CBSE Maths Questions 2007

Priyanka - EE21MTECH12002

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https://github.com/PeriPriyanka/cbsemathsquestions/2007/12/matrices/codes/solutions

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1) (CBSE 2007-Question 1) If $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}$, show that $\mathbf{A}^2 - 6\mathbf{A} + 17\mathbf{I} = 0$. Hence find $\mathbf{A}^{-1}$.
Solution: Consider the matrix given in the problem statement.

$$\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix} \tag{0.0.1}$$

Considering the characteristic equation:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{0.0.2}$$

From (0.0.2) we get,

$$\begin{vmatrix} 2-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = 0 \tag{0.0.3}$$

$$(2-\lambda)(4-\lambda) + 9 = 0 \tag{0.0.4}$$

$$\lambda^2 - 6\lambda + 17 = 0 \tag{0.0.5}$$

From the Cayley-Hamilton theorem (0.0.5) can be written as

$$\mathbf{A}^2 - 6\mathbf{A} + 17\mathbf{I} = 0 \tag{0.0.6}$$

Multiplying with $\mathbf{A}^{-1}$ on both sides of equation (0.0.6) We get,

$$\mathbf{A} - 6\mathbf{I} + 17\mathbf{A}^{-1} = 0 \tag{0.0.7}$$

$$\mathbf{A}^{-1} = \frac{6\mathbf{I} - \mathbf{A}}{17} \tag{0.0.8}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 4/17 & 3/17 \\ -3/17 & 2/17 \end{pmatrix} \tag{0.0.9}$$

2) (CBSE 2007-Question 3) Using the properties of determinants, prove the following:

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

Solution:

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \tag{0.0.10}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2 + R_3}$$

$$\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \tag{0.0.11}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \tag{0.0.12}$$

$$\xleftrightarrow{C_2 \leftarrow C_2 - C_1, C_3 \leftarrow C_3 - C_1}$$

$$(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & a+b+c & 0 \\ 2c & 0 & a+b+c \end{vmatrix} \tag{0.0.13}$$

$$= (a+b+c)(a+b+c)^2 \tag{0.0.14}$$

$$= (a+b+c)^3 \tag{0.0.15}$$

3) (CBSE 2007-Question 19) Using matrices, solve the following system of equation:

$$x + 2y - 3z = 6 \tag{0.0.16}$$

$$3x + 2y - 2z = 3 \tag{0.0.17}$$

$$2x - y + z = 2 \tag{0.0.18}$$

Solution: Consider the equations given in the problem statement. The solution can be found by solving the above system of linear equations.
System of linear equations are defined as

$$\mathbf{Ax} = \mathbf{B} \tag{0.0.19}$$

From the equations (0.0.16) ,(0.0.17) and

(0.0.18),

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{pmatrix} \tag{0.0.20}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{0.0.21}$$

$$\mathbf{B} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \tag{0.0.22}$$

Substituting the values of **A**, **x** and **B** in the equation (0.0.19) We get,

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \tag{0.0.23}$$

Considering the augmented matrix

$$\begin{pmatrix} 1 & 2 & -3 & 6 \\ 3 & 2 & -2 & 3 \\ 2 & -1 & 1 & 2 \end{pmatrix} \tag{0.0.24}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1, R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & -4 & 7 & -15 \\ 0 & -5 & 7 & -10 \end{pmatrix} \tag{0.0.25}$$

$$\xleftrightarrow{R_3 \leftarrow 4R_3 - 5R_2} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & -4 & 7 & -15 \\ 0 & 0 & -7 & 35 \end{pmatrix} \tag{0.0.26}$$

$$\xleftrightarrow{R_3 \leftarrow R_3/-7} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & -4 & 7 & -15 \\ 0 & 0 & 1 & -5 \end{pmatrix} \tag{0.0.27}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 7R_3} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & -4 & 0 & 20 \\ 0 & 0 & 1 & -5 \end{pmatrix} \tag{0.0.28}$$

$$\xleftrightarrow{R_2 \leftarrow R_2/-4} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{pmatrix} \tag{0.0.29}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 2R_2, R_1 \leftarrow R_1 + 3R_2} \begin{pmatrix} 1 & 0 & -0 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{pmatrix} \tag{0.0.30}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -5 \end{pmatrix} \tag{0.0.31}$$

By solving equation (0.0.31) we get,

$$x = 1 \tag{0.0.32}$$

$$y = -5 \tag{0.0.33}$$

$$z = -5 \tag{0.0.34}$$

Therefore, x=1, y= -5 and z=-5 are solutions to the given equations.

4) (CBSE 2007-Question 24) Find the projection of $\overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{c}}$ on $\overrightarrow{\mathbf{a}}$ where $\overrightarrow{\mathbf{a}} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}, \overrightarrow{\mathbf{b}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\overrightarrow{\mathbf{c}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$
Solution: Consider the given vectors,

$$\mathbf{A} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \tag{0.0.35}$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \tag{0.0.36}$$

$$\mathbf{C} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \tag{0.0.37}$$

Projection of $(\mathbf{B} + \mathbf{C})$ on **A** is given by

$$\frac{(\mathbf{B} + \mathbf{C})^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}\|} \tag{0.0.38}$$

By substituting **A**, **B** and **C** in (0.0.38) we get,

$$\frac{(\mathbf{B} + \mathbf{C})^{\mathrm{T}}\mathbf{A}}{\|\mathbf{A}\|} = \frac{\begin{pmatrix} 3 & 1 & 2 \end{pmatrix} . \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}{\sqrt{4 + 4 + 1}} = 2 \tag{0.0.39}$$

5) (CBSE 2007-Question 25) Find the value of λ which makes the vectors $\overrightarrow{\mathbf{a}}, \overrightarrow{\mathbf{b}}$ and $\overrightarrow{\mathbf{c}}$ coplanar, where $\overrightarrow{\mathbf{a}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}, \overrightarrow{\mathbf{b}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\overrightarrow{\mathbf{c}} = 3\hat{\mathbf{i}} - \lambda\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$
Solution: Consider the given vectors,

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{0.0.40}$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \tag{0.0.41}$$

$$\mathbf{C} = \begin{pmatrix} 3 \\ -\lambda \\ 5 \end{pmatrix} \tag{0.0.42}$$

For the vectors **A**, **B** and **C** to be coplanlar, the three vectors are linearly dependent. Therfore,

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & -\lambda & 5 \end{vmatrix} = 0 \tag{0.0.43}$$

$$= 2(10 - 3\lambda) + 1(5 + 9) + 1(-\lambda - 6) = 0 \tag{0.0.44}$$

$$\lambda = 4 \tag{0.0.45}$$

6) (CBSE 2007-Question 31) Find the equation of the plane which is perpendicular to the plane $5x+3y+6z+8 = 0$ and which contains the line of intersection of the planes $x+2y+3z-4 = 0$ and $2x + y - z + 5 = 0$.
Solution: consider the given planes as

$$\mathbf{A}^{\mathrm{T}}\mathbf{x} = \mathrm{c}_1$$

$$= \begin{pmatrix} 5 & 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -8 \tag{0.0.46}$$

$$\mathbf{B}^{\mathrm{T}}\mathbf{x} = \mathrm{c}_2$$

$$= \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4 \tag{0.0.47}$$

$$\mathbf{C}^{\mathrm{T}}\mathbf{x} = \mathrm{c}_3$$

$$= \begin{pmatrix} 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -5 \tag{0.0.48}$$

Plane $\perp$ to $5x + 3y + 6z + 8 = 0$ is given by,

$$(\mathbf{B} + \mathrm{k}\mathbf{C})\mathbf{x} = \mathrm{c} \tag{0.0.49}$$

$$\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mathrm{k}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right)^{\mathrm{T}} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4 - 5\mathrm{k} \tag{0.0.50}$$

The plane in equation (0.0.50) is $\perp$ to plane in equation (0.0.46). Therefore,

$$\mathbf{A}^{\mathrm{T}}.(\mathbf{B} + \mathrm{k}\mathbf{C}) = 0 \tag{0.0.51}$$

$$\begin{pmatrix} 5 & 3 & 6 \end{pmatrix} . \left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mathrm{k}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right] = 0 \tag{0.0.52}$$

$$\mathrm{k} = \frac{-29}{7} \tag{0.0.53}$$

The required plane is

$$\begin{pmatrix} 51 & 15 & -50 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -173 \tag{0.0.54}$$