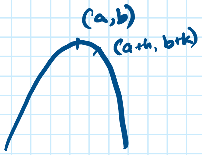
MAXIMA AND MINIMA

• Maxima

For some $F(x,y)$,



Considering value of $F(x,y)$ at (a,b) here,
(a,b) is a maxima of $F(x,y)$ when

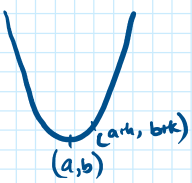
$$F(a+h,\ b+k) - F(a,b) < 0$$

$$\Delta F < 0$$

Irrespective of signs of h and k.

Here $F(a,b)$ gives the maximum value of the function.

• Minima



Again taking (a,b) here,
(a,b) is a minima of $F(x,y)$ when

$$F(a+h,\ b+k) - F(a,b) > 0$$

$$\Delta F > 0$$

Irrespective of signs of h and k.

Here, $F(a,b)$ gives the minimum value of the function.

• Saddle points

The surface ascends in all directions about the maximum point and descends in all directions about the minimum point.
However, there are some points on the surface where the surface ascends in one direction and descends in the other direction.
Such points are called saddle points.

CONDITIONS

Using Taylor's Series expansion of $F(x,y)$ about (a,b), where $x = a+h$ and $y = b+k$

$$F(a+h,\ b+k) = F(a,b) + \frac{1}{1!}\left[h F_x + k F_y\right] + \frac{1}{2!}\left[h^2 F_{xx} + 2hk F_{xy} + k^2 F_{yy}\right]$$

Neglecting higher powers of h and k, as h and k are very small:

$$F(a+h, b+k) - F(a,b) = hF_x + kF_y$$

$$\Delta F = hF_x + kF_y$$

Condition for (a,b) to be stationary point:

$$\left(\frac{\partial F}{\partial x}\right)_{(a,b)} = 0 \quad , \quad \left(\frac{\partial F}{\partial y}\right)_{(a,b)} = 0$$

Considering second order term,

$$\Delta F = \frac{1}{2}\left(h^2 F_{xx} + 2hk\, F_{xy} + k^2 F_{yy}\right)$$

Let $F_{xx} = r$, $F_{xy} = s$, $F_{yy} = t$

$$\Delta F = \frac{1}{2}\left(h^2 r + 2hks + k^2 t\right)$$

Multiplying and dividing by r

$$\Delta F = \frac{1}{2r}\left((hr)^2 + 2hrks + k^2 rt\right)$$

$$= \frac{1}{2r}\left(\underbrace{(hr)^2 + 2(hr)(ks) + (ks)^2} - (ks)^2 + k^2 rt\right)$$

$$\Delta F = \frac{1}{2r}\left((hr + ks)^2 + k^2(rt - s^2)\right)$$

Thus sign of Δ = sign of $\frac{1}{2r}\{(hr + ks)^2 + k^2(rt - s^2)\}$ ...(ii)

In (ii), $(hr + ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$. In this case, Δ will have the same sign as that of r for all values of h and k.

From the above, we find the following conclusions:

$rt - s^2 > 0$ and $r < 0$ $\longrightarrow$ (a,b) is a maximum point

$rt - s^2 > 0$ and $r > 0$ $\longrightarrow$ (a,b) is a minimum point

$rt - s^2 < 0$ $\longrightarrow$ (a,b) is a saddle point

$rt - s^2 = 0$ $\longrightarrow$ Insufficient information to draw any conclusion; further investigation required.

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

- For any number of variables that are not independent; variables are constrained by some equation called constraint equation.

Let $F(x,y,z)$ be a function whose extreme value is to be determined, subjected to a constraint $\phi(x,y,z) = c$.

a constraint $\phi(x,y,z) = c$.

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy + \frac{\partial F}{\partial z} dz$$

Condition for critical points: $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$.

$$\Rightarrow \frac{dF}{dx} = 0 + 0 + 0 = 0 \quad \text{—— } (1)$$

Illy,

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} \cdot dy + \frac{\partial \phi}{\partial z} \cdot dz = 0$$

Multiplying (2) by some constant λ

$$= \lambda \frac{\partial \phi}{\partial x} \cdot dx + \frac{\lambda \partial \phi}{\partial y} dy + \frac{\lambda \partial \phi}{\partial z} \cdot dz = 0 \quad \text{—— } (3)$$

(1) + (3)

$$\left(\frac{\partial F}{\partial x} + \frac{\lambda \partial \phi}{\partial x}\right) dx + \left(\frac{\partial F}{\partial y} + \frac{\lambda \partial \phi}{\partial y}\right) dy + \left(\frac{\partial F}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0$$

Comparing similar terms,

$$\boxed{\begin{aligned} \frac{\partial F}{\partial x} + \lambda \frac{\partial \phi}{\partial x} &= 0 \\ \frac{\partial F}{\partial y} + \lambda \frac{\partial \phi}{\partial y} &= 0 \\ \frac{\partial F}{\partial z} + \lambda \frac{\partial \phi}{\partial z} &= 0 \end{aligned}}$$

Lagrange's Equations: when solved, gives critical points

NOTE: You only get critical points using this; you cannot find the nature of the point