4. Relation between Beta and Gamma

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RELATION BETWEEN β AND γ

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

$$\Gamma m = 2\int_0^\infty e^{-x^2} x^{2m-1}\,dx \quad — (1)$$

$$\Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1}\,dy \quad — (2)$$

$$\Gamma m \, \Gamma n = 4\int_0^\infty e^{-x^2} x^{2m-1}\,dx \int_0^\infty e^{-y^2} y^{2n-1}\,dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\,dx\,dy \quad — (3)$$

$x = r\cos\theta$ $\qquad x^2+y^2 = r^2$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$r \longrightarrow 0$ to ∞

$\theta \longrightarrow 0$ to $\frac{\pi}{2}$ $\Big] \longrightarrow$ First quadrant $(x: 0 \rightarrow \infty,\ y: 0 \rightarrow \infty)$

Substituting in (3),

$$\Gamma m \, \Gamma n = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1}\, r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2} r^{2n+1+2n-1+1} \cos^{2m-1}\theta \cdot \sin^{2n-1}\theta\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta\,d\theta \cdot 2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\,dr$$

$$\Gamma m \, \Gamma n = \beta(m,n)\, \Gamma(m+n)$$

$$\boxed{\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}}$$

DUPLICATION FORMULA FOR GAMMA FUNCTION

$$\boxed{\Gamma n \, \Gamma\left(n+\tfrac{1}{2}\right) = \frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}}}$$

DUPLICATION FORMULA FOR BETA FUNCTION

$$\boxed{\beta\left(p, \tfrac{1}{2}\right) = 2^{2p-1}\, \beta(p,p)}$$