Laplace Transform

$$L\{f(t)\} = f(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

(i) Let $f(t) = 1$,

$$L\{1\} = \int_0^{\infty} e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \left(\frac{0-1}{-s}\right) = \frac{1}{s} \qquad (s>0)$$

(ii) Let $f(t) = t^n$,

$$L\{t^n\} = \int_0^{\infty} e^{-st} \cdot t^n\, dt$$

Let $st = u \Rightarrow s\,dt = du$

$$= \int_0^{\infty} e^{-u} \left(\frac{u}{s}\right)^n \left(\frac{du}{s}\right) = \frac{1}{s^{n+1}} \int_0^{\infty} u^n e^{-u}\, du = \frac{1}{s^{n+1}} \int_0^{\infty} u^{(n+1)-1} e^{u}\, du =$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \qquad s>0 \ \& \ n+1>0 \qquad \text{(Gamma Functions)}$$

Ex

$$L^{-1}\left(\frac{1}{s^9}\right) = \frac{t^8}{\Gamma 9}$$

$$\boxed{L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\Gamma n}}$$

$$\boxed{L(e^{at}) = \frac{1}{s-a}}$$

$$\boxed{L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}}$$

1.	$L(1) = \frac{1}{s}$	**2.**	$L(e^{at}) = \frac{1}{s-a}$, $L(e^{-at}) = \frac{1}{s+a}$, $L(c^{at}) = \frac{1}{s - a\log c}$
3.	$L(t^n) = \frac{\lfloor n+1}{s^{n+1}} = \frac{n!}{s^{n+1}}$ if $n \in N$	**4.**	$L(\cos at) = \frac{s}{s^2+a^2}$
5.	$L(\sin at) = \frac{a}{s^2+a^2}$	**6.**	$L(\cos h\ at) = \frac{s}{s^2-a^2}$
7.	$L(\sin h\ at) = \frac{a}{s^2-a^2}$		

- Change of scale property

  $L[f(t)] = \phi(s)$, then $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

- Error function (Error function integral or Probability integral)

  $$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2}\, du$$

- Complementary Error function

  $$\mathrm{erf}_c(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du$$

Q] $f(t) = \begin{cases} t & \text{when } 0 < t < 4 \\ 5 & \text{when } t > 4 \end{cases}$

→ $L\, f(t) = \int_0^{\infty} e^{-st} f(t).\,dt = \int_0^4 e^{-st}(t)\,dt + \int_4^{\infty} e^{-st}(5)\,dt$

$$= \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^4 + 5\left(\frac{e^{-st}}{-s}\right)_4^{\infty}$$

$$= \left[4\left(\frac{e^{-4s}}{-s}\right) - 1\left(\frac{e^{-4s} - 1}{s^2}\right)\right] + \frac{5}{s}\left[0 - e^{-4s}\right]$$

$$L[f(t)] = \frac{1}{s^2} + \left(\frac{1}{s} - \frac{1}{s^2}\right) e^{-4s}$$

Q] $f(t) = \begin{cases} \cos t & \text{when } 0 < t < \pi \\ \sin t & \text{when } t > \pi \end{cases}$

→

$$L[f(t)] = \int_0^{\pi} e^{-st}\cos(t)\,dt + \int_{\pi}^{\infty} e^{-st}\sin(t)\,dt$$

$$\left[\begin{array}{l} \int e^{ax}\sin(bx)\,dx = \dfrac{e^{ax}}{a^2+b^2}\left(a\sin(bx) - b\cos(bx)\right) \\ \int e^{ax}\cos(bx)\,dx = \dfrac{e^{ax}}{a^2+b^2}\left(a\cos(bx) + b\sin(bx)\right) \end{array}\right]$$

$$= \left[\frac{e^{-st}}{s^2+1}\left(-s\cos(t)+\sin(t)\right)\right]_0^{\pi} + \left[\frac{e^{-st}}{s^2+1}\left(-s\sin(t)-\cos(t)\right)\right]_{\pi}^{\infty}$$

$$= \left[\frac{e^{-\pi s}}{s^2+1}(-s(-1)) - \frac{1}{s^2+1}(-s)\right] + \left[0 - \frac{e^{-\pi s}}{s^2+1}(0-(-1))\right]$$

$$= \frac{s \cdot e^{-\pi s}}{s^2+1} + \frac{s}{s^2+1} - \frac{e^{-\pi s}}{s^2+1}$$

$$= \frac{1}{s^2+1}\left[s + (s-1)e^{-\pi s}\right]$$

Q] (i) $L\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$

(ii) $L\{\sin^3 t\}$ (iii) $L\{\cos^3 t\}$ (iv) $L\{(t^2+1)^2\}$ (v) $L\{\sin(\omega t + \alpha)\}$ ω & α being const

→

(i) Using the linearity property

$L\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$

$$= 3L[t^4] - 2L[t^3] + 4[e^{-3t}] - 2L[\sin 5t] + 3L[\cos 2t]$$

$$= 3\frac{4!}{s^5} - 2\frac{3!}{s^4} + \frac{4}{s+3} - \frac{2 \cdot 5}{s^2+5^2} + \frac{3 \cdot s}{s^2+2^2}$$

$$= \frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4}$$

(ii) $L\{\sin^3 t\} = L\left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$

$$= \frac{3}{4} L\{\sin t\} - \frac{1}{4} L\{\sin 3t\}$$

$$= \frac{3}{4} \frac{1}{s^2+1} - \frac{1}{4} \frac{1}{s^2+9} = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right] = \frac{6}{(s^2+1)(s^2+9)}$$

(iii) $L\{\cos^3 t\} = L\left[\frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right] = \frac{3}{4} L\{\cos t\} + \frac{1}{4} L\{\cos 3t\}$

$$= \frac{3}{4} \frac{s}{s^2+1} + \frac{1}{4} \frac{s}{s^2+9} = \frac{3}{4}\left[\frac{s}{s^2+1} + \frac{s}{s^2+9}\right] = \frac{s(s^2+7)}{(s^2+1)(s^2+9)}$$

(iv) $L\{(t^2+1)^2\} = L\{t^4 + 2t^2 + 1\} = L[t^4] + 2L[t^2] + L[1]$

$$= \frac{4!}{s^5} + 2 \cdot \frac{2!}{s^3} + \frac{1}{s} = \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}$$

(v) $\sin(\omega t + \alpha) = \sin \omega t \cos \alpha + \cos \omega t \sin \alpha$

$L\{\sin(\omega t + \alpha)\} = L[\sin \omega t \cos \alpha] + L[\cos \omega t \sin \alpha]$

$$= \cos \alpha \, L[\sin \omega t] + \sin \alpha \, L[\cos \omega t]$$

$$= \cos \alpha \left[\frac{\omega^2}{s^2+\omega^2}\right] + \sin \alpha \left[\frac{s}{s^2+\omega^2}\right] = \frac{\omega \cos \alpha + s \sin \alpha}{s^2+\omega^2}$$

Q] Evaluate $L[\sin 2t \cdot \sin 3t]$

$\rightarrow L[\sin 2t \cdot \sin 3t] = \frac{1}{2} L[\cos t - \cos 5t]$

$\left\{\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]\right\}$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]$$

Q] Evaluate $L[\cos t \cos 2t \cos 3t]$

→ $\cos A \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$

$$L\left[\cos t \cdot \frac{1}{2}(\cos 5t + \cos t)\right] = \frac{1}{2} L[\cos t \cos 5t + \cos^2 t]$$

$$\frac{1}{2} L\left[\frac{1}{2}(\cos 6t + \cos 4t) + \frac{1}{2}(1 + \cos 2t)\right.$$

$$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4} + \frac{1}{s}\right]$$

Q] P.T $L[\sin^5 t] = \dfrac{5!}{(s^2+1)(s^2+9)(s^2+25)}$

→ Let $x = \cos t + i\sin t \Rightarrow \frac{1}{x} = \cos t - i\sin t$

$\therefore \sin t = \frac{1}{2i}\left(x - \frac{1}{x}\right)$

Also $x^n = \cos nt + i\sin nt$ & $\frac{1}{x^n} = \cos nt - i\sin nt$

$\therefore x^n - \frac{1}{x^n} = 2i\sin nt$

$\therefore \sin^5 t = \left(\frac{1}{2i}\right)^5\left(x - \frac{1}{x}\right)^5$

$$= \frac{1}{32i}\left(x^5 - 5x^4\cdot\frac{1}{x} + 10x^3\cdot\frac{1}{x^2} - 10x^2\cdot\frac{1}{x^3} + 5x\cdot\frac{1}{x^4} - \frac{1}{x^5}\right)$$

$$= \frac{1}{32i}\left[\left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)\right]$$

$$= \frac{1}{32i}\left[2i\sin 5t - 5(2i\sin 3t) + 10(2i\sin t)\right]$$

$$= \frac{1}{16}\left[\sin 5t - 5\sin 3t + 10\sin t\right]$$

$$L\{\sin^5 t\} = \frac{1}{16}\left[\frac{5}{s^2+25} - 5\cdot\frac{3}{s^2+9} + 10\,\frac{1}{s^2+1}\right]$$

$$= \frac{5}{16}\left[\frac{1}{s^2+25} - \frac{3}{s^2+9} + \frac{2}{s^2+1}\right]$$

$$= \frac{5!}{(s^2+1)(s^2+9)(s^2+1)}$$

Q] If $f(t) = (\sin 2t - \cos 2t)^2$ then find $L[f(t)]$, Hence find $L[f(2t)]$

$\rightarrow$ $f(t) = (\sin 2t - \cos 2t)^2$

$$= \sin^2 2t - 2\sin 2t\cos 2t + \cos^2 2t$$

$$= 1 - \sin 4t$$

$$L(f(t)) = L[1 - \sin 4t] = \frac{1}{s} - \frac{4}{s^2+16}$$

$$= \frac{s^2+16-4s}{s(s^2+16)}$$

Now using change of scale property

If $L(f(t)) = \phi(s)$ then $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

$$\therefore L[f(2t)] = \frac{1}{2}\left[\frac{\left(\frac{s}{2}\right)^2 - 4\left(\frac{s}{2}\right)+16}{\left(\frac{s}{2}\right)\left[\left(\frac{s}{2}\right)^2+16\right]}\right] = \frac{s^2-8s+64}{s(s^2+64)}$$

Q] If $L(f(t)) = \log\left(\frac{s+3}{s+1}\right)$. find $L[f(2t)]$

→ Using change of scale property

If $L(f(t)) = \phi(s)$ then $L(f(at)) = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

Now $L[f(t)] = \log\left(\frac{s+3}{s+1}\right)$

$$\therefore L[f(2t)] = \frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right) = \frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

Q] If $L[\text{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$ find $L[\text{erf}\ 3\sqrt{t}]$

→

Using change of scale property

$$L[\text{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}} = \phi(s)$$

$$L[\operatorname{erf} 3\sqrt{t}] = L[\operatorname{erf}\sqrt{9t}]$$

$$= \frac{1}{9}\phi\left(\frac{s}{9}\right) = \frac{1}{9}\cdot\frac{1}{\frac{s}{9}\sqrt{\frac{s}{9}+1}} = \frac{3}{s\sqrt{s+9}}$$

Q] Find Laplace transform of $\sin\sqrt{t}$. Hence find $L[\sin 2\sqrt{t}]$

→ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \ldots\ldots$$

$$\therefore L[\sin\sqrt{t}] = L[t^{1/2}] - \frac{1}{3!}L[t^{3/2}] + \frac{1}{5!}L[t^{5/2}] - \frac{1}{7!}L[t^{7/2}] + \ldots$$

Now $L[t^n] = \frac{\Gamma n+1}{s^{n+1}}$. $\Gamma n = (n-1)\Gamma n-1$ & $\Gamma\frac{1}{2} = \sqrt{\pi}$

$$L[\sin\sqrt{t}] = \frac{\Gamma 3/2}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma 5/2}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma 7/2}{s^{7/2}} - \frac{1}{7!}\frac{\Gamma 9/2}{s^{9/2}} + \ldots$$

$$= \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} - \frac{1}{3!}\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\frac{1}{2}}{s^{5/2}} + \frac{1}{5!}\frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\frac{1}{2}}{s^{7/2}} \ldots\ldots$$

$$= \frac{\Gamma\frac{1}{2}}{2s^{3/2}}\left[1 - \left(\frac{1}{2^2\cdot s}\right) + \frac{1}{2!}\left(\frac{1}{2^2\cdot s}\right)^2 \ldots\ldots\right]$$

$$L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s} \qquad \left[e^{-x} = 1 - 4 + \frac{x^2}{2!} \ldots\ldots\right]$$

Using change of scale property

$$L[\sin 2\sqrt{t}] = L[\sin\sqrt{4t}] = \frac{1}{4}\cdot\frac{\sqrt{\pi}}{2\left(\frac{s}{4}\right)^{3/2}}\, e^{-1/4\left(\frac{s}{4}\right)} = \frac{\sqrt{\pi}}{s^{3/2}}\, e^{-1/s}$$

* First Shifting Theorem :—

$L|f(t)| = \phi(s)$, $L|e^{-at} f(t)| = \phi(s+a)$, $L|e^{at} f(t)| = \phi(s-a)$

$$L|e^{-bt}\sin at| = \frac{a}{(s+b)^2+a^2}$$

Q] If $L|f(t)| = \dfrac{s}{s^2+s+4}$. find $L[e^{-3t} f(2t)]$

→

By change of scale property

$$L[f(2t)] = \frac{1}{2}\,\frac{\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)^2+\left(\frac{s}{2}\right)+4} = \frac{s}{s^2+2s+16} = \phi(s)$$

Now , using first shifting property

$$L[e^{-3t} f(2t)] = \phi(s+3)$$

$$= \frac{s+3}{(s+3)^2+2(s+3)+16} = \frac{s+3}{s^2+8s+31}$$

Q] find $L[\cosh 2t \cos 2t]$

→ $= L\left[\frac{1}{2}\left(e^{2t}+e^{-2t}\right)\cos 2t\right]$

$$= \frac{1}{2}\left[L\,(e^{2t}\cos 2t) + L\,(e^{-2t}\cos 2t) \right]$$

$$L|\cos 2t| = \frac{s}{s^2+4}$$

By shifting theorem,

$$L|\cosh 2t \cos 2t| = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+4} + \frac{s+2}{(s+2)^2+4} \right] = \frac{s^3}{s^4+64}$$

Q] Find $L|(t^2 \sinh t)^2|$

$$\rightarrow (t^2 \sinh t)^2 = t^4 \left(\frac{e^t - e^{-t}}{2} \right)^2 = \frac{t^4}{4}\left[e^{2t} - 2 + e^{-2t} \right]$$

$$\therefore\ L|(t^2 \sinh t)^2| = L\left[\frac{t^4}{4}(e^{2t} - 2 + e^{-2t}) \right]$$

$$= \frac{1}{4}\left[L\,(e^{2t}t^4) - 2L\,(t^4) + L\,(e^{-2t}t^4) \right]$$

$$L\,(t^4) = \frac{4!}{s^5}$$

$$\therefore\ L|(t^2 \sinh t)^2| = \frac{1}{4}\left[\frac{4!}{(s-2)^5} - \frac{2\cdot 4!}{s^5} + \frac{4!}{(s+2)^5} \right]$$

$$= 6\left[\frac{1}{(s-2)^5} - \frac{2}{s^5} + \frac{1}{(s+2)^5} \right]$$

Q] P.T. $L\left[\sinh\left(\frac{t}{2}\right) \sin\left(\frac{\sqrt{3}t}{2}\right)\right] = \frac{\sqrt{3}}{2} \cdot \frac{s}{s^4+s^2+1}$

$\rightarrow$ $$\sinh\left(\frac{t}{2}\right) \sin\left(\frac{\sqrt{3}t}{2}\right) = \left(\frac{e^{t/2} - e^{-t/2}}{2}\right) \sin\left(\frac{\sqrt{3}t}{2}\right)$$

Now, $$L\left[\sin\left(\frac{\sqrt{3}t}{2}\right)\right] = \frac{\sqrt{3}/2}{s^2 + 3/4}$$

By First shifting theorem,

$$L\left|e^{t/2} \sin\left(\frac{\sqrt{3}t}{2}\right)\right| = \frac{\sqrt{3}/2}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{\sqrt{3}/2}{s^2 - s + 1}$$

$$L\left|e^{-t/2} \sin\left(\frac{\sqrt{3}t}{2}\right)\right| = \frac{\sqrt{3}/2}{\left(s+\frac{1}{2}\right)^2-\frac{3}{4}} = \frac{\sqrt{3}/2}{s^2 + s - 1}$$

$$L\left(\sinh\left(\frac{t}{2}\right) \sin\left(\frac{\sqrt{3}t}{2}\right)\right) = \frac{1}{2}\left[\frac{\sqrt{3}/2}{(s^2+1) - s} - \frac{\sqrt{3}/2}{(s^2+1) + s}\right]$$

$$= \frac{\sqrt{3}}{2}\left[\frac{s}{s^4+s^2+1}\right]$$

* Second Shifting Theorem :-

$L|f(t)| = \phi(s)$ & $g(t) = f(t-a)$ when $t>a$ & $g(t)=0$ when $t<a$

then $L|g(t)| = e^{-as}\phi(s)$

Q] (i) $L\{f(t)\}$ where $f(t) = \cos(t-\alpha)$, $t > \alpha$ & $f(t) = 0$, $t < \alpha$

(ii) $L\{f(t)\}$ where $f(t) = e^{t-k}$, $t > k$ & $f(t) = 0$ $t < k$

→ (i) $L(\cos t) = \dfrac{s}{s^2+1}$

Hence by second shifting theorem

$$L(\cos(t-\alpha)) = e^{-\alpha s} \cdot \frac{s}{s^2+1}$$

(ii) $L(e^t) = \dfrac{1}{s-1}$

Hence by second shifting theorem

$$L(e^{t-k}) = e^{-ks} \cdot \frac{1}{s-1}$$

* Effect of Multiplication by t :-

Let $f(t)$ be a function & if $L\{f(t)\} = f(s)$ then $L\{t\, f(t)\} = \dfrac{-d}{ds} f(s)$

& $L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} f(s)$

Q] Find $L\{t\, e^{-t}\cosh 2t\}$

$\rightarrow$ $e^{-t}\cosh 2t = e^{-t}\left[\dfrac{e^{2t}+e^{-2t}}{2}\right] = \dfrac{e^{t}+e^{-3t}}{2}$

$$L(e^{-t}\cosh 2t) = \frac{1}{2}\left[L(e^{t}) + L(e^{-3t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+3}\right]$$

$$\therefore\ L\{t\, e^{-t}\cosh 2t\} = -\frac{d}{ds}\cdot\frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+3)^2}\right]$$

Q] Find $L[(1+te^{-t})^3]$

$\rightarrow$ $L[(1+te^{-t})] = L[1+3te^{-t}+3t^2e^{-2t}+t^3e^{-3t}]$

$$= L[1] + 3L[te^{-t}] + 3L[t^2e^{-2t}] + L[t^3e^{-3t}]$$

$$= \frac{1}{s} - 3\cdot\frac{d}{ds}[L(e^{-t})] + 3\frac{d^2}{ds^2}[L(e^{-2t})] - \frac{d^3}{ds^3}[L(e^{-3t})]$$

$$= \frac{1}{s} - 3\frac{d}{ds}\left[\frac{1}{s+1}\right] + 3\frac{d^2}{ds^2}\left[\frac{1}{s+2}\right] - \frac{d^3}{ds^3}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^2} + \frac{6}{(s+3)^4}$$

Q] Find $L[te^{-4t}\sin 3t]$

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$L[t\sin 3t] = -\frac{d}{ds}\cdot\left[\frac{3}{s^2+9}\right]$$

$$= \frac{6s}{(s^2+9)^2}$$

$$L[e^{-4t}t\sin 3t] = \frac{6(s+4)}{[(s+4)^2+9]^2}$$ [Using First Shifting]

$$= \frac{6(s+4)}{(s^2+8s+25)^2}$$

Q] Find $L[t^5\cosh t]$

$\rightarrow$ $$L[t^5\cosh t] = L\left[t^5\left(\frac{e^t+e^{-t}}{2}\right)\right]$$

$$= \frac{1}{2}L\left[e^t t^5 + e^{-t}t^5\right]$$

But $L[t^5] = \frac{5!}{s^6}$ & using first shifting property

$$L\left[t^5\cosh t\right] = \frac{1}{2}\left[\frac{5!}{(s-1)^6} + \frac{5!}{(s+1)^6}\right] = 60\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

Q] Find $L[t\sqrt{1+\sin t}]$

$\rightarrow$ $$\sqrt{1+\sin t} = \sqrt{\sin^2\left(\frac{t}{2}\right)+\cos^2\left(\frac{t}{2}\right)+2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}$$

$$= \sqrt{\left(\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)\right)^2}$$

$$= \sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)$$

$$L[\sqrt{1+\sin t}] = L\left[\sin\left(\frac{t}{2}\right)\right] + L\left[\cos\left(\frac{t}{2}\right)\right]$$

$$= \frac{1/2}{s^2+\frac{1}{4}} + \frac{s}{s^2+\frac{1}{4}}$$

$$= \frac{2}{4s^2+1} + \frac{4s}{4s^2+1} = \frac{2(2s+1)}{(4s^2+1)}$$

Now using multiplication by t property

$$L[t\sqrt{1+\sin t}] = \frac{-d}{ds}\left[\frac{2(2s+1)}{(4s^2+1)}\right]$$

$$= -2\left[\frac{(4s^2+1).2 - (2s+1)8s}{(4s^2+1)^2}\right]$$

$$L[t\sqrt{1+\sin t}] = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

Q] Find $L[te^{3t}\,\text{erf}\sqrt{t}]$

$\rightarrow$ $L[\text{erf}\sqrt{t}] = \dfrac{1}{s\sqrt{s+1}}$

Using Multiplication by t

$$L[t\,\text{erf}\sqrt{t}] = -\frac{d}{ds}\left[\frac{1}{s\sqrt{s+1}}\right]$$

$$= -\left[\frac{-1}{s^2(s+1)} \cdot \frac{d}{ds}(s\sqrt{s+1})\right]$$

$$= \frac{1}{s^2(s+1)}\left[s \cdot \frac{1}{2\sqrt{s+1}} + \sqrt{s+1}\right]$$

$$= \frac{1}{s^2(s+1)}\left[\frac{s+2(s+1)}{2\sqrt{s+1}}\right]$$

$$L[t\,\text{erf}\sqrt{t}] = \frac{3s+2}{2s^2(s+1)^{3/2}}$$

Now using First shifting theorem,

$$L[e^{3t}\,t\,\text{erf}\sqrt{t}] = \frac{3(s-3)+2}{2(s-3)^2(s-3+1)^{3/2}} = \frac{3s-7}{2(s-3)^2(s-2)^{3/2}}$$

Q] $L\left(t\left(\dfrac{\sin t}{e^t}\right)^2\right)$

$\rightarrow$

$$f(t) = t\left(\frac{\sin t}{e^t}\right)^2 = te^{-2t}\sin^2 t = te^{-2t}\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2}te^{-2t}(1-\cos 2t)$$

$$L(1-\cos 2t) = L(1) - L(\cos 2t)$$

$$= \frac{1}{s} - \frac{s}{s^2+4}$$

$$L(t(1-\cos 2t)) = \frac{-d}{ds}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= -\left[\frac{-1}{s^2} - \frac{(s^2+4)1 - s(2s)}{(s^2+4)^2}\right]$$

$$= -\left[\frac{-1}{s^2} - \frac{4-s^2}{(s^2+4)^2}\right] = \frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}$$

Now using first shifting theorem,

$$L[e^{-2t}t\sin^2 t] = \frac{1}{(s+2)^2} + \frac{4-(s+2)^2}{[(s+2)^2+4]^2} = \frac{1}{(s+2)^2} - \frac{s^2+4s}{(s^2+4s+8)^2}$$

* Effect of division by t

If $L|f(t)| = \phi(s)$, then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty \phi(s)\, ds$

Q] $L\left(\frac{e^{-at} - e^{-bt}}{t}\right)$

→ $L(e^{-at} - e^{-bt}) = \frac{1}{s+a} - \frac{1}{s-b}$

Now effect by division by t

$$L = \left[\frac{1}{t}(e^{-at} - e^{-bt})\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s-b}\right) ds$$

$$= \Big[\log(s+a) - \log(s-b)\Big]_s^\infty$$

$$= \log\left(\frac{s+a}{s+b}\right)\Bigg|_s^\infty$$

$$= \log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)_s^\infty$$

$$= \log(1) - \log\left(\frac{1 + a/s}{1 + b/s}\right)$$

$$L\left(\frac{e^{-at} - e^{-bt}}{t}\right) = \log\left(\frac{s+b}{s+a}\right)$$

Q] $L\left[\frac{\sin^2 2t}{t}\right]$

$\rightarrow$ $L[\sin^2 2t] = L\left[\frac{1-\cos 4t}{2}\right] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$

$$L\left[\frac{\sin^2 2t}{t}\right] = \int_s^\infty \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right] ds$$

$$= \frac{1}{2}\int_s^\infty \frac{1}{s} ds - \frac{1}{4}\int_s^\infty \frac{2s}{s^2+16} ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+16)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+16}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log(1) - \log\left(\frac{1}{\sqrt{1+\frac{16}{s^2}}}\right)\right.$$

$$L\left(\frac{\sin^2 2t}{t}\right) = \frac{1}{2}\log\left(\frac{\sqrt{s^2+16}}{s}\right)$$

Q] $L\left[\frac{e^{-2t}\sin 2t\cosh t}{t}\right]$

$\rightarrow$ $e^{-2t}\sin 2t\cosh t = e^{-2t}\sin 2t\left(\frac{e^t+e^{-t}}{2}\right) = \frac{1}{2}\left[e^{-t}\sin 2t + e^{-3t}\sin 2t\right]$

$$L[\sin 2t] = \frac{2}{s^2+4}$$

By shifting theorem,

$$L(e^{-t}\sin 2t) = \frac{2}{(s+1)^2+4} \quad \& \quad L[e^{-3t}\sin 2t] = \frac{2}{(s+3)^2+4}$$

$$L[e^{-2t}\sin 2t\cosh t] = \frac{1}{2}\left[\frac{2}{(s+1)^2+4} + \frac{2}{(s+3)^2+4}\right]$$

$$= \frac{1}{(s+1)^2+2^2} + \frac{1}{(s+3)^2+2^2}$$

By effect of division by t

$$L\left[\frac{e^{-2t}\sin 2t\cosh t}{t}\right] = \int_s^{\infty} \frac{1}{(s+1)^2+2^2} + \frac{1}{(s+3)^2+2^2}\, ds$$

$$= \left[\frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right) + \frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right)\right]_s^{\infty}$$

$$= \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right)\Bigg] + \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{(s+3)}{2}\Bigg]$$

Q] $L\left[\frac{\sin at}{t}\right]$ Also does $L\left[\frac{\cos at}{t}\right]$ exist?

$\rightarrow$ $L[\sin at] = \frac{a}{s^2+a^2}$

$$L\left[\frac{\sin at}{t}\right] = \int_s^{\infty} \frac{a}{s^2+a^2}\, ds = \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

Now,

$$L[\cos at] = \frac{s}{s^2+a^2}$$

$$L\left[\frac{\cos at}{t}\right] = \int_s^{\infty} \frac{s}{s^2+a^2}\, ds = \frac{1}{2}\int_s^{\infty} \frac{2s}{s^2+a^2}\, ds = \left[\frac{1}{2}\log(s^2+a^2)\right]_s^{\infty}$$

since $\log(s^2+a^2)$ is infinite when $s \longrightarrow \infty$,

$L\left[\frac{\cos at}{t}\right]$ does not exist.

* Laplace Transforms of derivatives

$$L(f'(t)) = -f(0) + sL(f(t))$$

$$L(f''(t)) = s^2 L(f(t)) - sf(0) - f'(0)$$

Q] Given $f(t) = t+1$, $0 \le t \le 2$ & $f(t) = 3$, $t > 2$. Find $Lf(t)$, $Lf'(t)$, $Lf''(t)$

$\rightarrow$ $L[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt$

$$= \int_0^2 e^{-st}(t+1)\,dt + \int_2^\infty e^{-st}(3)\,dt$$

$$= \left[(t+1)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^2 + 3\left(\frac{e^{-st}}{-s}\right)_2^\infty$$

$$= 3\frac{e^{-2s}}{-s} - \frac{e^{-2s}}{s^2} - (1)\left(-\frac{1}{s}\right) + \frac{1}{s^2} + 3\left(0 - \frac{e^{-2s}}{-s}\right)$$

$$L|f(t)| = \frac{1}{s} + \frac{1}{s^2} - \frac{e^{-2s}}{s^2}$$

Now, $L[f'(t)] = -f(0) + sL[f(t)]$

But by data $f(0) = 1$

$$\therefore L f'(t) = -1 + s\left[\frac{1}{s} + \frac{1}{s^2}(1-e^{-2s})\right] = \frac{1}{s}(1-e^{-2s})$$

$$L[f''(t)] = s^2 L[f(t)] - s[f(0)] - f'(0)$$

$$= s^2\left[\frac{1}{s} + \frac{1}{s^2}(1-e^{-2s})\right] - s. - 1$$

$$= s + (1-e^{-2s}) - s - 1 = -e^{-2s}$$

Q] $L\left[\frac{d}{dt}\left(\frac{\sin 3t}{t}\right)\right]$

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty \frac{3}{s^2+9}\,ds = \tan^{-1}\left(\frac{s}{3}\right)\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right) = \cot^{-1}\left(\frac{s}{3}\right)$$

$$L f'(t) = -f(0) - sL[f(t)]$$

$$f(0) = \lim_{t \to 0} \frac{\sin 3t}{t} = 3 \lim_{t \to 0} \frac{\sin 3t}{3t} = 3.(1) = 3$$

$$\therefore L[f'(t)] = -3 - s\cot^{-1}\left(\frac{s}{3}\right)$$

* Laploce Transforms of Integrals

$$L|f(t)| = \phi(s) \quad , \quad L\int_0^t f(u)\,du = \frac{1}{s}\phi(s)$$

$\boxed{Q}$ $L\left[\int_0^t \sin 2u\,du\right]$

$\rightarrow$

$$L[\sin 2t] = \frac{2}{s^2+4} = \phi(s)$$

$$L\left[\int_0^t \sin 2u\,du\right] = \frac{1}{s}\phi(s) = \frac{2}{s(s^2+4)}$$

$$\frac{1}{2}[1+\cos 4u]$$

$\boxed{Q}$ $L\left[\int_0^t ue^{-3u}\cos^2 2u\,du\right]$

$$\rightarrow \cos^2 2u = \frac{1+\cos 4u}{2}$$

$$L[\cos^2 2u] = \frac{1}{2} L\left[1+\cos 4u\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$$

$$L[u\cos^2 2u] = -\frac{d}{ds}\,\frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$$ [Multiplication by u]

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{(s^2+16)(1) - s(2s)}{(s^2+16)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{s^2-16}{(s^2+16)^2}\right]$$

$$\therefore L[e^{-3u} u\cos^2 2u] = \frac{1}{2}\left[\frac{1}{(s+3)^2} + \frac{(s+3)^2-16}{[(s+3)^2+16]^2}\right]$$ [First shifting theorem]

$$= \frac{1}{2}\left[\frac{1}{(s+3)^2} + \frac{s^2+6s-7}{(s^2+6s+25)^2}\right] = \phi(s)$$

$$L\left[\int_0^t e^{-3u} u\cos^2 2u\,du\right] = \frac{1}{s}\phi(s) = \frac{1}{2s}\left[\frac{1}{(s+3)^2} + \frac{s^2+6s-7}{(s^2+6s+25)^2}\right]$$

Q $L\left[t\int_0^t e^{-4u}\sin 3u\,du\right]$

$\rightarrow$ $$L[\sin 3u] = \frac{3}{s^2+9}$$

$$L[e^{-4u}\sin 3u] = \frac{3}{(s+4)^2+9}$$ [By first shifting theorem]

$$L\left[\int_0^t e^{-4u}\sin 3u\, du\right] = \frac{1}{s}\cdot\frac{3}{(s+4)^2+9}$$ [Laplace Transform of Integral]

$$L\left[t\int_0^t e^{-4u}\sin 3u\, du\right] = (-1)\frac{d}{ds}\left[\frac{3}{s^3+8s^2+25s}\right]$$

$$L\left[t\int_0^t e^{-4u}\sin 3u\, du\right] = \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

Q] $L\left[e^{-t}\int_0^t e^{u}\cosh u\, du\right]$

$\rightarrow$ $$L[\cosh u] = \frac{s}{s^2-1}$$

$$L[e^{u}\cosh u] = \frac{s-1}{(s-1)^2-1} = \frac{s-1}{s^2-2s+1-1} = \frac{s-1}{s(s-2)}$$ [First shifting theorem]

$$L\left[\int_0^t e^{u}\cosh u\, du\right] = \frac{1}{s}\cdot\frac{s-1}{s(s-2)} = \frac{s-1}{s^2(s-2)}$$ [Laplace of Integral]

$$L\left[e^{-t}\int_0^t e^{u}\cosh u\, du\right] = \frac{(s+1)-1}{(s+1)^2[(s+1)-2]} = \frac{s}{(s+1)^2(s-1)}$$ [First shifting theorem]

Q] $L[\operatorname{erf}\sqrt{t}]$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$$

$$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}du$$

put $u^2 = v$

$\therefore u = \sqrt{v} \Rightarrow du = \dfrac{1}{2\sqrt{v}} \, dv$

u	0	$\sqrt{t}$
v	0	t

$$\therefore \operatorname{erf} \sqrt{t} = \frac{2}{\sqrt{\pi}} \int_0^t e^{-v} \frac{1}{2\sqrt{v}} \, dv = \frac{1}{\sqrt{\pi}} \int_0^t e^{-v} \underline{v^{-1/2}} \, dv$$

$$\therefore L[v^{-1/2}] = \frac{\Gamma 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$L[e^{-v} v^{-1/2}] = \frac{\sqrt{\pi}}{\sqrt{s+1}}$$

$$L\left[\int_0^t e^{-v} v^{-1/2} \, dv\right] = \frac{\sqrt{\pi}}{s\sqrt{s+1}}$$

$$L[\operatorname{erf} \sqrt{t}] = \frac{1}{\sqrt{\pi}} L\left[\int_0^t e^{-v} v^{-1/2} \, dv\right] = \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{s\sqrt{s+1}}$$

$$\therefore L[\operatorname{erf} \sqrt{t}] = \frac{1}{s\sqrt{s+1}}$$

Q] Evaluate $\int_0^\infty e^{-2t} \sin^3 t \, dt$

$\rightarrow$ $L[\sin^3 t] = L\left[\frac{3}{4} \sin t - \frac{1}{4} \sin 3t\right]$

$$= \frac{3}{4} \frac{1}{s^2+1} \cdot \frac{-1}{4} \cdot \frac{3}{s^2+9} = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

This means that $\int_0^\infty e^{-st} \sin^3 t \, dt = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$

Now put $s=2$

$$\therefore \int_0^\infty e^{-2t} \sin^3 t \, dt = \frac{3}{4}\left[\frac{1}{5} - \frac{1}{13}\right] = \frac{3}{4}\left[\frac{8}{65}\right] = \frac{6}{65}$$

Q] $\int_0^\infty e^{-2t} \sin(t+\alpha) \cos(t-\alpha) \, dt = \frac{1}{4}$, find α

$\rightarrow$ $\sin(t+\alpha)\cos(t-\alpha) = \frac{1}{2}[\sin(2t) + \sin(2\alpha)]$

$$L[\sin(t+\alpha)\cos(t-\alpha)] = \frac{1}{2}\left[\left[\frac{2}{s^2+4}\right] + \sin 2\alpha \left(\frac{1}{s}\right)\right]$$

$$= \frac{1}{s^2+4} + \frac{1}{2} \sin 2\alpha \left(\frac{1}{s}\right)$$

Put $s=2$

$$\int_0^\infty e^{-2t} \sin(t+\alpha)\cos(t-\alpha) \, dt = \frac{1}{8} + \frac{1}{4} \sin 2\alpha$$

$$\frac{1}{4} = \frac{1}{8} + \frac{1}{4} \sin 2\alpha$$

$$\sin 2\alpha = \frac{1}{2} \Rightarrow 2\alpha = \frac{\pi}{6} \Rightarrow \alpha = \frac{\pi}{12}$$

Inverse Laplace Transforms

1. Table of Inverse Laplace Transforms:

$L(1) = \dfrac{1}{s}$	$L^{-1}\left(\dfrac{1}{s}\right) = 1$
$L(e^{-at}) = \dfrac{1}{s+a}$	$L^{-1}\left(\dfrac{1}{s+a}\right) = e^{-at}$
$L(e^{at}) = \dfrac{1}{s-a}$	$L^{-1}\left(\dfrac{1}{s-a}\right) = e^{at}$
$L(t^{n-1}) = \dfrac{\lvert n}{s^n}$	$L^{-1}\left(\dfrac{1}{s^n}\right) = \dfrac{t^{n-1}}{\lvert n}$
$L(t^{n-1}) = \dfrac{(n-1)!}{s^n}$	$L^{-1}\left(\dfrac{1}{s^n}\right) = \dfrac{t^{n-1}}{(n-1)!}$
$L(\sin at) = \dfrac{a}{s^2+a^2}$	$L^{-1}\left(\dfrac{1}{s^2+a^2}\right) = \dfrac{1}{a}\sin at$
$L(\cos at) = \dfrac{s}{s^2+a^2}$	$L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \cos at$
$L(\sin h\, at) = \dfrac{a}{s^2-a^2}$	$L^{-1}\left(\dfrac{1}{s^2-a^2}\right) = \dfrac{1}{a}\sin h\, at$
$L(\cos h\, at) = \dfrac{s}{s^2-a^2}$	$L^{-1}\left(\dfrac{s}{s^2-a^2}\right) = \cos h\, at$

(i) distinct linear factor $\dfrac{px+q}{(x-a)(x-b)}$	Express : $\dfrac{px+q}{(x-a)(x-b)} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-b)}$
(ii) distinct linear factor $\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	Express : $\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} +$
(iii) repetitive linear factor $\dfrac{px+q}{(x-a)^2}$	Express : $\dfrac{px+q}{(x-a)^2} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2}$
(iv) repetitive linear factor $\dfrac{px^2+qx+r}{(x-a)^3}$	Express : $\dfrac{px^2+qx+r}{(x-a)^3} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-a)^3}$
(v) repetitive linear factor $\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	Express : $\dfrac{px^2+qx+r}{(x-a)^2(x-b)} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-b)}$
(vi) Linear & quadratic factor $\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	Express: $\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \dfrac{A}{(x-a)} + \dfrac{Bx+C}{(x^2+bx+c)}$

Q] Find $L^{-1}\left[\frac{1-\sqrt{s}}{s^2}\right]^2$

$\rightarrow$ $L^{-1}\left[\frac{1-\sqrt{s}}{s^2}\right]^2 = L^{-1}\left[\frac{1-2\sqrt{s}+s}{s^4}\right] = L^{-1}\left(\frac{1}{s^4}\right) - 2L^{-1}\left(\frac{1}{s^{7/2}}\right) + L^{-1}\left(\frac{1}{s^3}\right)$

But $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma n}$ or $\frac{t^{n-1}}{(n-1)!}$

$\therefore L^{-1}\left[\frac{1-\sqrt{s}}{s^2}\right]^2 = \frac{t^3}{3!} - 2\cdot\frac{t^{5/2}}{\Gamma\frac{7}{2}} + \frac{t^2}{2!}$

$= \frac{t^3}{6} - \frac{16\,t^{5/2}}{15\sqrt{\pi}} + \frac{t^2}{2}$

Q] If $L(f(t)) = \frac{s+2}{s^2+2}$. Find $L[f'(t)]$

$\rightarrow$

$L[f'(t)] = -f(0) + sL[f(t)]$

$f(t) = L^{-1}\left[\frac{s+2}{s^2+2}\right] = L^{-1}\left[\frac{s}{s^2+2}\right] + 2L^{-1}\left[\frac{1}{s^2+2}\right]$

$= \cos\sqrt{2}t + 2\cdot\frac{1}{\sqrt{2}}\sin\sqrt{2}t$

$f(t) = \cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t$

$f(0) = \cos(0) + \sqrt{2}\sin(0) = 1$

$$L[f'(t)] = -f(0) + sL[f(t)]$$

$$= -1 + s\left[\frac{s+2}{s^2+2}\right]$$

$$L[f'(t)] = \frac{2(s-1)}{s^2+2}$$

Q] i] $\frac{2s+3}{s^2+9}$

$$\rightarrow L^{-1}\left[\frac{2s+3}{s^2+9}\right] = 2L^{-1}\left[\frac{s}{s^2+9}\right] + 3L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= 2\cos 3t + 3\cdot\frac{1}{3}\sin 3t$$

$$L^{-1}\left[\frac{2s+3}{s^2+9}\right] = 2\cos 3t + \sin 3t$$

(ii) $L^{-1}\left[\frac{1}{4s+5}\right] = \frac{1}{4}L^{-1}\left[\frac{1}{s+5/4}\right] = \frac{1}{4}e^{-\frac{5}{4}t}$

(iii) $L^{-1}\left[\frac{4s+15}{16s^2-25}\right] = \frac{4}{16}L^{-1}\left[\frac{s}{s^2-\frac{25}{16}}\right] + \frac{15}{16}L^{-1}\left[\frac{1}{s^2-\frac{25}{16}}\right]$

$$= \frac{4}{16}L^{-1}\left[\frac{s}{s^2-\left(\frac{5}{4}\right)^2}\right] + \frac{15}{16}L^{-1}\left[\frac{1}{s^2-\left(\frac{5}{4}\right)^2}\right]$$

$$= \frac{\cancel{4}^{1}}{\cancel{16}_{4}} \cosh\left[\frac{5}{4}\right] t + \frac{\cancel{15}^{3}}{\cancel{16}_{4}} \frac{\cancel{4}^{1}}{\cancel{5}_{1}} \sinh\left(\frac{5}{4}\right) t$$

$$L^{-1}\left[\frac{4s+15}{16s^2-25}\right] = \frac{1}{4}\cosh\left(\frac{5}{4}\right)t + \frac{3}{4}\sinh\left(\frac{5}{4}\right)t$$

Q] $L^{-1}\left[\frac{1}{(s-3)^3}\right]$

$$= e^{3t} L^{-1}\left[\frac{1}{s^3}\right] = e^{3t} \cdot \frac{t^2}{2!} = \frac{e^{3t} t^2}{2}$$

Q] Find $L^{-1}\left[\frac{s}{s^2+2s+2}\right]$

$$\rightarrow L^{-1}\left[\frac{s+1-1}{(s+1)^2+1}\right] = L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t} L^{-1}\left[\frac{s}{s^2+1}\right] - e^{-t}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

$$= e^{-t}(\cos t - \sin t)$$

Q] Inverse Laplace Transform of

$$\left[\frac{1}{(s+1)^2} + \frac{s-2}{s^2-4s+5} + \frac{s-2}{s^2+4s+3}\right]$$

$$\rightarrow L^{-1}\left(\frac{1}{(s+1)^2}\right) + L^{-1}\left(\frac{s-2}{s^2-4s+5}\right) + L^{-1}\left(\frac{s-2}{s^2+4s+3}\right)$$

$$= e^{-1} L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left[\frac{s-2}{(s-2)^2+1}\right] + L^{-1}\left[\frac{s-2}{(s-2)^2-1}\right]$$

$$= e^{-t} \cdot \frac{t}{1} + e^{2t}\cos t + e^{2t}\cosh t$$

$$= te^{-t} + e^{2t}\cos t + e^{2t}\cosh t$$

Q] Find $L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^2}\right]$

Let $\frac{5s^2-15s-11}{(s+1)(s-2)^2} = \frac{a}{s+1} + \frac{b}{s-2} + \frac{c}{(s-2)^2}$

$\therefore\ 5s^2-15s-11 = a(s-2)^2 + b(s+1)(s-2) + c(s+1)$

Put $s=2$, $20-30-11 = c(3) \Rightarrow -21 = 3c \Rightarrow c=-7$

Put $s=-1$, $5+15-11 = 9a \Rightarrow 9a = 9 \Rightarrow a=1$

Put $s=0$, $-11 = 4a-2b+c \Rightarrow b=4$

$$\therefore \quad \frac{5s^2-15s-11}{(s+1)(s-2)^2} = \frac{1}{s+1} + \frac{4}{s-2} - \frac{7}{(s-2)^2}$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^2}\right] = L^{-1}\left[\frac{1}{s+1}\right] + 4L^{-1}\left[\frac{1}{s-2}\right] - 7L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$$= e^{-t} + 4e^{2t} - 7e^{2t}.t$$

Q] $L^{-1}\left[\dfrac{s+29}{(s+4)(s^2+9)}\right]$

$\rightarrow$ Let $\dfrac{s+29}{(s+4)(s^2+9)} = \dfrac{a}{s+4} + \dfrac{bs+c}{s^2+9}$

$$s+29 = a(s^2+9) + (bs+c)(s+4)$$

$$\therefore \quad s+29 = (a+b)s^2 + (4b+c)s + (9a+4c)$$

Comparing the coefficients of similar powers of s

$a+b=0$, $4b+c=1$, $9a+4c=29$

$$4a+4b=0$$

$$\underline{c+4b=1}$$

$$4a-c=-1$$

$$16a-\cancel{4c}=-4$$

$$\underline{9a+\cancel{4c}=29}$$

$$25a = 25 \Rightarrow \boxed{a=1}$$

$\therefore \boxed{b=-1}$ also $c = 1-4b = 1+4 \Rightarrow \boxed{c=5}$

$$\therefore \frac{s+29}{(s+4)(s^2+9)} = \frac{1}{s+4} + \frac{(-s+5)}{s^2+9}$$

$$\therefore L^{-1}\left[\frac{s+29}{(s+4)(s^2+9)}\right] = L^{-1}\left[\frac{1}{s+4}\right] - L^{-1}\left[\frac{s}{s^2+9}\right] + 5L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= e^{-4t} - \cos 3t + \frac{5}{3}\sin 3t$$

Q) $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

$\rightarrow$ $s^2 = x$

Let $\dfrac{x}{(x+a^2)(x+b^2)} = \dfrac{A}{x+a^2} + \dfrac{B}{x+b^2}$

$\therefore$ $x = A(x+b^2) + B(x+a^2)$

Put $x=-a^2$, $-a^2 = A(-a^2+b^2) \Rightarrow A = \dfrac{a^2}{a^2-b^2}$

Put $x=-b^2$, $-b^2 = B(-b^2+a^2) \Rightarrow B = \dfrac{-b^2}{a^2-b^2}$

$$\therefore \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a^2-b^2}\left[\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right]$$

$$= \frac{1}{a^2-b^2}\left[a^2 \frac{1}{a}\sin at - b^2 \cdot \frac{1}{b}\sin bt\right]$$

$$\therefore \quad L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

Q] Find $L^{-1}\left[\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right]$

$$\rightarrow \quad = L^{-1}\left[\frac{(s+1)^2+2}{[(s+1)^2+4][(s+1)^2+1]}\right] = e^{-t} L^{-1}\left[\frac{s^2+2}{(s^2+4)(s^2+1)}\right]$$

Assume, $s^2 = x$

Let $\dfrac{x+2}{(x+4)(x+1)} = \dfrac{A}{(x+4)} + \dfrac{B}{(x+1)}$

$$x+2 = A(x+1) + B(x+4)$$

Put $x=-1$, $\quad 1 = B(3) \quad \Rightarrow \quad B = \dfrac{1}{3}$

Put $x=-4$, $\quad -2 = -3A \quad \Rightarrow \quad A = \dfrac{2}{3}$

$$L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = e^{-t} L^{-1}\left[\frac{2}{3}\cdot\frac{1}{s^2+4} + \frac{1}{3}\cdot\frac{1}{s^2+1}\right]$$

$$= e^{-t}\left\{\frac{2}{3}\cdot\frac{1}{2}\sin 2t + \frac{1}{3}\sin t\right\}$$

$$= \frac{e^{-t}}{3}(\sin 2t + \sin t)$$

Q] $L^{-1}\left[\dfrac{s}{(s^2+a^2)\,(s^2+b^2)}\right]$

→ First we consider,

$$\frac{1}{(s^2+a^2)\,(s^2+b^2)} = \frac{1}{b^2-a^2}\left[\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right]$$

$$\frac{s}{(s^2+a^2)\,(s^2+b^2)} = \frac{1}{b^2-a^2}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)\,(s^2+b^2)}\right] = \frac{1}{b^2-a^2}\left[\cos at - \cos bt\right]$$

* Inverse by Convolution theorem:

$$f_1(t) * f_2(t) = \int_0^t f_1(t)\, f_2(t-u)\, du$$

$L\,|f_1(t)| = \phi_1(s)$ & $L\,|f_2(t)| = \phi_2(s)$, $L^{-1}|\phi_1(s)\ \phi_2(s)| = \int_0^t f_1(u)\,.\,f_2(t-u)\,du$

Procedure of Applying Convolution Theorem:

To find $L^{-1}(\emptyset_1(s).\emptyset_2(s))$

1. Find $L^{-1}|\emptyset_1(s)| = f_1(u)$, say putting u in place to t.
2. Find $L^{-1}|\emptyset_2(s)| = f_2(t-u)$, say putting $(t-u)$ in place to t.
3. Find $L^{-1}|\emptyset_1(s).\emptyset_2(s)| = \int_0^t f_1(u) f_2(t-u)du$

Q] Find inverse Laplace transform using convolation theorem

① $\dfrac{s}{(s^2+a^2)^2}$

→ Let $\phi_1(s) = \dfrac{s}{s^2+a^2}$, $\phi(s) = \dfrac{1}{s^2+a^2}$

$\therefore f_1(t) = L^{-1}|\phi_1(s)| = \cos at$, $f_2(t) = L^{-1}|\phi_2(s)| = \dfrac{\sin at}{a}$

$\therefore L^{-1}|\phi(s)| = L^{-1}[\phi_1(s) \cdot \phi_2(s)]$

$$= \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$= \int_0^t \cos au \cdot \frac{\sin a(t-u)}{a} du$$

$$= \frac{1}{a} \int_0^t \cos au \, \sin(at-au)$$

$\cos A \sin B = 1/2\,(\sin(A+B) - \sin(A-B))$

$$= \frac{1}{2a} \int_0^t \sin at - \sin(au - at + au)\, du$$

$$= \frac{1}{2a} \int_0^t \sin at - \sin(2au - at)\, du$$

$$= \frac{1}{2a} \left[\sin at(u) + \frac{\cos(2au - at)}{2a} \right]_0^t$$

$$= \frac{1}{2a} \left[t \sin at + \cancel{\frac{\cos at}{2a}} - 0 - \cancel{\frac{\cos at}{2a}} \right]$$

$$= t \sin at$$

$\boxed{Q}$ $\phi(s) = \dfrac{1}{(s-2)(s+2)^2}$

$\rightarrow$ $\phi(s) = \dfrac{1}{(s-2)(s+2)^2} = \dfrac{1}{(s+2)^2} \cdot \dfrac{1}{s-2}$

$$= \phi_1(s) \quad \phi_2(s)$$

$$f_1(t) = L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}$$

$$f_2(t) = L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$$

By convolution theorem,

$$L^{-1}[\phi(s)] = L^{-1}[\phi_1(s)\,\phi_2(s)]$$

$$= \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$= \int_0^t ue^{-2u} \cdot e^{2(t-u)}\, du$$

$$= \int_0^t ue^{2t-4u}\, du$$

$$= \left[u\left(\frac{e^{2t-4u}}{-4}\right) - (1)\left(\frac{e^{2t-4u}}{16}\right)\right]_0^t$$

$$= \left[t\left(\frac{e^{-2t}}{-4}\right) - \left(\frac{e^{-2t}}{16}\right)\right] - \left[0 - \frac{e^{2t}}{16}\right]$$

$$= \left(\frac{e^{2t} - e^{-2t}}{16}\right) - \frac{te^{-2t}}{4} = \frac{1}{16}\left[e^{2t} - 4te^{-2t} - e^{-2t}\right]$$

$\boxed{Q}$ $\phi(s) = \dfrac{s}{(s^2+1)^2}$

$\rightarrow$ $\phi(s) = \dfrac{s}{(s^2+1)^2} = \dfrac{s}{s^2+1} \cdot \dfrac{1}{s^2+1}$

$$= \phi_1(s) \quad \phi_2(s)$$

$$f_1(t) = L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$$

$$f_2(t) = L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

By convolution theorem,

$$L^{-1}[\phi(s)] = L^{-1}[\phi_1(s) \cdot \phi_2(s)]$$

$$= \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$= \int_0^t \cos u \, \sin(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \sin t + \sin(t-2u)\, du$$

$$= \frac{1}{2}\left[u \sin t - \frac{\cos(t-2u)}{-2}\right]_0^t$$

$$= \frac{1}{2}\left[(t\sin t + \cos t) - (0 + \cos t)\right]$$

$$= \frac{t \sin t}{2}$$

Q] $\phi(s) = \dfrac{1}{(s-a)(s-b)}$

$$= \frac{1}{s-a} \cdot \frac{1}{s-b}$$

$$= \phi_1(s) \cdot \phi_2(s)$$

$$f_1(t) = L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$f_2(t) = L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s-b}\right] = e^{bt}$$

By Convolution theorem,

$$L^{-1}[\phi(s)] = L^{-1}[\phi_1(s) \cdot \phi_2(s)]$$

$$= \int_0^t f_1(u) \cdot f_2(t-u)\, du$$

$$= \int_0^t e^{au} \cdot e^{b(t-u)}\, du$$

$$= \int_0^t e^{bt+(a-b)u}\, du = \left[\frac{e^{bt+(a-b)u}}{(a-b)}\right]_0^t$$

$$= \frac{1}{a-b}\left[e^{at} - e^{bt}\right]$$

* Use of differentiation of $\phi(s)$

$$L^{-1}\phi(s) = \frac{-1}{t} L^{-1}[\phi'(s)]$$

Q] $L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right]$

$\rightarrow$ $$L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(\log\left(\frac{s+a}{s+b}\right)\right)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left[\log(s+a) - \log(s+b)\right]\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$= \frac{-1}{t}\left[e^{-at} - e^{-bt}\right]$$

Q] $L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right]$

$\rightarrow$ $$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left(\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left[\log(s^2+a^2) - \frac{1}{2}\log(s+b)\right]\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{1}{2}\cdot\frac{1}{s+b}\right]$$

$$= \frac{-1}{t}\left[2\cos at - \frac{1}{2}e^{-bt}\right]$$

$$= \frac{1}{t}\left[\frac{1}{2}e^{-bt} - 2\cos at\right]$$

Q] $L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$

$\rightarrow$ $$= \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(\tan^{-1}\left(\frac{1}{s}\right)\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{1+\left(\frac{1}{s}\right)^2}\cdot\left(\frac{-1}{s^2}\right)\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{1}{1+s^2}\right]$$

$$= \frac{1}{t}\sin t$$

Q] $L^{-1}[\cot^{-1}(s)]$

$\rightarrow$ $$= \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\cot^{-1}(s)\right]$$

$$= \frac{-1}{t}\left[\frac{-1}{1+s^2}\right] = \frac{1}{t}\sin t$$

Q] $L^{-1}\left[\tan^{-1}\left(\frac{s+a}{b}\right)\right]$

$\rightarrow$ $= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[\tan^{-1}\left(\frac{s+a}{b}\right)\right]\right\}$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+\left(\frac{s+a}{b}\right)^2} \cdot \frac{1}{b}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{b}{(s+a)^2+b^2}\right] = -\frac{1}{t} e^{-at} L^{-1}\left[\frac{b}{s^2+b^2}\right]$$

$$= -\frac{1}{t} e^{-at} \sin bt$$

* Periodic Functions

If $f(t)$ is a periodic function of period a

$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\, dt$$

Q] $f(t) = \begin{cases} \frac{t}{a}, & 0 < t \leq a \\ \frac{1}{a}(2a-t), & a < t < 2a \end{cases}$ & $f(t) = f(t+2a)$

→ As $f(t)$ is periodic function with period $2a$,

$$L(f(t)) = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left\{ \int_0^a \frac{t}{a} e^{-st} dt + \int_a^{2a} \frac{1}{a}(2a-t)\, e^{-st}\, dt \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \frac{1}{a}\left[t\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \frac{1}{a}\left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{-s^2}\right)\right]_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \frac{1}{a}\left[a\left(\frac{e^{-as}}{-s}\right) - \left(\frac{e^{-as}}{s^2}\right) - 0 + \frac{1}{s^2}\right] + \frac{1}{a}\left[0 + \frac{e^{-2as}}{s^2} - (a)\left(\frac{e^{-as}}{-s}\right) - \left(\frac{e^{-as}}{s^2}\right)\right]\right\}$$

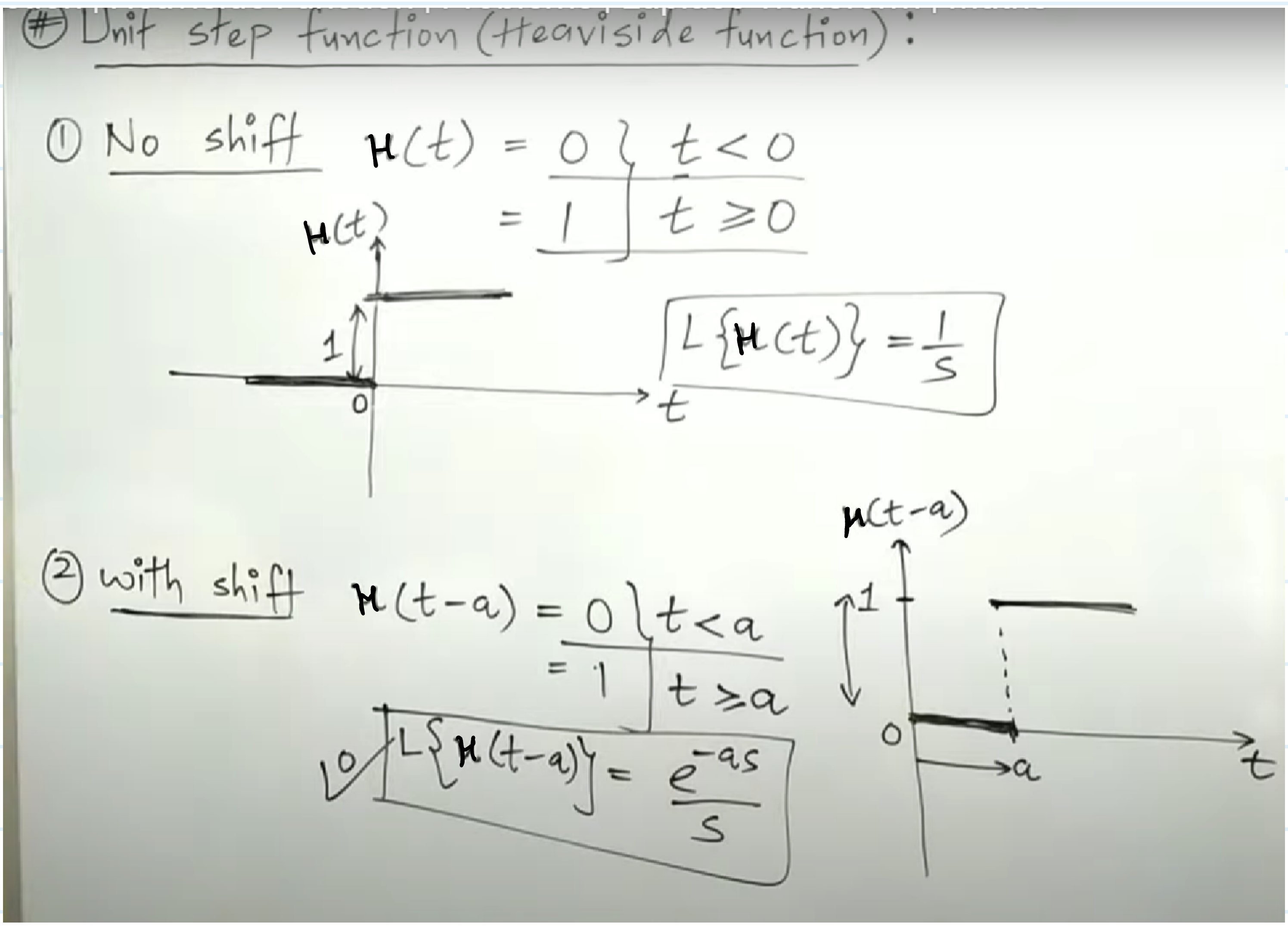
$$= \frac{1}{1-e^{-2as}} \left\{ \frac{1}{as^2}\left[1 - 2e^{-as} + e^{-2as}\right]\right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \frac{1}{as^2} (1-e^{-as})^2 \right\}$$

$$= \frac{1}{as^2}\left(\frac{1-e^{-as}}{1+e^{-as}}\right) = \frac{1}{as^2}\left(\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right)$$

$$= \frac{1}{as^2}\tanh\left(\frac{as}{2}\right)$$

* Heaviside Functions



Q] $L[(1+2t-t^2+t^3)\,H(t-1)]$

$\rightarrow \quad L[f(t)H(t-a)] = e^{-as} L[f(t+a)]$

$f(t) = 1 + 2t - t^2 + t^3 \quad \& \quad a = 1$

$\therefore L[(1+2t-t^2+t^3)\,H(t-1)] = e^{-1s} L[f(t+1)]$

Now, $f(t+1) = 1 + 2(t+1) - (t+1)^2 + (t+1)^3$

$= 1 + 2t + 2 - (t^2+2t+1) + (t^3+3t^2+3t+1)$

$f(t+1) = 3 + 3t + 2t^2 + t^3$

$\therefore L[(1+2t-t^2+t^3)\,H(t-1)] = e^{-s} L[3+3t+2t^2+t^3]$

$= e^{-s}\left[\frac{3}{s} + \frac{3}{s^2} + \frac{2\cdot 2!}{s^3} + \frac{3!}{s^4}\right]$

$L[(1+2t-t^2+t^3)\,H(t-1)] = e^{-s}\left[\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right]$

Q] $L[e^{-t}\sin t\; H(t-\pi)]$

$\rightarrow \quad L[f(t)H(t-a)] = e^{-as} L[f(t+a)]$

$a = \pi, \quad f(t) = e^{-t}\sin t$

$L[e^{-t}\sin t\; H(t-\pi)] = e^{-\pi s} L[f(t+\pi)]$

$f(t+\pi) = e^{-t-\pi}\sin(t+\pi)$

$= -e^{-\pi}[e^{-t}\sin t]$

$L[e^{-t}\sin t\; H(t-\pi)] = -e^{-\pi(s+1)}\dfrac{1}{(s+1)^2+1} = \dfrac{-e^{-\pi(s+1)}}{s^2+2s+2}$