1.

18 April 2024 11:33

Odd & Even Functions

Given some $f(x)$ in some interval (a,b), to check if its odd or even:

Substitute $x \longrightarrow a+b-x$

$f(a+b-x) = f(x) \longrightarrow$ Even

$f(a+b-x) = -f(x) \longrightarrow$ Odd

$f(a+b-x) \neq f(x)$, $\neq -f(x) \longrightarrow$ Neither even nor odd

Fourier Series

It is a finite series representation of periodic function in terms of trigonometric sine & cosine functions. It is a very powerful method to solve ODE & PDE, particularly with periodic functions appearing as non-homogeneous.

> **Note:** Condition for a Fourier expansion - Dirichlet's condition
> - $f(x)$ is periodic
> - $f(x)$ has finite number of discontinuities
> - $f(x)$ has finite number of maxima and minima.

For a function $f(x)$ in the interval $(c, c+2l)$:

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)}$$

where $a_0, a_n, b_n \longrightarrow$ Fourier coefficients

They are given by

$$\boxed{a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx} \qquad \boxed{a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx} \qquad \boxed{b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx}$$

Case 1

Suppose $f(x)$ is defined in the interval $[0, 2\pi]$

$\Rightarrow c = 0, \; l = \pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where the Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

When $f(x) \longrightarrow$ odd function

$a_0 = a_n = 0$ } Integral of odd function

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

When $f(x) \longrightarrow$ even function

$b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx)$$

Case 2 : $f(x)$ defined in the interval $[-\pi, \pi]$

$c = -\pi \qquad l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

where the Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

HALF RANGE FOURIER SERIES

In some problems, we require a Fourier expansion in the interval $[0, l]$, i.e., to expand $f(x)$ in the range $[0, l]$ which is half of the period $[-l, l]$.
In this case, function will not be defined in the range $[-l, 0]$. Hence we expand $f(x)$ arbitrarily to include the interval $[-l, 0]$.

If we expand the function to cover the range $[-l, l]$, function may be even or odd.

If we extend $f(x)$ such that $f(-x) = f(x)$, then the new function is even $(b_n = 0)$.
If we extend $f(x)$ such that $f(-x) = -f(x)$, then the new function is odd $(a_n, a_0 = 0)$.

Half Range Fourier Cosine Series for $f(x) \in [0, l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$, $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

Half Range Fourier Sine Series for $f(x) \in [0, l]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$