ is even function



$$F(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(u) \cdot \cos\lambda u \, du$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{1} \cos\lambda u \, du$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin\lambda u}{\lambda} \right]_0^1$$

$$F(\lambda) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin\lambda}{\lambda}$$

Inverse :

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(\lambda) \cdot \cos\lambda x \, d\lambda$$

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left[\sqrt{\frac{2}{\pi}} \left(\frac{\sin\lambda}{\lambda}\right) \cos\lambda x \, d\lambda \right]$$

$$\frac{\pi}{2} F(x) = \int_0^{\infty} \frac{\sin\lambda}{\lambda} \cdot \cos\lambda x \, d\lambda$$

$|x| < 1$

$$\frac{\pi}{2} \cdot (1) = \int_0^{\infty} \frac{\sin\lambda}{\lambda} \cdot \cos\lambda x \, d\lambda$$

Put $x = 0 \Rightarrow \frac{\pi}{2} = \int_0^{\infty} \frac{\sin\lambda}{\lambda} \, d\lambda$