Homogenous Functions, Euler's Theorem

08 September 2023 09:39

HOMOGENEOUS FUNCTIONS

A function $u(x,y)$ is said to be homogeneous of degree n, if it can be written as:

$$\boxed{\begin{gathered} u(x,y) = x^n \cdot \phi\left(\frac{y}{x}\right) \\ [\text{OR}] \\ u(x,y) = y^n \cdot \phi_1\left(\frac{x}{y}\right) \end{gathered}}$$

Eg: $u(x,y) = \dfrac{x^3+y^3}{\sqrt{x+y}} = \dfrac{x^3\left(1+\left(\frac{y}{x}\right)^3\right)}{\sqrt{x}\left(\sqrt{1+\left(\frac{y}{x}\right)}\right)}$

$u(x,y) = x^{\frac{5}{2}} \cdot \phi\left(\frac{y}{x}\right)$ $\therefore$ It is homogeneous

Eg: $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

$= x^2\left[\tan^{-1}\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2 \tan^{-1}\left(\frac{1}{y/x}\right)\right]$

$u = x^2 \cdot \phi\left(\frac{y}{x}\right)$

$\therefore$ It is homogeneous

NOTE:

Sum/difference of 2 homogeneous = Homogeneous w/ same degree

$\Updownarrow$

The degrees of both are the same

NOTE:

Product of 2 homogeneous functions = Homogeneous w/ degree = sum of degrees of 2 functions

- **Function w/ 3 variables**

A function $u(x,y,z)$ is said to be homogenous with degree n if:

$$\boxed{\begin{gathered} u(x,y,z) = x^n \cdot \phi\left(\frac{y}{x}, \frac{z}{x}\right) \\ [\text{OR}] \\ u(x,y,z) = y^n \cdot \phi\left(\frac{x}{y}, \frac{z}{y}\right) \\ [\text{OR}] \\ u(x,y,z) = z^n \cdot \phi\left(\frac{x}{z}, \frac{y}{z}\right) \end{gathered}}$$

This can be extended for any number of independent variables

EULER'S THEOREM (for homogeneous functions)

If $u(x,y)$ is a homogeneous function of degree n,

$$\boxed{x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu}$$

PROOF

Given $u(x,y)$ is homogeneous fn. of degree n,

$$u(x,y) = x^n \cdot \phi\left(\frac{y}{x}\right) \quad \text{——} \; (1)$$

Differentiating partially w.r.t. x,

$$\frac{\partial u}{\partial x} = n x^{n-1}\left[\phi\left(\frac{y}{x}\right)\right] + x^n \left[\phi'\left(\frac{y}{x}\right)\right]\left[\frac{-y}{x^2}\right]$$

Multiplying by x on both sides,

$$x\frac{\partial u}{\partial x} = n x^n\left[\phi\left(\frac{y}{x}\right)\right] - y x^{n-1}\phi'\left(\frac{y}{x}\right) \quad \text{——} \; (2)$$

Multiplying by x on both sides,

$$x\frac{\partial u}{\partial x} = nx^n\left[\phi\left(\frac{y}{x}\right)\right] - yx^{n-1}\phi'\left(\frac{y}{x}\right) \quad \text{—— } (2)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = x^n\left[\phi'\left(\frac{y}{x}\right)\right]\left(\frac{1}{x}\right)$$

Multiplying by y,

$$y\frac{\partial u}{\partial y} = yx^{n-1}\left[\phi'\left(\frac{y}{x}\right)\right] \quad \text{—— } (3)$$

(2) + (3)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\left[\phi\left(\frac{y}{x}\right)\right]$$

$$= nu$$

NOTE:

If $u(x,y,z)$ is homogeneous w/ degree n,

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z} = nu$$

and so on for any no. of independent variables

Eg: $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\sin\left(\frac{x^2+y^2}{xy}\right)$

Find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Soln: $x^2\left[\tan^{-1}\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2\sin\left(\frac{x^2+y^2}{xy}\right)\right]$

$\Rightarrow$ It is homogeneous, $n=2$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2\tan^{-1}\left(\frac{y}{x}\right) - 2y^2\sin\left(\frac{x^2+y^2}{xy}\right)$$

$\partial x \qquad \partial y \qquad (x) \qquad (xy)$

EXTENSION OF EULER'S THEOREM FOR 2^{nd} ORDER DERIVATIVES

If $u(x,y)$ is a homogeneous fn. of degree n, then

$$\boxed{x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial y \partial x} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} = n(n-1)u}$$

Proof

Since $u(x,y)$ is homogeneous fn. of degree n,

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu \quad \text{—— (1)}$$

Differentiating partially w.r.t. x:

$$\frac{\partial u}{\partial x} + x \cdot \frac{\partial^2 u}{\partial x^2} + y\left(\frac{\partial^2 u}{\partial x \partial y}\right) = n\left(\frac{\partial u}{\partial x}\right)$$

Multiplying by x,

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + x \cdot \frac{\partial u}{\partial x} = nx\left(\frac{\partial u}{\partial x}\right) \quad \text{—— (2)}$$

IIIly, differentiating eqn (1) w.r.t. y and multiplying by y,

$x \leftrightarrow y$

$$y^2 \cdot \frac{\partial^2 u}{\partial y^2} + xy\left(\frac{\partial^2 u}{\partial y \partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = ny\left(\frac{\partial u}{\partial y}\right) \quad \text{—— (3)}$$

Adding eqns. (2), (3):

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \partial y} + nu = n \cdot nu$$

$$x^2 \cdot u_{xx} + y^2 \cdot u_{yy} + 2xy \cdot u_{xy} = n^2 u - nu$$

$$x^2 \cdot u_{xx} + y^2 \cdot u_{yy} + 2xy \cdot u_{xy} = n^2u - nu$$

$$\boxed{x^2 \cdot u_{xx} + y^2 \cdot u_{yy} + 2xy \cdot u_{xy} = n(n-1)u}$$

Hence proved

If $f(u)$ is a homogeneous fn. of degree n,

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n\left[\frac{f(u)}{f'(u)}\right]$$

Elaborate

[3rd order]

If $n \cdot \dfrac{f(u)}{f'(u)} = g(u)$,

$$x^2 u_{xx} + 2xy \cdot u_{xy} + y^2 u_{yy} = g(u)(g'(u) - 1)$$