Potentials and challenges in the application of peridynamics for the determination of virtual allowables

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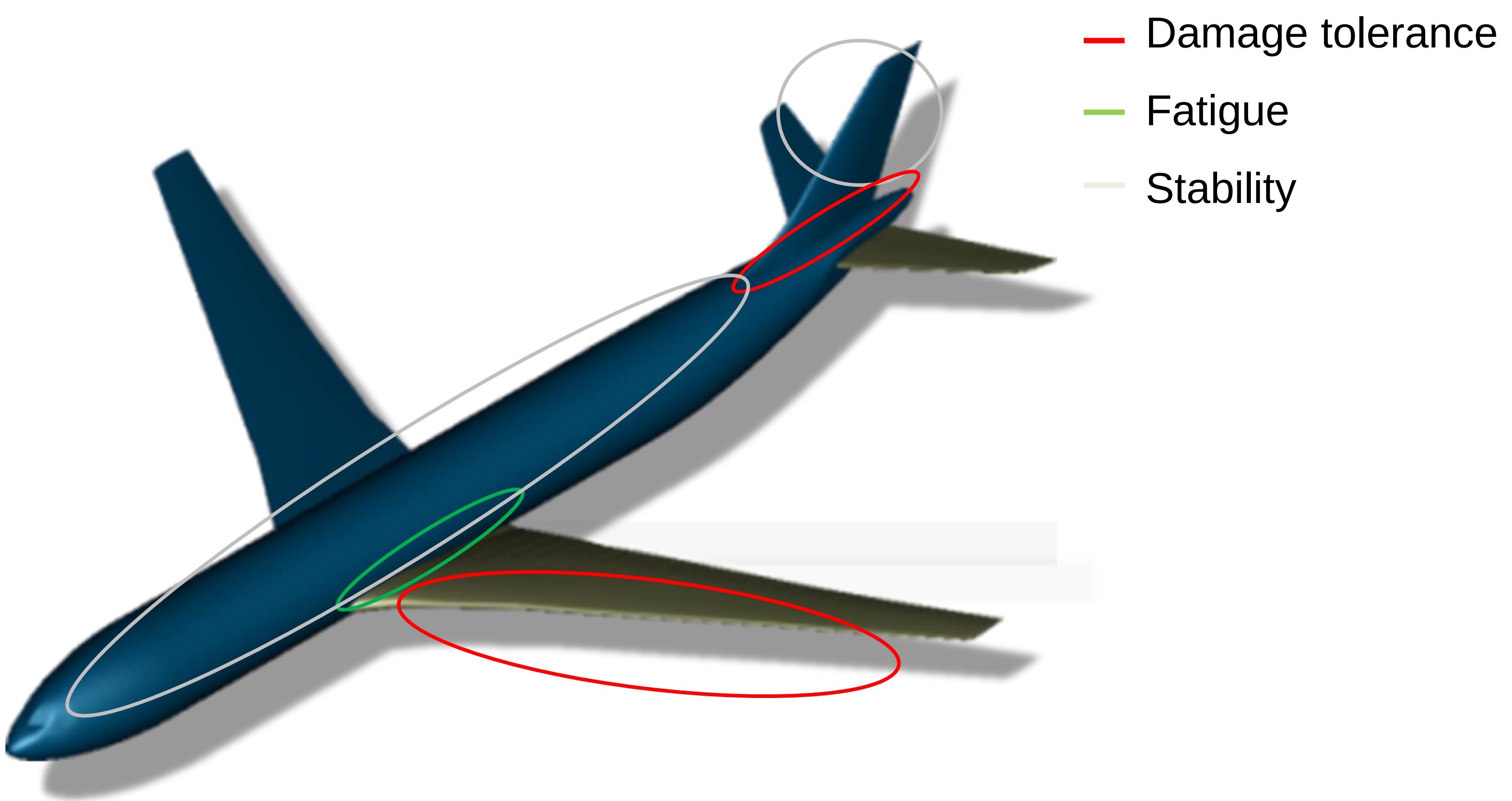
Motivation – State-of-the-art design process in aeronautics

- **Design criteria**
    - **Fatigue**
    - Stability
    - ***Damage tolerance***
    - Plain and bearing strength
    - ...

- **Example: Impact Analysis**
    - Development of an experimental database of the most frequently used laminates
    - Not tested laminates are charged with safety factors
    - **Model-based characteristic values are only used to a limited extent, since trust in the simulations is lacking.**
- fail safe (alternative load paths) vs. **safe life (no damage)**

Motivation - Main Design Driver



Motivation - Summary

- Micromechanical or damage models are not directly used in the design process
- These models can be used to verify simplified criteria
- Robustness of damaged structures can be evaluated
- Reduction of cost-intensive experiments

- **A better understanding of damage initiation can be used to improve criteria and avoid expensive experiments.**

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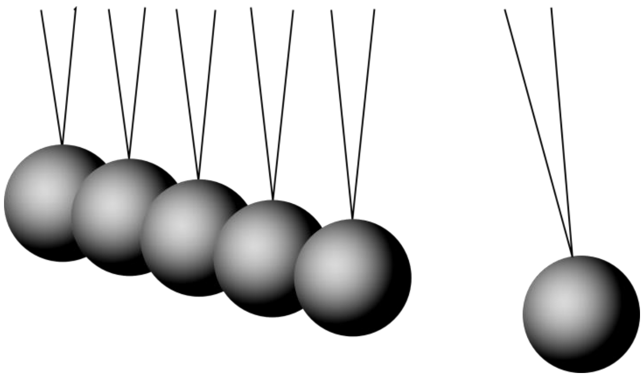
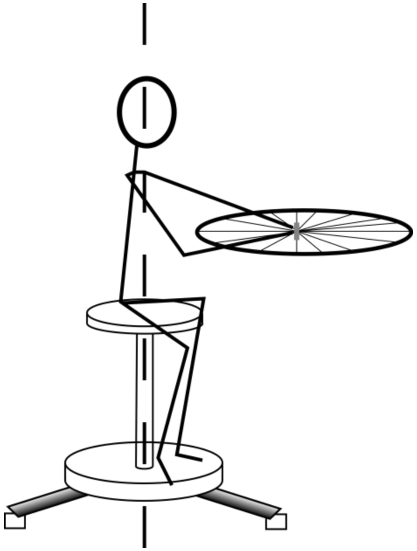
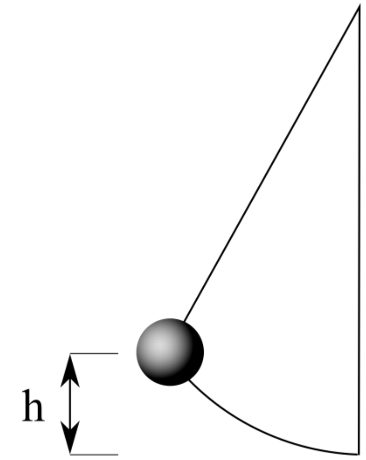
Physically motivated material modeling







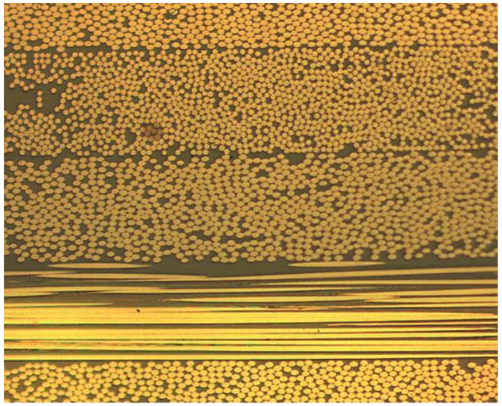
Conservation of momentum, angular momentum and energy

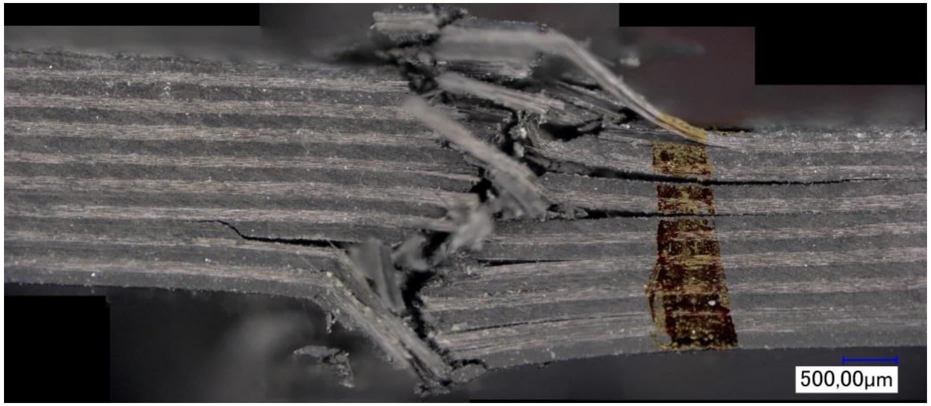
- If the conservation equations are fulfilled + if the material behaviour is described, it is a physically motivated modelling

Assumptions in classical continuum mechanics[1]

1. The medium is continuous
2. Internal forces are contact forces (interaction only with the neighbourhood)
3. Deformations are twofold continuously derivable (in the weak formulation only simple)
4. The conservation equations are fulfilled

Points 1 and 3 are not fulfilled for heterogeneous materials and in case of damage





[1]Bobaru, F.; Foster, J. T.; Geubelle, P. H. & Silling, S. A. „Handbook of Peridynamic Modeling" *CRC Press,* **2016**

Conservation equation

- Momentum conservation in continuum mechanics

$$\mathrm{div}(\boldsymbol{\sigma}) + \mathbf{b} = \rho\ddot{\mathbf{u}}$$

$$\mathrm{div}(\boldsymbol{\sigma}) = \begin{pmatrix} \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z} \\ \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z} \\ \frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} \end{pmatrix}$$

$$\boldsymbol{\sigma} = \mathbf{C} \cdot\cdot\, \mathbf{E} \qquad \mathbf{E} = 0.5(Grad\,\mathbf{u} + Grad^T\mathbf{u})$$

If points 1 and 3 are violated, the conservation of impulses is no longer strictly fulfilled!

Bertram, A. & Glüge, R. „Festkörpermechanik“ *Otto-von-Guericke-Universität Magdeburg, ISBN 978-3-940961-88-4,* **2013**

Implications

- The continuum mechanics is not able to model damages
- FEM is a method for the numerical solution of a differential equation
- If the model based on a differential equation loses its validity, the FEM is also not able to model damage.

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{C}_{Voigt} \mathbf{B} dv$$

$$\mathbf{B} = \boldsymbol{D}\mathbf{N}$$

$\boldsymbol{D}$ - Differential operator, which requires continuously differentiable ansatz functions **N**

Derivation only required in the element, i.e. discontinuities can be represented at the element boundaries. Continuum mechanics is not strictly fulfilled.

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Damages

- Strictly speaking, they cannot be represented in FEM.
    - Damages are implemented via additional elements based on fracture mechanics.
- Problems with consistency!
- Additional assumptions to enable the jump between the theories (continuum mechanics and fracture mechanics).

Peridynamics

1. ~~The medium is continuous~~
2. ~~Internal forces are contact forces (interaction only with the neighbourhood)~~
3. ~~Deformations are twofold continuously derivable (in the weak formulation only simple)~~
4. **The conservation equations are fulfilled**

$$\operatorname{div}(\boldsymbol{\sigma}) + \mathbf{b} = \rho\ddot{\mathbf{u}}$$

$$\int_H \left(\underline{\mathbf{T}}(\mathbf{x}, t)\langle\mathbf{q} - \mathbf{x}\rangle - \underline{\mathbf{T}}(\mathbf{q}, \mathrm{t})\langle\mathbf{x} - \mathbf{q}\rangle\right) dV + \mathbf{b} = \rho\ddot{\mathbf{u}}$$

$$\lim_{H\to 0} \int_H \left(\underline{\mathbf{T}}\,(\mathbf{x}, t)\langle\mathbf{q} - \mathbf{x}\rangle - \underline{\mathbf{T}}(\mathbf{q}, \mathrm{t})\langle\mathbf{x} - \mathbf{q}\rangle\right) dV = \operatorname{div}(\boldsymbol{\sigma})$$

Peridynamic formulations

- bond based (Poisson’s ratio of 0.25 for 3D & 2D plane strain)
  - too easy
  - a non-local spring formulation
  - should be avoided
- Extension to States
  - States are no longer feathers and should not be interpreted as such!
  - ordinary state based
    - Balance of forces is fulfilled at integral, however not at every bond potential.
  - non-ordinary state based
    - Force and moment equilibrium is fulfilled in the integral, but not in every bond potential.

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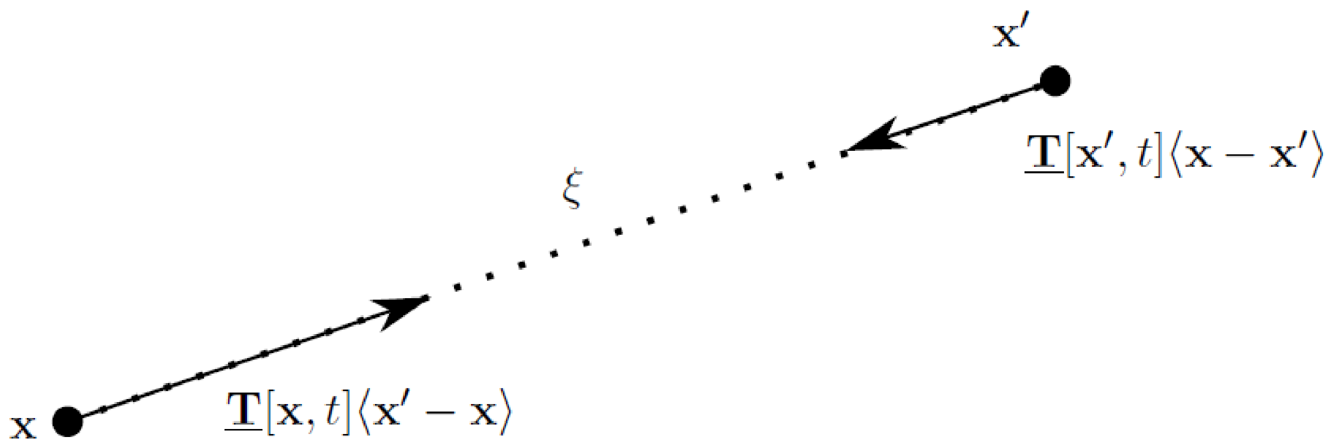
Peridynamics – ordinary state based formulation



$$\rho\left(\mathbf{x}\right)\ddot{\mathbf{u}}\left(\mathbf{x},t\right) = \int_{\mathcal{H}}\left(\underline{\mathbf{T}}\left[\mathbf{x},t\right]\left\langle\mathbf{x}'-\mathbf{x}\right\rangle - \underline{\mathbf{T}}\left[\mathbf{x}',t\right]\left\langle\mathbf{x}-\mathbf{x}'\right\rangle\right)\mathrm{d}V + \mathbf{b}\left(\mathbf{x},t\right)$$

Peridynamics – ordinary state based formulation

$$W_{CM} = \frac{1}{2} K \left[\epsilon_{kk}\right]^2 \delta_{ij} + 2G \left[\epsilon_{ij}^d\right]^2 \overset{!}{=} W_{PD}$$

$$\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle = \mathbf{F}\boldsymbol{\xi} = \mathbf{F}\langle\mathbf{x}' - \mathbf{x}\rangle \quad \forall\boldsymbol{\xi} \in \mathcal{H}$$

- For small deformations and isotropic material

$$\underline{x} = |\underline{\mathbf{X}}\langle\boldsymbol{\xi}\rangle| \qquad \underline{y} = |\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| \qquad \underline{e}\langle\boldsymbol{\xi}\rangle = \underline{y} - \underline{x}$$

$$\underline{e}\langle\boldsymbol{\xi}\rangle = |\mathbf{F}\boldsymbol{\xi}| - |\boldsymbol{\xi}| = \epsilon_{ij}\xi_i \frac{\xi_j}{|\boldsymbol{\xi}|}$$

$$\underline{e}^d\langle\boldsymbol{\xi}\rangle = \epsilon_{ij}^d \xi_i \frac{\xi_j}{|\boldsymbol{\xi}|} \qquad \underline{e}^i\langle\boldsymbol{\xi}\rangle = \epsilon_{ii}\xi_i \frac{\xi_i}{|\boldsymbol{\xi}|}$$

$$W_{PD} = \frac{A}{2}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left[\epsilon_{ij}^d \xi_i \frac{\xi_j}{|\boldsymbol{\xi}|}\right]^2 \mathrm{d}V_{\boldsymbol{\xi}} + \frac{B}{2}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \left[\epsilon_{ii}\xi_i \frac{\xi_i}{|\boldsymbol{\xi}|}\right]^2 \mathrm{d}V_{\boldsymbol{\xi}}$$

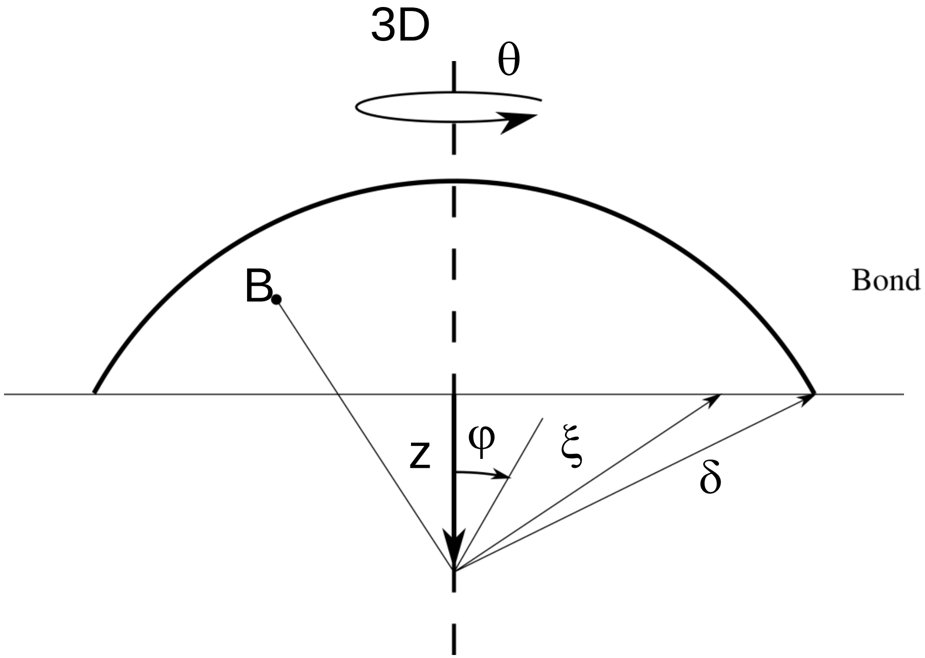
Peridynamics – ordinary state based formulation

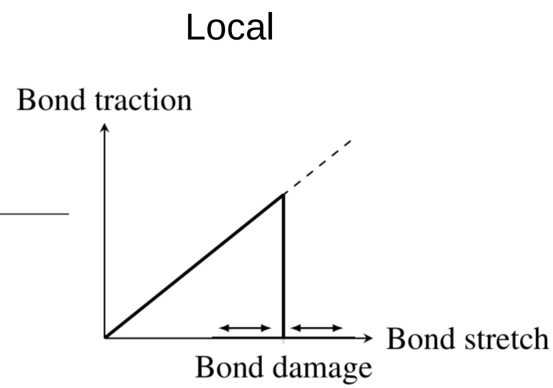
$$A = \frac{3K}{m_V} \quad \text{and} \quad B = \frac{15G}{m_V}$$

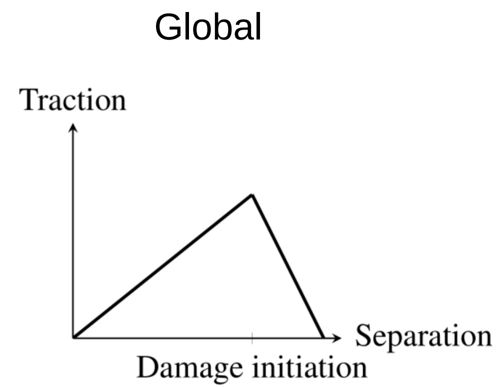
$$m_V = \int_{\mathcal{H}(\mathbf{x})} \underline{\omega}\langle\boldsymbol{\xi}\rangle \underline{x}\,\underline{x}\,\mathrm{d}V_{\boldsymbol{\xi}} \qquad \theta = \frac{3}{m_V}\int_{\mathcal{H}(\mathbf{x})} \underline{\omega}\langle\boldsymbol{\xi}\rangle \underline{x}\,\underline{e}\langle\boldsymbol{\xi}\rangle\,\mathrm{d}V_{\boldsymbol{\xi}}$$

$$\underline{t}\langle\boldsymbol{\xi}, t\rangle = \frac{\underline{\omega}\langle\boldsymbol{\xi}\rangle}{m_V}\left[3K\theta\underline{x} + 15G\underline{e}^d\right]$$

$$\underline{\mathbf{T}} = \underline{t}\frac{\underline{\mathbf{Y}}}{|\underline{\mathbf{Y}}|}$$



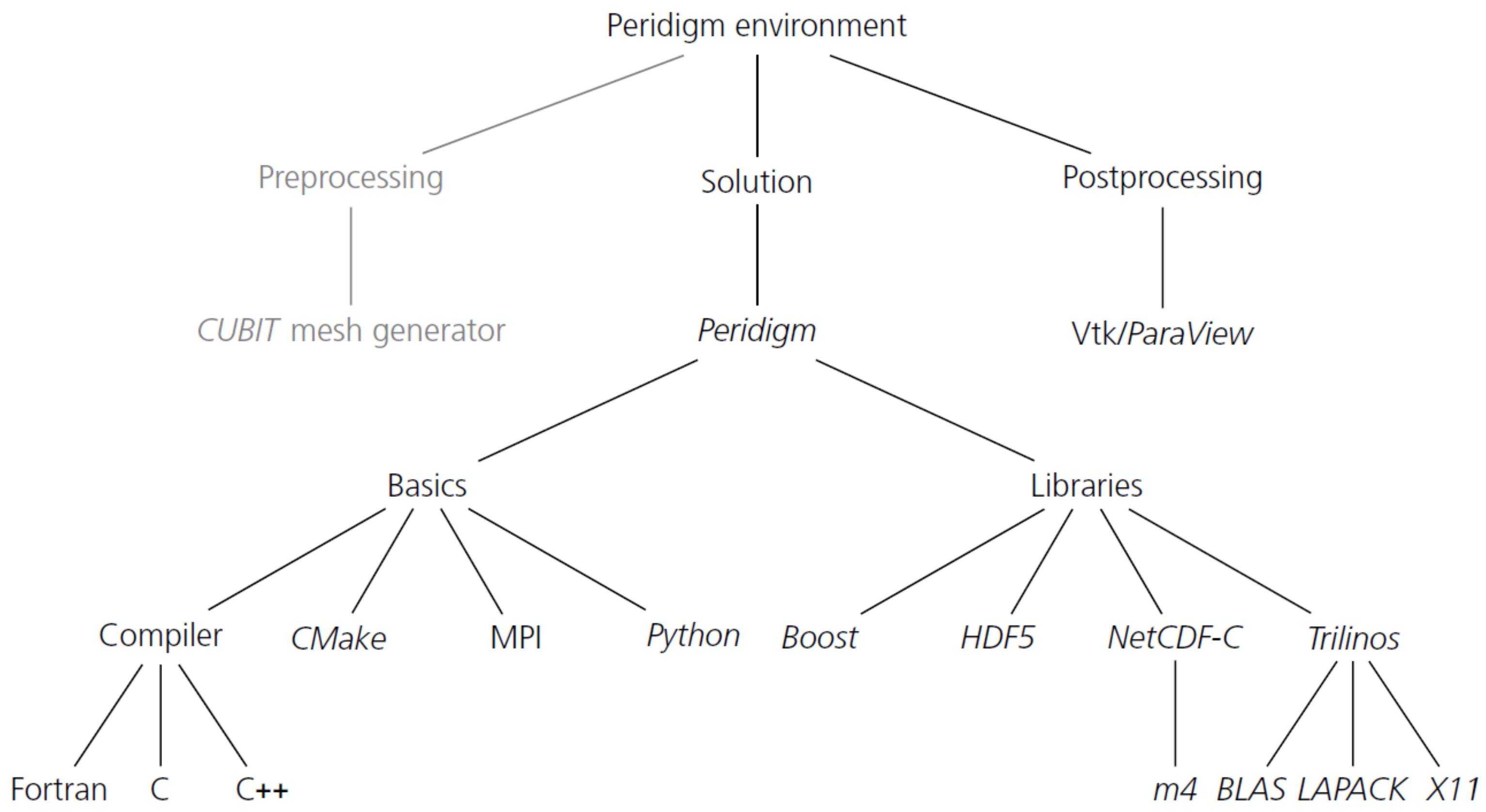




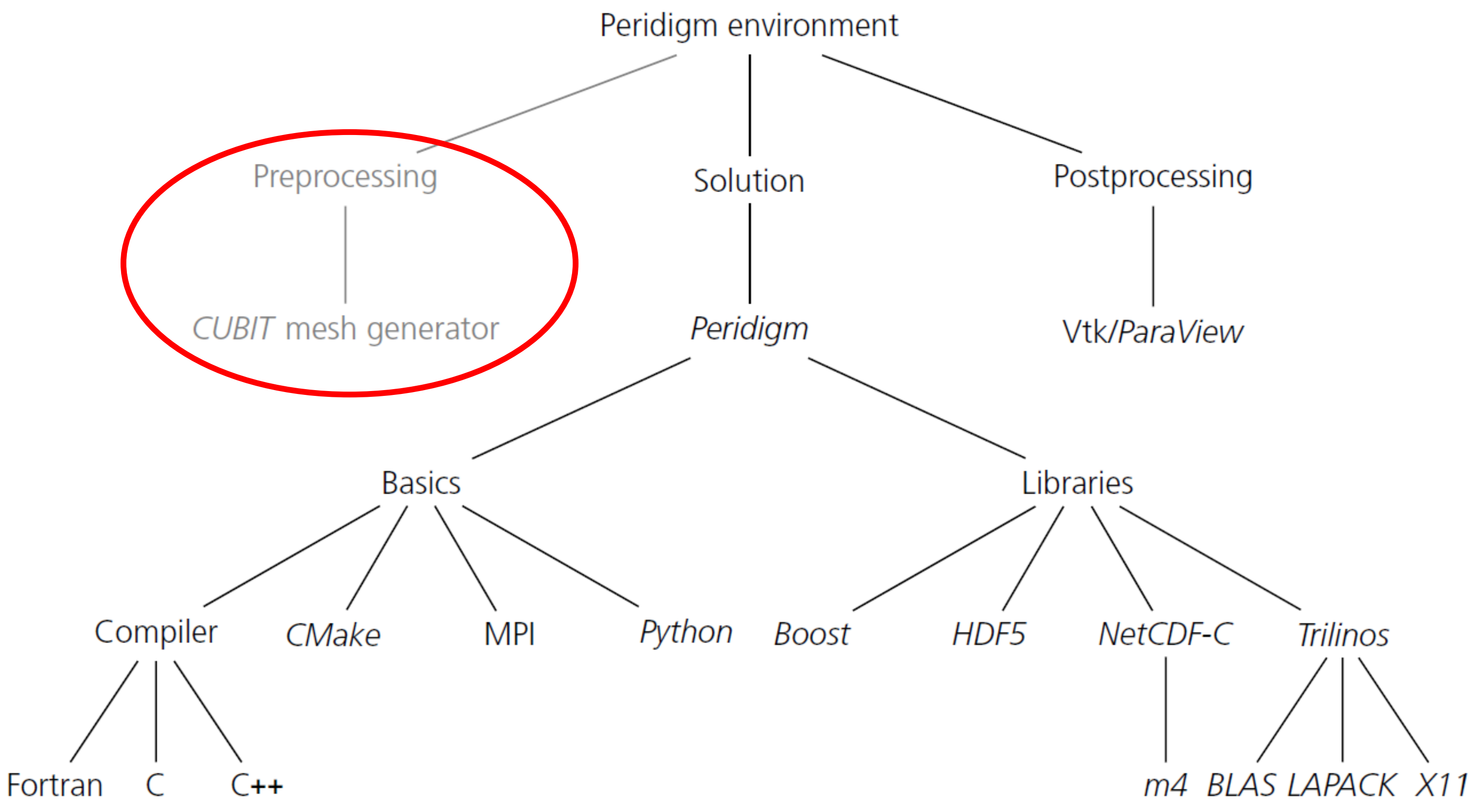
$$G = 2\int_{0}^{\delta}\int_{H_r} w_c dV_\xi \qquad G = \int_{0}^{\delta}\int_{0}^{2\pi}\int_{z}^{\delta}\int_{0}^{\cos^{-1} z/\xi} w_c \xi^2 \sin\varphi \, d\varphi d\xi d\theta dz$$

$$w_c = \frac{4G}{\pi\delta^4}$$

Framework


Peridigm environment
Preprocessing
Solution
Postprocessing
CUBIT mesh generator
Peridigm
Vtk/ParaView
Basics
Libraries
Compiler
CMake
MPI
Python
Boost
HDF5
NetCDF-C
Trilinos
Fortran
C
C++
m4
BLAS
LAPACK
X11

Framework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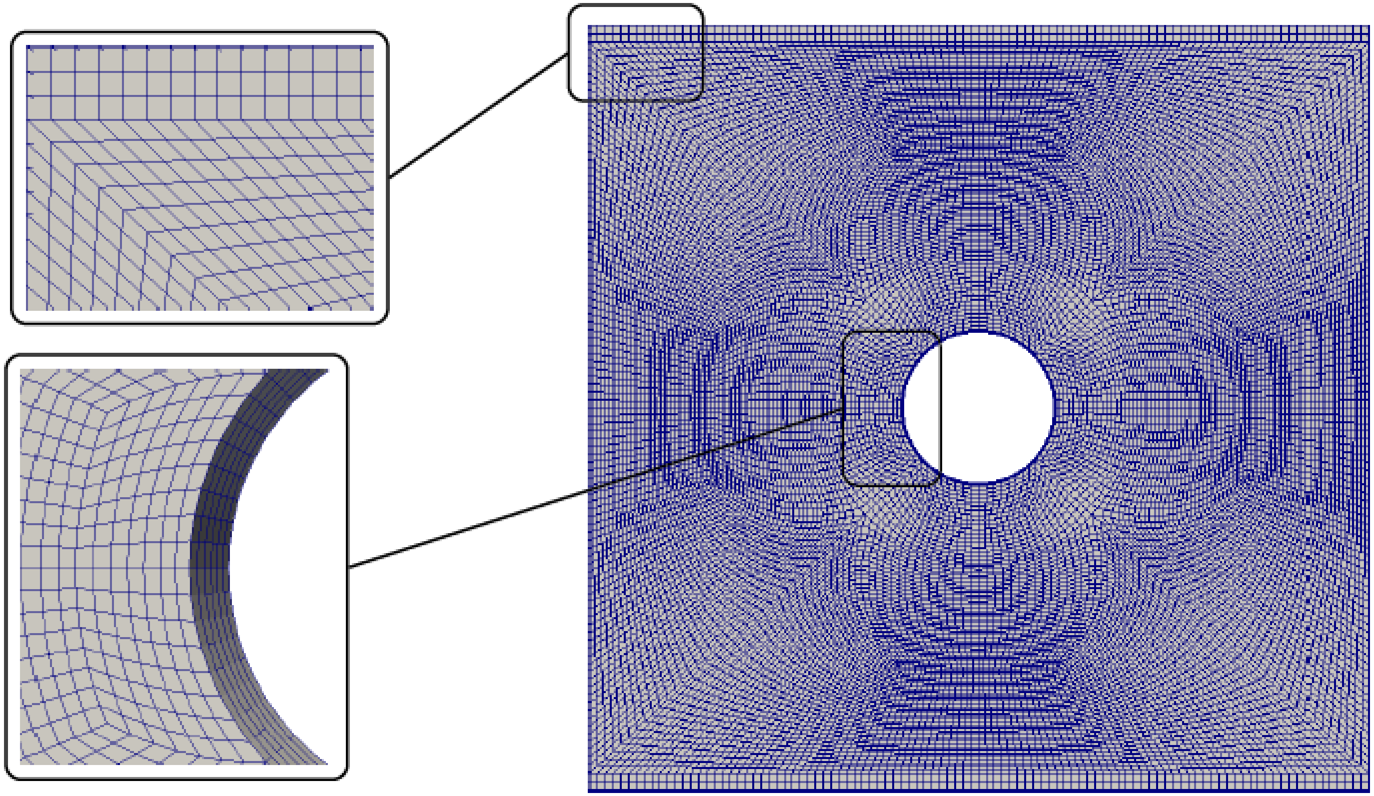
Preprozessor



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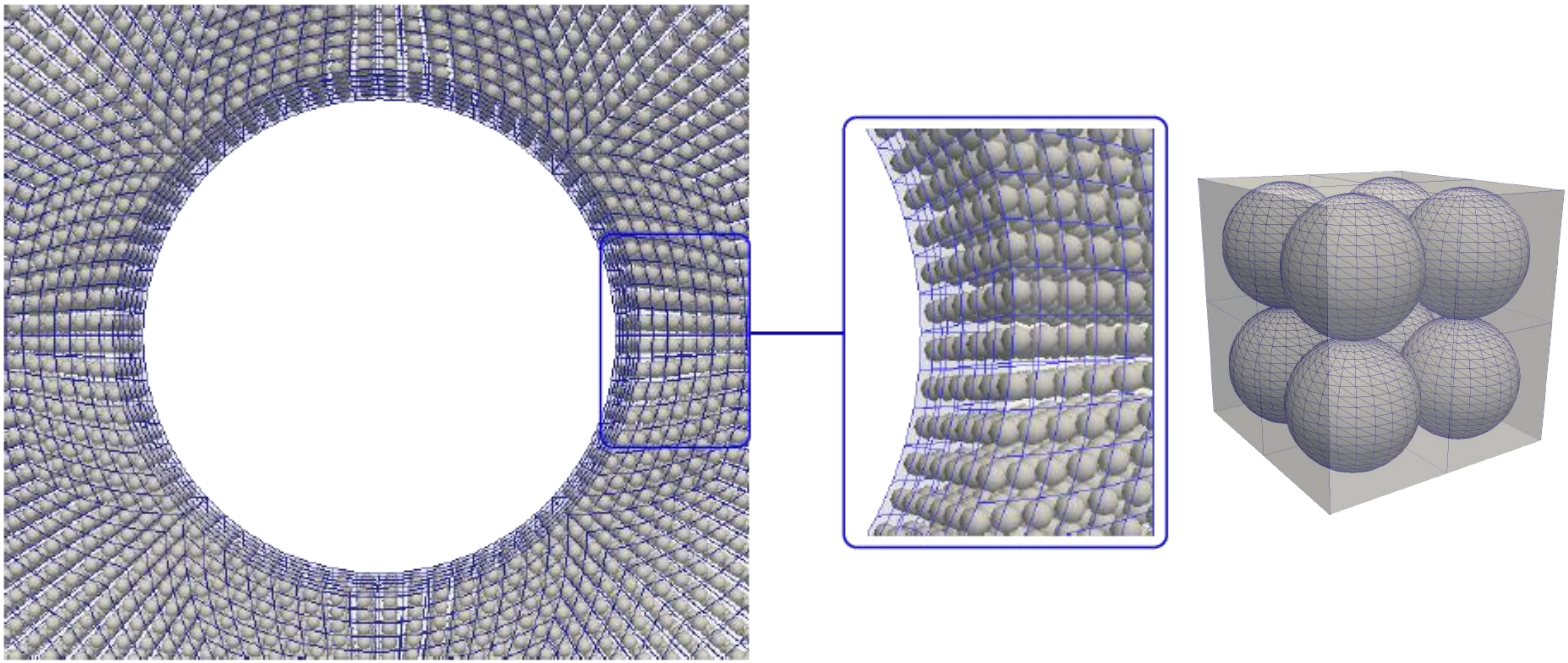

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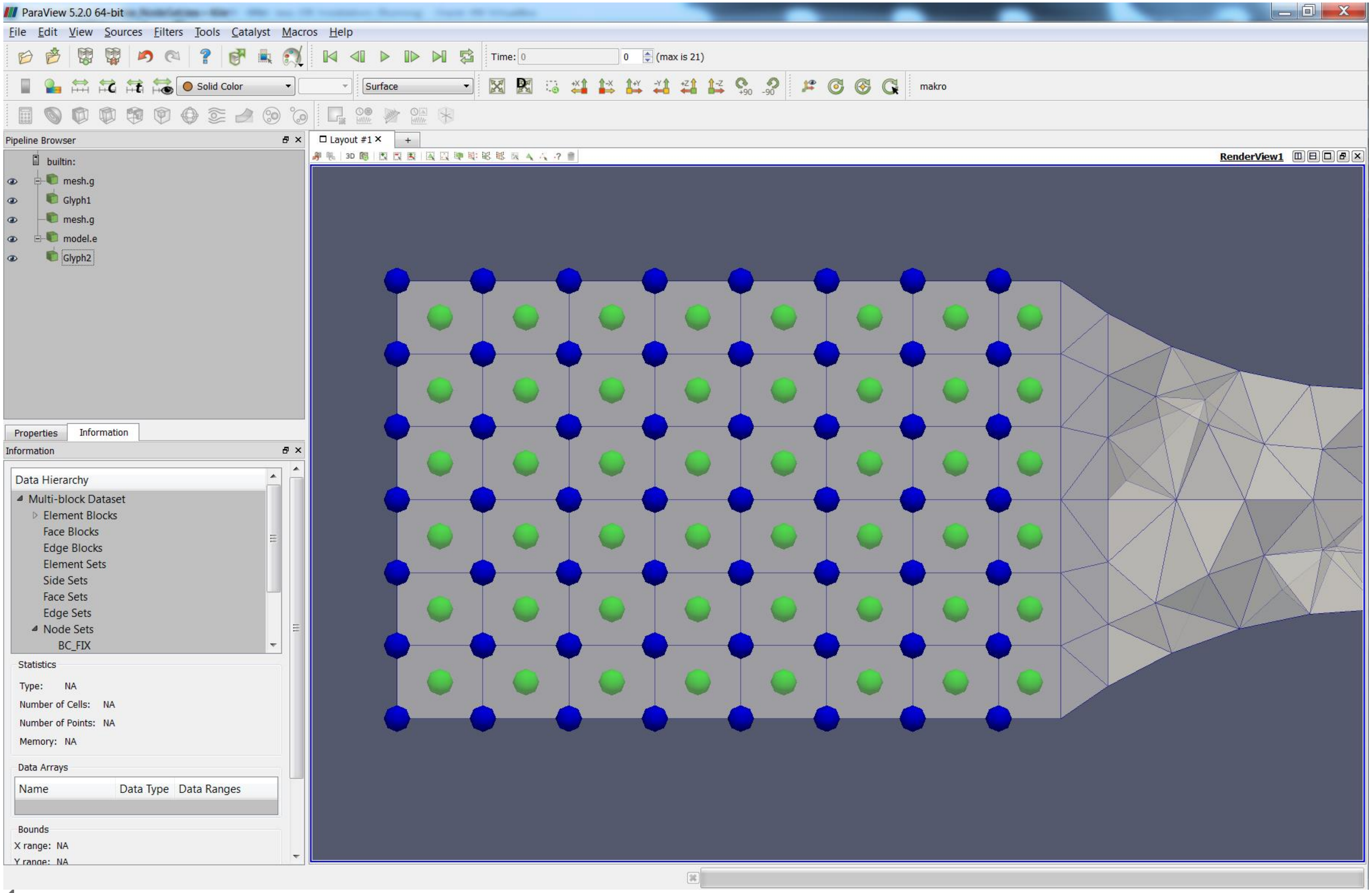

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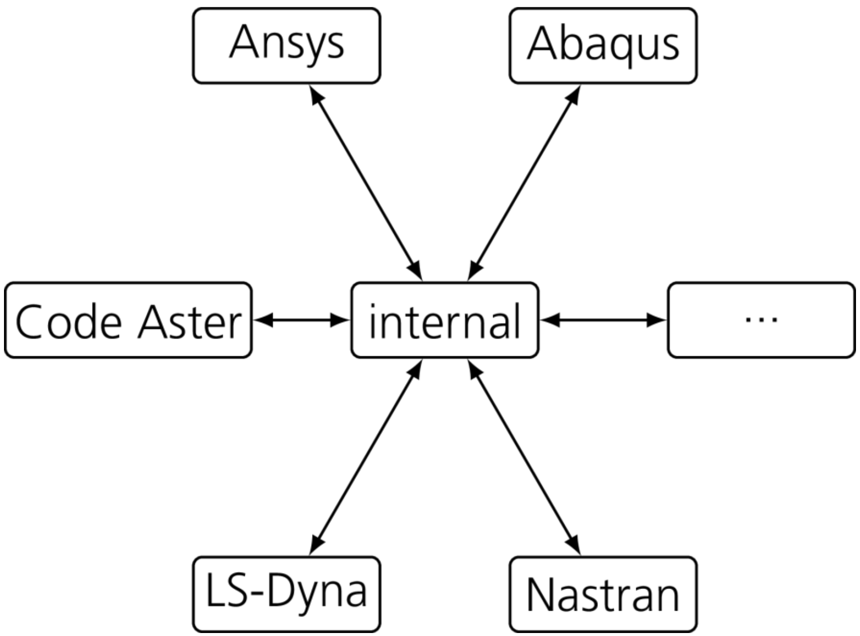
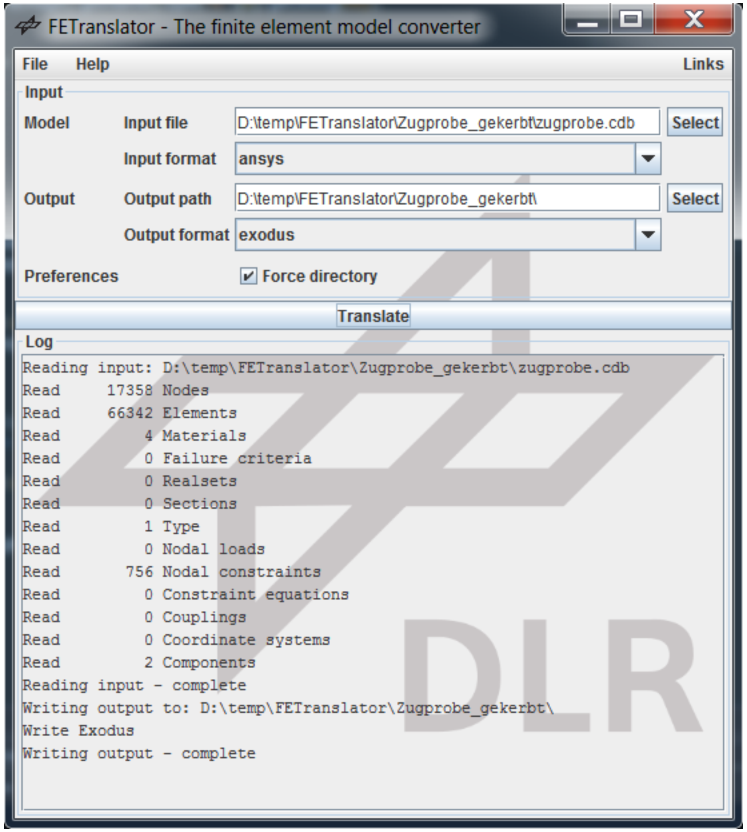
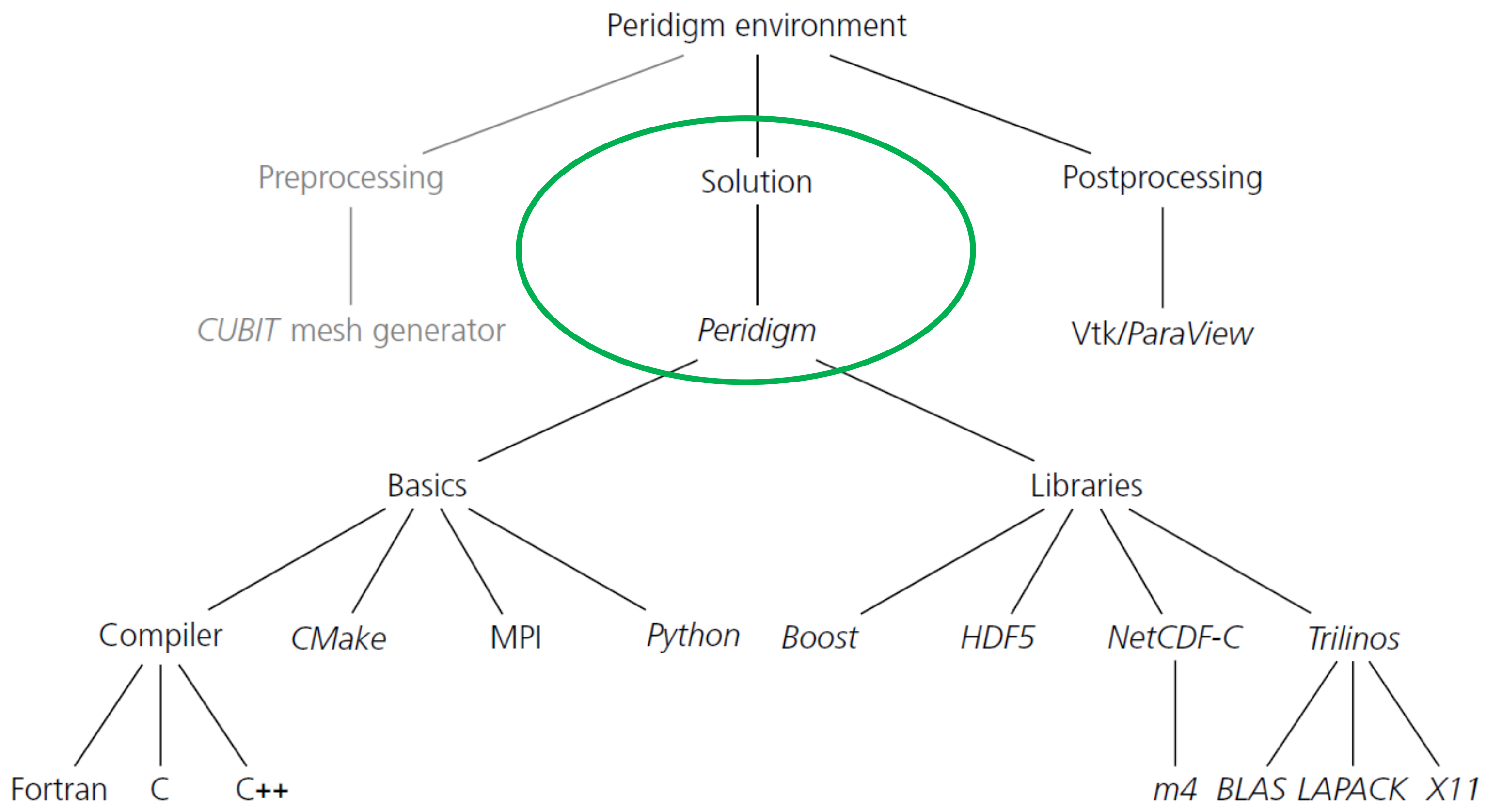




Framework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Damage models

$$\chi\left(\boldsymbol{\xi}, t\right) = \begin{cases} 1 & \text{no failure} \\ 0 & \text{failure} \end{cases}$$

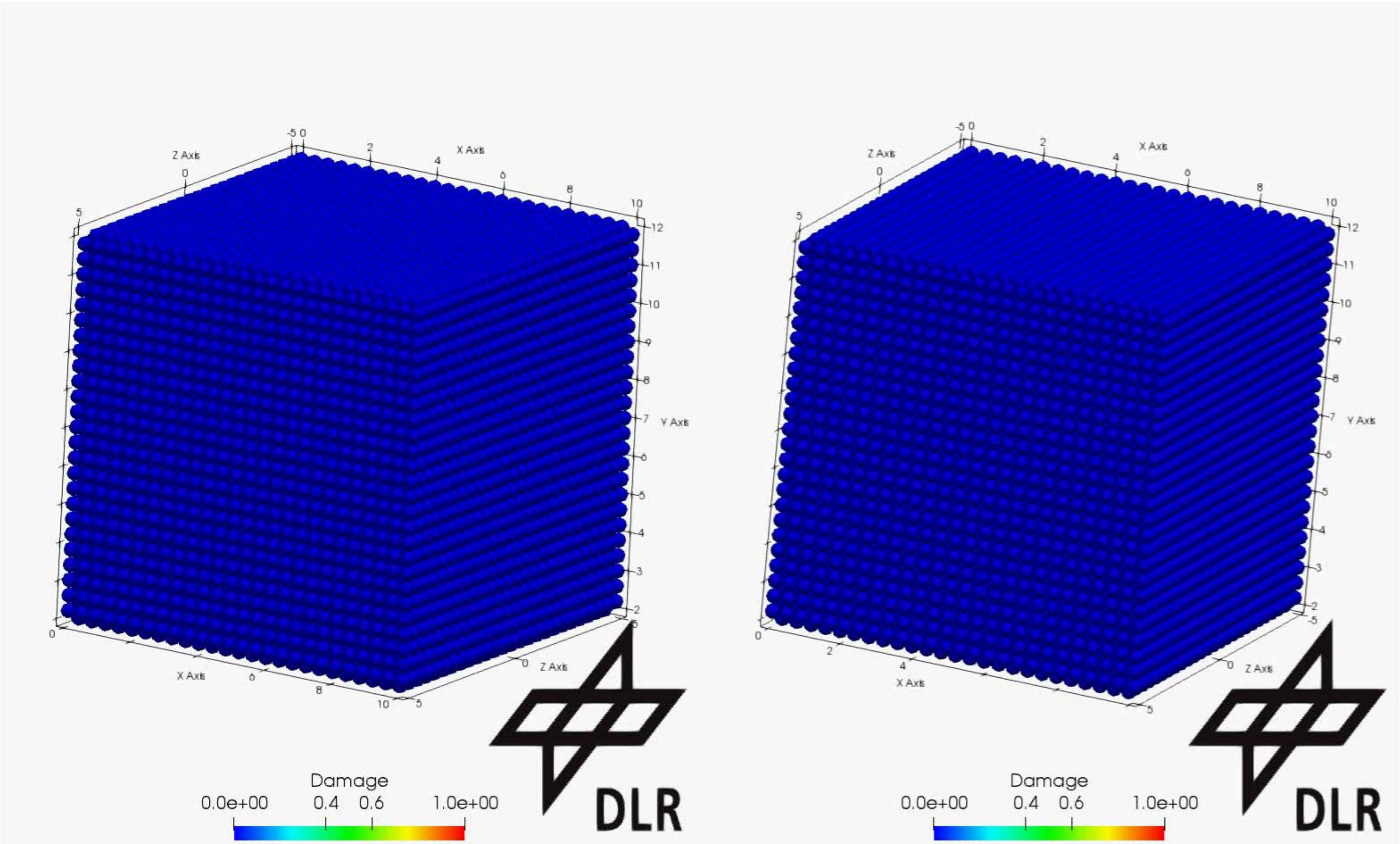
- Critical energy model by Foster et al.

$$W_C = \frac{4G_{0C}}{\pi\delta^4}$$

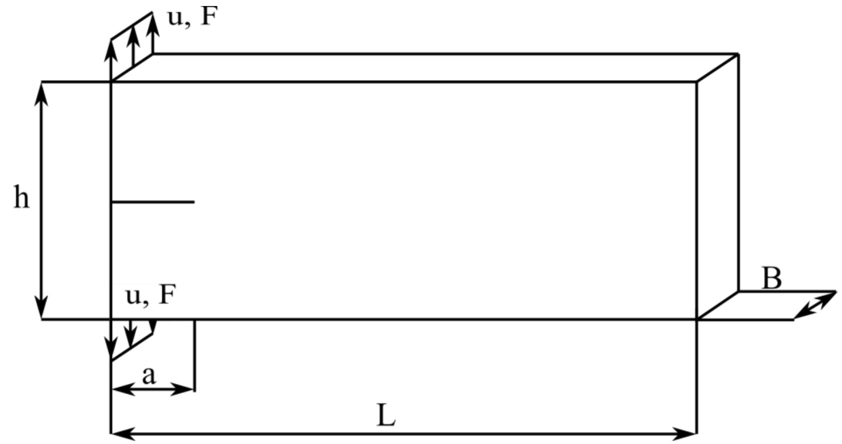
$$W_{\text{bond}} = 0.25\chi(\underline{e}\langle\boldsymbol{\xi}\rangle, t)\left\{\underline{t}\left[\mathbf{x}, t\right] - \underline{t}\left[\mathbf{x}', t\right]\right\}\underline{e} > W_C$$

- Critical stretch model

$$s_C = \sqrt{\frac{G_{0C}}{\left[3G + \left(\frac{3}{4}\right)^4\left(K - \frac{5G}{3}\right)\right]\delta}}$$


Z Axis
X Axis
Y Axis
Damage
0.0e+00
0.4
0.6
1.0e+00
DLR
Z Axis
X Axis
Y Axis
Damage
0.0e+00
0.4
0.6
1.0e+00
DLR

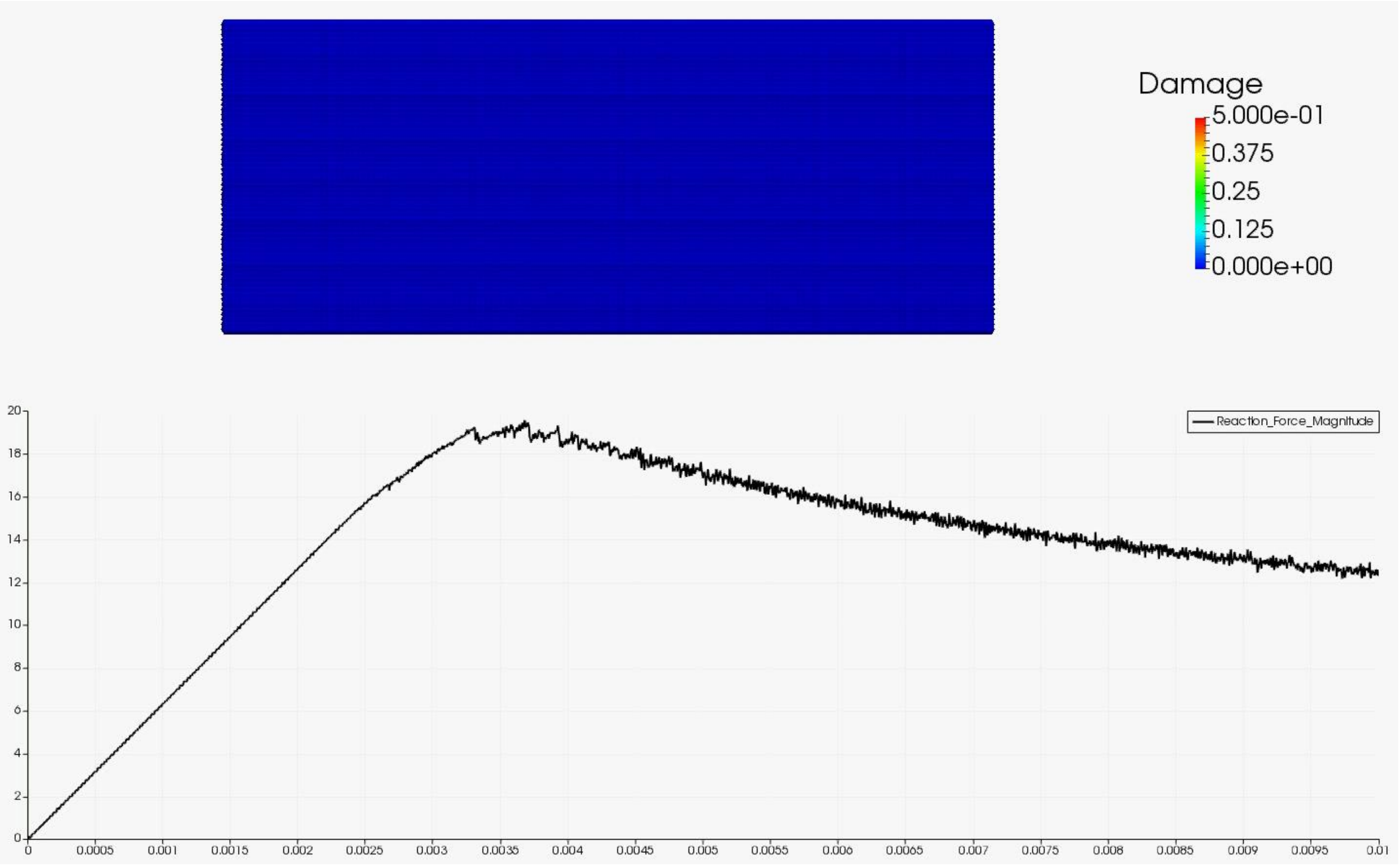
Verification



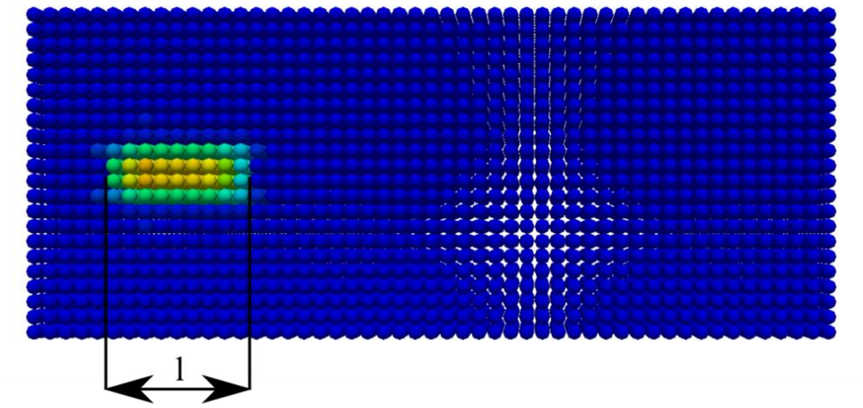
Geometry	a	h	L	B
	0.005m	0.02m	0.05m	0.003m
Material	**Bulk Modulus**	**Shear Modulus**	**Density**	**G_0**
	1.75E+09 Nm^{-2}	8.08E+08 Nm^{-2}	2000 kgm^{-3}	12 Nm^{-1}

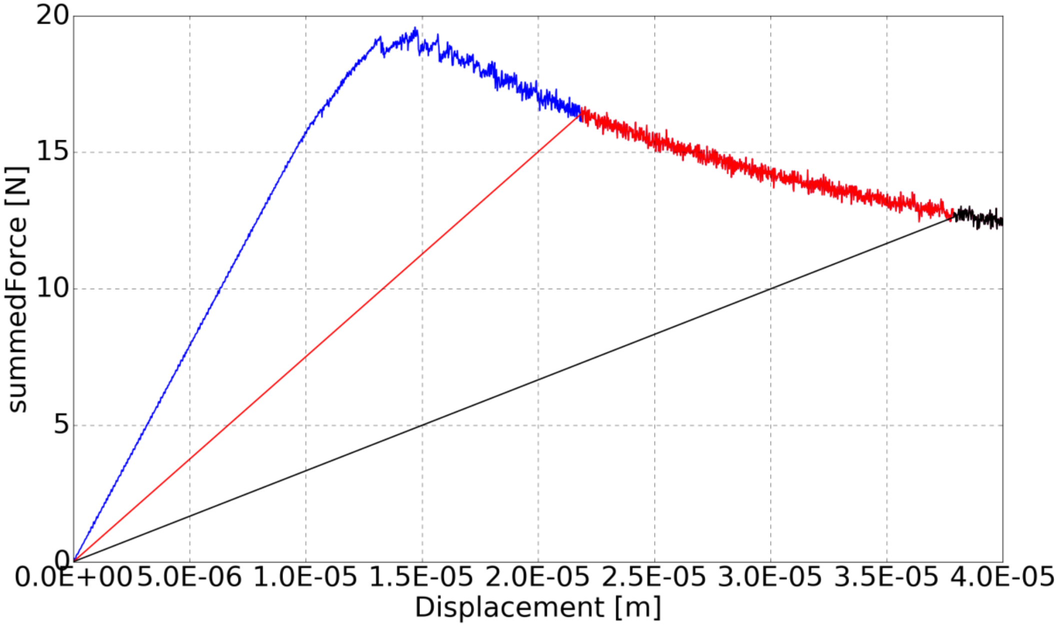
Mesh	2.01dx	3.01dx	4.01dx	5.01dx
0.0005	0.001005	0.001505	0.002005	0.002505
0.00033	0.000663	0.000993	0.001323	0.001653
0.00025	0.000503	0.000753	0.001003	0.001253
0.000125	0.000251		0.000501	

Verification: Double Cantilever Beam (DCB)



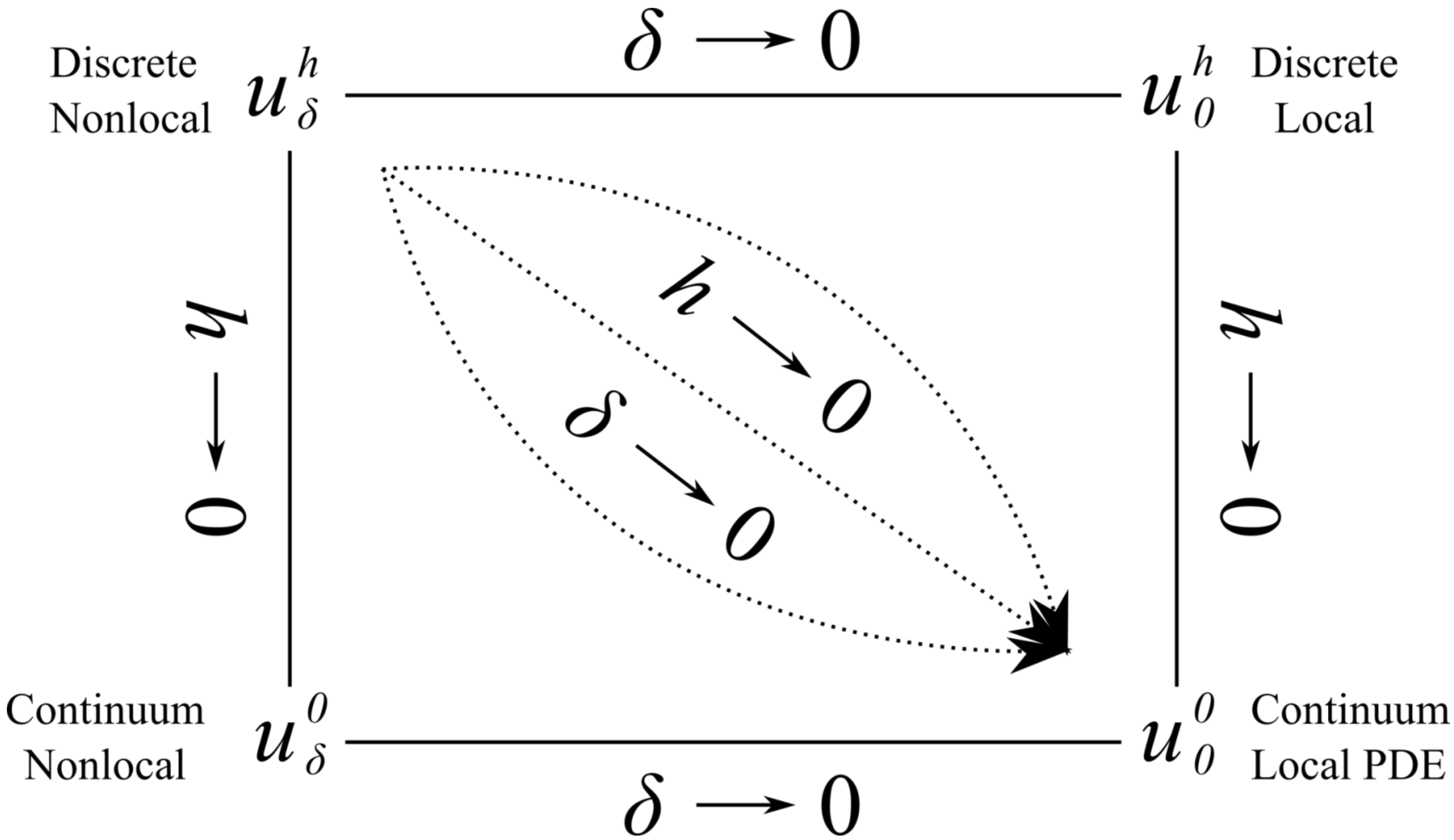
Verification





δ [m]	Line 1 G_0 [N/m]	Line 2 G_0 [N/m]
$2.015 \cdot 10^{-3}$	12.8	11.4
$3.015 \cdot 10^{-3}$	13.1	12.9
$4.015 \cdot 10^{-3}$	11.1	11.3
$5.015 \cdot 10^{-3}$	11.2	11.9

Verification: Convergence



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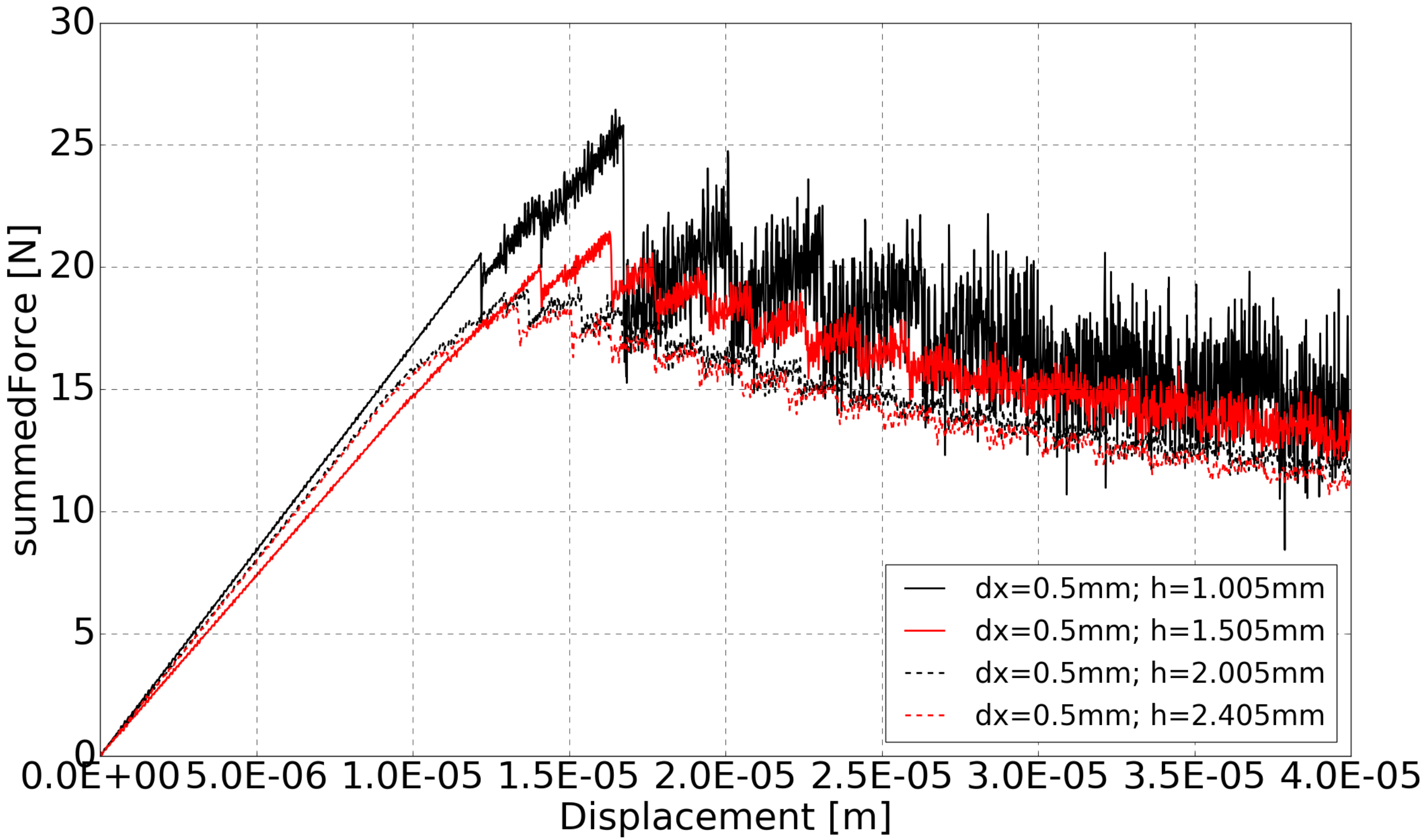

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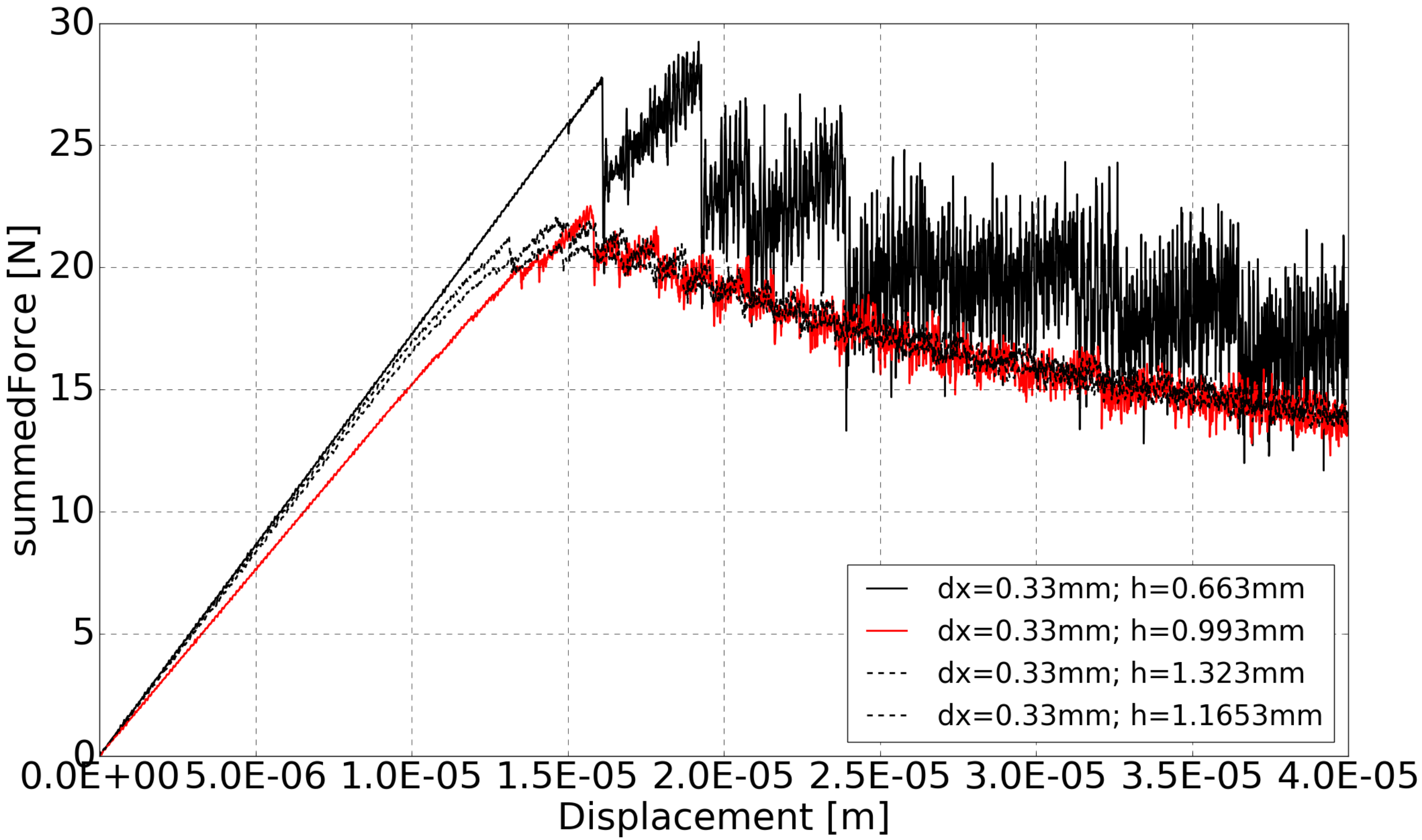

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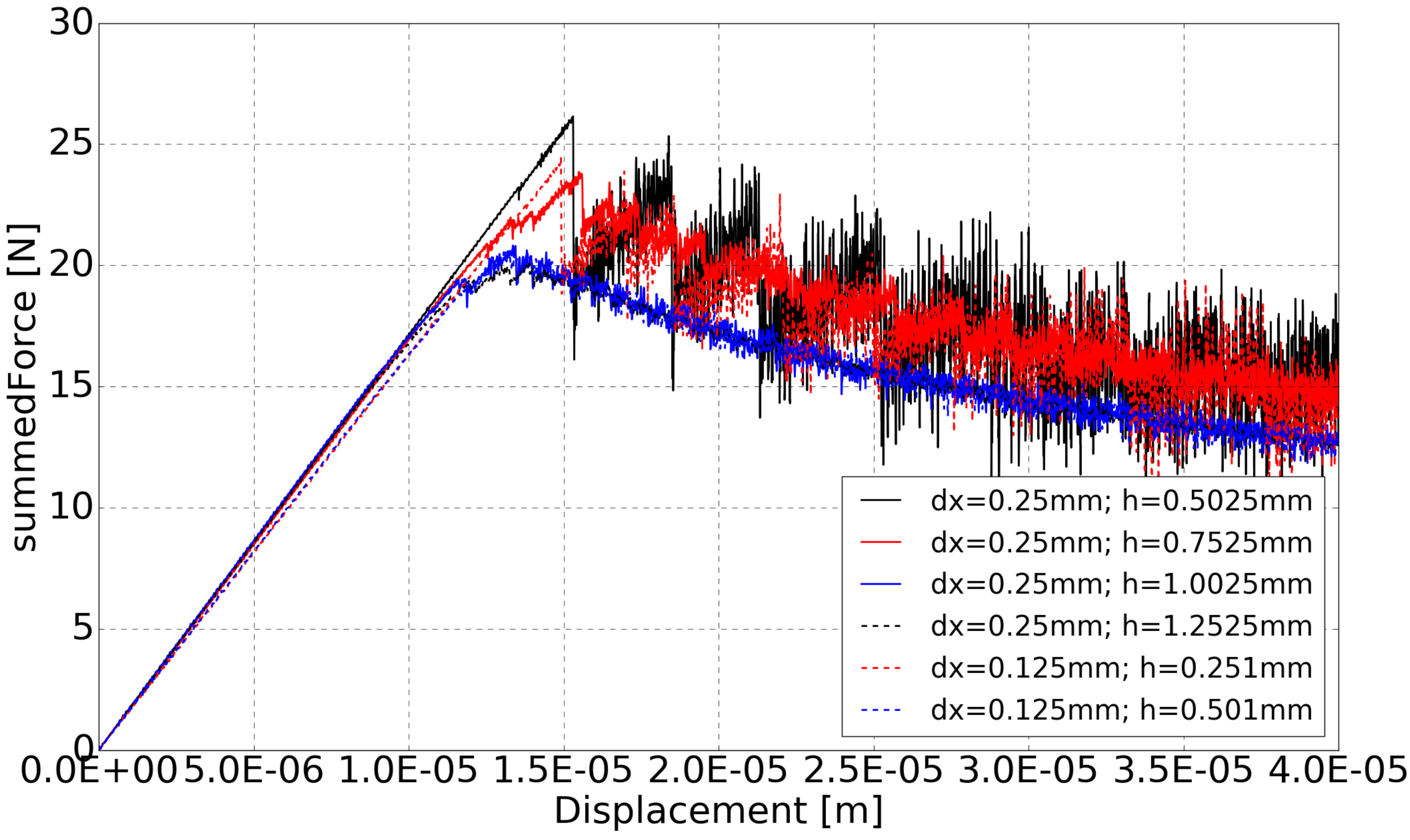

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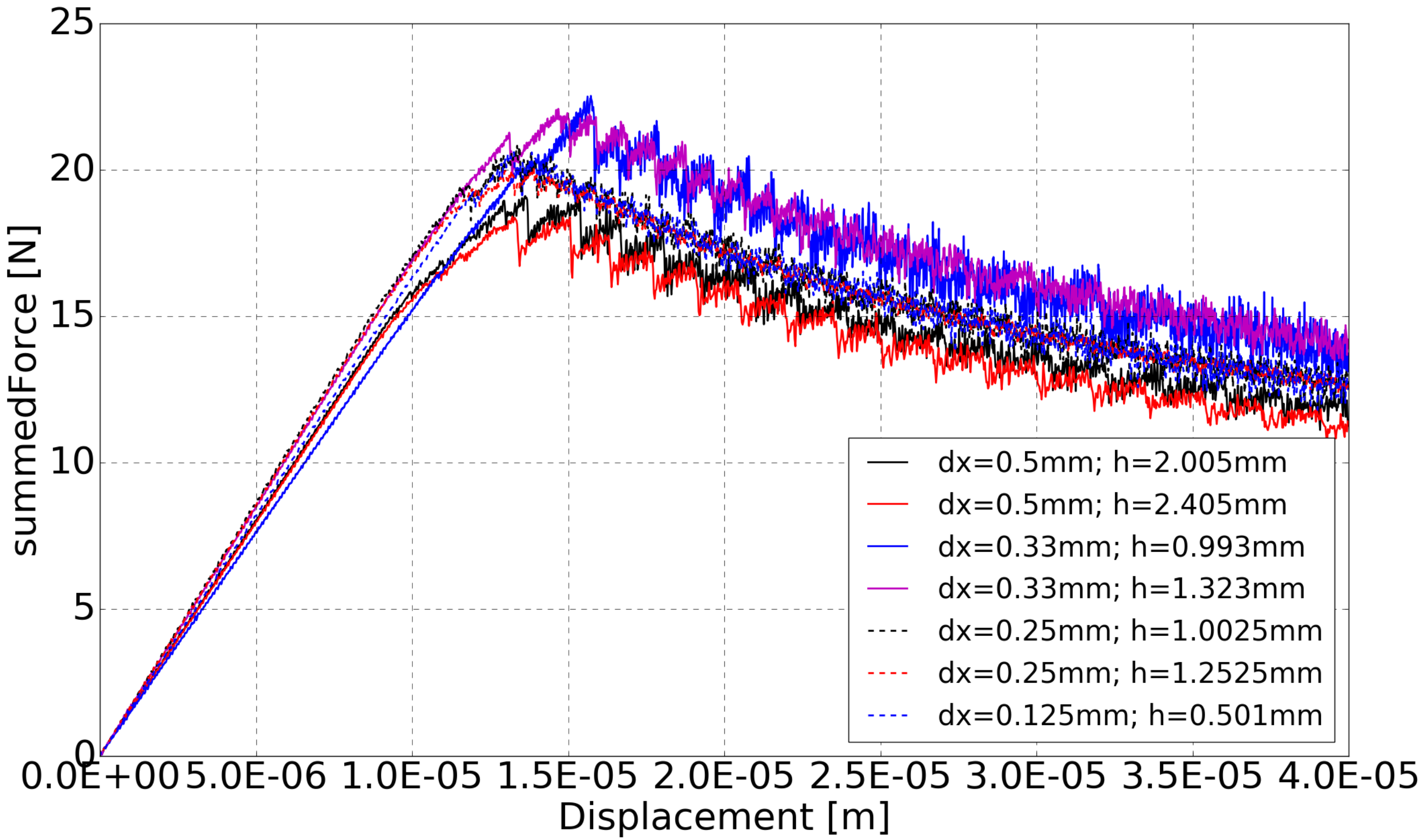

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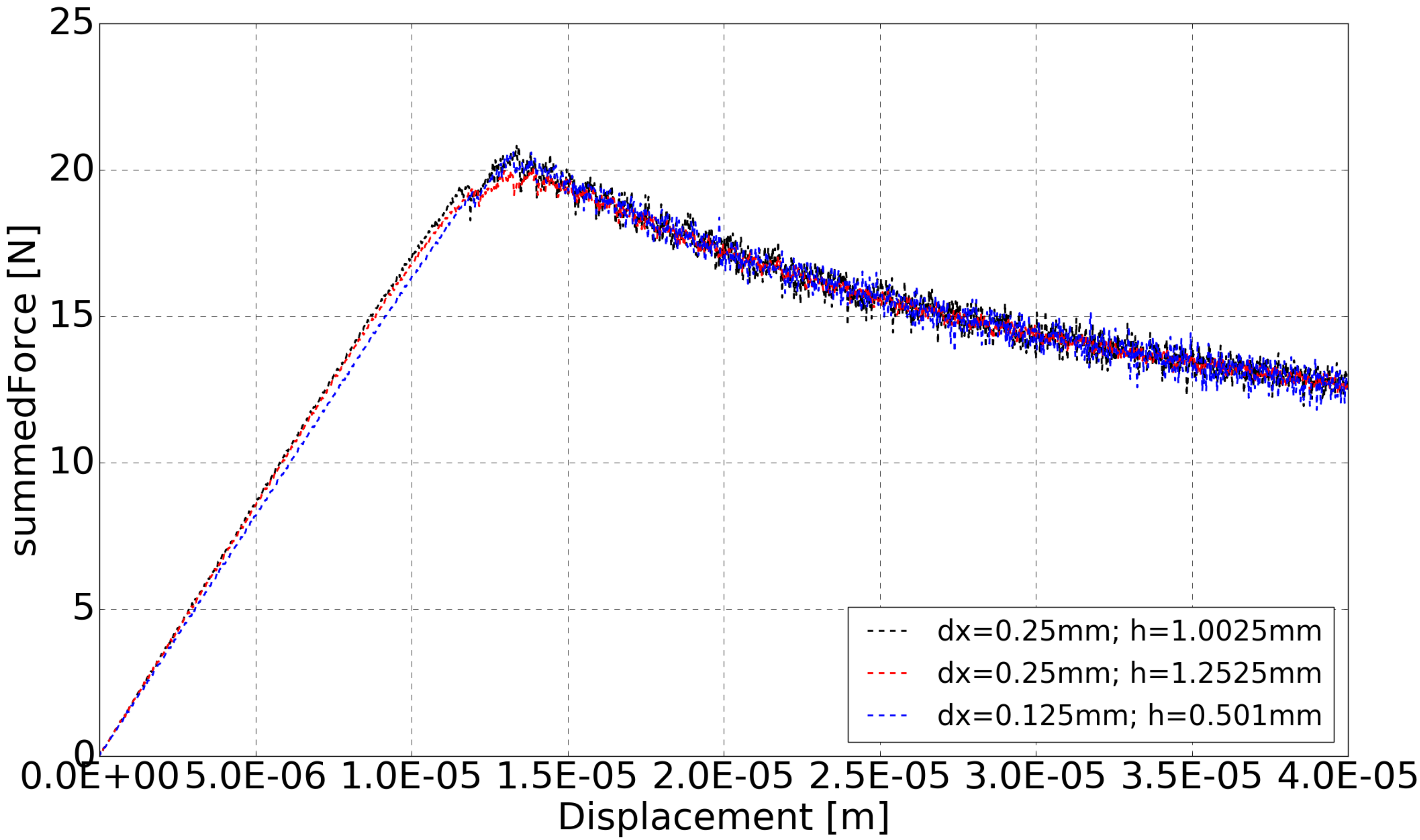
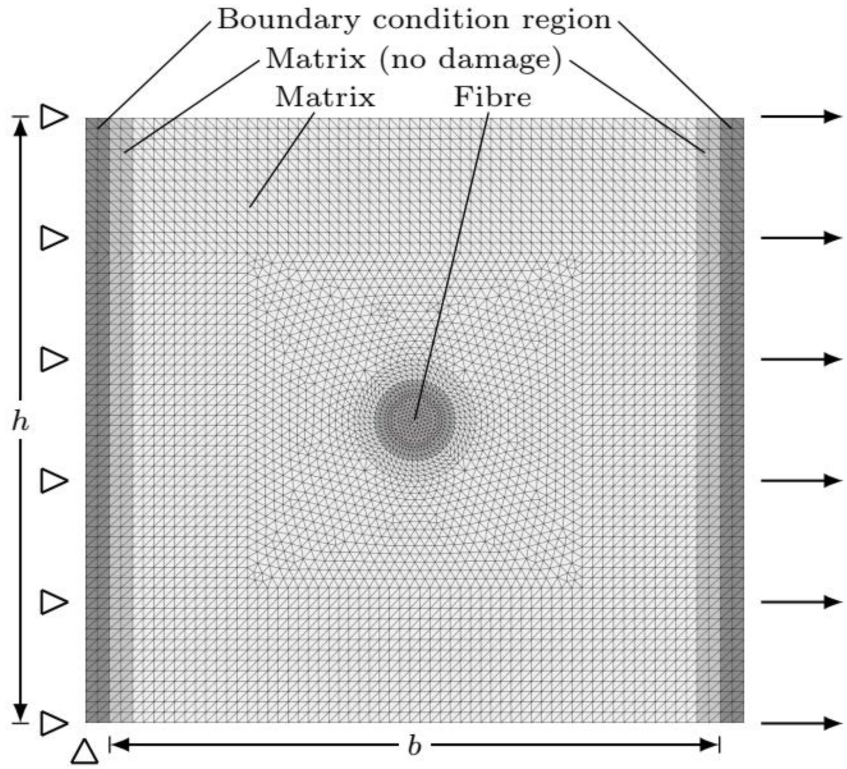
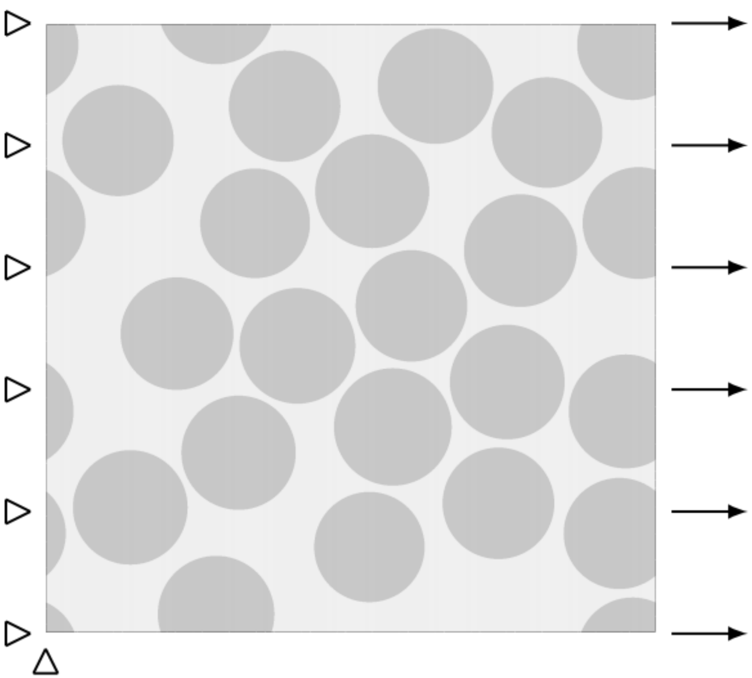


Comparison: Fibre-matrix models

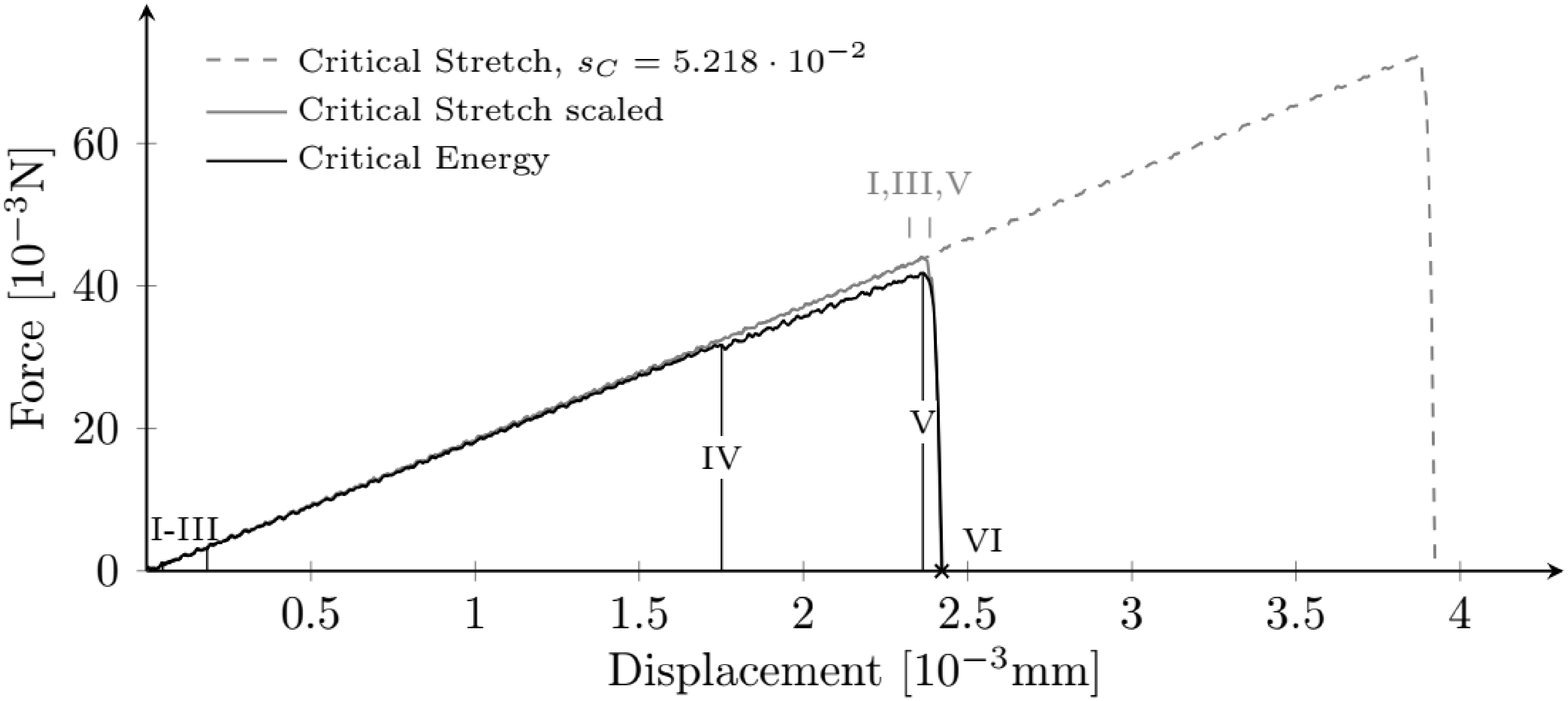


Single fibre

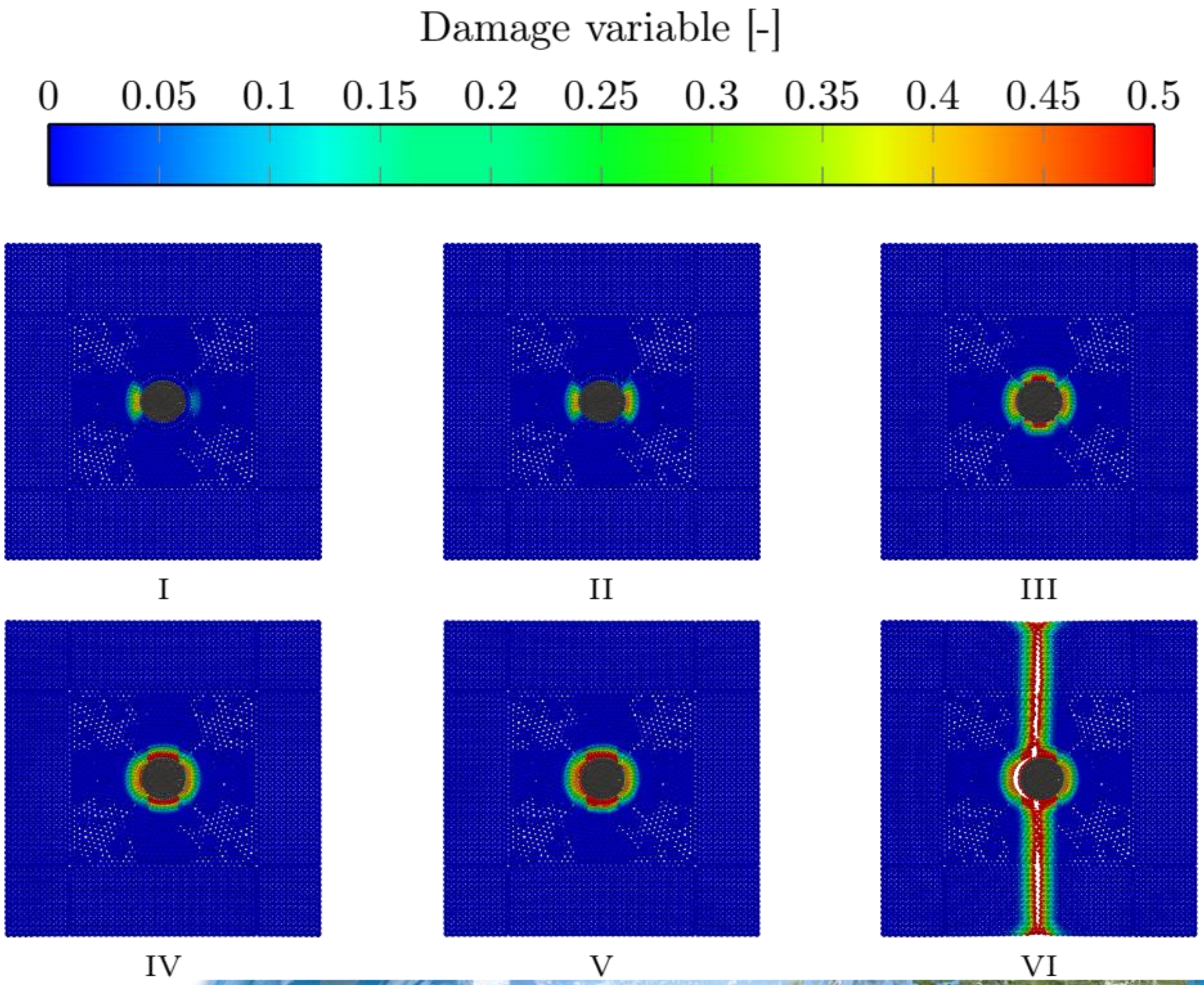


RVE

Comparison: Single fibre – Force-Displacement



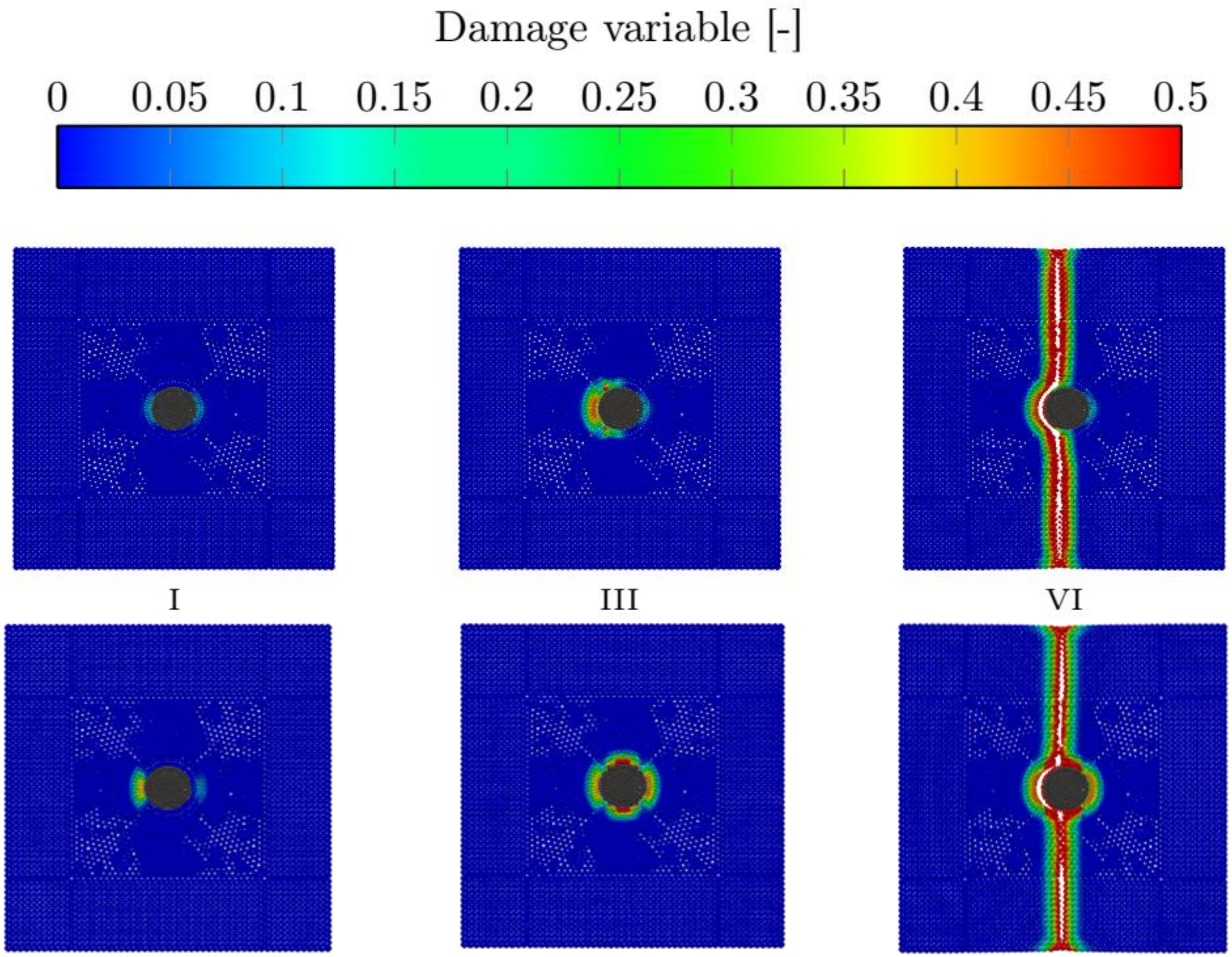
Comparison: Single Fibre – Energy Criterion



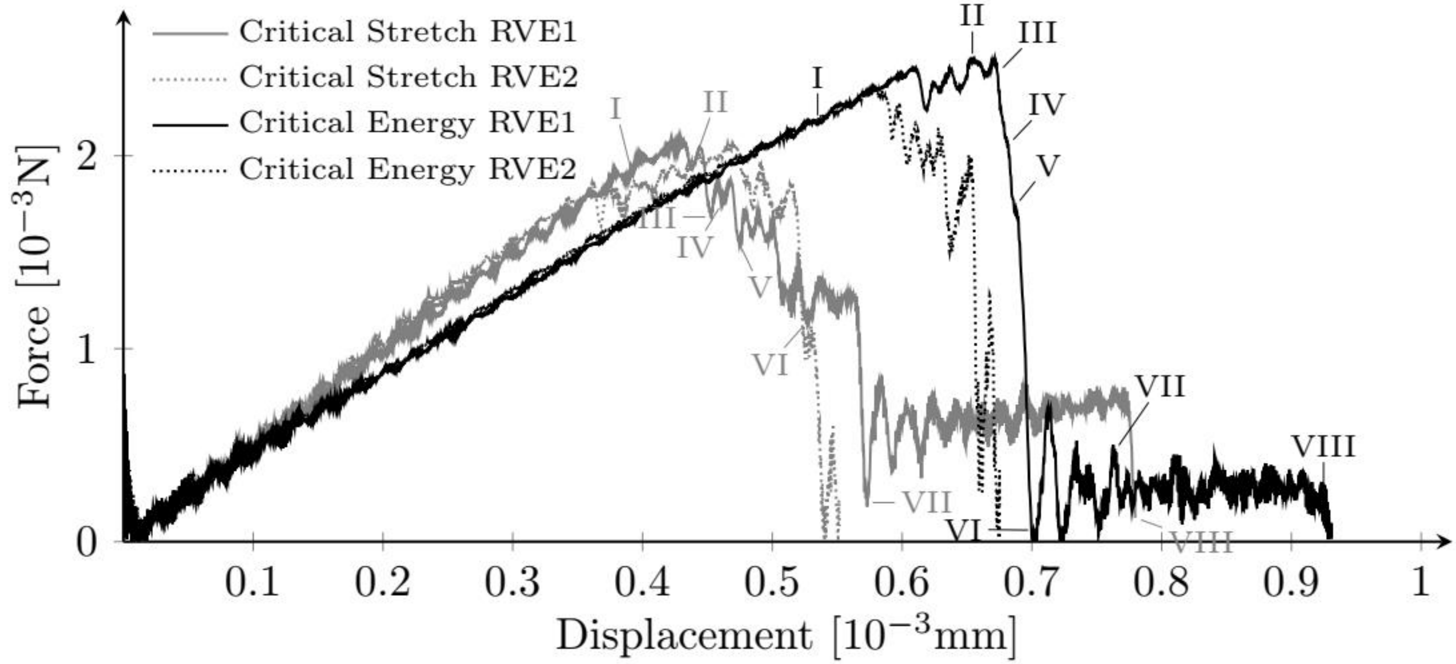
Comparison: Single fibre

Critical Stretch

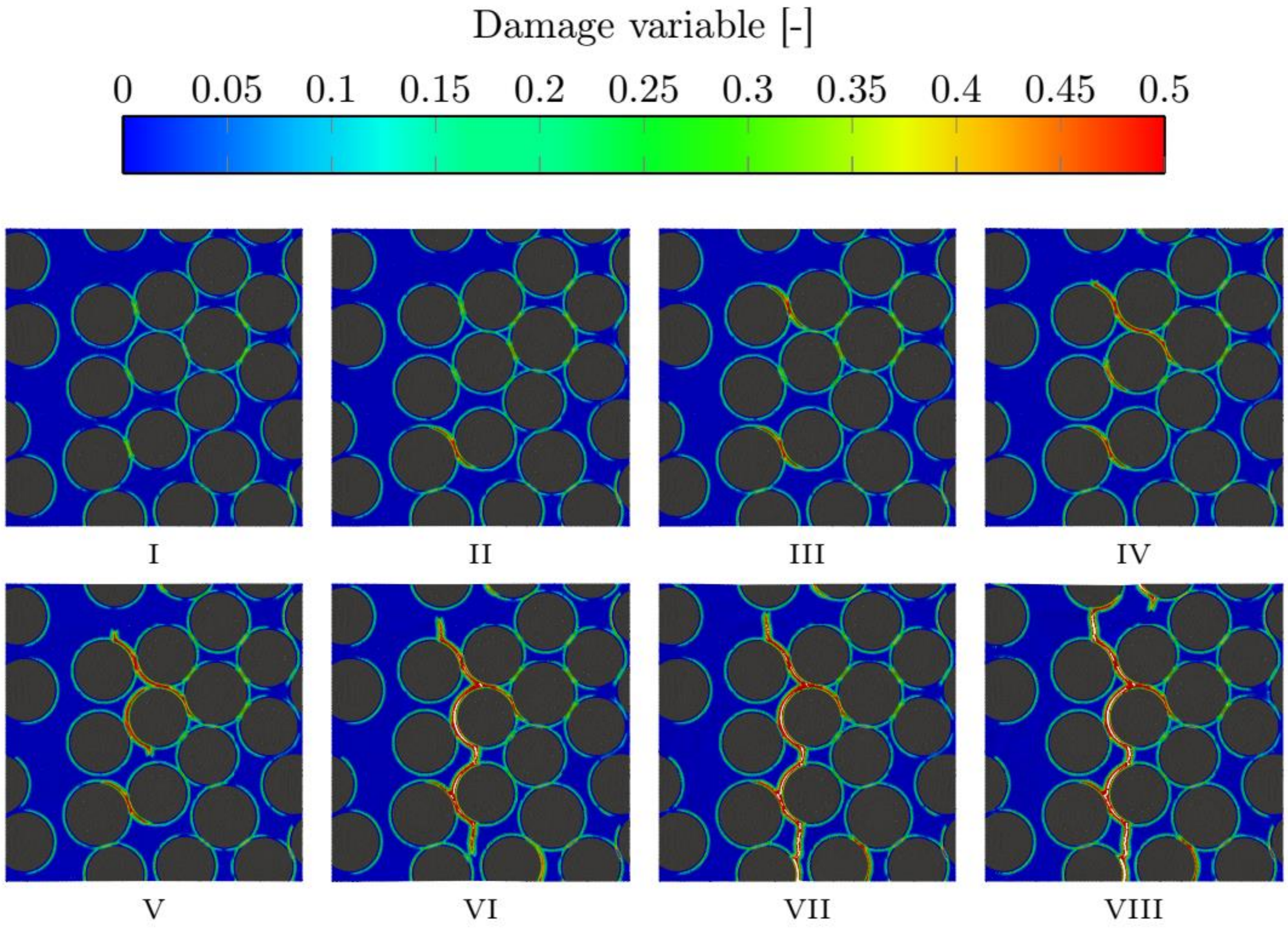
Critical Energy



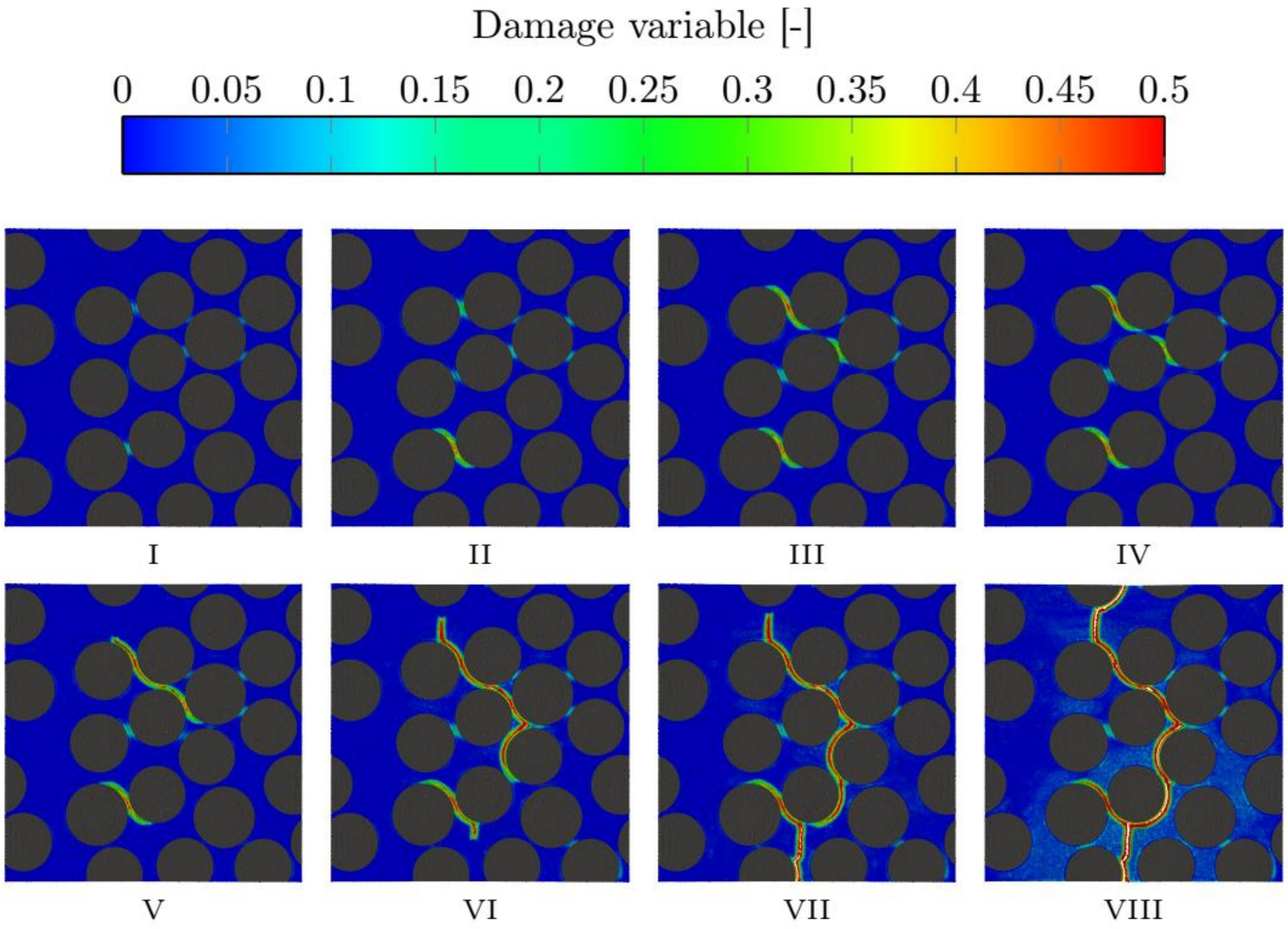
Comparison: RVE-Force-Displacement



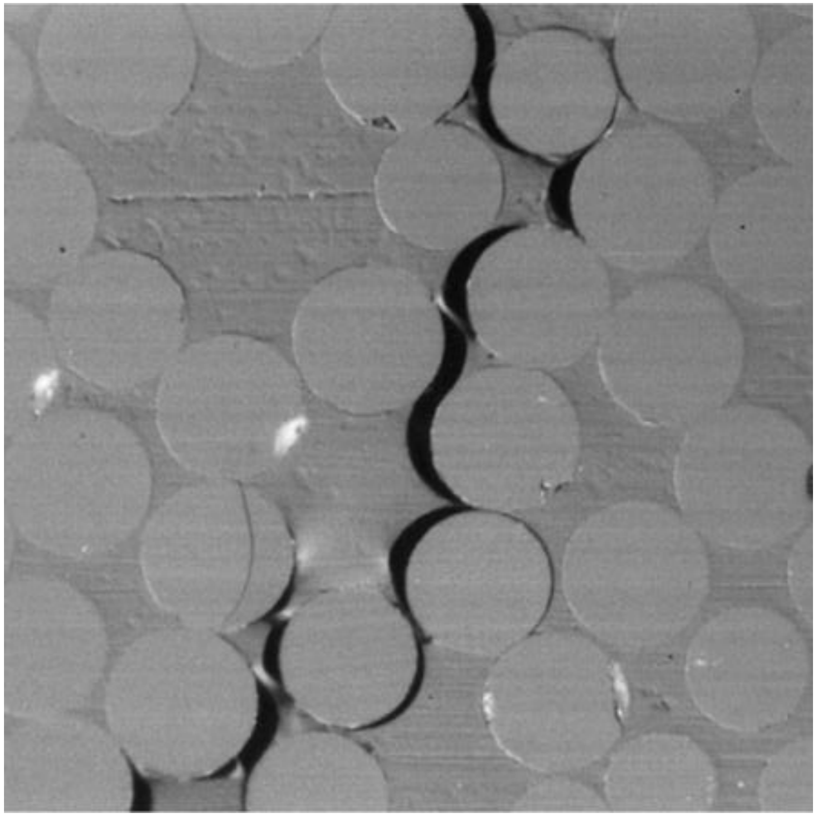
Comparison: RVE – Energy Criterion



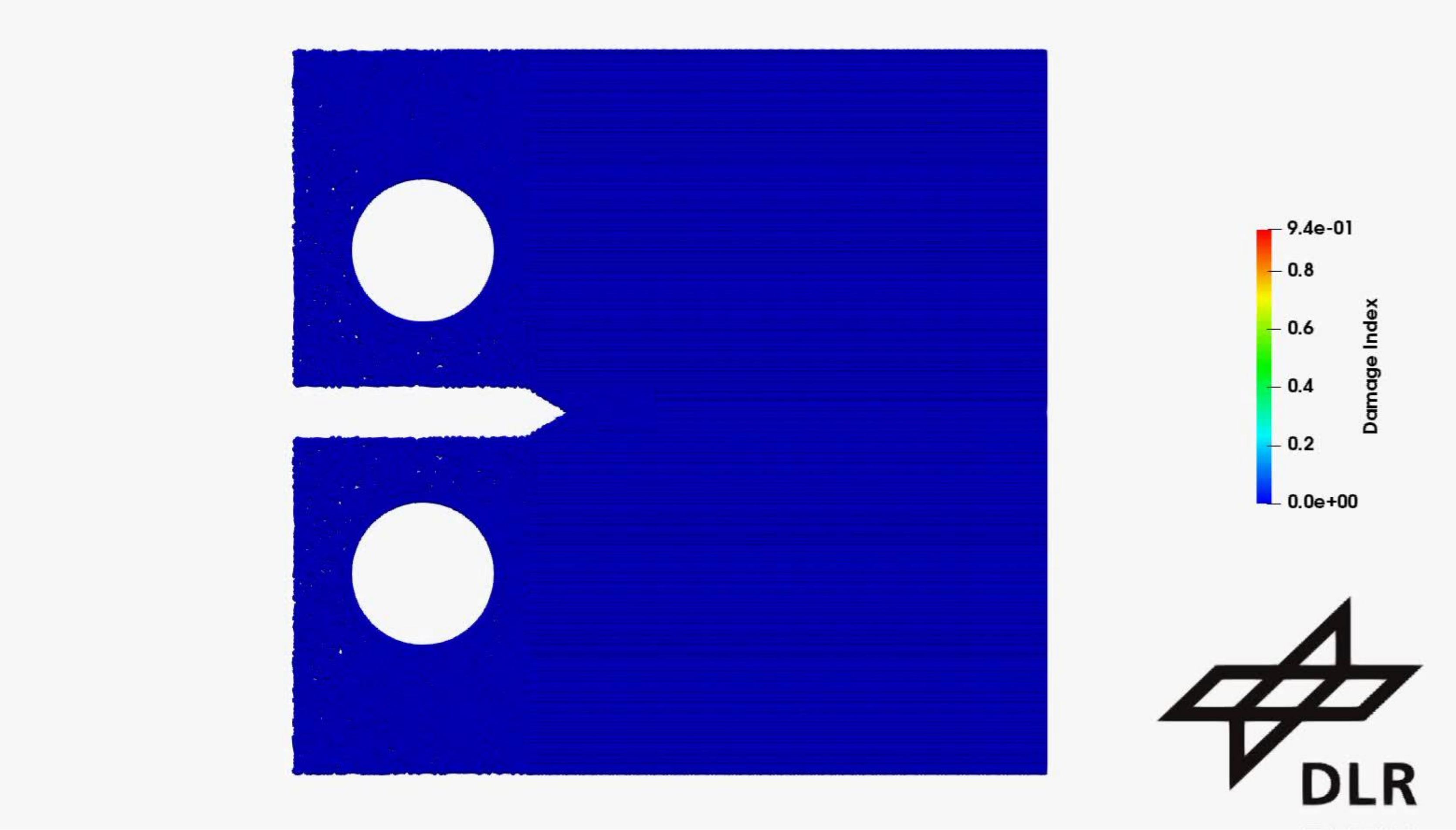
Comparison: RVE - Critical Stretch



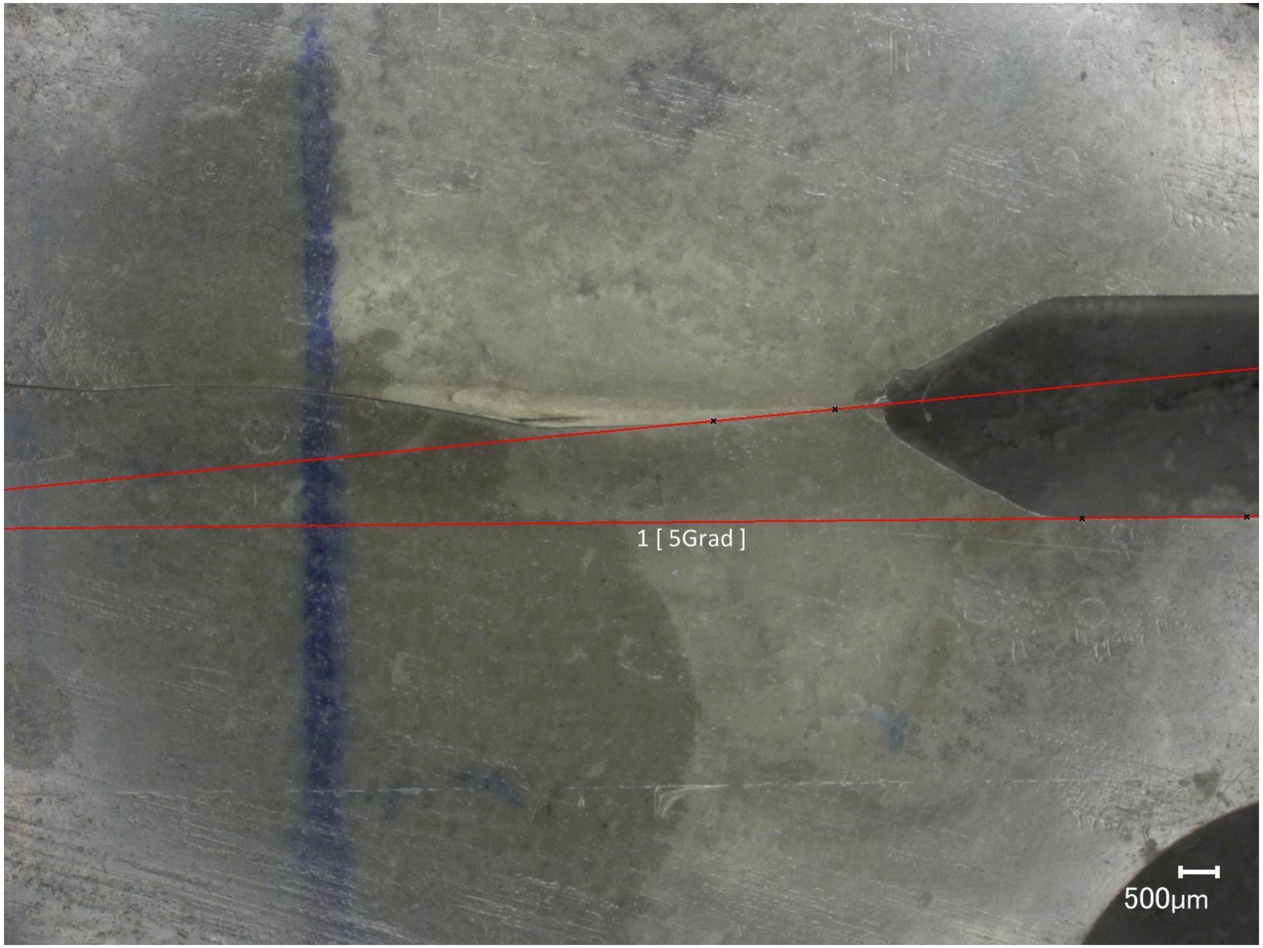
Experimental comparison



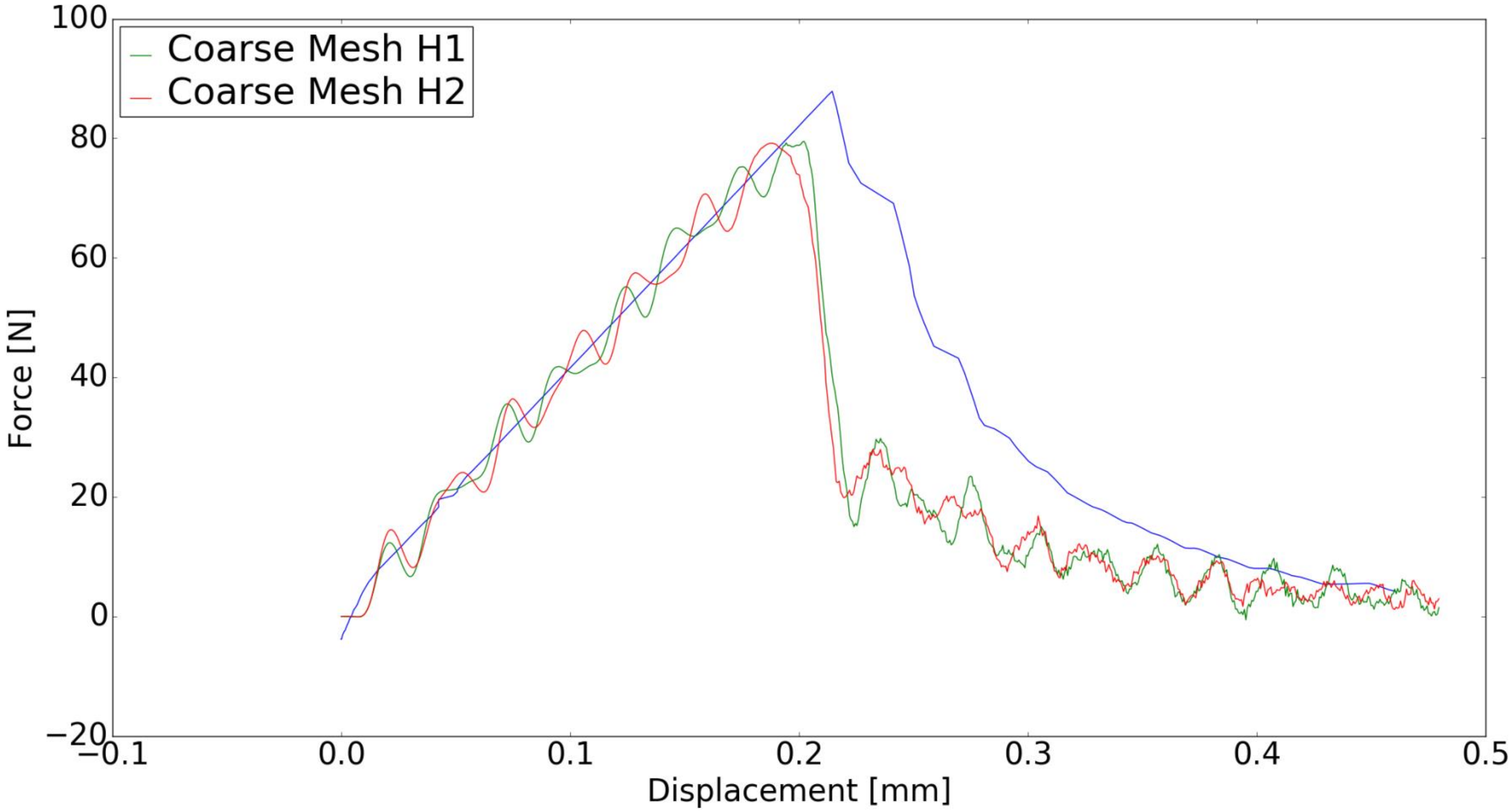
Validation



Validation



Validation


Coarse Mesh H1
Coarse Mesh H2
100
80
60
40
20
0
−20
−0.1
0.0
0.1
0.2
0.3
0.4
0.5
Force [N]
Displacement [mm]

Conclusion

Peridynamics is an interesting approach, but there is still a lot of work to do...

Especially in terms of validation…

Thank you!

Dr.-Ing. Christian Willberg

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Email: christian.willberg@dlr.de

All presented models and source code can be found here
Rädel, M. & Willberg, C. PeriDoX Repository
https://github.com/PeriDoX/PeriDoX

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