Dynamic model of Otbot

This live script develops the dynamic model of Otbot and the solution to its forward and inverse dynamics problems.

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Recall that the equation of motion of a robot takes the form

$$\mathbf{M}(\mathbf{q})\ \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\ \dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) + \mathbf{J}(\mathbf{q})^\top \boldsymbol{\lambda} = \mathbf{Q}_a(\mathbf{u}) + \mathbf{Q}_f$$

where:

- $\mathbf{q}$ is the configuration vector of the robot (of size $n_q = 6$ in our case)
- $\mathbf{u} = [\boldsymbol{\tau}_r, \boldsymbol{\tau}_l, \boldsymbol{\tau}_p]^\top$ is the vector of motor torques (the right and left wheel torques and the platform torque).
- $\mathbf{M}(\mathbf{q})$ is the mass matrix of the unconstrained system (positive-definite of size $n_q \times n_q$)
- $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ is the generalized Coriolis and centrifugal force matrix
- $\mathbf{G}(\mathbf{q})$ is the generalized gravity force
- $\mathbf{J}(\mathbf{q})$ is the constraint Jacobian of the robot
- $\boldsymbol{\lambda}$ is a vector of Lagrange mulitpliers.
- $\mathbf{Q}_a(\mathbf{u})$ is the generalized force of actuation, which can be written in the form $\mathbf{E}(\mathbf{q}) \cdot \mathbf{u}$
- $\mathbf{Q}_f$ is the generalised force modelling all friction forces in the system

Note that, since the Otbot will move on flat terrain, $\mathbf{G}(\mathbf{q}) = \mathbf{0}$. Friction forces will also be neglected for the moment, so $\mathbf{Q}_f$ will be zero initially. The constraint Jacobian $\mathbf{J}(\mathbf{q})$ was already obtained in kinematic_model.mlx and we will simply load it from an appropriate mat file.

Our task thus boils down to obtaining $\mathbf{M}(\mathbf{q})$, $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$, and $\mathbf{E}(\mathbf{q})$ initially. After that, we will assemble the full dynamic model using the acceleration-level constraint (the time derivative of $\mathbf{J} \cdot \dot{\mathbf{q}} = \mathbf{0}$).

Hereafter, when we say "chassis" we mean the robot chassis **including** the wheels.

1. Initializations

```
% Clear all variables, close all figures, and clear the command window
clearvars
close all
clc

% Start stopwatch
tic;

% Display the matrices with rectangular brackets
sympref('MatrixWithSquareBrackets',true);

% Turn off abbreviated output format to avoid Matlab's own
% substitution of long expressions)
% sympref('AbbreviateOutput',false);

% Symbolic variables to be used (see the figures and explanations below)
syms x y            % Absolute coords of the pivot joint
syms x_dot y_dot    % Absolute velocity components of the pivot joint
syms alpha          % Absolute angle of the platform
syms alpha_dot      % Absolute angular velocity of the platform
syms varphi_r       % Angle of right wheel relative to chassis
syms varphi_l       % Angle of left wheel relative to chassis
syms varphi_p       % Pivot joint angle (platform wrt chassis)
syms varphi_dot_l   % Angular velocity of left wheel
syms varphi_dot_r   % Angular velocity of right wheel
syms varphi_dot_p   % Angular velocity of the platform relative to the chassis
syms l_1            % Pivot offset relative to the wheels axis
syms l_2            % One half of the wheels separation
syms m_c            % Mass of the chassis (including wheels)
syms m_p            % Mass of the platform
syms x_F y_F        % Coords of F (c.o.m. of the platform) in platform frame
syms x_B y_B        % Coords of B (c.o.m. of the chassis) in chassis frame
syms I_p            % Vertical moment of inertia of the platform at its c.o.m.
syms I_a            % Axial moment of inertia of one wheel
syms I_c            % Vertical moment of inertia of the chassis at its c.o.m.
```

2. Computation of the mass matrix

Recall that the robot configuration is given by

$$\mathbf{q} = (x, y, \alpha, \varphi_r, \varphi_l, \varphi_p)$$

To find the mass matrix $\mathbf{M}(\mathbf{q})$ we first write the kinetic energy T of the robot as a function of $\mathbf{q}$ and $\dot{\mathbf{q}}$ and then express it as:

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\,\dot{\mathbf{q}}^{\top} \cdot \mathbf{M}(\mathbf{q}) \cdot \dot{\mathbf{q}}$$

We have to compute the translational and rotational kinetic energies of all bodies in the Otbot, and add them all to obtain T.

```
% Chassis angle theta, and its derivative, in terms of qdot
theta = alpha - varphi_p;
theta_dot = alpha_dot - varphi_dot_p;

% Translational kinetic energies

    % Of the chassis
    T_tra_c = kin_en_trans(x_B, y_B, m_c, x_dot, y_dot, theta, theta_dot);

    % Of the platform
    T_tra_p = kin_en_trans(x_F, y_F, m_p, x_dot, y_dot, alpha, alpha_dot);

% Rotational kinetic energies

    % Whole chassis including wheels
    T_rot_c = 1/2 * I_c * (theta_dot)^2;

    % Right wheel when turning about its axis alone
    T_rot_r = 1/2 * I_a * (varphi_dot_r)^2;

    % Left wheel when turing about its axis alone
    T_rot_l = 1/2 * I_a * (varphi_dot_l)^2;

    % Platform
    T_rot_p = 1/2 * I_p * (alpha_dot)^2;

% Total kinetic energy
T = T_tra_c + T_tra_p + ...
    T_rot_c + T_rot_r + T_rot_l + T_rot_p;

% Display the results
display(T_tra_c); display(T_tra_p);
```

```
display(T_rot_c); display(T_rot_r); display(T_rot_l); display(T_rot_p);
display(T);
```

Since T is a quadratic form, the Hessian of T gives the desired mass matrix:

```
Mmat = hessian(T, ...
     [x_dot y_dot alpha_dot varphi_dot_r varphi_dot_l varphi_dot_p])
```

3. Computation of the Coriolis matrix

Recall that the (i, j) element of $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is given by the following formula (see Murray, Sastry, Lee)

$$\mathbf{C}_{\mathrm{ij}} = \frac{1}{2}\sum_{\mathrm{k}=1}^{\mathrm{n}} \left(\frac{\partial \mathbf{M}_{\mathrm{ij}}}{\partial \mathbf{q}_{\mathrm{k}}} + \frac{\partial \mathbf{M}_{\mathrm{ik}}}{\partial \mathbf{q}_{\mathrm{j}}} - \frac{\partial \mathbf{M}_{\mathrm{kj}}}{\partial \mathbf{q}_{\mathrm{i}}} \right) \dot{\mathbf{q}}_{\mathrm{k}},$$

which in Matlab can be implemented as follows:

```
qvec = [x y alpha varphi_r varphi_l varphi_p].';
qdotvec = [x_dot y_dot alpha_dot varphi_dot_r varphi_dot_l varphi_dot_p].';

nq = length(qvec);
Cmat = sym(zeros(nq,nq));

for i=1:nq
    for j=1:nq
        for k=1:nq
            Cmat(i,j) = Cmat(i,j) + qdotvec(k) * 0.5 * ( ...
                diff(Mmat(i,j),qvec(k)) + ...
                diff(Mmat(i,k),qvec(j)) - ...
                diff(Mmat(k,j),qvec(i))                   );
        end
    end
end
Cmat = simplify(Cmat);
display(Cmat)
```

4. Generalized force of actuation

Our vector $\mathbf{u}$ of motor torques is defined as

$$\mathbf{u} = \begin{bmatrix} \tau_r \\ \tau_l \\ \tau_p \end{bmatrix}$$

```
syms tau_r tau_l tau_p
uvec = [tau_r; tau_l; tau_p];
```

Each of the torques in $\mathbf{u}$ acts directly on a $\dot{\mathbf{q}}_i$ coordinate and, therefore, the generalized force of actuation is given by

$$\mathbf{Q}_a = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tau_r \\ \tau_l \\ \tau_p \end{bmatrix}$$

To rewrite this force in the form $\mathbf{Q}_a = \mathbf{E} \cdot \mathbf{u}$ we only have to define

$$\mathbf{E}(\mathbf{q}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
Ematrix = [zeros(3);eye(3)];
```

This completes our derivation of all matrices involved in the equation of motion.

5. Acceleration constraint

We now wish to obtain the robot model in the form $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u})$. This requires writing $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ as a function of $\mathbf{q}$, $\dot{\mathbf{q}}$, and $\mathbf{u}$. Recall that the Euler-Lagrange equation is

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{C}\,\dot{\mathbf{q}} + \mathbf{J}^{\mathsf{T}}\,\boldsymbol{\lambda} = \mathbf{E}\,\mathbf{u}$$ (we omit the dependencies on $\mathbf{q}$ and $\dot{\mathbf{q}}$ for simplicity)

Since this equation is a system of 6 equations in 9 unknowns (6 coordinates in $\ddot{\mathbf{q}}$ and 3 in $\boldsymbol{\lambda}$) we need 3 additional equations to determine $\ddot{\mathbf{q}}$ and $\boldsymbol{\lambda}$ for a given $\mathbf{u}$. These equations can be obtained by taking the time derivative of the kinematric constraint

$$\mathbf{J} \cdot \dot{\mathbf{q}} = \mathbf{0},$$

which gives

$$\mathbf{J}\,\ddot{\mathbf{q}} + \dot{\mathbf{J}}\,\dot{\mathbf{q}} = \mathbf{0},$$

or, equivalently,

$$\mathbf{J}\,\ddot{\mathbf{q}} = -\dot{\mathbf{J}}\,\dot{\mathbf{q}}$$

The latter equation is called the acceleration constraint of the robot.

Recall that $\mathbf{J}$ was obtained in `kinematic_model.mlx. Therefore,` we only need to calculate its time derivative now:

```
% We first import J_of_q (obtained by an earlier run of kinematic_model.mlx)
load("constraint_jacobian.mat")

% Substitute its variables by functions of t and take the time derivative
syms f1(t) f2(t)
J_of_t = subs(J_of_q,[alpha varphi_p],[f1(t) f2(t)]);
dJdt = diff(J_of_t,t);

% Substitute d/dt of f1 and f2 by the original variables using dot notation
Jdot = subs(dJdt,[f1(t) f2(t) diff(f1,t) diff(f2,t)], ...
    [alpha varphi_p alpha_dot varphi_dot_p])
```

6. Final model in explicit first-order form

The equations

$$\begin{cases} \mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{C}\,\dot{\mathbf{q}} + \mathbf{J}^{\top}\,\boldsymbol{\lambda} = \mathbf{E}\,\mathbf{u} \\ \mathbf{J}\,\ddot{\mathbf{q}} = -\dot{\mathbf{J}}\,\dot{\mathbf{q}} \end{cases}$$

can be written as a linear system with the form

$$\begin{bmatrix} \mathbf{M} & \mathbf{J}^{\top} \\ \mathbf{J} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{E}\,\mathbf{u} - \mathbf{C}\,\dot{\mathbf{q}} \\ -\dot{\mathbf{J}}\,\dot{\mathbf{q}} \end{bmatrix}$$

and since $\mathbf{M}$ is positive-definite and $\mathbf{J}$ is full row rank, the matrix on the left-hand side can be inverted to write

$$\begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{M} & \mathbf{J}^{\top} \\ \mathbf{J} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{E}\,\mathbf{u} - \mathbf{C}\,\dot{\mathbf{q}} \\ -\dot{\mathbf{J}}\,\dot{\mathbf{q}} \end{bmatrix}.$$

Therefore

$$\ddot{\mathbf{q}} = \begin{bmatrix}\mathbf{I}_6 & \mathbf{0}\end{bmatrix} \begin{bmatrix}\mathbf{M} & \mathbf{J}^\top \\ \mathbf{J} & \mathbf{0}\end{bmatrix}^{-1} \begin{bmatrix}\mathbf{E}\,\mathbf{u} - \mathbf{C}\,\dot{\mathbf{q}} \\ -\dot{\mathbf{J}}\,\dot{\mathbf{q}}\end{bmatrix}.$$

Finally, by considering the trivial equation $\dot{\mathbf{q}} = \dot{\mathbf{q}}$ in conjunction with the earlier equation we arrive at

$$\begin{bmatrix}\dot{\mathbf{q}} \\ \ddot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\dot{\mathbf{q}} \\ \begin{bmatrix}\mathbf{I}_6 & \mathbf{0}\end{bmatrix} \begin{bmatrix}\mathbf{M} & \mathbf{J}^\top \\ \mathbf{J} & \mathbf{0}\end{bmatrix}^{-1} \begin{bmatrix}\mathbf{E}\,\mathbf{u} - \mathbf{C}\,\dot{\mathbf{q}} \\ -\dot{\mathbf{J}}\,\dot{\mathbf{q}}\end{bmatrix}\end{bmatrix},$$

which gives the robot model in the usual control form

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}).$$

7. Solution to the forward dynamics problem

The forward dynamics problem consists in finding the acceleration $\ddot{\mathbf{q}}$ that corresponds to a given $\mathbf{u} = [\tau_r, \tau_l, \tau_p]^\top$. Such a $\ddot{\mathbf{q}}$ is given by the earlier expression

$$\ddot{\mathbf{q}} = \begin{bmatrix}\mathbf{I}_6 & \mathbf{0}\end{bmatrix} \begin{bmatrix}\mathbf{M} & \mathbf{J}^\top \\ \mathbf{J} & \mathbf{0}\end{bmatrix}^{-1} \begin{bmatrix}\mathbf{E}\,\mathbf{u} - \mathbf{C}\,\dot{\mathbf{q}} \\ -\dot{\mathbf{J}}\,\dot{\mathbf{q}}\end{bmatrix}.$$

8. Solution to the inverse dynamics problem

The inverse dynamics problem consists in finding the torques $\mathbf{u} = [\tau_r, \tau_l, \tau_p]^\top$ that produce a desired $\ddot{\mathbf{q}}$. These torques are obtained by solving

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{C}\,\dot{\mathbf{q}} + \mathbf{J}^\top\,\boldsymbol{\lambda} = \mathbf{E}\,\mathbf{u}$$

for $\mathbf{u}$ and $\boldsymbol{\lambda}$ (6 equations and 6 unknowns). For this we define $\boldsymbol{\tau}_{\mathbf{ID}} = \mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{C}\,\dot{\mathbf{q}}$ and rewrite the previous equation as

$$\mathbf{E}\,\mathbf{u} - \mathbf{J}^\top\,\boldsymbol{\lambda} = \boldsymbol{\tau}_{\mathbf{ID}},$$

or, equivalently, as

$$\begin{bmatrix}\mathbf{E} & -\mathbf{J}^\top\end{bmatrix} \begin{bmatrix}\mathbf{u} \\ \boldsymbol{\lambda}\end{bmatrix} = \boldsymbol{\tau}_{\mathbf{ID}}.$$

It is easy to see that $\begin{bmatrix}\mathbf{E} & -\mathbf{J}^\top\end{bmatrix}$ is a 6×6 full rank matrix irrespectively of $\mathbf{q}$. This follows directly from the expressions of $\mathbf{E}$ and $\mathbf{J}$ in the Otbot. Thus, we can write

$$\begin{bmatrix}\mathbf{u} \\ \boldsymbol{\lambda}\end{bmatrix} = \begin{bmatrix}\mathbf{E} & -\mathbf{J}^\top\end{bmatrix}^{-1} \boldsymbol{\tau}_{\mathbf{ID}},$$

so that

$$\mathbf{u} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{E} & -\mathbf{J}^\top \end{bmatrix}^{-1} \boldsymbol{\tau}_{\mathbf{ID}}$$

provides the desired value for $\mathbf{u}$.

9. Save wokspace and main matrices to file

```
save("dynamic_model_workspace")
save('Mmatrix.mat',"Mmat")
save('Cmatrix.mat',"Cmat")
```

10. Print ellapsed time

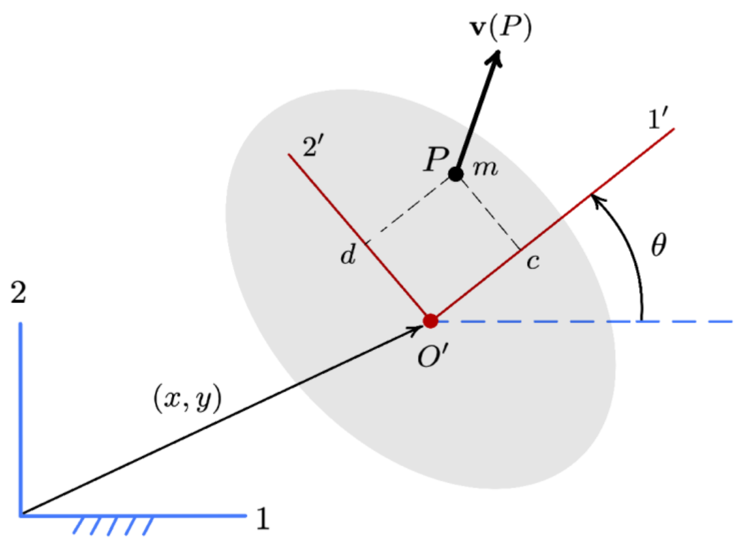
```
toc
```

Appendix: Function kin_en_trans

This function computes the translational kinetic energy of a point mass located at P on the grey body, in terms of the velocity coordinates $\dot{x}$, $\dot{y}$, and $\dot{\theta}$ of the body.

The following notation is used:

- (c, d) = Local coordinates of P in the body-fixed frame {1',2'}
- (x, y) = Absolute coordinates of the origin of the body-fixed frame
- θ = absolute angle of the body
- m = mass of P



This function can be used to compute the translational kinetic energy of any body in planar motion. Just let P be the c.o.m. of the body, and m be the total mass of the body.

```
function T = kin_en_trans(c,d,m,xdot,ydot,theta,thetadot)
    cth = cos(theta);
    sth = sin(theta);

    A = [1,    0,    -cth * d - sth * c;
         0,    1,     cth * c - sth * d;
         0,    0,              1           ];

    Mp = [m, 0, 0; 0, m, 0; 0, 0, 0];
    M = simplify(transpose(A) * Mp * A);
    w = [xdot;ydot;thetadot];
    T = expand(1/2 * transpose(w) * M * w,'ArithmeticOnly',true);
end
```