ELECTRODYNAMICS

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Fields :-

A field is a region of space where some physical quantity takes different values at different points in the region.

At each point of the region there exists a corresponding value of physical quantity.

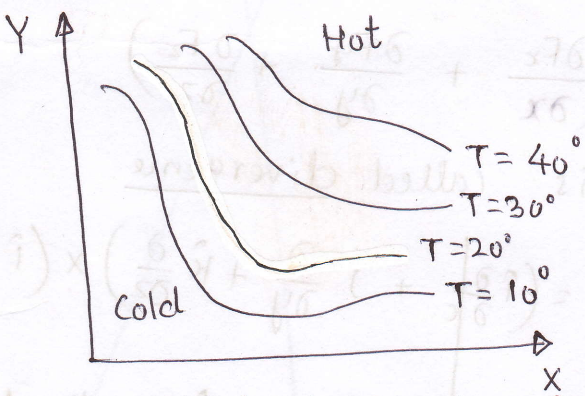
A field is a mathematical function of position and time. Depending upon the type of physical quantity, fields are classified into - scalar fields and vector fields.

Scalar field : If the value of a physical quantity at each point is a scalar quantity, then the field is said to scalar field.

Ex: temperature.

If a body is hot at some point and cold at some other point then temperature of body changes from point to point in complex way and function of x, y, z. The temperature of body may also vary with time 't'.

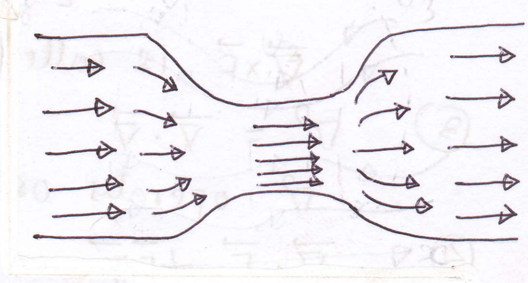
The temperature field can be represented as below.



Vector field : A field is said to be vector field when physical quantity at each point is vector quantity. The vector field has both magnitude and direction.

A vector is associated with each point in the region which varies from point to point.

Ex. The field of liquid flowing in a constricted pipe is an example of vector field. The flow of liquid at different points in the pipe has a direction and magnitude. We can denote the flow of liquid at different points in the pipe by vectors.



The Operator Del ($\vec{\nabla}$):

The vector differential operator del($\vec{\nabla}$) is written as

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$\vec{\nabla}$ is a vector operator, it possesses all prop. of ordinary vector.

The operator $\vec{\nabla}$ can be operated on scalar and vector. when $\vec{\nabla}$ is operated on scalar field, it is called Gradient. when $\vec{\nabla}$ is operated on vector via dot product, then it is called Divergence.

when $\vec{\nabla}$ is operated on vector via cross product, it is called Curl.

Gradient:- If $\phi(x,y,z)$ is a scalar function then

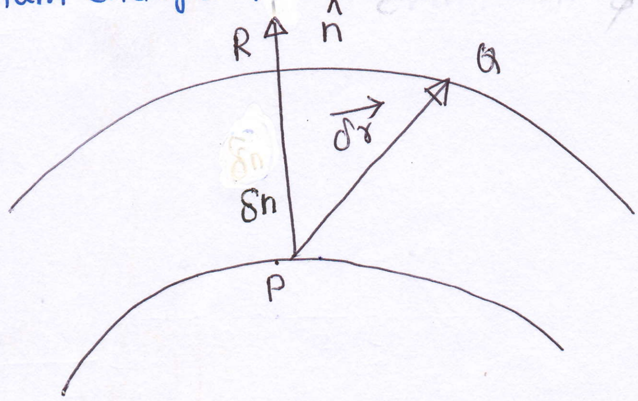
$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x,y,z)$$

$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

Gradient is also called Directional derivative

It gives the direction of maximum change in ϕ.

Meaning of Gradient:-



If a surface $\phi(x,y,z) = c$ passes through a point P, then the value of function at each point on the surface is same as P. such a surface is called level surface through P. Two level surfaces can not intersect.

Proof: Let the level surface pass through point P at which the functional value is ϕ. The another level surface passing through Q, where functional value is $(\phi + d\phi)$.

Let $\vec{r}$ and $\vec{r}+\vec{dr}$ be the position vector of P and Q

$$\therefore \quad \vec{PQ} = \delta \vec{r}$$

$$\vec{\nabla}\phi \cdot \vec{dr} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= d\phi$$

If Q and P lies on same level surface then $d\phi = 0$,

$$\nabla\phi \cdot \vec{dr} = 0$$

and $\vec{\nabla}\phi$ is normal to $\vec{dr}$ (tangent)

ie $\vec{\nabla}\phi$ is normal to surface $\phi(x,y,z) = C$

Divergence :

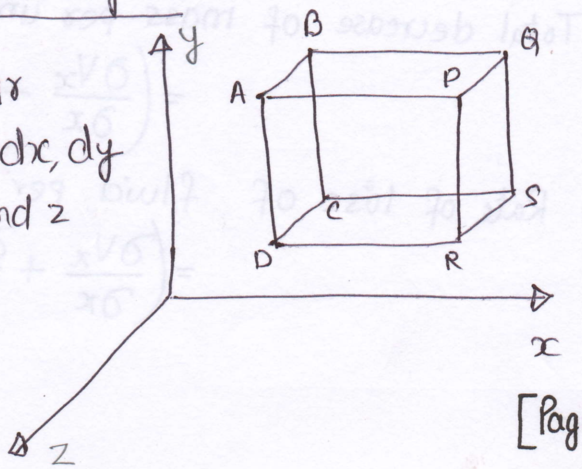
If $\vec{V}(x,y,z) = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ is a vector function defined and differentiable at each point (x,y,z) in certain region in space then, the divergence of $\vec{V}$ is scalar product with $\vec{\nabla}$ ie $\vec{\nabla}\cdot\vec{V}$

$$\vec{\nabla}\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)$$

$$\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

Physical Significance of Divergence :

Consider a small rectangular parallelopiped of dimensions dx, dy and dz parallel to x, y and z respectively.



Let $\overline{V} = \hat{i} V_x + \hat{j} V_y + \hat{k} V_z$ represent the velocity of fluid.

Fluid enters through face ABCD and and comes out from PQRS.

Mass of fluid flowing through face ABCD in unit time = (velocity) (area of face)

$$= (V_x)\ (dy)\ (dz) \longrightarrow (1)$$

Mass of fluid flowing out across face PQRS per unit time $= V_x(x+dx)\ (dy)(dz)$

$$= \left[V_x + \frac{\partial V_x}{\partial x} dx\right] [dy)(dz) \longrightarrow (2)$$

$\therefore$ Net decrease in mass along x-axis

$$= (V_x)(dy)(dz) - \left[V_x + \frac{\partial V_x}{\partial x} dx\right] (dy)\ (dz)$$

$$= -\frac{\partial V_x}{\partial x} dx\ dy\ dz$$

Similarly decrease in mass along y-axis

$$= \frac{\partial V_y}{\partial y} (dx)\ (dy)(dz)$$

Also decrease along z-axis

$$= \frac{\partial V_z}{\partial z} (dx)\ (dy)(dz)$$

$\therefore$ Total decrease of mass per unit time

$$= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dx\ dy\ dz$$

$\therefore$ Rate of loss of fluid per unit volume

$$= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z\right)$$

$$= \vec{\nabla}\cdot\vec{V} \quad (\text{div. of } \vec{V})$$

If the fluid is incompressible, then there is no gain or loss of fluid in volume

$$\therefore \vec{\nabla}\cdot\vec{V} = 0$$

$\vec{V}$ is also called solenoid vector function.

<u>curl:</u> The curl of a vector is a vector point function. If $\vec{V}(x,y,z)$ is a differentiable vector field, then curl of $\vec{V}$ (also called rotation of $\vec{V}$) is written as $\vec{\nabla}\times\vec{V}$.

$$\text{curl}\,\vec{V} = \vec{\nabla}\times\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right] + \hat{j}\left[\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right] + \hat{k}\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]$$

<u>Physical interpretation of curl:</u>

Curl of vector field represents rotational motion if vector field represents flow of a fluid.

A vector field $\vec{V}$ is called irrotational if $\vec{\nabla}\times\vec{V} = 0$ this means, flow of fluid is free from rotational motion i.e no whirlpool.

If $\vec{\nabla}\times\vec{V} \neq 0$ then $\vec{V}$ is not a conservative field.

for any scalar function f

$$\text{curl}(\text{grad}.f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= 0$$

ie gradient fields describing the motion are irrotational.

Fundamental theorems and Continuity Equation:-

(a) fundamental theorem of gradient:

$$\int_a^b (\vec{\nabla}\phi) \cdot \vec{dl} = \phi(b) - \phi(a)$$

(b) fundamental theorem of divergence

$$\int_{Vol} (\vec{\nabla} \cdot \vec{V}) \, dv = \int_{surface} \vec{V} \cdot \vec{ds}$$

(c) fundamental theorem of curl:-

$$\int_{surface} (\vec{\nabla} \times \vec{V}) \cdot \vec{ds} = \int_{line} \vec{V} \cdot \vec{dl}$$ (also called stoke's thm)

(d) Continuity Equation

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Electric field :- A region of space around a charge in which any other charge experiences force of attraction or repulsion is called electric field.

The electric field of a charge is measured in terms of vector quantity called Electric field Intensity ($\vec{E}$).

The electric field intensity of a charge at any given point (P) is defined as force acting on unit positive charge at that point.

$$\vec{E} = \frac{\vec{F}}{q_0}$$

SI unit : N/C

for point charge q, electric intensity at distance 'r' is given by

$$\vec{E} = \left(\frac{1}{4\pi\epsilon_0\epsilon_r}\right)\frac{q}{r^2}\,\hat{r}$$

$\hat{r} \Rightarrow$ unit vector

In magnitude

$$E = \left(\frac{1}{4\pi\epsilon_0\epsilon_r}\right)\left(\frac{q}{r^2}\right)$$

Electric field due to a continuous charge distribution.

(a) for line charge :

$\because$ linear charge density $\lambda = \frac{dq}{dl}$

$\therefore dq = (\lambda)(dl)$

$\therefore q = \int_{line} \lambda \, dl$

$$\vec{E} = \frac{1}{4\pi\epsilon_0\epsilon_r} \int_{line} \frac{\hat{r}}{r^2} \lambda \, dl$$

(b) for Surface charge :

$\because$ Surface charge density, $\sigma = \frac{dq}{ds}$

$\therefore dq = \sigma \, ds$

$q = \int_{surface} \sigma \, ds$

$$\therefore \vec{E} = \left(\frac{1}{4\pi\epsilon_0\epsilon_r}\right) \int_{Surface} \frac{\hat{r}}{r^2} \sigma \, ds$$

(c) for Volume charge :-

$\because$ Volume charge density $\rho = \frac{dq}{dv}$

$\therefore dq = \rho \, dv$

$q = \int_{Vol} \rho \, dv$

$$\vec{E} = \left(\frac{1}{4\pi\epsilon_0\epsilon_r}\right) \int_{Vol} \frac{\hat{r}}{r^2} \rho \, dv$$