3. Partial Differential Equations

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PARTIAL DIFFERENTIAL EQUATIONS

An equation containing dependent variable, independent variables and partial derivatives of dependent variable w.r.t. independent variable.

Eg: $z = f(x,y)$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = xy$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} + 2z$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Convenient Notations

$$\frac{\partial z}{\partial x} = p \qquad \frac{\partial z}{\partial y} = q$$

$$\frac{\partial^2 z}{\partial x^2} = r \qquad \frac{\partial^2 z}{\partial x \partial y} = s \qquad \frac{\partial^2 z}{\partial y^2} = t$$

FIRST ORDER LINEAR PDE

$$f(x, y, z, p, q) = 0$$

- $p, q, z \longrightarrow$ Degree should be one
- Equation should not contain $z \times p$, $z \times q$

SECOND ORDER LINEAR PDE

$$f_1(x,y)p + f_2(x,y) + q + f_3(x,y)r + f_4(x,y)s + f_5(x,y)t + f_6(x,y)z = 0$$

FORMATION OF PDE

① By eliminating arbitrary constants

$$Z = (x-a)^2 + (y-b)^2$$

$$p = 2(x-a) \Rightarrow x-a = \frac{p}{2}$$

$$q = 2(y-b) \Rightarrow y-b = \frac{q}{2}$$

$$z = \frac{p^2}{4} + \frac{q^2}{4}$$

$$\boxed{p^2+q^2 = 4z}$$

② By eliminating arbitrary functions

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$z = x^n \phi\left(\frac{y}{x}\right) \quad \text{— ①}$$

$$\frac{\partial z}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n$$

TYPES OF SOLUTIONS

① General solution: Contains arbitrary constants

② Complete solution: Contains arbitrary functions

③ Particular solution.

SOLVING PDEs

① By direct integration

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$$

Integrating w.r.t. x

$$\frac{\partial z}{\partial y} = \frac{1}{y}\cdot\left(\frac{x^2}{2}\right) + ax + \phi_1(y)$$

$$z = \frac{x^2}{2}\log y + axy + \int \phi_1(y)\,dy + \phi_2(x)$$

$$z = \frac{x^2 \cdot \log y}{2} + axy + F_1(y) + \phi_2(x)$$

② Lagrange's Method : I order PDE of the form $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$

<u>Auxiliary equations</u> :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Form two different DEs of first order and solve them.

Let the solutions be

$$u(x,y,z) = c_1 \qquad v(x,y,z) = c_2$$

<u>General solution</u>

$$\boxed{\underbrace{\phi(u,v)}_{\phi\text{: some arbitrary function}} = 0} \xrightarrow{\text{alternate}} u = \phi(v), \quad v = \phi(u)$$