Beta, gamma functions:

1. $\Gamma(n+1)=(m+1)^{n+1}(-1)^n\int_0^1 x^m(\log x)^n dx$, where m is positive integer and $m>-1$
2. Prove that $\int_0^\infty e^{-x^4}dx=\frac{1}{4}\Gamma\left(\frac{1}{4}\right)$
3. Show that $\int_0^\infty xe^{-x^8}dx\times\int_0^\infty x^2e^{-x^4}dx=\frac{\pi}{16\sqrt{2}}$
4. Show that $\int_0^\infty \sqrt{y}e^{-y^2}dy\times\int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}dy=\frac{\pi}{2\sqrt{2}}$
5. Prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx\times\int_0^1\frac{x^2}{\sqrt{1+x^4}}dx=\frac{\pi}{4\sqrt{2}}$
6. Show that $\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\sin\theta}}\times\int_0^{\frac{\pi}{2}}\sqrt{\sin\theta}\,d\theta=\pi$
7. Show that $\int_0^{\frac{\pi}{2}}\left(\sqrt{\tan\theta}+\sqrt{\sec\theta}\right)d\theta=\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\Gamma\left(\frac{3}{4}\right)+\frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}\right\}$
8. Show that $\int_{-1}^1(1+x)^{p-1}(1-x)^{q-1}dx=2^{p+q-1}\beta(p,q)$
9. Express $\int_0^1 x^m(1-x^n)^p dx$ in terms of gamma function and hence evaluate $\int_0^1 x^5(1-x^3)^{10}dx$
10. Show that $\int_a^b(x-a)^{m-1}(b-x)^{n-1}dx=(b-a)^{m+n-1}\beta(m,n)$
11. Prove that $\int_0^\infty\frac{x^3(1-x^5)}{(1+x)^{13}}dx=0$
12. Show that $\int_0^\infty\frac{x^2}{(1+x^4)^3}dx=\frac{5\pi\sqrt{2}}{128}$
13. Evaluate :

a) $\int_0^2(4-x^2)^{3/2}dx$; Ans: $\frac{3\pi}{2}$

b) $\int_0^\infty\frac{e^{-\sqrt{x}}}{x^{7/4}}dx$; Ans: $\frac{8\sqrt{\pi}}{3}$

c) $\int_0^1\frac{1}{\sqrt{-x\log x}}dx$; Ans: $\sqrt{2\pi}$

d) $\int_0^1 x^4\left(\log\frac{1}{x}\right)^3 dx$; Ans:6/625

e) $\int_0^\infty 3^{-4z^2}\,dz$; Ans: $\frac{1}{2}\sqrt{\frac{\pi}{\log 3}}$

f) $\int_0^4 x^{\frac{3}{2}}(4-x)^{\frac{5}{2}}\,dx$; Ans: 20π

g) $\int_0^2 x\sqrt[3]{8-x^3}\,dx$; Ans: $\frac{16\pi}{9\sqrt{3}}$

h) $\int_0^1 x^4(1-x)^3\,dx$: Ans: $\frac{1}{280}$

i) $\int_0^1 \frac{dx}{\sqrt{-\log x}}$;Ans: $\sqrt{\pi}$

Bessel Function:

1. Prove that $J_n(-x)=(-1)^n J_n(x)$ where n is a positive integer.
2. $J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos x$
3. Find the values of $J_{\frac{5}{2}}$ and $J_{-\frac{5}{2}}$.
4. Express J_3, J_4 in terms of J_0 and J_1.
5. Using recurrence relations, show that $\frac{d}{dx}(xJ_1(x)) = xJ_0(x)$ and hence prove that $\int_0^b xJ_0(ax)dx = \frac{b}{a}(J_1(ab))$
6. Prove that $\frac{d}{dx}\left(J_n^2(x)\right)=\frac{x}{2n}\left(J_{n-1}^2 - J_{n+1}^2\right)$
7. Prove that $\frac{d}{dx}\left(J_n^2(x)+J_{n+1}^2(x)\right)=\frac{2}{x}\left(nJ_n^2-(n+1)J_{n+1}^2\right)$
8. Express $\int x^4 J_1 dx$ in terms of J_0 and J_1 .
9. Prove that $J_0(x)=\frac{1}{\pi}\int_0^\pi \cos(x\cos\phi)\,d\phi$
10. Prove that $J_0^2+2J_1^2+2J_2^2+--------=1$
11. Prove the following;

    a) $x\sin x = 2(2^2 J_2 - 4^2 J_4 + 6^2 J_6 + ------)$

    b) $x\cos x = 2(1^2 J_1 - 3^2 J_3 + 5^2 J_5 + ------)$

    c) $\frac{x}{2}=J_1+3J_3+5J_5+-------$

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