线性方程组II

Rectangular Systems and Echelon Forms

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Row Echelon Form and Rank

- We are now ready to analyze more general linear systems consisting of m linear equations involving n unknowns where m may be different from n.
- The system is said to be **rectangular**.
- The first goal is to extend the Gaussian elimination technique from square systems to completely general rectangular systems.
- Recall that for a square system with a unique solution:
  - The pivotal positions are always located along the main diagonal.
  - The diagonal line from the upper-left-hand corner to the lower-right- hand corner.
  - Gaussian elimination results in a reduction of the coefficient matrix $\mathbf{A}$ to a triangular matrix.
- However, in the case of a general rectangular system, it is not always possible to have the pivotal positions lying on a straight diagonal line in the coefficient matrix.
- This means that the final result of Gaussian elimination will not be triangular in form.

- For example, consider the following system:

$$\begin{aligned} x_1 + 2x_2 + x_3 + 3x_4 + 3x_5 &= 5, \\ 2x_1 + 4x_2 + \quad + 4x_4 + 4x_5 &= 6, \\ x_1 + 2x_2 + 3x_3 + 5x_4 + 5x_5 &= 9, \\ 2x_1 + 4x_2 + \quad + 4x_4 + 7x_5 &= 9. \end{aligned}$$

- Applying Gaussian elimination to the coefficient matrix $\mathbf{A}$ yields the following result:

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix}.$$

- In the basic elimination process, the strategy is to move down and to the right to the next pivotal position.
- However, in this example, it is clearly impossible to bring a nonzero number into the (2, 2) -position by interchanging the second row with a lower row.

- In order to handle this situation, the elimination process is modified as follows.

Modified Gaussian Elimination

Suppose that $\mathbf{U}$ is the augmented matrix associated with the system after $i-1$ elimination steps have been completed. To execute the i^{th} step, proceed as follows:

- Moving from left to right in $\mathbf{U}$, locate the first column that contains a nonzero entry on or below the i^{th} position—say it is $\mathbf{U}_{*j}$.
- The pivotal position for the i^{th} step is the (i,j)-position.
- If necessary, interchange the i^{th} row with a lower row to bring a nonzero number into the (i,j)-position, and then annihilate all entries below this pivot.
- If row $\mathbf{U}_{i*}$ as well as all rows in $\mathbf{U}$ below $\mathbf{U}_{i*}$ consist entirely of zeros, then the elimination process is completed.

Problem: Apply modified Gaussian elimination to the following matrix and circle the pivot positions:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- Notice that the final result of applying Gaussian elimination in the above example is not a purely triangular form but rather a jagged or "stair-step" type of triangular form.

- Hereafter, a matrix that exhibits this stair-step structure will be said to be in **row echelon form**.

Row Echelon Form

An $m \times n$ matrix $\mathbf{E}$ with rows $\mathbf{E}_{i*}$ and columns $\mathbf{E}_{*j}$ is said to be in ***row echelon form*** provided the following two conditions hold.

- If $\mathbf{E}_{i*}$ consists entirely of zeros, then all rows below $\mathbf{E}_{i*}$ are also entirely zero; i.e., all zero rows are at the bottom.
- If the first nonzero entry in $\mathbf{E}_{i*}$ lies in the j^{th} position, then all entries below the i^{th} position in columns $\mathbf{E}_{*1}, \mathbf{E}_{*2}, \ldots, \mathbf{E}_{*j}$ are zero.

These two conditions say that the nonzero entries in an echelon form must lie on or above a stair-step line that emanates from the upper-left-hand corner and slopes down and to the right. The pivots are the first nonzero entries in each row. A typical structure for a matrix in row echelon form is illustrated below with the pivots circled.

$$\begin{pmatrix} * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Because of the flexibility in choosing row operations to reduce a matrix $\mathbf{A}$ to a row echelon form $\mathbf{E}$, the entries in $\mathbf{E}$ are not uniquely determined by $\mathbf{A}$.
- Nevertheless, the "form" of $\mathbf{E}$ is unique in the sense that the positions of the pivots in $\mathbf{E}$ (and $\mathbf{A}$) are uniquely determined by the entries in $\mathbf{A}$.
- The number of pivots, is also uniquely determined by entries in $\mathbf{A}$.
- This number is called the **rank** of $\mathbf{A}$, which is the same as the number of nonzero rows in $\mathbf{E}$.

Rank of a Matrix

Suppose $\mathbf{A}_{m\times n}$ is reduced by row operations to an echelon form $\mathbf{E}$. The ***rank*** of $\mathbf{A}$ is defined to be the number

$$\begin{aligned} rank\,(\mathbf{A}) &= \text{number of pivots} \\ &= \text{number of nonzero rows in } \mathbf{E} \\ &= \text{number of basic columns in } \mathbf{A}, \end{aligned}$$

where the ***basic columns*** of $\mathbf{A}$ are defined to be those columns in $\mathbf{A}$ that contain the pivotal positions.

Problem: Determine the rank, and identify the basic columns in

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{pmatrix}.$$

Solution: Reduce $\mathbf{A}$ to row echelon form as shown below:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{E}.$$

Consequently, $rank\,(\mathbf{A}) = 2$. The pivotal positions lie in the first and fourth columns so that the basic columns of $\mathbf{A}$ are $\mathbf{A}_{*1}$ and $\mathbf{A}_{*4}$. That is,

$$\text{Basic Columns} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right\}.$$

Pay particular attention to the fact that the basic columns are extracted from $\mathbf{A}$ and not from the row echelon form $\mathbf{E}$.

Reduced Row Echelon Form

- If the Gauss-Jordan technique is applied to a general $m \times n$ matrix, the final result is not necessarily the same as the square case.

Problem: Apply Gauss–Jordan elimination to the following 4×5 matrix and circle the pivot positions.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- The row echelon form produced by the Gauss-Jordan method contains a reduced number of nonzero entries, so it seems only natural to refer to this as a **reduced row echelon form**.

Reduced Row Echelon Form

A matrix $\mathbf{E}_{m\times n}$ is said to be in ***reduced row echelon form*** provided that the following three conditions hold.

- $\mathbf{E}$ is in row echelon form.
- The first nonzero entry in each row (i.e., each pivot) is 1.
- All entries above each pivot are 0.

A typical structure for a matrix in reduced row echelon form is illustrated below, where entries marked $*$ can be either zero or nonzero numbers:

$$\begin{pmatrix} 1 & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- If $\mathbf{A}$ is transformed by row operations to a reduced row echelon form $\mathbf{E_A}$, both the form as well as the individual entries in $\mathbf{E_A}$ are uniquely determined by $\mathbf{A}$.
- In other words, the reduced row echelon form $\mathbf{E_A}$ produced from $\mathbf{A}$ is independent of whatever elimination scheme is used.
- Producing an unreduced form is computationally more efficient, but the uniqueness of $\mathbf{E_A}$ makes it more useful for theoretical purposes.

$\mathbf{E_A}$ Notation

For a matrix $\mathbf{A}$, the symbol $\mathbf{E_A}$ will hereafter denote the unique reduced row echelon form derived from $\mathbf{A}$ by means of row operations.

- The relationships between the nonbasic and basic columns in a general matrix $\mathbf{A}$ are usually obscure, but the relationships among the columns in $\mathbf{E_A}$ are absolutely transparent.
- $\mathbf{E_A}$ can be used as a "map" or "key" to discover or unlock the hidden relationships among the columns of $\mathbf{A}$.

Column Relationships in $\mathbf{A}$ and $\mathbf{E_A}$

- Each nonbasic column $\mathbf{E}_{*k}$ in $\mathbf{E_A}$ is a combination (a sum of multiples) of the basic columns in $\mathbf{E_A}$ to the left of $\mathbf{E}_{*k}$. That is,

$$\mathbf{E}_{*k} = \mu_1\mathbf{E}_{*b_1} + \mu_2\mathbf{E}_{*b_2} + \cdots + \mu_j\mathbf{E}_{*b_j}$$

$$= \mu_1\begin{pmatrix}1\\0\\\vdots\\0\\\vdots\\0\end{pmatrix} + \mu_2\begin{pmatrix}0\\1\\\vdots\\0\\\vdots\\0\end{pmatrix} + \cdots + \mu_j\begin{pmatrix}0\\0\\\vdots\\1\\\vdots\\0\end{pmatrix} = \begin{pmatrix}\mu_1\\\mu_2\\\vdots\\\mu_j\\\vdots\\0\end{pmatrix},$$

  where the $\mathbf{E}_{*b_i}$'s are the basic columns to the left of $\mathbf{E}_{*k}$ and where the multipliers μ_i are the first j entries in $\mathbf{E}_{*k}$.

- The relationships that exist among the columns of $\mathbf{A}$ are exactly the same as the relationships that exist among the columns of $\mathbf{E_A}$. In particular, if $\mathbf{A}_{*k}$ is a nonbasic column in $\mathbf{A}$, then

$$\mathbf{A}_{*k} = \mu_1\mathbf{A}_{*b_1} + \mu_2\mathbf{A}_{*b_2} + \cdots + \mu_j\mathbf{A}_{*b_j},$$

  where the $\mathbf{A}_{*b_i}$'s are the basic columns to the left of $\mathbf{A}_{*k}$, and where the multipliers μ_i are as described above—the first j entries in $\mathbf{E}_{*k}$.

Problem: Write each nonbasic column as a combination of basic columns in

$$\mathbf{A} = \begin{pmatrix} 2 & -4 & -8 & 6 & 3 \\ 0 & 1 & 3 & 2 & 3 \\ 3 & -2 & 0 & 0 & 8 \end{pmatrix}.$$

Solution: Transform $\mathbf{A}$ to $\mathbf{E_A}$ as shown below.

$$\begin{pmatrix} 2 & -4 & -8 & 6 & 3 \\ 0 & 1 & 3 & 2 & 3 \\ 3 & -2 & 0 & 0 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -4 & 3 & \frac{3}{2} \\ 0 & 1 & 3 & 2 & 3 \\ 3 & -2 & 0 & 0 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -4 & 3 & \frac{3}{2} \\ 0 & 1 & 3 & 2 & 3 \\ 0 & 4 & 12 & -9 & \frac{7}{2} \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 2 & 7 & \frac{15}{2} \\ 0 & 1 & 3 & 2 & 3 \\ 0 & 0 & 0 & -17 & -\frac{17}{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 7 & \frac{15}{2} \\ 0 & 1 & 3 & 2 & 3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

The third and fifth columns are nonbasic. Looking at the columns in $\mathbf{E_A}$ reveals

$$\mathbf{E}_{*3} = 2\mathbf{E}_{*1} + 3\mathbf{E}_{*2} \quad \text{and} \quad \mathbf{E}_{*5} = 4\mathbf{E}_{*1} + 2\mathbf{E}_{*2} + \frac{1}{2}\mathbf{E}_{*4}.$$

The relationships that exist among the columns of $\mathbf{A}$ must be exactly the same as those in $\mathbf{E_A}$, so

$$\mathbf{A}_{*3} = 2\mathbf{A}_{*1} + 3\mathbf{A}_{*2} \quad \text{and} \quad \mathbf{A}_{*5} = 4\mathbf{A}_{*1} + 2\mathbf{A}_{*2} + \frac{1}{2}\mathbf{A}_{*4}.$$

You can easily check the validity of these equations by direct calculation.

- In summary, the utility of $\mathbf{E_A}$ lies in its ability to reveal dependencies in data stored as columns in an array $\mathbf{A}$.

Consistency of Linear Systems

- A system of m linear equations in n unknowns is said to be a **consistent** system if it possesses at least one solution.
- If there are no solutions, then the system is called **inconsistent**.
- Stating conditions for consistency of systems involving only two or three unknowns is easy.
  - A linear system of m equations in two unknowns is consistent if and only if the m lines defined by the m equations have at least one common point of intersection.
  - Similarly, a system of m equations in three unknowns is consistent if and only if the associated m planes have at least one common point of intersection.
- However, when m is large, these geometric conditions may not be easy to verify visually.
- When $n > 3$, the generalizations of intersecting lines or planes are impossible to visualize with the eye.

Consistency

Each of the following is equivalent to saying that $[\mathbf{A}|\mathbf{b}]$ is consistent.

- In row reducing $[\mathbf{A}|\mathbf{b}]$, a row of the following form never appears:

$$(\,0 \quad 0 \quad \cdots \quad 0 \quad | \quad \alpha\,), \quad \text{where} \quad \alpha \neq 0.$$

- $\mathbf{b}$ is a nonbasic column in $[\mathbf{A}|\mathbf{b}]$.
- $rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A})$.
- $\mathbf{b}$ is a combination of the basic columns in $\mathbf{A}$.

Problem: Determine if the following system is consistent:

$$\begin{aligned}
x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 1,\\
2x_1 + 2x_2 + 4x_3 + 4x_4 + 3x_5 &= 1,\\
2x_1 + 2x_2 + 4x_3 + 4x_4 + 2x_5 &= 2,\\
3x_1 + 5x_2 + 8x_3 + 6x_4 + 5x_5 &= 3.
\end{aligned}$$

Solution: Apply Gaussian elimination to the augmented matrix $[\mathbf{A}|\mathbf{b}]$ as shown:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 2 & 4 & 4 & 3 & 1 \\ 2 & 2 & 4 & 4 & 2 & 2 \\ 3 & 5 & 8 & 6 & 5 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Because a row of the form $(0 \quad 0 \quad \cdots \quad 0 \quad | \quad \alpha)$ with $\alpha \neq 0$ never emerges, the system is consistent. We might also observe that $\mathbf{b}$ is a nonbasic column in $[\mathbf{A}|\mathbf{b}]$ so that $rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A})$. Finally, by completely reducing $\mathbf{A}$ to $\mathbf{E_A}$, it is possible to verify that $\mathbf{b}$ is indeed a combination of the basic columns $\{\mathbf{A}_{*1}, \mathbf{A}_{*2}, \mathbf{A}_{*5}\}$.

Homogeneous Systems

- A system of m linear equations in n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0,\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0.
\end{aligned}
$$

  is said to be a **homogeneous system**.
- If there is at least one nonzero number on the right-hand side, then the system is called **nonhomogeneous**.
- Consistency is never an issue for homogeneous systems.
- $x_1 = x_2 = \cdots = x_n = 0$ is one solution regardless of the values of the coefficients, called as the **trivial solution**.
- "Are there solutions other than the trivial solution, and if so, how can we best describe them?"

Summary

Let $\mathbf{A}_{m\times n}$ be the coefficient matrix for a homogeneous system of m linear equations in n unknowns, and suppose $rank\,(\mathbf{A}) = r$.

- The unknowns that correspond to the positions of the basic columns (i.e., the pivotal positions) are called the ***basic variables***, and the unknowns corresponding to the positions of the nonbasic columns are called the ***free variables.***
- There are exactly r basic variables and $n-r$ free variables.
- To describe all solutions, reduce $\mathbf{A}$ to a row echelon form using Gaussian elimination, and then use back substitution to solve for the basic variables in terms of the free variables. This produces the ***general solution*** that has the form

$$\mathbf{x} = x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r},$$

  where the terms $x_{f_1}, x_{f_2}, \ldots, x_{f_{n-r}}$ are the free variables and where $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}$ are $n\times 1$ columns that represent particular solutions of the homogeneous system. The $\mathbf{h}_i$'s are independent of which row echelon form is used in the back substitution process. As the free variables x_{f_i} range over all possible values, the general solution generates all possible solutions.
- A homogeneous system possesses a unique solution (the trivial solution) if and only if $rank\,(\mathbf{A}) = n$—i.e., if and only if there are no free variables.

The homogeneous system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0, \\ 2x_1 + 5x_2 + 7x_3 &= 0, \\ 3x_1 + 6x_2 + 8x_3 &= 0, \end{aligned}$$

has only the trivial solution because

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 3 & 6 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} = \mathbf{E}$$

shows that $rank\,(\mathbf{A}) = n = 3$. Indeed, it is also obvious from $\mathbf{E}$ that applying back substitution in the system $[\mathbf{E}|\mathbf{0}]$ yields only the trivial solution.

Problem: Explain why the following homogeneous system has infinitely many solutions, and exhibit the general solution:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0, \\ 2x_1 + 5x_2 + 7x_3 &= 0, \\ 3x_1 + 6x_2 + 6x_3 &= 0. \end{aligned}$$

Solution:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 3 & 6 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{E}$$

shows that $rank\,(\mathbf{A}) = 2 < n = 3$. Since the basic columns lie in positions one and two, x_1 and x_2 are the basic variables while x_3 is free. Using back substitution on $[\mathbf{E}|\mathbf{0}]$ to solve for the basic variables in terms of the free variable produces $x_2 = -3x_3$ and $x_1 = -2x_2 - 2x_3 = 4x_3$, so the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}, \quad \text{where} \quad x_3 \text{ is free.}$$

That is, every solution is a multiple of the one particular solution $\mathbf{h}_1 = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$.

Nonhomogeneous Systems

- Unlike homogeneous systems, a nonhomogeneous system may be inconsistent.
- To describe the set of all possible solutions of a consistent nonhomogeneous system, construct a general solution by exactly the same method used for homogeneous systems as follows.
  - Use Gaussian elimination to reduce the associated augmented matrix $[\mathbf{A}|\mathbf{b}]$ to a row echelon form $[\mathbf{E}|\mathbf{c}]$.
  - Identify the basic variables and the free variables.
  - Apply back substitution to $[\mathbf{E}|\mathbf{c}]$ and solve for the basic variables in terms of the free variables.
  - Write the result in the form

$$\mathbf{x} = \mathbf{p} + x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r},$$

    where $x_{f_1}, \cdots, x_{f_{n-r}}$ are the free variables and $\mathbf{p}$, $\mathbf{h}_1, \cdots, \mathbf{h}_{n-r}$ are $n \times 1$ columns.
- This is the **general solution** of the nonhomogeneous system.

Problem: Determine the general solution of the following nonhomogeneous sy tem and compare it with the general solution of the associated homogeneo system:

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 1, \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 3x_5 &= 1, \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 2x_5 &= 2, \\ 3x_1 + 5x_2 + 8x_3 + 6x_4 + 5x_5 &= 3. \end{aligned}$$

Solution: Reducing the augmented matrix $[\mathbf{A}|\mathbf{b}]$ to $\mathbf{E}_{[\mathbf{A}|\mathbf{b}]}$ yields

$$\mathbf{A} = \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 2 & 4 & 4 & 3 & 1 \\ 2 & 2 & 4 & 4 & 2 & 2 \\ 3 & 5 & 8 & 6 & 5 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = \mathbf{E}_{[\mathbf{A}|\mathbf{b}]}.$$

Observe that the system is indeed consistent because the last column is nonbasic. Solve the reduced system for the basic variables x_1, x_2, and x_5 in terms of the free variables x_3 and x_4 to obtain

$$\begin{aligned} x_1 &= 1 - x_3 - 2x_4, \\ x_2 &= 1 - x_3, \\ x_3 &\text{ is “free,”} \\ x_4 &\text{ is “free,”} \\ x_5 &= -1. \end{aligned}$$

The general solution to the nonhomogeneous system is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 - x_3 - 2x_4 \\ 1 - x_3 \\ x_3 \\ x_4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The general solution of the associated homogeneous system is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_3 - 2x_4 \\ -x_3 \\ x_3 \\ x_4 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Summary

Let $[\mathbf{A}|\mathbf{b}]$ be the augmented matrix for a consistent $m \times n$ nonhomogeneous system in which $rank\,(\mathbf{A}) = r$.

- Reducing $[\mathbf{A}|\mathbf{b}]$ to a row echelon form using Gaussian elimination and then solving for the basic variables in terms of the free variables leads to the ***general solution***

  $$\mathbf{x} = \mathbf{p} + x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}.$$

  As the free variables x_{f_i} range over all possible values, this general solution generates all possible solutions of the system.
- Column $\mathbf{p}$ is a particular solution of the nonhomogeneous system.
- The expression $x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}$ is the general solution of the associated homogeneous system.
- Column $\mathbf{p}$ as well as the columns $\mathbf{h}_i$ are independent of the row echelon form to which $[\mathbf{A}|\mathbf{b}]$ is reduced.
- The system possesses a unique solution if and only if any of the following is true.
  - ▷ $rank\,(\mathbf{A}) = n =$ number of unknowns.
  - ▷ There are no free variables.
  - ▷ The associated homogeneous system possesses only the trivial solution.

Exercises

1. Reduce each of the following matrices to row echelon form, determine the rank, and identify the basic columns.

(a) $\begin{pmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \\ 2 & 6 & 0 \\ 1 & 2 & 5 \\ 3 & 8 & 6 \end{pmatrix}$

2. How many different forms are possible for a 3×4 matrix that is in row echelon form?

3. Determine the general solution for each of the following homogeneous systems

(a) $\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 &= 0, \\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 0, \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 0, \end{aligned}$ (b) $\begin{aligned} 2x + y + z &= 0, \\ 4x + 2y + z &= 0, \\ 6x + 3y + z &= 0, \\ 8x + 4y + z &= 0. \end{aligned}$

4. Determine the general solution for each of the following nonhomogeneous systems

(a) $$\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 &= 3, \\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 4, \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 5, \end{aligned}$$

(b) $$\begin{aligned} 2x + y + z &= 4, \\ 4x + 2y + z &= 6, \\ 6x + 3y + z &= 8, \\ 8x + 4y + z &= 10. \end{aligned}$$

5. If columns s_1 and s_2 are particular solutions of the same nonhomogeneous system, must it be the case that the sum $s_1 + s_2$ is also a solution?