2. Beta Function

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BETA FUNCTION - $\beta(m,n)$ / $\beta(p,q)$

$\beta(m,n)$ is defined as follows:

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx}$$

NOTE: Properties of definite integrals

① $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

② $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

③ $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

④ $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{WHEN: } f(x) = f(2a-x) \\ 0 & \text{WHEN: } f(x) = -f(2a-x) \end{cases}$$

⑤ $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{WHEN: } f(x) = f(-x) \\ 0 & \text{WHEN: } f(x) = -f(-x) \end{cases}$$

⑥ $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

⑦ $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ ...

PROPERTIES OF BETA FUNCTION

① Symmetry property

$$\beta(m,n) = \beta(n,m)$$

PROOF

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Using change in variable property,

$$\beta(m,n) = \int_0^1 (1-x)^{m-1}\left(1-(1-x)\right)^{n-1}dx$$

$$= \int_0^1 (1-x)^{m-1}\, x^{n-1} \qquad = \beta(n,m)$$

② Expressing in terms of trigonometric functions

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x = \sin^2\theta$ $\qquad\qquad x=0 \longrightarrow \theta = 0$

$dx = 2\sin\theta\cos\theta\,d\theta$ $\qquad\qquad x = 1 \longrightarrow \theta = \frac{\pi}{2}$

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$\boxed{\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \,.\, \cos^{2n-1}\theta \,.\, d\theta}$$

Put $2m-1 = p$, $2n-1 = q$

$m = \frac{p+1}{2}$, $n = \frac{q+1}{2}$

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^p\theta \,.\, \cos^q\theta \,.\, d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \quad \cos^q\theta \quad d\theta = \frac{1}{2}\,\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^p\theta \cdot \cos^q\theta \cdot d\theta = \frac{1}{2}\,\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$

where $\frac{p+1}{2} > 0$, $\frac{q+1}{2} > 0$

$p > -1$, $q > -1$

Example

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta} \cdot d\theta = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \cdot \cos^{-\frac{1}{2}}\theta \cdot d\theta$$

$$p = \frac{1}{2}, \quad q = \frac{-1}{2}$$

$$\Rightarrow \quad = \frac{1}{2}\,\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

(3) As an improper integral

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x = \frac{1}{1+y} \Rightarrow 1+y = \frac{1}{x}$ $\Big|$ $dx = \frac{-1}{(1+y)^2}\,dy$

$$y = \frac{1}{x} - 1$$

Put $x = 0 \longrightarrow y = \infty$

$x = 1 \longrightarrow y = 0$

Thus,

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1 - \frac{1}{1+y}\right)^{n-1}\left(\frac{-1}{(1+y)^2}\right)dy$$

$$\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{\ldots}\,dy$$

$$\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

Change of variables, symmetry:

$$\boxed{\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx}$$

Example

Express $\beta(2,\frac{5}{2})$ as an improper integral

$$\beta\left(2,\frac{5}{2}\right) = \int_0^\infty \frac{x}{(1+x)^{\frac{9}{2}}}\,dx$$