Optimitzation
Modelling for Science and Engineering, 2023-2024

1 Introduction

Finding the optimum solution to a problem that was previously known to be expressed in a certain mathematical form and to involve one or more criteria constitutes the resolution of a mathematical optimization problem in practice. This or these criteria are expressed in the form of a mathematical function, called an objective function. Finding an extreme value for this objective function, also known as an extremum, i.e. a maximum or a minimum, is an optimal solution to the problem.

1.1 Objectives of this delivery

The objectives of this delivery are:

- To understand and formulate the optimization problems.
- To characterize the problems.
- To identify the numerical algorithm that will solve it most effectively.
- To solve it and verify and analyze the results obtained.

1.2 Optimization with MATLAB

In MATLAB, a number of specific functions are available to address optimization problems. As we saw in class, the software's basic edition includes access to some features like `fsolve`, `fminsearch`, and `fminbnd`, but the optimization toolbox is the only way to use the rest of the features. Here below, you have a table summary of the MATLAB optimization functions that were covered in class organized according to the problem to be treated and the numerical method programmed. The last column lists the functions accessible via the optimization toolbox:

Problem to be treated	**Numerical method**	**Function**
Unconst. monodim.	Golden section search and parabolic interpolation	`fminbnd`
Unconst. multidim.	Nedler and Mead	`fminsearch`
Unconst. multidim.	Quasi-Newton	`fminunc`
Const. linear multidim.	Dual simplex Interior-point legacy	`linprog`
Const. nonlinear multidim.	Sequential quadratic programming method (SQP) Interior-point Trust-region-reflective Active-set	`fmincon`

We saw that:

- `fminbnd` finds the minimum of a monodimensional (monodim.) unconstrained (unconst.) problem within a fixed interval. It applies an algorithm for minimization without derivatives from Richard P. Brent, based on golden section search and parabolic interpolation.

- `fminsearch` finds the minimum of an unconstrained multidimensional (multidim.) problem using the Nedler and Mead method. Nedler and Mead method is a zero-order method, it does not use any information about the gradient of the objective function.

- `fminunc` finds the minimum of an unconstrained multivariable function using a quasi-Newton method, a second-order method where the Hessian is approximated.

- `linprog` finds the minimum of a constrained (const.) linear multidimensional problem. A method of the dual simplex is used by default. However an interior-point method can be also used, see this link for more details.

- `fmincon` finds the minimum of a constrained nonlinear multidimensional problem. A number of techniques are coded by the `fmincon` function for multidimensional optimization problems with constraints. A knowledgeable user can select the numerical approach (by setting the options of the function) or let the `fmincon` function pick the best approach for them. Some recommendations for choosing the numerical approach can be found in this mathworks link.

2 Exercises to solve

1. In an artistic workshop, we want to organize the teams' work in order to maximize benefit. The captor decides to organize two teams, one of which would work on ceramics and the other on carvings in copper. We are interested in learning how many products each team will produce.
   An art shop buys the entire product of the workshop. The artistic workshop and the art shop's managers came to an arrangement that caps the overall number of things the workshop can make each day at 80. On the other hand, regardless of the maximum number of items produced by the workshop, the shop requires that the total number of ceramic items produced must not exceed the total number of metal things by more than 30 items. One hour per person was needed to produce a ceramic item, while four hours per person were needed to produce a piece of metal. However, the maximum number of hours per day that can be devoted to pottery is 160 due to the availability of the craftspeople expert on ceramic.
   The artisan workshop makes a net profit of 200 euros for pottery and 600 euros for leather per unit.

2. We know that the inflationary tax function is:

$$I = xM_0e^{-\alpha x} \tag{1}$$

   Where x is the inflation rate, M_0 is the initial amount of money, and α is the inflation-affected money demand elasticity. Find the optimal inflation rate for an initial amount among [0, 100] euros and a 0.2 elasticity.

3. Consider a river that is diverted to three companies that use water and are owned by the same corporation. Each company produces a good. The crucial resource for producing the product is water. The total amount of water that is accessible is restricted due to the river flow, Q. And the water allocations plus the amount required to be kept in the river, R, cannot flow farther than the whole volume of water accessible, Q. (You can try different possibilities of the value $Q - R$).

The three companies are represented by the indexes $j = 1, 2, 3$, respectively, and each water-using company's benefit, $B_j(x_j)$, is influenced in part by the number of goods it produces pj, and the cost per unit, cj, of the product that is charged. The amount of product produced by each company is dependent on the amount of water, x_j, allocated to it.
Let the function $P_j(x_j)$ represent the maximum amount of product, p_j, that can be produced by the company j. These are named production functions, and they are usually concave, meaning that when x_j increases the slope, $dP_j(x_j)/dx_j$ decreases.
Assume the production functions for the three companies are:

$$P_1(x_1) = 0.4(x_1)^{0.9} \tag{2}$$
$$P_2(x_2) = 0.5(x_2)^{0.8} \tag{3}$$
$$P_3(x_3) = 0.6(x_3)^{0.7} \tag{4}$$

and the respective cost of production can be expressed by the following convex functions:

$$C_1 = 3(P_1(x_1))^{1.3} \tag{5}$$
$$C_2 = 5(P_2(x_2))^{1.2} \tag{6}$$
$$C_3 = 6(P_3(x_3))^{1.15} \tag{7}$$

Each company creates a patented, distinctive product, so it is able to establish and regulate the price per unit of that product. The greater the demand, the more each company can sell, and the cheaper the unit price. Each company has established the connection between the unit pricing and the anticipated demand and sales. The unit price, c_j that guarantees the sale of the produced amounts of products, p_j, are respectively:

$$c1 = 12 - p1 \tag{8}$$
$$c2 = 20 - 1.5p2 \tag{9}$$
$$c3 = 28 - 2.5p3 \tag{10}$$

We are assuming linear demand functions in order to simplify the problem, although this assumption is not a must.
We want to find the water allocations, the production amounts, and the unit price that maximize the total net benefit obtained from all three companies.
Please solve it by using one of the Matlab toolbox functions we have seen in class.

3 Tasks

In this two-person teamwork you need:

- To write the optimization problem for each exercise and describe how did you formulate the objective function, and constraints if needed.
- To explain which kind of problem you want to treat (is it constrained, unconstrained, linear, non-linear?) and justify the MATLAB function you will use to solve it.
- Solve it by providing the code needed.
- Verify and analyze the results obtained.

- Additionally, for the last exercise (exercise 3) please provide also a Lagrange multiplier approach to solve the problem and compare the solution with the one you found using the optimization toolbox. To do this, you can formulate a modified optimization problem (with four equality constraints) by assuming that the availability of water is a binding constraint and that all variables are positive. Then, you can construct the respective Lagrangian function, by adding a Lagrange multiplier λ_i for each constraint i. Afterward, you will need to find the gradient of the Lagrange function and set it to zero (remember that you can use the MATLAB function `fsolve` to solve a system of nonlinear equations of the form $F(x) = 0$, where x is a vector and F is a function that returns a vector value. You can also use the Symbolic Toolbox of MATLAB and solve the system of equations using the function `vpasolve`.
  Finally, you can try different values for the difference $Q - R$ and see what happens with the Lagrange multipliers, as well as the net benefit. What happens for $Q - R > 38.2$ and $Q - R < 38.2$, for example?

Please, provide a small report containing these tasks for all three exercises and the code of each exercise in independent folders.

Remark: It will be a plus if you made an effort to complete at least one of the exercises using the CVX toolbox, and compare the result with the respective function of table 1.2.