Classwork Problems

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① Solve:

$$(x^2+y^2)\,dx + 3xy\,dy = 0$$

Soln: $M = x^2+y^2$ $\quad N = 3xy$

$\frac{\partial M}{\partial y} = 2y$ $\quad \frac{\partial N}{\partial x} = 3y$; $\frac{\partial N}{\partial y} \neq \frac{\partial N}{\partial x} \longrightarrow$ non-exact

$$\left.\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-y}{3xy} = \frac{-1}{3x} = \phi(x)\right] \longrightarrow \text{Type ①}$$

$$IF = e^{\int \frac{-1}{3x}dx}$$
$$= e^{\frac{-1}{3}\log x}$$
$$= x^{\frac{-1}{3}}$$

Multiplying IF:

$$(x^2+y^2)\,x^{\frac{-1}{3}}\,dx + 3xy\,(x)^{\frac{-1}{3}}\,dy = 0$$
$$\left(x^{\frac{5}{3}} + y^2 x^{\frac{-1}{3}}\right)dx + 3y(x)^{\frac{2}{3}}\,dy = 0$$

$$\frac{\partial M_1}{\partial y} = 2y\,x^{\frac{-1}{3}}, \quad \frac{\partial N_1}{\partial x} = \cancel{3}y\left(\frac{2}{\cancel{3}}\right)x^{\frac{-1}{3}}$$

Eqn is now exact.

$$\text{Solution} = \int_{y\text{ const.}} \left(x^{\frac{5}{3}} + y^2 x^{\frac{-1}{3}}\right)dx + \int 0\,dy = C$$

$$\frac{3}{8}x^{\frac{8}{3}} + \frac{y^2 \cdot 3x^{\frac{2}{3}}}{2} = C$$

② Solve:

$$(x^2y^2 + xy + 1)\,y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0$$

Soln. $M = x^2y^3 + xy^2 + y$

$N = x^3y^2 - x^2y + x$

$$\frac{\partial M}{\partial y} = x^2(3y^2) + x(2y) + 1$$

$$\frac{\partial N}{\partial x} = y^2(3x^2) - y(2x) + 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \implies \text{Eqn is non exact}$$

Considering case ① and ②,

Denominator (M or N) will make it so that the result will not be a pure function of x or y.

Considering case ③,

M and N are not homogeneous

Considering case ④

$$IF = \frac{1}{Mx - Ny} = \frac{1}{x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy}$$

$$= \frac{1}{2x^2y^2}$$

Multiplying the eqn. by IF.

$$\left(\frac{x^2y^3 + xy^2 + y}{2x^2y^2}\right)dx + \left(\frac{x^3y^2 - x^2y + x}{2x^2y^2}\right)dy = 0$$

$$\left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \left(\frac{x}{2} - \frac{1}{2y} + \frac{1}{2y^2}\right)dy = 0$$

G.S. is:

$$\int_{y\,\text{const}} M \cdot dx + \int N(y)\, dy = C$$

$$\int\left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \int \frac{-1}{2y}\, dy = C$$

$$\frac{xy}{2} + \frac{1}{2}\log x + \frac{1}{2y}\left(\frac{-1}{x}\right) - \frac{1}{2}\log y = C$$

$$\frac{xy}{2} + \frac{1}{2}\log\left(\frac{x}{y}\right) - \frac{1}{2xy} = C$$ } Represents a family of plane curves

③ Solve:

③ Solve:

$$\underbrace{(5x^3+12x^2+6y^2)}_{M}dx + \underbrace{6xy}_{N}\,dy = 0$$

Soln $\frac{\partial M}{\partial y} = 12y \neq \frac{\partial N}{\partial x} = 6y$

$\Rightarrow$ Eqn is non-exact.

Considering case ①

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\cancel{6y}}{\cancel{6}xy} = \frac{1}{x}$$

$$IF = e^{\int \frac{1}{x} \cdot dx}$$

$$= e^{\log x} = x$$

Multiplying IF,

$$(5x^4+12x^3+6xy^2)dx + 6x^2y\,dy = 0$$

GS:

$$\int_{y\ \text{const.}} (5x^4+12x^3+6xy^2)dx + \int 0 \cdot dy = C$$

$$\frac{\cancel{5}x^5}{\cancel{5}} + \frac{\cancel{12}^{3}x^4}{\cancel{4}} + \frac{\cancel{6}^{3}y^2x^2}{\cancel{2}} = C$$

$$x^5 + 3x^4 + 3x^2y^2 = C$$

④ $\underbrace{(xy+y^2)}_{M}dx + \underbrace{(x+2y-1)}_{N}dy = 0$

$$\frac{\partial M}{\partial y} = x+2y \neq \frac{\partial N}{\partial x} = 1$$

$\Rightarrow$ Eqn is non exact

Considering case ①

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\cancel{x+2y-1}}{\cancel{x+2y-1}} = 1$$

$$IF = e^{\int 1 \cdot dx} = e^x$$

Multiplying by IF,

$$(xy + y^2)e^x \, dx + (x + 2y - 1)e^x \, dy = 0$$

GS:

$$\int_{y \text{ const.}} (y \cdot xe^x + y^2 e^x) \, dx + \int$$