Taylor's and McLaurin's Series

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TAYLOR'S & McLAURIN'S SERIES

special case of Taylor's infinite series that progresses in powers of x

↳ infinite series about a point a, that progresses in powers of $(x-a)$

Taylor's Series for $y = f(x)$ about $x = a$

$$\boxed{f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots}$$

McLaurin's Series

Put $x = a = 0$.

$$\boxed{f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots}$$

EXAMPLE:

Find McLaurin's series expansion for $f(x) = e^x$

Soln:

$f(x) = e^x$ $\quad$ $f(0) = 1$

$f'(x) = e^x$ $\quad$ $f'(0) = 1$

$f''(x) = e^x$ $\quad$ $f''(0) = 1$

Thus we have

$$e^x = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \dots$$

Find McLaurins series expansion for $f(x) = \tan^{-1}(x)$

Soln:

$f(x) = \tan^{-1}(x)$ $\quad$ $f(0) = 0$

$f'(x) = \frac{1}{1+x^2}$ $\quad$ $f'(0) = 1$

$f''(x) = \frac{-2x}{(1+x^2)^2}$ $\quad$ $f''(0) = 0$

Taylor Series Expansion of $f(x,y)$ about the point (a,b)

- Expansion in terms of $(x-a), (y-b)$

$$F(x,y) = F(a,b) + \frac{1}{1!}\Big[(x-a)[F_x(a,b)] + (y-b)[F_y(a,b)]\Big]$$
$$+ \frac{1}{2!}\Big[(x-a)^2[F_{xx}(a,b)] + 2(x-a)(y-b)[F_{xy}(a,b)] + (y-b)^2[F_{yy}(a,b)]\Big]$$
$$+ \frac{1}{3!}\Big[(x-a)^3[F_{xxx}(a,b)] + 3(x-a)^2(y-b)[F_{xxy}(a,b)] + 3(x-a)(y-b)^2[F_{xyy}(a,b)] + (y-b)^3[F_{yyy}(a,b)]\Big]$$
$$+ \dots$$

McLaurin's Series of $F(x,y)$

Put $(a,b) = 0$

$$F(0,0) = F(0,0) + \frac{1}{1!}\Big[x[F_x(0,0)] + y[F_y(0,0)]\Big]$$
$$+ \frac{1}{2!}\Big[x^2[F_{xx}(0,0)] + 2xy[F_{xy}(0,0)] + y^2[F_{yy}(0,0)]\Big]$$
$$+ \frac{1}{3!}\Big[x^3[F_{xxx}(0,0)] + 3x^2y[F_{xxy}(0,0)] + 3xy^2[F_{xyy}(0,0)] + y^3[F_{yyy}(0,0)]\Big]$$
$$+ \dots$$

Problems

① Expand:

$F(x,y) = \sin x \sin y$ in Taylors series about $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ upto second term

Soln:

$F_x =$	$\sin y \cos x$	$\frac{1}{2}$
$F_y =$	$\sin x \cos y$	$\frac{1}{2}$
$F_{xx} =$	$-\sin y \sin x$	$-\frac{1}{2}$
$F_{yy} =$	$-\sin x \sin y$	$-\frac{1}{2}$
$F_{xy} =$	$\cos y \cos x$	$\frac{1}{2}$
$F(x,y) =$	$\sin x \sin y$	$\frac{1}{2}$

$$\sin x \sin y = \frac{1}{2} + \frac{1}{1!}\left[\left(x-\frac{\pi}{4}\right)\left(\frac{1}{2}\right) + \left(y-\frac{\pi}{4}\right)\left(\frac{1}{2}\right)\right] + \frac{1}{2}\left[\left(x-\frac{\pi}{4}\right)^2\left(\frac{-1}{2}\right) + \left(y-\frac{\pi}{4}\right)^2\left(\frac{-1}{2}\right)\right.$$
$$\left. + 2\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)\left(\frac{1}{2}\right)\right]$$

② Expand $F(x,y) = \sin(x+2y)$ in Taylor's series upto 3^{rd} degree term, about the point $(0,0)$

Soln:

Function	At the point
$F(x,y) = \sin(x+2y)$	0
$F_x = \cos(x+2y)(1)$	1
$F_y = \cos(x+2y)(2)$	2
$F_{xx} = -\sin(x+2y)(1)$	0
$F_{xy} = -\sin(x+2y)(2)$	0
$F_{yy} = -2(\sin(x+2y))(2)$	0
$F_{xxx} = -(\cos(x+2y))$	-1
$F_{xxy} = -[\cos(x+2y)(2)]$	-2
$F_{xyy} = -4[\cos(x+2y)]$	-4
$F_{yyy} = -4(\cos(x+2y))(2)$	-8

$$\sin(x+2y) = 0 + \frac{1}{1!}\left[x + 2y\right] + \frac{1}{3!}\left[-x^3 - 2x^2y - 4xy^2 - 8y^3\right]$$

③ Expand y^x about point $(1,1)$, hence find value of $(1.1)^{1.1}$

④ Expand $e^x \cdot \log(1+y)$ about $(0,0)$