Chapter 16

Kinematics of Rigid Bodies

16.1 Introduction

In this chapter we will do motion analysis of rigid bodies without involving the forces responsible for the motion. We will learn to find the position, velocity and acceleration of the different particles which together form a rigid body.

In kinematics of particles the size or dimensions of the body were not taken into consideration during motion analysis, since we treated the entire body irrespective of its size (as large as a car, lift, train or airplane) as one single particle. In rigid body kinematics, we shall involve the size and dimensions of the body also. Therefore now the body would be considered as made up of several particles, connected to each other, such that their relative positions do not change, as the body performs its motion.

There are various types of rigid body motion. We will begin with first classifying them and then we will study each type in detail and thereby develop our base for study of kinetics of rigid bodies.

16.2 Types of Rigid body Motion

The motion of the rigid body can be classified under the following types,

1) Translation
2) Rotation about fixed axis
3) General plane motion
4) Motion about a fixed point
5) General motion

16.3 Translation Motion

In this type of motion all the particles forming the body travel along parallel paths. Also the orientation of the body does not change during motion. Translation motion is further classified as

a) Rectilinear Translation
b) Curvilinear Translation

16.4.1 Important Terms

i) Angular Position

Let O be the centre of rotation and z axis be the fixed axis of rotation. Let the x axis be the reference axis with respect to which we measure the position of the rotating body. Let us mark an arbitrary point P on the body.

The angle θ measured in the anticlockwise direction which the line OP makes with the x axis would then be the angular position of the particle.

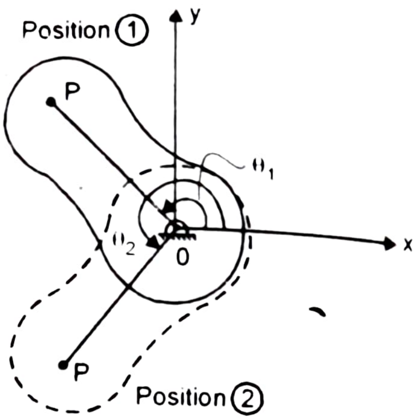


Fig. 16.5

As the body rotates and occupies position (2), the angular position of the body also changes and would have a new value of θ. Angular position is measured in units of radians (rad). It may also be measured in units of revolution or degree. These are related as

$$1 \text{ revolution} = 2\pi \text{ rad} = 360^\circ$$

For example if the body's initial angular position is $30^\circ = 0.5236$ rad and it completes two revolutions, its new angular position would be, $750^\circ = 13.09$ rad.

ii) Angular Displacement

The change in angular position of the body during its motion is known as the angular displacement of the body.

If θ_1, is the angular position of the body in position (1) and if this changes to θ_2 at position (2), the angular displacement of the body is

$$\Delta\theta = \theta_2 - \theta_1$$

iii) Angular Velocity

The rate of change of angular position with respect to time is the angular velocity of the rotating body.

$$\omega = \frac{d\theta}{dt} \qquad \text{.......... [16.2]}$$

The magnitude of angular velocity is denoted by notation ω (omega) and its direction acts along the axis of rotation, the sense being defined by right hand rule. i.e. for a body in the x–y plane rotating about a fixed axis parallel to the z axis in the anticlockwise direction, will have positive angular velocity. The same body, if it rotates in the clockwise direction will have negative angular velocity. Angular velocity is represented by curved arrow representing clockwise or anticlockwise sense.

Units of angular velocity are rad/s, though other unit like revolutions per minute (rpm) is also commonly used. They are related as

$$1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

iv) Angular Acceleration –

The rate of change of angular velocity with respect to time is the angular acceleration of the rotating body.

$$\alpha = \frac{d\omega}{dt} \quad \text{.......... [16.3]}$$

The magnitude of angular acceleration is denoted by notation α (alpha) and its direction acts along the axis of rotation. The sense of angular acceleration is same as the sense of angular velocity, if the angular velocity increases with time, and is opposite to the sense of angular velocity, if the angular velocity decreases with time. Angular acceleration is represented by curved arrow, representing clockwise or anticlockwise sense.

Units of angular acceleration are rad/s^2.

16.4.2 *Types of Rotation Motion about Fixed Axis*

Rotation about fixed axis can be classified under three categories,

1) Uniform Angular Velocity Motion

In this case of rotation motion the angular velocity of the rotating body remains constant during motion. For such motions, we use a simple relation relating ω, θ and t as

$$\omega = \frac{\theta}{t} \quad \text{.......... [16.4]}$$

2) Uniform Angular Acceleration Motion

In this case of rotation motion, the angular velocity of the rotating body is not constant, but increases or decreases at a constant rate.

If ω_0 = initial angular velocity
ω = final angular velocity
α = angular acceleration
θ = angular displacement of the body
t = time interval, then

we relate ω_0, ω, α, θ and t as

$$\omega = \omega_0 + \alpha t \qquad \text{.......... [16.5 (a)]}$$

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 \qquad \text{.......... [16.5 (b)]}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta \qquad \text{.......... [16.5 (c)]}$$

3) <u>Variable Angular Acceleration Motion</u>

In this case of rotation motion the angular velocity changes during motion and the rate of change of angular velocity is variable.
To solve problems on variable angular acceleration motion we make use of the basic differential equations discussed earlier.

$$\omega = \frac{d\theta}{dt} \qquad \text{.......... [16.2]}$$

$$\alpha = \frac{d\omega}{dt} \qquad \text{.......... [16.3]}$$

From equations 16.2 and 16.3, we have

$$\alpha = \frac{\omega\, d\omega}{d\theta} \qquad \text{.......... [16.6]}$$

16.4.3 Relation between Linear Velocity and Angular Velocity

Consider a body rotating about a fixed axis passing through O.
Let at the given instant, the angular velocity of the body be ω rad/s anticlockwise.

All the particles on the rotating body will have the same angular velocity but different linear velocity. If v_P is the linear velocity of a particle P and also r_{PO} is the radial distance from P to O, then

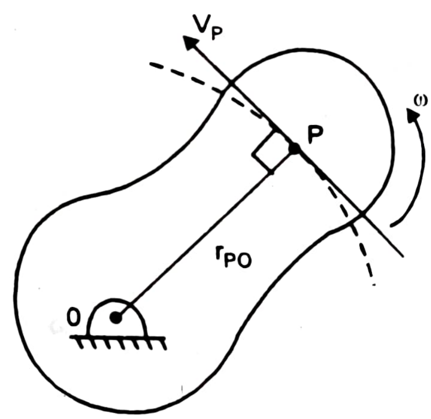


Fig. 16.6

$$v_P = r_{PO} \times \omega$$

The sense of v_P would be consistent with the direction of ω.

In general linear velocity v of any particle located at a radial distance r from the axis of rotation O is related to the angular velocity ω of the body by a relation

$$v = r\,\omega \qquad \text{.......... [16.7]}$$

16.4.4 Relation between Linear Acceleration and Angular Acceleration

Consider a body rotating about a fixed axis passing through O, having an angular velocity ω and angular acceleration α, at the given instant as shown.

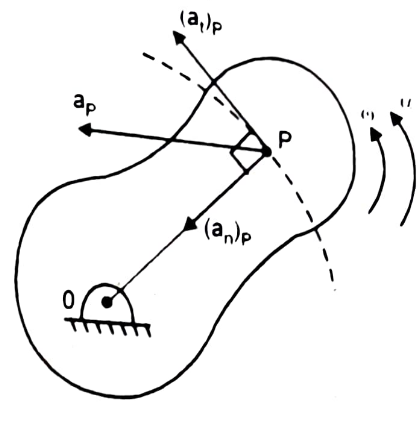


Fig. 16.7

We have studied in Chapter 12 that a particle in curvilinear motion, has acceleration 'a' which can be resolved into normal component a_n and tangential component a_t. Similarly if a_P is the acceleration of particle P and if $(a_n)_P$ and $(a_t)_P$ are its components, then

$$(a_n)_P = \frac{v^2}{\rho} \qquad \text{...... from 12.12}$$

$$= \frac{r_{PO} \times \omega^2}{r_{PO}}$$

$$\therefore \quad (a_n)_P = r_{PO} \times \omega^2$$

also

$$(a_t)_P = \frac{dv}{dt} \qquad \text{...... from 12.13}$$

$$= \frac{d(r_{PO} \times \omega)}{dt} = r_{PO}\frac{d\omega}{dt}$$

$$\therefore \quad (a_t)_P = r_{PO}\,\alpha$$

In general the linear acceleration a of any particle located at a radial distance r from the axis of rotation O, has its components related to angular velocity and angular acceleration by the relation

$$\mathbf{a_n = r.\omega^2} \qquad \text{.......... [16.8 (a)]}$$

$$\mathbf{a_t = r.\alpha} \qquad \text{.......... [16.8 (b)]}$$

also total linear acceleration $a = \sqrt{a_n^2 + a_t^2}$

16.4.5 Special case 1 of rotation motion: A block connected to a rotating pulley

Consider a pulley of radius r, hinged at the centre and supporting a block by a string wound over it. The rotational motion of the pulley is related to translation motion of the block. This relation can be worked out.

Let at a given instant, the pulley have an angular position of θ rad, angular velocity of ω rad/s and angular acceleration of α rad/s^2

Let x_A, v_A and a_A be the corresponding position, velocity and acceleration respectively of the block A at this instant.

Since the block translates, the string connecting it also translates. We find point P on the pulley is common to the rotating pulley and the translating block.

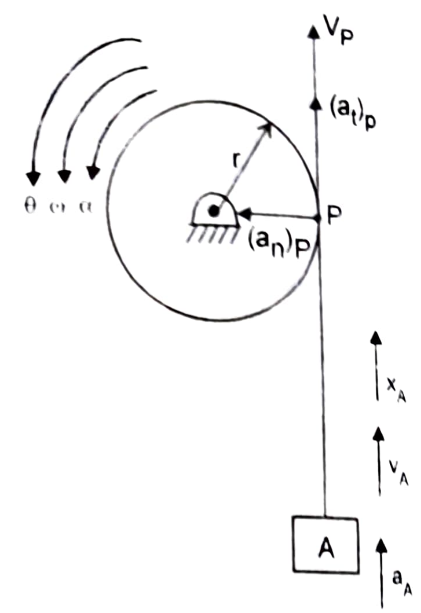


Fig. 16.8

Refer figure 16.8. If point P belongs to the pulley, its position, linear velocity and tangential acceleration is given by

$$s_P = r\,\theta$$
$$v_P = r\,\omega$$
$$(a_t)_P = r\,\alpha$$

Relating these parameters to the motion of the connected block, we have

$$x_A = s_P$$
$$\therefore \quad \mathbf{x_A = r\,\theta} \qquad \text{..........} \; [16.9\ (a)]$$

$$v_A = v_P$$
$$\therefore \quad \mathbf{v_A = r\,\omega} \qquad \text{..........} \; [16.9\ (b)]$$

and
$$a_A = (a_t)_P$$
$$\therefore \quad \mathbf{a_A = r\,\alpha} \qquad \text{..........} \; [16.9\ (c)]$$

Equations 16.9 (a), (b) and (c) relate the motion of a block hanging from a rotating pulley.

16.4.6 Special case 2 of rotation motion: A pulley coupled to another pulley and they rotate without slip

Consider a pulley A of radius r_A, be coupled to another pulley B of radius r_B, and the two rotate without slip. The rotational motion of pulley A is related to the rotational motion of pulley B. This relation can be worked out.

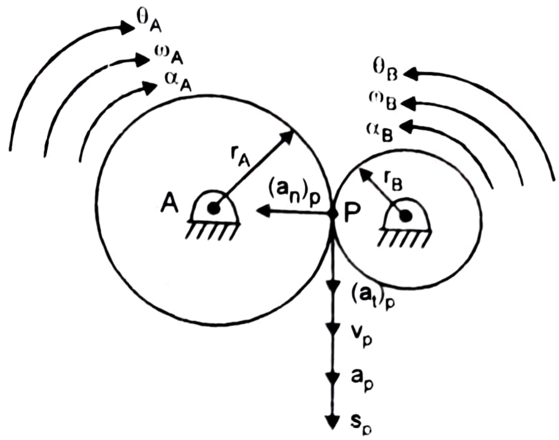


Fig. 16.9

Let at a given instant, pulley A have an angular position of θ_A rad, angular velocity of ω_A rad/s and angular acceleration of α_A rad/s^2.

Let θ_B, ω_B and α_B be the corresponding angular position, angular velocity and angular acceleration of pulley B at this instant.

Since the pulley A rotates, it causes pulley B to also rotate. We find point P is a common point to the two pulleys.

If point P belongs to pulley A, its position, linear velocity and tangential acceleration is given by

$$s_P = r_A \theta_A$$
$$v_P = r_A \omega_A$$
$$(a_t)_P = r_A \alpha_A$$

Similarly point P also belongs to pulley B, therefore if the above parameters are related to the pulley B, we have

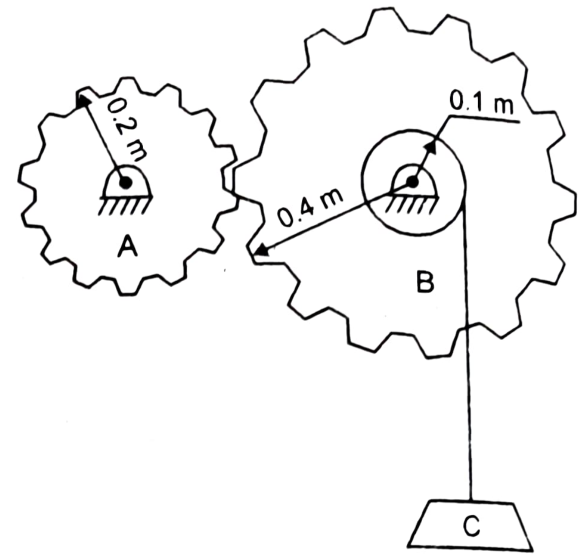
$$s_P = r_A \theta_A = r_B \theta_B \quad \text{......... [16.10 (a)]}$$

$$v_P = r_A \omega_A = r_B \omega_B \quad \text{......... [16.10 (b)]}$$

$$(a_t)_P = r_A \alpha_A = r_B \alpha_B \quad \text{......... [16.10 (c)]}$$

Equations 16.10 (a), (b) and (c) relate the motions of two pulleys or gears engaged with one another.

Ex. 16.1 Figure shows a hoisting gear arrangement. If gear A has an initial angular velocity of 5 rad/sec clockwise and a constant angular acceleration of 2 rad/sec². Find the velocity, acceleration and displacement of the load C in t = 4 sec.



Solution: The system consists of three bodies in motion. Gear A and gear B perform rotation motion, while the load C translates vertically up.

<u>Motion of gear A</u>

It performs rotation motion with uniform angular acceleration

ω_0 = 5 rad/s, ω = ?, α = 2 rad/s², θ = ?, t = 4 sec.

Using $\omega = \omega_0 + \alpha t$

$\omega = 5 + 2 \times 4 = 13$ rad/sec

Using $\theta = \omega_0 t + \frac{1}{2} \alpha t^2$

$\theta = 5 \times 4 + \frac{1}{2} \times 2 \times (4)^2 = 36$ rad

i.e at t = 4 sec, α_A = 2 rad/sec², ω_A = 13 rad/s and θ_A = 36 rad

<u>Motion of gear B</u>

It also performs rotation motion with uniform angular acceleration

Gear B is coupled to gear A ....(*similar to special case 2 of rotation motion*)

$\therefore$ $r_A \omega_A = r_B \omega_B$

$0.2 \times 13 = 0.4 \times \omega_B$

or $\omega_B = 6.5$ rad/s

also $r_A \alpha_A = r_B \alpha_B$

$0.2 \times 2 = 0.4 \times \alpha_B$

$\therefore$ $\alpha_B = 1$ rad/s²,

also $r_A \theta_A = r_B \theta_B$

$0.2 \times 36 = 0.4 \times \theta_B$

or $\theta_B = 18$ rad

$\therefore$ at t = 4 sec, α_B = 1 rad/s². ω_B = 6.5 rad/s, θ_B = 18 rad

<u>Motion of load C</u>

It performs rectilinear translation motion. Since gear B has uniform angular acceleration, the load also performs uniform acceleration rectilinear translation motion. Load C is coupled to gear B (*similar to special case 1 of rotation motion*)

$v_C = r\,\omega_B$

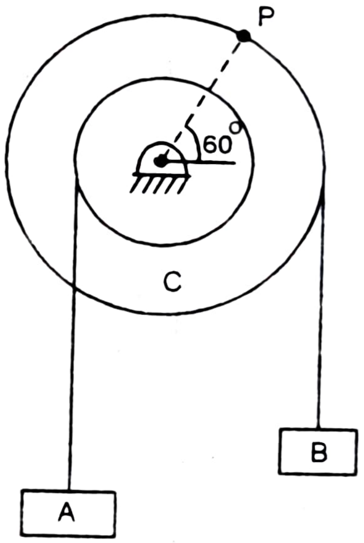
$= 0.1 \times 6.5 = 0.65$ m/s .... **Ans.**

$a_C = r\,\alpha_B$
$= 0.1 \times 1 = 0.1\ m/s^2$ .... **Ans.**

$s_C = r\,\theta_B$
$= 0.1 \times 18 = 1.8\ m$ .... **Ans.**

Ex. 16.2 A double pulley C of inner and outer radius being 150 mm and 200 mm respectively supports two blocks A and B. Block A is moving downwards and has an acceleration given by a relation $a = 0.3\ t\ m/s^2$ and an initial velocity of 0.5 m/s find at $t = 5$ sec

a) the number of revolutions turned by the pulley
b) normal and tangential components of acceleration of a point P on the rim of the pulley.
c) The position, velocity and acceleration of block B.



Solution: The system consists of three bodies in motion. Blocks A and B perform rectilinear translation motion while double pulley C performs rotation motion.

Motion of block A

It performs rectilinear motion with variable acceleration

$$a_A = 0.3\ t$$

using $a = \dfrac{dv}{dt}$ $\quad\therefore\ dv = 0.3\ t\ dt$

Integrating taking lower limits as v = 0.5 m/s and t = 0

$$\int_{0.5}^{v} dv = \int_0^t 0.3\,t\,dt$$

$$[v]_{0.5}^{v} = \left[\frac{0.3t^2}{2}\right]_0^t$$

$$v - 0.5 = 0.15\ t^2$$

or

$$v = 0.15\ t^2 + 0.5\ \ m/s \qquad \text{...... (2)}$$

using $v = \dfrac{dx}{dt}$ $\quad\therefore\ dx = v\ dt$

$\therefore$

$$dx = (0.15\ t^2 + 0.5)dt$$

Integrating taking lower limits as x = 0 and t = 0

$$\int_0^x dx = \int_0^t 0.15\,t^2 + 0.5\,dt$$

$$[x]_0^x = \left[\frac{0.15\,t^3}{3} + 0.5\,t\right]_0^t$$

$$x = 0.05\,t^3 + 0.5\,t \text{ m} \qquad \text{.........} (3)$$

At t = 5 sec, substituting in equation (1), (2) and (3)

$$a_A = 0.3 \times 5 = 1.5 \text{ m/s}^2 \downarrow$$
$$v_A = 0.15\,(5)^2 + 0.5 = 4.25 \text{ m/s} \downarrow$$
$$x_A = 0.05\,(5)^3 + 0.5\,(5) = 8.75 \text{ m} \downarrow$$

$\therefore$ at t = 5 sec, $a_A = 1.5 \text{ m/s}^2 \downarrow$, $v_A = 4.25 \text{ m/s} \downarrow$ and $x_A = 8.75 \text{ m} \downarrow$

Motion of double pulley C

Pulley C is coupled to block A ....*(similar to special case 1 of rotation motion)*
As the block A moves down, the pulley rotates in anticlockwise direction

$$v_A = r\,\omega_C$$
$$4.25 = 0.15 \times \omega_C$$
$$\therefore \quad \omega_C = 28.33 \text{ rad/s}$$

$$a_A = r\,\alpha_C$$
$$1.5 = 0.15 \times \alpha_C$$
$$\therefore \quad \alpha_C = 10 \text{ rad/s}^2$$

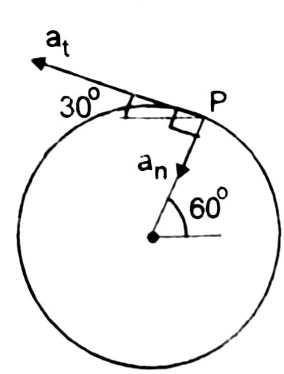
$$x_A = r\,\theta_C$$
$$8.75 = 0.15 \times \theta_C$$
$$\therefore \quad \theta_C = 58.33 \text{ rad}$$

or revolutions turned $N = \dfrac{58.33}{2\pi} = 9.28$ .... **Ans.**

$\therefore$ at t = 5 sec, $\omega_C = 28.33$ rad/s, $\alpha_C = 10$ rad/s², $\theta_C = 58.33$ rad

Acceleration of point P

The point P performs curvilinear motion along a circle of radius 0.2 m



The normal component of acceleration $a_n = \dfrac{v^2}{\rho}$

Here velocity of point P $= r\,\omega$
$= 0.2 \times 28.33 = 5.66$ m/s

$$\therefore \quad a_n = \frac{(5.66)^2}{0.2} = 160.52 \text{ m/s}^2$$

$$\therefore \quad a_n = 160.52 \text{ m/s}^2,\ \theta = 60^\circ$$

The tangential component of acceleration

$$a_t = r\,\alpha = 0.2 \times 10 = 2 \text{ m/s}^2$$ .... **Ans.**

$$\therefore \quad a_t = 2 \text{ m/s}^2,\ \theta = 30^\circ$$

Motion of block B

It performs rectilinear motion which is dependent on the motion of the pulley C. Block B is coupled to pulley C (*similar to special case 2 of rotation motion*)

$v_B = r\,\omega_C$

$\therefore \quad = 0.2 \times 28.33 = 5.66\ m/s^2$ .... **Ans.**

$a_B = r\,\alpha_C$

$= 0.2 \times 10 = 2\ m/s^2$ .... **Ans.**

$s_B = r\,\theta_C$

$= 0.2 \times 58.33 = 11.66\ m$ .... **Ans.**

Ex. 16.3 A wheel has an angular acceleration given by the relation $\alpha = 36 - 4\,t\ rad/s^2$. If $\omega = 4\ rad/s$ at $t = 0$, find

a) the maximum angular velocity and the corresponding time.
b) the total time taken for it to come to rest.
c) the total number of revolutions executed by the wheel.

Solution:

a) The wheel performs variable angular acceleration motion. The acceleration of the rotating wheel is given by

$$\alpha = 36 - 4\,t \quad rad/s^2 \qquad \text{........ (1)}$$

Using $\alpha = \dfrac{d\omega}{dt}$

$d\omega = \alpha\,dt$

$\therefore \quad d\omega = 36 - 4t\ dt$

Integrating taking lower limits as $\omega = 4\ rad/s$ and $t = 0$

$$\int_4^{\omega} d\omega = \int_0^{t} 36 - 4\,t\ dt$$

$$[\omega]_4^{\omega} = \left[36\,t - 2\,t^2\right]_0^t$$

$$\omega - 4 = 36\,t - 2\,t^2$$

or $\quad \omega = -2\,t^2 + 36\,t + 4$ .... (2)

For maximum angular velocity condition put $\alpha = 0$

$\therefore \quad \alpha = 36 - 4\,t = 0$

$\therefore \quad t = 9\ sec$

i.e angular velocity is maximum at $t = 9\ sec$ .... **Ans.**

Substituting $t = 9\ sec$ in equation (2)

$$\omega_{max} = -2 \times (9)^2 + 36\,(9) + 4$$
$$= 166 \text{ rad/s}$$

.... **Ans.**

b) When the wheel comes to a halt, $\omega = 0$

$\therefore$ $$\omega = -2\,t^2 + 36\,t + 4 = 0$$

Solving the quadratic and taking + ve value of t, we get

$\therefore$ $$t = 18.11 \text{ sec}$$

.........**Ans.**

c) From equation (2), we get

$$\omega = -2\,t^2 + 36\,t + 4$$

using $$\omega = \frac{d\theta}{dt}$$

$\therefore$ $$d\theta = \omega\,dt$$
$$d\theta = (-2\,t^2 + 36\,t + 4)\,dt$$

Integrating taking lower limits as $\theta = 0$ and $t = 0$

$$\int_0^\theta d\theta = \int_0^t -2t^2 + 36t + 4\,dt$$

$$\theta = \frac{-2}{3}t^3 + 18t^2 + 4t$$

Knowing the wheel comes to a halt at $t = 18.11$ sec

$\therefore$ $$\theta = \frac{-2}{3}(18.11)^3 + 18 \times (18.11)^2 + 4 \times 18.11$$

$$\theta = 2016.2 \text{ rad}$$

or Revolutions $N = 320.9$

.........**Ans.**

16.5 General Plane Motion

Translation motion and rotation about fixed axis motion are plane motions since the motion of the body can be analysed by taking a representative slab or a plane of the body. They may also be referred to as a Two Dimensional Motion.

Any plane motion which does not fall under the category of rotation about fixed axis or translation motion can be put under the category of general plane motion. In fact a general plane motion is a combination of translation motion and rotation motion. Three examples of body performing general plane motion are shown below.

In Fig. 16.10 (a), Two blocks A and B travel in fixed slot performing translation motion. The blocks are pin-connected by a link AB. The link AB moves from position (1) to position (2) performing general plane motion. We observe that the link AB rotates, but not about a fixed axis. Thus the centre of rotation of the link AB performing general plane motion keeps on moving at every instant.

In Fig. 16.10 (b), rod AB, rod BC and block C, form a pin-connected mechanism. The rod AB performs rotation motion about fixed axis at A. Piston C is free to perform translation motion in the fixed slot. It is the rod BC which neither performs pure translation or pure rotation motion. It is therefore said to perform general plane motion. The centre of rotation of rod BC keeps on changing as it performs general plane motion.

In Fig. 16.10 (c), a wheel rolls without slip on the ground. The rolling wheel rotates as well as translates. It therefore performs general plane motion. In general any body which rolls without slip performs general plane motion.

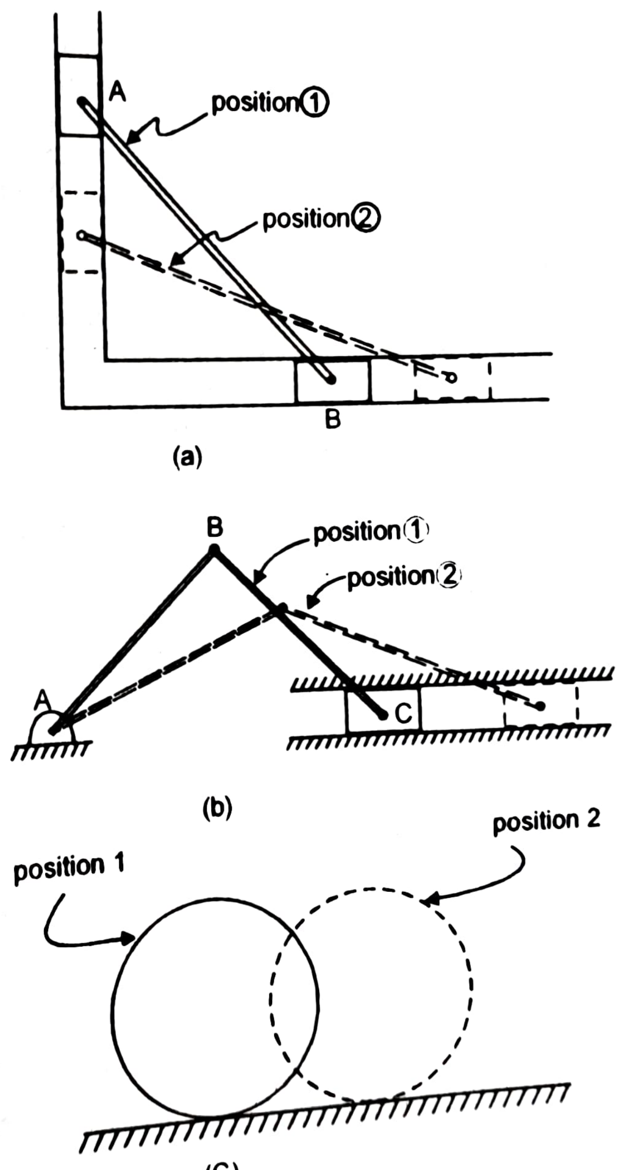


Fig. 16.10

Though G.P. bodies don't actually translate and then rotate in succession, but the motion can be duplicated by first translating the body and then rotating it.

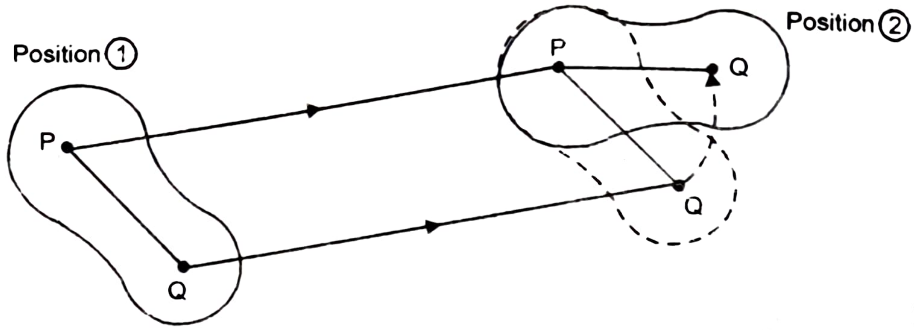


Fig. 16.11
General Plane Motion = Translation Motion + Rotation Motion

Consider a body which has moved from position (1) to position (2) performing G.P. motion. Refer Fig. 16.11. Let P and Q be two arbitrary points chosen on it. We may dublicate the motion by first translating the body to its position (2), i.e segment PQ maintains its orientation and remains parallel. Now we can rotate the body about P to get the true orientation of the body. Hence G.P. Motion is said to be a sum of Translation Motion and Rotation Motion

16.5.1 *Relative Velocity Method*

We know that a G.P. Motion is a sum of translation and rotation motion. To find the angular velocity of a body performing G.P. Motion, we may use the Relative Velocity Method, which is one of the methods for analysing G.P. motion. To understand this method we note down the required steps and simultaneously take up an example and apply the procedure.

Consider a rod AB pin-connected to pistons A and B which move in fixed slots as shown. Let piston B have a known velocity v_B to the right. Let us learn to find angular velocity ω_{AB} of the G.P body AB.

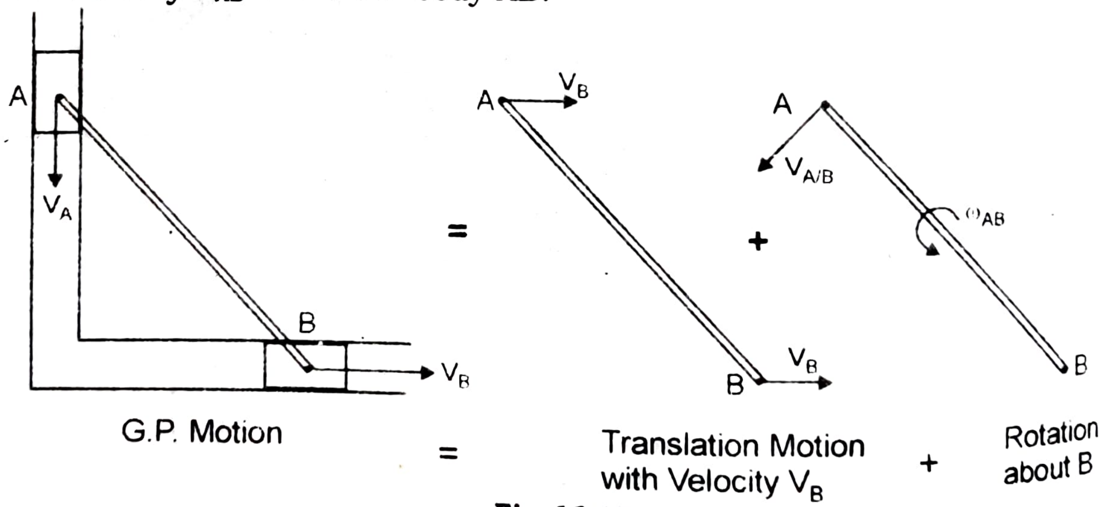


Fig. 16.12

Step 1 – Locate a point on the G.P body whose magnitude and direction of velocity is known. Such a point is referred to as the reference point and the known velocity is referred to as the translating velocity.

In our example, the velocity of point B i.e. v_B is known. Hence B is the reference point and v_B is the translating velocity.

Step 2 – Locate another point on the G.P body whose direction of velocity is known. Such a point is referred to as translation point.

In our example, the direction of velocity of point A is known, since it is constrained to move in the vertical slot. Point A is therefore the translation point.

Step 3 – Translate the body with the translating velocity and then rotate it about the reference point with the angular velocity of the G.P body.

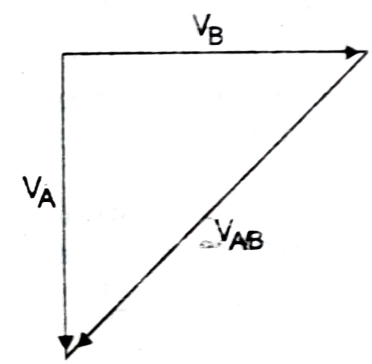
In our example, the bar is translated with translating velocity v_B and then rotated about reference point B with angular velocity ω_{AB}.

Step 4 – Write the relation for the relative linear velocity of translation point.

In our example, we write the relative linear velocity of translation point A as

$$v_{A/B} = r_{AB} \times \omega_{AB} \quad \text{......... (1)}$$

Step 5 – Write the relation for the absolute velocity of the translation point. Simultaneously draw the vector diagram for the same. Obtain the relative velocity of the translation point and hence obtain the angular velocity of the G.P body.



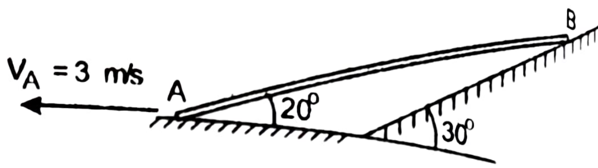
Vector Diagram
Fig. 16.13

In our example,

$$v_A = v_B + v_{A/B} \quad \text{......... (2)}$$

From relation (2), $v_{A/B}$ can be found out, which in turn when subsituted in relation (1) gives the angular velocity ω_{AB}.

Ex. 16.4 A bar AB 2 m long slides down the plane as shown. The end A slides on the horizontal floor with a velocity of 3 m/s. Determine the angular velocity of the rod AB and the velocity of end B for the position shown.



Solution: The system consists of a single bar AB in general plane motion.

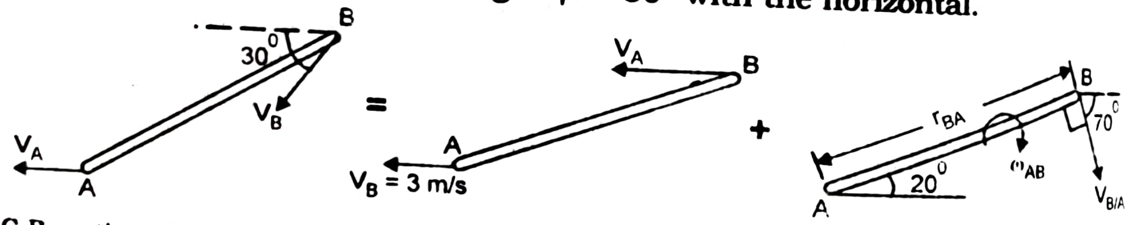
<u>General Plane Motion of bar AB</u>

We shall use relative velocity method.

Here, since velocity of A is known, it is the reference point and its velocity i.e. v_A = 3 m/s it the translating velocity. The other end B is the translating point, since i direction of velocity is known.

Resolving the G.P. motion of rod AB into a sum of translation motion wit velocity v_A and rotation about reference point A.

Since end B is constrained to slide down the plane, direction of velocity of end is along and down the plane i.e. at angle β = 30° with the horizontal.



G.P. motion of AB

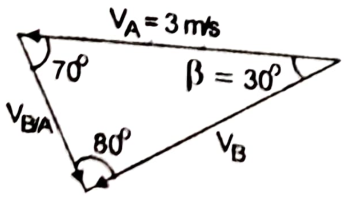
Translation with velocity v_A

Rotation about reference point A

We therefore have,

$$\bar{v}_B = \bar{v}_A + \bar{v}_{B/A}$$

To solve the above equation, let us draw the vector diagram.



Vector Diagram

Solving the triangle
Using sine rule

$$\frac{3}{\sin 80} = \frac{v_B}{\sin 70} = \frac{v_{B/A}}{\sin 30}$$

$\therefore$ v_B = 2.862 m/s .... **Ans.**

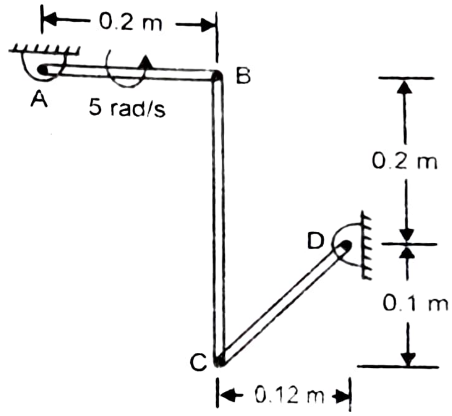
and $v_{B/A}$ = 1.523 m/s

Now, $v_{B/A} = r_{BA} \times \omega_{AB}$

$$1.523 = 2 \times \omega_{AB}$$

$\therefore$ $\omega_{AB} = 0.7615 \text{ rad/s}$ .... **Ans.**

Ex. 16.5 Figure shows three bars AB, BC and CD, internally pin-connected at B and C and externally hinge supported at A and D.
In the position shown, bar AB has a constant angular velocity of 5 rad/s anticlockwise. Find the angular velocities of bars BC and CD for the instant shown.



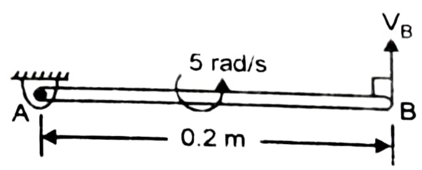
Solution: The system has three bodies in motion.
Rod AB performs Rotation Motion about A.
Rod CD performs Rotation Motion about D.
Rod BC performs General Plane Motion.

Rotation Motion of rod AB about A.

Direction of velocity of end B would be ⊥ to radial distance AB.

$v_B = r_{BA} \times \omega_{AB}$

$0.2 \times 5 = 1 \text{ m/s} \uparrow$



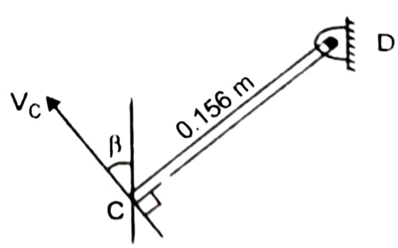
Rotation Motion of rod CD about D

Direction of velocity of point C would be ⊥ to the radial distance CD.

Let β be the angle made by v_C with the vertical from geometry $\beta = 39.8°$

Now $v_C = r_{CD} \times \omega_{CD}$

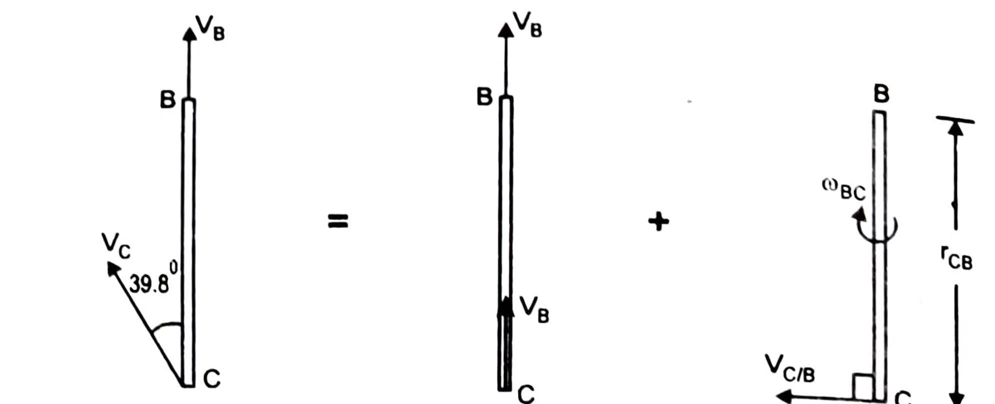
$\therefore$ $v_C = 0.156 \times \omega_{CD}$ .... (1)



General Plane Motion of rod BC

We shall use relative velocity method.
Here since velocity of B is known, it is the reference point and its velocity i.e. $v_B = 1$ m/s is the translating velocity. The other end C is the translating point, since its direction of velocity is known.

Resolving the G.P. Motion of rod BC into a sum of translation motion with translating velocity v_B and rotation motion about reference point B.

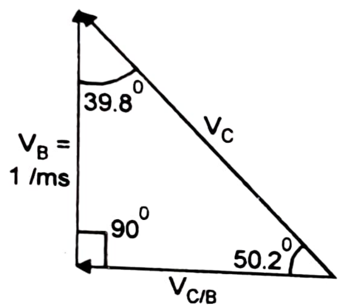


G.P. Motion of BC — Translation with velocity v_B — Rotation about B

We therefore have

$$\bar{v}_C = \bar{v}_B + \bar{v}_{C/B}$$

Let us draw the vector diagram corresponding to the above equation.



Solving the triangle

$$\text{Sin } 50.2 = \frac{v_B}{v_C} = \frac{1}{v_C}$$

$\therefore$ $$v_C = 1.3 \text{ m/s}$$

also $$\cos 50.2 = \frac{v_{C/B}}{v_C} = \frac{v_{C/B}}{1.3}$$

$\therefore$ $$v_{C/B} = 0.833 \text{ m/s}$$

Now $$v_{C/B} = r_{CB} \times \omega_{BC}$$

$$0.833 = 0.3 \times \omega_{BC}$$

$\therefore$ $$\omega_{BC} = 2.777 \text{ rad/s}$$ .... **Ans.**

Also substituting value of v_C in equation (1)

$$1.3 = 0.156 \times \omega_{CD}$$

$\therefore$ $$\omega_{CD} = 8.34 \text{ rad/s}$$ .... **Ans.**

16.5.2 *Instantaneous Centre Method*

Instantaneous Centre is defined as the point about which the G.P. body rotates at the given instant. This point keeps on moving as the G.P. body performs its motion. The locus of the instantaneous centres during the motion is known as *centrode*. Instantaneous Centre may be denoted by letter I.

Let us understand the Instantaneous Centre Method to find the angular velocity of a G.P body. Let us work with the earlier example we took in article 16.5.1

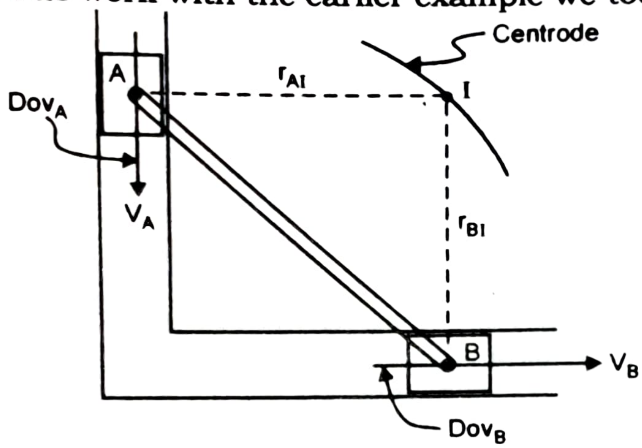


Fig. 16.14

Given – v_B i.e. velocity of block B
To find – Angular velocity of rod AB at given instant

Step – 1 Locate a point on the G.P body whose magnitude and direction of velocity is known and another point whose direction of velocity is known. Mark the direction of velocity (Dov) of these two points.

In our example, the magnitude and Dov of point B (Dov_B) is known and also Dov of point A is known (Dov_A)

Step – 2 Draw perpendiculars to the direction of velocities (Dov) and extend them to intersect at a point. Call this point as I.

Step – 3 Point I i.e. the instantaneous centre, is the centre of rotation of the G.P body at the given instant. Now treating the G.P body as a rotating body about I, and using $v = r\,\omega$ relation, the angular velocity of the G.P body can be found out.

In our example, the radial length r_{BI} can be found out by geometry of Δ ABI.

Next using $v_B = r_{BI} \times \omega_{AB}$

The angular velocity ω_{AB} can be found out
Also now knowing ω_{AB}, velocity of position A can be found out, using

$$v_A = r_{AI} \times \omega_{AB}$$

Step – 4 To find the angular velocity at a new instant, the same procedu steps 1 to 3 are followed and we get a new location of instantaneous centre The locus of the instantaneous centre is known as centrode.

16.6 Rotation about Fixed Point

In this type of rigid body motion, the body rotates about a fixed point, but the axis of rotation passing through the fixed point is not stationary as its direction keeps on changing. Such motion is a *three dimensional motion.*

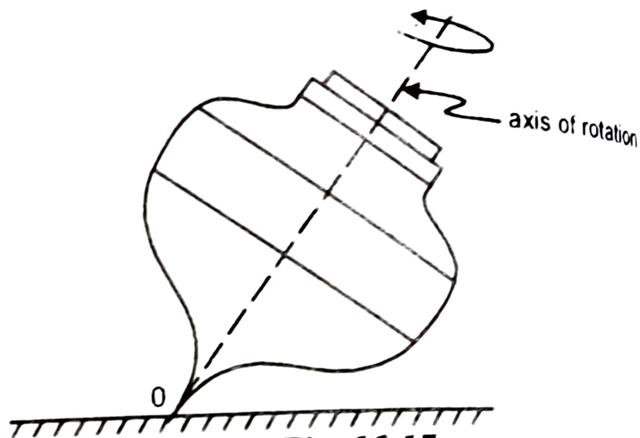


Fig. 16.15

Example of this type of motion is of a top rotating about the pivot a The axis of rotation is not fixed but changes its direction as the top rot about fixed pivot point O. Refer Fig. 16.15

Another example of rotation about a fixed point is of the motion of a boom which is ball and socket supported at a point O on the crane. Fig. 16.16 shows the boom of a crane being raised up by rotation about the z axis, and at the same time the crane itself rotates about the y axis to position the boom over its target.

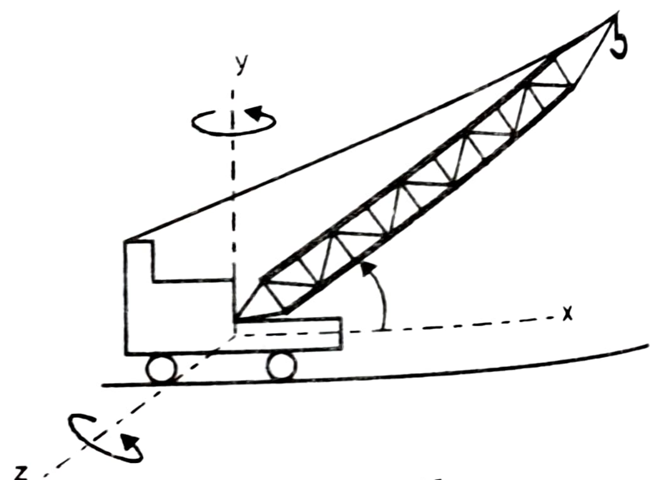


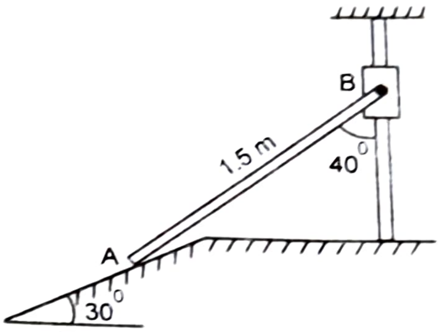
Fig. 16.16

16.7 General Motion

Any other motion of the rigid bodies which do not fit in any of the above types of rigid body motion, may be classified as a General Motion.

Motion analysis of a rigid body having Rotation Motion about a Fixed Poi General Motion of a rigid body are beyond the scope of this book.

Ex. 16.6 Figure shows a collar B which moves up with constant velocity of 2 m/s. To the collar is pinned a rod AB, the end A of which slides freely against a 30° sloping ground. For this instant, determine the angular velocity of the rod and velocity of end A of the rod.



Solution: The system consists of two bodies in motion. Rod AB performs General Plane Motion and collar B performs Rectilinear Translation Motion.

General Plane Motion of rod AB

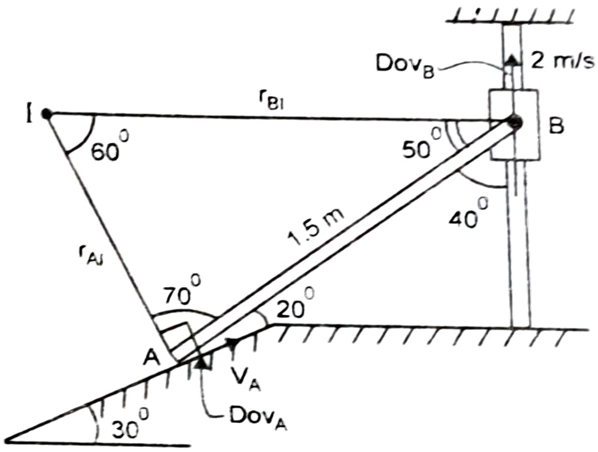
Let us use Instantaneous Centre Method. Working as per the procedure discussed earlier.

(1) Velocity of point B on the G.P. body = the velocity of collar = 2 m/s

Also direction of velocity (Dov_B) is vertical.

Direction of velocity (Dov_A) of end A is along the inclined plane.

(2) Drawing the perpendiculars to the direction of velocity at A and B. Let the point of intersection be I.



(3) From geometry the radial lengths r_{AI} and r_{BI} need to be worked out.

Using sine rule to solve Δ ABI

$$\frac{1.5}{\sin 60} = \frac{r_{AI}}{\sin 50} = \frac{r_{BI}}{\sin 70}$$

$\therefore \quad r_{AI} = 1.327 \text{ m} \quad \text{and} \quad r_{BI} = 1.627 \text{ m}$

4) Since I is the instantaneous centre of rotation of the G.P body AB, we have

$$v_B = r_{BI} \times \omega_{AB}$$
$$2 = 1.627 \times \omega_{AB}$$

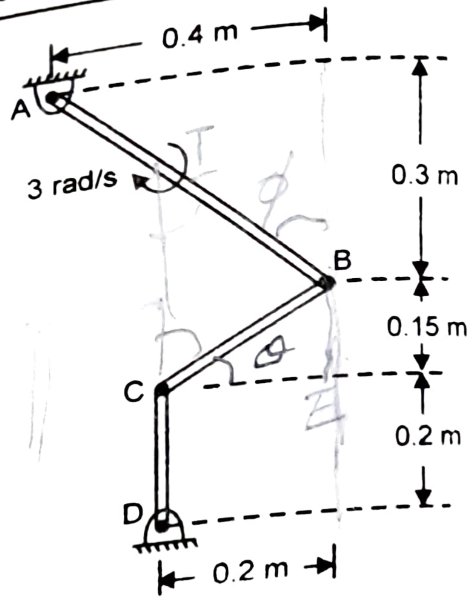
$\therefore \quad \omega_{AB} = 1.229 \text{ rad/s}$ ↺ .... **Ans.**

also

$$v_A = r_{AI} \times \omega_{AB}$$
$$= 1.327 \times 1.229$$

$\therefore \quad v_A = 1.63 \text{ m/s}$

$\therefore \quad v_A = 1.63 \text{ m/s} \ \theta = 30° \angle$ .... **Ans.**



Ex. 16.7 Figure shows a mechanism in motion. Rod AB has a constant angular velocity of 3 rad/s clockwise. Find angular velocity of rod BC and rod CD.

Solution: The system consists of three bodies in motion. Rods AB and CD perform rotation motion and rod BC performs General Plane Motion.

$\tan\theta = \frac{0.15}{0.2} =$

$\theta = 36.87°$

$\therefore \angle BCI = 53.13°$

<u>Rotation Motion of rod AB</u>

Rod AB rotates about A

$\therefore$ velocity of end B $= v_B = r_{BA} \times \omega_{AB}$

$= 0.5 \times 3$

$\therefore \quad v_B = 1.5$ m/s

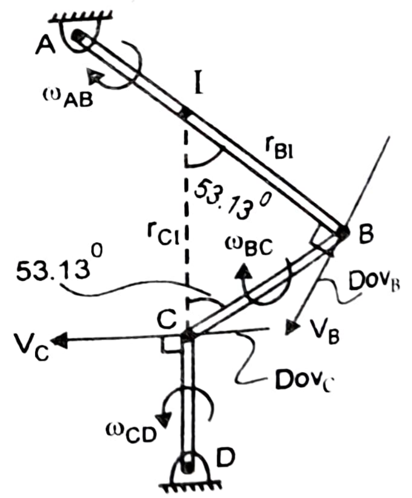
also the direction of velocity of v_B is $\perp$ to radial length AB.

$\tan\phi = \frac{0.4}{0.3} \Rightarrow \theta = 53°$

$\therefore \angle CIB = 53°$

<u>General Plane Motion of rod BC</u>

Let us use Instantaneous Centre Method. Working as per the procedure discussed earlier.



1) Velocity of end B, v_B = 1.5 m/s and Dov_B is $\perp$ to radial length AB. Also direction of velocity of end C i.e. Dov_C is $\perp$ to radial length CD since C is also common to rod CD, which rotates about D.

2) Drawing the perpendiculars to the directions of velocity, Dov_B and Dov_C. Let the point of intersection be I.

3) From geometry the radial lengths r_{BI} and r_{CI} need to be worked out.

Solving Δ BCI ΔBCE

L (BC) = 0.25 m

Since Δ BCI is isosceles, r_{BI} = L (BC) = 0.25 m

also r_{CI} = 0.3 m

4) Since I is the instantaneous centre of rotation of the G.P body BC, we have

$$v_B = r_{BI} \times \omega_{BC}$$
$$1.5 = 0.25 \times \omega_{BC}$$
$$\therefore \quad \omega_{BC} = 6 \text{ rad/s} \quad \text{.........}\textbf{Ans.}$$

also
$$v_C = r_{CI} \times \omega_{BC}$$
$$= 0.3 \times 6$$
$$\therefore \quad = 1.8 \text{ m/s} \leftarrow$$

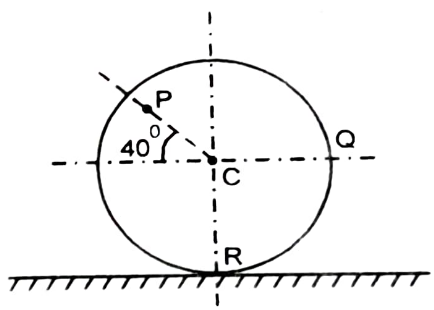
<u>Rotation Motion of rod CD</u>

Rod CD rotates about D

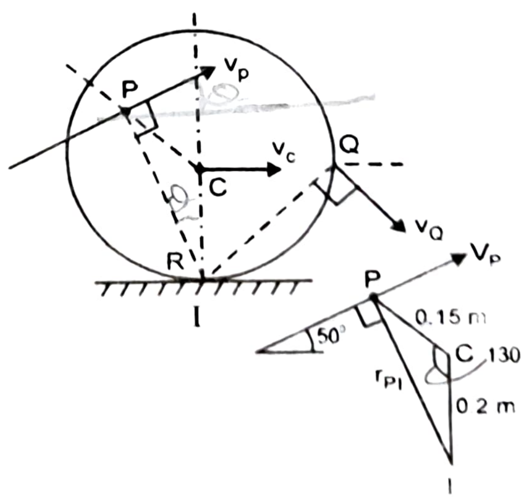
Knowing C is a common end to both rods BC and rod CD, we use v_C = 1.8 m/s from above

$$v_C = r_{CD} \times \omega_{CD}$$
$$1.8 = 0.2 \times \omega_{CD}$$
$$\therefore \quad \omega_{CD} = 9 \text{ rad/s} \quad \text{.........}\textbf{Ans.}$$

Ex. 16.8 A 0.4 m diameter wheel rolls on a horizontal plane without slip, such that its centre has a velocity of 10 m/s towards right. Find the angular velocity of the wheel and also velocities of points P, Q and R shown on the wheel. Given L (CP) = 0.15 m.



Solution: The wheel performs General Plane Motion. For a wheel which rolls without slip, the instantaneous centre of rotation is the point of contact with the ground (since the centre of rotation should have zero velocity, and the point in contact with the ground has velocity of the ground, which is zero). Therefore point R is the instantaneous centre of rotation I.



Now,
$$v_C = r_{CI} \times \omega$$
$$10 = 0.2 \times \omega$$
$$\therefore \quad \omega = 50 \text{ rad/s}$$

Also
$$v_P = r_{PI} \times \omega \quad \text{........ (1)}$$

From geometry of Δ PCI

The radial length r_{PI} can be found out

Using cosine rule

$$(r_{PI})^2 = (0.15)^2 + (0.2)^2 - 2 \times 0.15 \times 0.2 \cos 130$$
$$\therefore \quad r_{PI} = 0.3179 \text{ m}$$

$$\frac{PC}{\sin\theta} = \frac{PR}{\sin 130}$$
$$\theta = 21.18$$

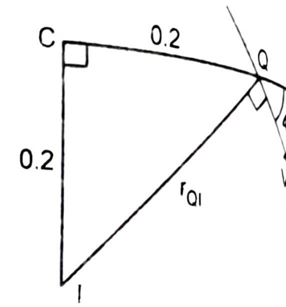
Substituting in equation (1)

$$v_P = 0.3179 \times 50$$
$$= 15.895 \text{ m/s}$$
$$\therefore \quad v_P = 15.895 \text{ m/s}, \quad \theta = 21.19° \measuredangle$$

Also $\quad v_Q = r_{QI} \times \omega$

From geometry of ΔCQI the radial length r_{QI} can be found out

$$(r_{QI})^2 = (0.2)^2 + (0.2)^2$$
$$\therefore \quad r_{QI} = 0.2828 \text{ m}$$



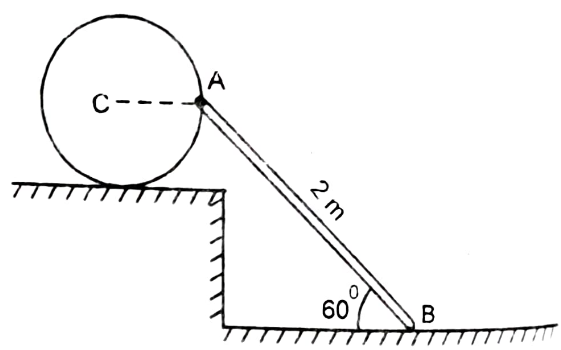
$$\therefore \quad v_Q = 0.2828 \times 50$$
$$= 14.14 \text{ m/s}$$

or $\quad v_Q = 14.14 \text{ m/s} \;\; \theta = 45°$ .... **Ans.**

Since point R coincides with the instantaneous centre I, velocity of point R is zero because the instantaneous centre of rotation has zero velocity.

$$\therefore \quad v_R = 0$$ .... **Ans.**

Ex. 16.9 Figure shows a disc of diameter 1 m which rolls on the horizontal plane with an angular velocity of 3 rad/s clockwise. Rod AB is hinged to the disk such that for the instant shown, line CA is horizontal. The end B of the rod slides on the lower horizontal plane. Determine
a) the angular velocity of rod AB
b) the velocity of end B of the rod AB



Solution: The system has two bodies in motion. Both the disk and the rod AB perform General Plane Motion.

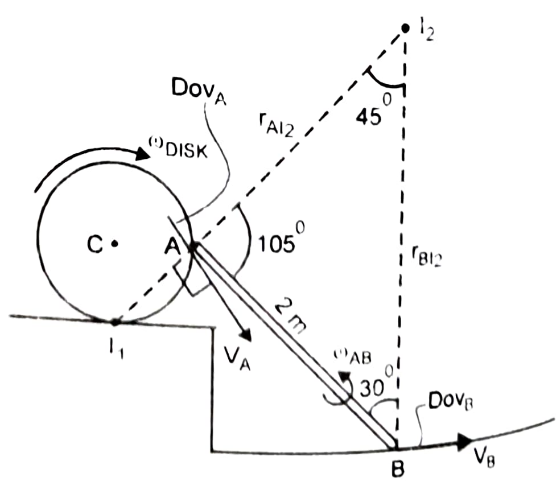
General Plane Motion of the Disk

Since the disk performs Rolling Motion, its instantaneous centre of rotation will be at the point of contact with ground. Let it be referred to as I_1.

Here $\quad v_A = r_{AI_1} \times \omega_{disk}$

From geometry of Rt. Angled $\Delta\, ACI_1$

$$r_{AI_1} = 0.707 \text{ m}$$
$$\therefore \quad v_A = 0.707 \times 3$$
$$= 2.121 \text{ m/s}$$



General Plane Motion of the rod AB

(1) Direction of velocity of end A

i.e. Dov_A is $\perp$ to the radial length r_{AI_1}

Direction of velocity of end B i.e. Dov_B is along the plane on which end B slides.

2) Drawing the perpendiculars to Dov_A and Dov_B. Let their point of intersection be I_2.

3) From geometry the radial lengths r_{AI_2}, and r_{BI_2} need to be worked out.

To solve ΔABI_2, Using sine rule, we get

$$\frac{2}{\sin 45} = \frac{r_{AI_2}}{\sin 30} = \frac{r_{BI_2}}{\sin 105}$$

$\therefore$ $r_{AI_2} = 1.414$ m and $r_{BI_2} = 2.732$ m

here $v_A = r_{AI_2} \times \omega_{AB}$

$2.121 = 1.414 \times \omega_{AB}$

$\therefore$ $\omega_{AB} = 1.5$ rad/s ↻ ....**Ans.**

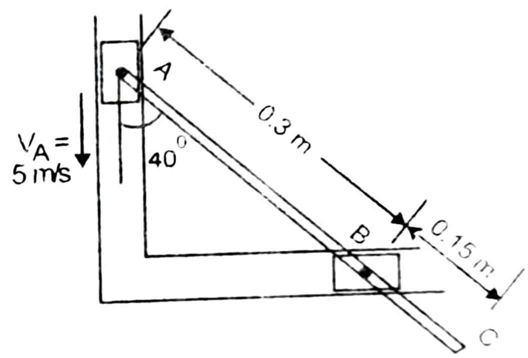
also $v_B = r_{BI_2} \times \omega_{AB}$

$= 2.732 \times 1.5$

$\therefore$ $v_B = 4.1$ m/s → ....**Ans.**

Excercise 16.2

P1. The rod ABC is guided by two blocks A and B which move in channels as shown. At the given instant, velocity of block A is 5 m/s downwards. Determine
a) the angular velocity of rod ABC
b) velocities of block B and end C of rod.



P2. The bar AB has an angular velocity of 6 rad/sec clockwise when $\theta = 50°$. Determine the corresponding angular velocities of bars BC and CD at this instant.