1. Legendre's LDE, Cauchy's LDE

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LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Legendre's LDE Cauchy's LDE

LEGENDRE'S LDE

$$\boxed{a_0(ax+b)^2\frac{d^2y}{dx^2}+a_1(ax+b)\frac{dy}{dx}+a_2y=\phi(x)}$$

} General form of degree = 2
Can be extended to any degree

First reduce to LDE with constant coefficient and then solve.

Let

$$\boxed{t=\log(ax+b)} \quad \text{--- (1)}$$

$$ax+b=e^t \implies x=\frac{e^t-b}{a}$$

Taking derivative,

$$\frac{dy}{dx}=\frac{dy}{dt}\left(\frac{dt}{dx}\right)=\frac{dy}{dt}\left(\frac{a}{ax+b}\right)$$

$$\boxed{(ax+b)\frac{dy}{dx}=a\left(\frac{dy}{dt}\right)=aD_1(y)} \quad \text{--- (2)}$$

$$D_1=\frac{d}{dt}$$

Differentiating w.r.t. x,

$$(ax+b)\frac{d^2y}{dx^2}+\frac{dy}{dx}(a)=a\left(\frac{d}{dx}\left(\frac{dy}{dt}\right)\right)$$

$$(ax+b)\frac{d^2y}{dx^2}+a\left(\frac{dy}{dx}\right)=a\left(\frac{d}{dt}\left(\frac{dy}{dt}\right)\right)\left(\frac{dt}{dx}\right)$$

} $\because \frac{d}{dx}=\frac{d}{dt}\left(\frac{dt}{dx}\right)$ by chain rule

$$(ax+b)\frac{d^2y}{dx^2}+a\left(\frac{dy}{dx}\right)=\frac{d^2y}{dt^2}\left(\frac{a^2}{ax+b}\right)$$

$$(ax+b)^2\left(\frac{d^2y}{dx^2}\right)+a(ax+b)\left(\frac{dy}{dx}\right)=a^2\cdot\frac{d^2y}{dt^2}$$

$$(ax+b)^2\left(\frac{d^2y}{dx^2}\right)=a^2\left(\frac{d^2y}{dt^2}\right)-\underline{a(ax+b)\frac{dy}{dx}}$$

$$(ax+b)\frac{dy}{dx}=a\left(\frac{dy}{dt}\right)$$

$$(ax+b)^2\left(\frac{d^2y}{dx^2}\right)=a^2\left(\frac{d^2y}{dt^2}\right)-a^2\left(\frac{dy}{dt}\right)$$

$$=a^2\left(D_1^2-D_1\right)y$$

$$= a^2 (D_1^2 - D_1) y$$

$$\boxed{(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D_1 (D_1 - 1) y} \longrightarrow (3)$$

Use eqns. (1), (2), (3) to solve the original equation

IIIly, for third order derivative:

$$\boxed{(ax+b)^3 \frac{d^3y}{dx^3} = a^3 D_1 (D_1-1)(D_1-2) y}$$

CAUCHY'S LDE

- Special case of Legendre's LDE
- In $ax+b$, $a=1$ and $b=0$

$$\Rightarrow ax+b = x$$

$$a_0 \cdot x^2 \cdot \frac{d^2y}{dx^2} + a_1 \cdot x \cdot \frac{dy}{dx} + a_2 y = \phi(x)$$

$$t = \log x \Rightarrow x = e^t$$

$$x \frac{dy}{dx} = D_1(y)$$

$$D_1 = \frac{d}{dt}$$

$$\boxed{x^2 \frac{d^2y}{dx^2} = D_1 (D_1 - 1) y}$$