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FOURIER TRANSFORMS

Used to solve DEs, especially PDEs

Integral transform of a function $f(x)$ is defined by

$$F(s) = \int_{x_1}^{x_2} f(x)\,\underbrace{K(s,x)}_{\text{kernel}}\,dx$$

If $K(s,x) = e^{-sx} \longrightarrow$ Laplace transform

$= x^{s-1} \longrightarrow$ Mellin transform

$= x\,J_n(x) \longrightarrow$ Hankel transform

$= e^{isx} \longrightarrow$ Fourier transform

The Fourier transform (or) complex Fourier transform of $f(t)$ is given by

$$F[f(t)] = F[\omega] = \int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt$$

The inverse Fourier transform of $f(t)$ is given by

$$f(t) = F^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{i\omega t}\,dt$$

Properties

① Linear property:

$$F[a f(t) \pm b g(t)] = a\,F[f(t)] \pm b\,F[g(t)]$$

② Change of scale:

$$F[f(at)] = \frac{1}{a}\,F\left(\frac{\omega}{a}\right)$$

③ Shifting property:

(a) Line shifting property

(a) Time shifting property

$$F[f(t-a)] = e^{-i\omega a} F[\omega]$$

(b) Frequency shifting property

$$F[e^{iat} f(t)] = F(\omega - a)$$

④ Time reversal property

$$F[f(-t)] = F[-\omega]$$

⑤ Duality / Symmetry

$$F[f(t)] = 2\pi F[-\omega]$$

⑥ Fourier transform of derivative

$$F[f^n(t)] = (i\omega)^n F[f(t)]$$

⑦ Fourier transform of basis function

(a) Unit impulse function

$$F[\delta(t)] = 1$$

(b) Constant function

$$F[k] = 2\pi K \delta(\omega)$$

(c) $f(t) = e^{iat}$

$$F[e^{iat}] = 2\pi \delta(\omega - a)$$

(d) Unit step function

$$F[u(t)] = \frac{1}{i\omega}$$

(e) Sine / cosine functions

$$F[\sin at] = i\pi[\delta(\omega + a) - \delta(\omega - a)]$$

$$F[\cos at] = \pi[\delta(\omega - a) + \delta(\omega + a)]$$

$$F[f(t)\cos at] = \frac{1}{2}\left[F(\omega+a) + F(\omega-a)\right]$$

Fourier Sine & Cosine Transforms

Fourier sine transform of $f(t)$ is

$$F_s[f(t)] = \int_0^\infty f(t)\sin \omega t \, dt = F_s(\omega)$$

Its inverse is given by

$$f(t) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin \omega t \, d\omega$$

Fourier cosine series of $f(t)$ is

$$F_c[f(t)] = \int_0^\infty f(t)\cos \omega t \, dt = F_c(\omega)$$

Its inverse is given by

$$f(t) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos \omega t \, d\omega$$

Finite Fourier Transform

Finite Fourier cosine transform

$$F_c[f(t)] = F_c(n) = \int_0^l f(t)\cos\left(\frac{n\pi t}{l}\right) dt$$

Inverse finite Fourier cosine transform

$$f(t) = \frac{F_c(0)}{l} + \frac{2}{l}\sum_{n=1}^{\infty} F_c(n)\cos\left(\frac{n\pi t}{l}\right)$$

Finite Fourier sine transform

$$F_s[f(t)] = F_s(n) = \int_0^l f(t)\sin\left(\frac{n\pi t}{l}\right) dt$$

Inverse finite Fourier sine transform

$$f(t) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n)\sin\left(\frac{n\pi t}{l}\right)$$