Function that generates Bessels functions of different order

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

We prove that the coefficient of e^n in the power series expansion of $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ is $J_n(x)$.

$$LHS = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{xt}{2}} \cdot e^{\frac{-x}{2t}}$$

$$= \left(1 + \frac{xt}{2}\left(\frac{1}{1!}\right) + \left(\frac{xt}{2}\right)^2\left(\frac{1}{2!}\right) + \dots + \left(\frac{xt}{2}\right)^n\left(\frac{1}{n!}\right) + \left(\frac{xt}{2}\right)^{n+1}\left(\frac{1}{n+1!}\right) + \dots\right)$$

$$\times$$

$$\left(1 + (-1)^1 \frac{x}{2t}\left(\frac{1}{1!}\right) + (-1)^2\left(\frac{x}{2t}\right)^2\left(\frac{1}{2!}\right) + \dots + (-1)^n\left(\frac{x}{2t}\right)^n\left(\frac{1}{n!}\right) + (-1)^{n+1}\left(\frac{x}{2t}\right)^{n+1}\left(\frac{1}{n+1!}\right) + \dots\right)$$

Case (1) : $n \in Z^+$

Collecting terms that contain e^n

$$\text{Coefficient of } t^n : \left(\left(\frac{x}{2}\right)^n\left(\frac{1}{n!}\right) - \left(\frac{x^{n+1}}{2^{n+1}}\right)\left(\frac{1}{(n+1)!}\right) \cdot \left(\frac{x}{2}\right)\left(\frac{1}{1!}\right) + \left(\frac{x^{n+2}}{2^{n+2}}\right)\left(\frac{1}{(n+2)!}\right) \cdot \left(\frac{x}{2}\right)^2\left(\frac{1}{2!}\right) + - \dots\right.$$

$$= \frac{(-1)^0\left(\frac{x}{2}\right)^{n+2(0)}}{0!\,\Gamma(n+0+1)} + \frac{(-1)^1\left(\frac{x}{2}\right)^{n+2(1)}}{1!\,\Gamma(n+1+1)} + \frac{(-1)^2\left(\frac{x}{2}\right)^{n+2(2)}}{2!\,\Gamma(n+2+1)} + \dots$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)} \underline{\underline{= J_n(x)}}$$

Case (2) : $n = 0$

Collecting constant terms

$$\text{Coefficient of } t^0 : 1 - \left(\frac{x}{2}\right)^2\left(\frac{1}{1!}\right)^2 + \left(\frac{x}{2}\right)^4\left(\frac{1}{2!}\right)^2 + \left(\frac{x}{3}\right)^6\left(\frac{1}{3!}\right)^2 + \dots$$

$$= \frac{(-1)^0\left(\frac{x}{2}\right)^{0+2(0)}}{0!\,\Gamma(0+0+1)} + \frac{(-1)^1\left(\frac{x}{2}\right)^{0+2(1)}}{1!\,\Gamma(0+1+1)} + \frac{(-1)^2\left(\frac{x}{2}\right)^{0+2(2)}}{2!\,\Gamma(0+2+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\left(\frac{x}{2}\right)^{0+2r}}{r!\,\Gamma(0+r+1)}$$

$$= J_0(x)$$

Case (3): $n \in \mathbb{Z}^-$

$$\text{Coefficients of } t^{-n} = \left(\frac{x}{2}\right)^n \left(\frac{1}{n!}\right) + \frac{(-1)^1 (x/2)^{n+2}}{1!\,(n+1)!} + \frac{(-1)^2 (x/2)^{n+4}}{2!\,(n+2)!} \Big] \text{ FIX}$$

$$= (-1)^n J_n(x)$$

$$= J_{-n}(x)$$

Combining the expressions from all cases:

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = t^0 J_0 + t^1 J_1 + t^2 J_2 + \ldots + t^n J_n + t^{-1} J_{-1} + t^{-2} J_{-2} + \ldots + t^{-n} J_{-n}$$

$$= \sum_{n=1}^{\infty} t^n J_n + \sum_{n=-\infty}^{-1} t^n J_n + \left[t^n J_n \right]_{n=0}$$

$$\boxed{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} t^n J_n(x)}$$