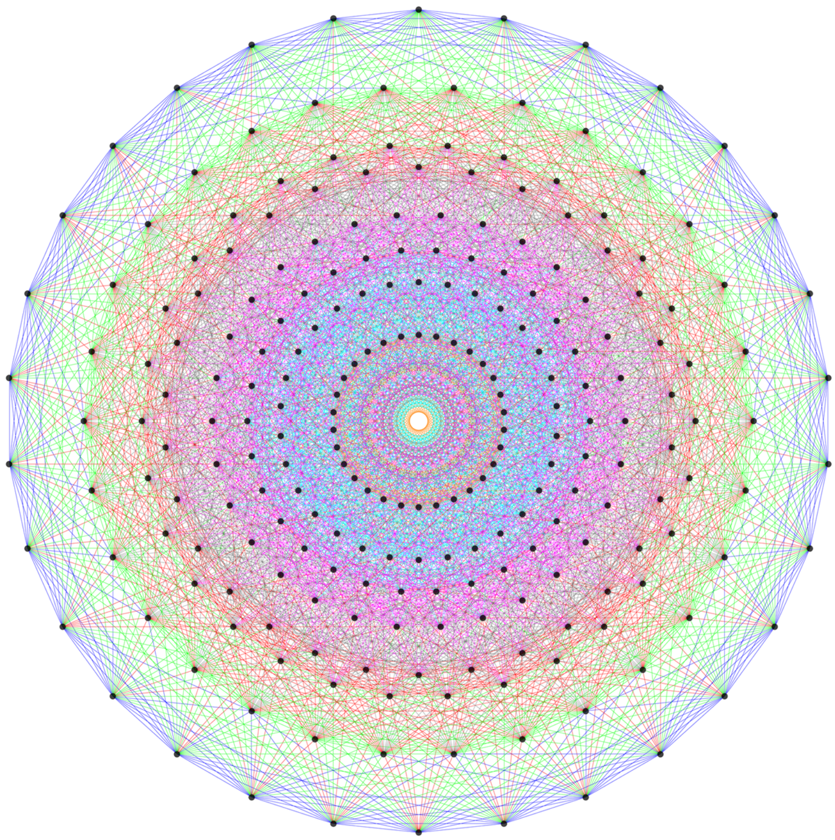
level I:mathematics

University of Technology and Applied Sciences

SULTANATE OF OMAN

GENERAL FOUNDATION PROGRAM



FPMB0001

BASIC MATHEMATICS

The content of this course material is compiled and organized by Dr.Selvagopal and Dr.Kulandaivel of UTAS, Al-Musannah, while updates and corrections are done by Dr.Naseer Ahmed, UTAS, Nizwa.

Majority of the content of this material is taken from https://openstax.org/. The topic on *Testing Equations for Symmetry* (Chapter 9.4) was taken from http://www.onemathematicalcat.org.

Resources from this material can be directly accessed for free from the following websites:

1. Prealgebra at https://openstax.org/details/books/prealgebra-2e
2. College Algebra at https://openstax.org/details/books/college-algebra
3. Intermediate Algebra at https://openstax.org/details/books/intermediate-algebra-2e
4. Algebra and Trigonometry at https://openstax.org/details/books/algebra-and-trigonometry
5. Elementary Algebra at https://openstax.org/details/books/elementary-algebra-2e

Correspondingly, each page of the material is provided with proper attribution, the link of which will direct to the original source.

Contents

Basic Mathematics
Learning Outcomes

A student who satisfactorily complete the course should be able to:

(a) Describe the set of real numbers, all its subsets and their relationship.

(b) Identify and use the arithmetic properties of subsets of integers, rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable.

(c) Demonstrate an understanding of the exponent laws, and apply them to simplify expression and manipulate fractions, ratios, decimals, and percentages.

(d) Understand measurements and conversion from one unit to another.

(e) Simplify rational expressions and rationalize numerators or denominators.

(f) Translate worded problems into mathematical expression and model simple real life problems with equations and inequalities.

(g) Solve linear equations, equations involving radicals, fractional expression and inequalities.

(h) Use coordinate plane to solve algebraic and geometric problem, and understand geometric concepts such as equation of circle, perpendicular, parallel, and tangent lines.

(i) Use the three types of symmetry of an equation to sketch its graph.

(j) Perform operations on polynomials and manipulate numerical and polynomial expressions and solve first degree equations.

(k) Use the quadratic formula to find roots of a second-degree polynomial.

(l) Know the relationship between degree and radian measure of an angle and find the length of a circular arc and the area of a sector.

(m) Understand trigonometric and circular functions and use the fundamental trigonometric identities in various problems.

(n) Solve a right angle triangles using angle of elevation and depression.

(o) Apply knowledge of basic algebra and trigonometry in real life problems.

Chapter 1

REAL NUMBERS

Contents

Learning outcome covered:

a. Describe the set of real numbers, all its subsets and their relationship.

b. Identify and use the arithmetic properties of subsets of integers, rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable.

c. Manipulate fractions, ratios, decimals, and percentages.

o. Apply knowledge of basic algebra and trigonometry in real life problems.

Learning Objectives

By the end of this chapter, the students will be able to:

- Classify different types of real numbers
- Use the commutative and associative properties
- Simplify expressions using the distributive property
- Recognize and use the identity and inverse properties of addition and multiplication
- Use the properties of zero

- Simplify expressions using the properties of identities, inverses, and zero
- Add and Subtract Fractions
- Convert percents to fractions and decimals
- Convert decimals and fractions to percents

Introduction

Even though counting is first taught at a young age, mastering mathematics, which is the study of numbers, requires constant attention. If it has been a while since you have studied math, it can be helpful to review basic topics. In this chapter, we will focus on numbers as well as four arithmetic operationsaddition, subtraction, multiplication and division. We will also discuss some vocabulary that we will use throughout this book.

1.1 Classification of Real Numbers

Algebra uses numbers and symbols to represent words and ideas. Let us look at the numbers first. In this section we describe the set of real numbers, all its subsets and their relationship.

1.1.1 Identify Counting Numbers and Whole Numbers

The most basic numbers used in algebra are those we use to count objects: 1, 2, 3, 4, 5, and so on. These are called the **counting numbers**. Counting numbers are also called **natural numbers**.

Counting Numbers or Natural Numbers(N)

The counting numbers start with 1 and continue.

$$N = \{1, 2, 3, 4, 5, \cdots\}$$

The discovery of the number zero was a big step in the history of mathematics. Including zero with the counting numbers gives a new set of numbers called the **whole numbers**.

Whole Numbers (W)

The whole numbers are the counting numbers and zero.

$$W = \{0, 1, 2, 3, 4, 5, \cdots\}$$

Introduction to Integers

A negative number is a number that is less than 0. Both positive and negative numbers can be represented on a number line. We could write a plus sign, +, before a positive number such as +2 or +3, but it is customary to omit the plus sign and write only the number. If there is no sign, the number is assumed to be positive.

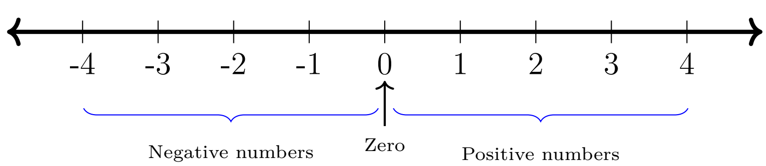


Figure 1

The arrows at either end of the line indicate that the number line extends forever in each direction. There is no greatest positive number and there is no smallest negative number.

Integers

The set of counting numbers, their opposites, and 0 is the set of integers.

Integers (Z)

The set of counting numbers, their opposites, and 0 is the set of integers.

$$Z = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

1.1.2 Rational and Irrational Numbers

Rational Numbers

Rational Numbers (Q)

A rational number is a number that can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

$$Q = \left\{ \frac{p}{q} : p \text{ and } q \text{ are integers and } q \neq 0 \right\}$$

Are integers rational numbers? To decide if an integer is a rational number, we try to write it as a ratio of two integers. An easy way to do this is to write it as a fraction with denominator one. $3 = \frac{3}{1}$, $-8 = \frac{-8}{1}$ and $0 = \frac{0}{1}$. Since any integer can be written as the ratio of two integers, all integers are rational numbers. Remember that all the counting numbers and all the whole numbers are also integers, and so they, too, are rational.

What about decimals? Are they rational? Think about the decimal 7.3. We can write it as $\frac{73}{10}$. It is a rational number.

Every rational number can be written both as a ratio of integers and as a decimal that either **stops** [e.g., 4.275] **or repeats** [e.g., 2.757575...]. The table below shows the numbers we looked at expressed as a ratio of integers and as a decimal.

Rational Numbers		
	Fractions	Integers
Number	$\frac{4}{5}, -\frac{7}{8}, \frac{13}{4}, \frac{-20}{3}$	$-2, -1, 0, 1, 2, 3$
Ratio of Integer	$\frac{4}{5}, \frac{-7}{8}, \frac{13}{4}, -\frac{20}{3}$	$\frac{-2}{1}, \frac{-1}{1}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}$
Decimal number	$0.8, -0.875, 3.25, -6.\overline{6}\,[-6.666\cdots]$	$-2.0, -1.0, 0.0, 1.0, 2.0, 3.0$

Irrational Numbers

Are there any decimals that do not stop or repeat? Yes. The number π (the Greek letter π, pronounced pi), which is very important in describing circles, has a decimal form that does not stop or repeat.

$$\pi = 3.141592654\cdots$$

Similarly, the decimal representations of square roots of numbers that are not perfect squares never stop and never repeat. For example,

$$\sqrt{5} = 2.236067978\cdots$$

A decimal that does not stop and does not repeat cannot be written as the ratio of integers. We call this kind of number an irrational number.

Irrational Numbers (I)

An irrational number is a number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.

Example 1.1: Identify each of the following as rational or irrational:

a. $0.58\overline{3}$ **b.** 0.475 **c.** 3.605551275...

Solution:

(a) The bar above the 3 indicates that it repeats. Therefore, $0.58\overline{3} = 0.583333...$ is a repeating decimal, and is therefore a rational number.

(b) This decimal stops after the 5 , so it is a rational number.

(c) The ellipsis (...) means that this number does not stop. There is no repeating pattern of digits. Since the number doesn't stop and doesn't repeat, it is irrational.

Let us think about square roots now. Square roots of perfect squares are always whole numbers, so they are rational. But the decimal forms of square roots of numbers that are not perfect squares never stop and never repeat, so these square roots are irrational.

Example 1.2: Identify each of the following as rational or irrational:

a. $\sqrt{36}$ **b.** $\sqrt{44}$

Solution:

(a) The number 36 is a perfect square, since $6^2 = 36$. So $\sqrt{36} = 6$. Therefore $\sqrt{36}$ is rational.

(b) Remember that $6^2 = 36$ and $7^2 = 49$, so 44 is not a perfect square. This means $\sqrt{44}$ is irrational.

Classify Real Numbers

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. Irrational numbers are a separate category of their own. When we put together the rational numbers and the irrational numbers, we get the set of real numbers.

Figure 2 illustrates how the number sets are related.

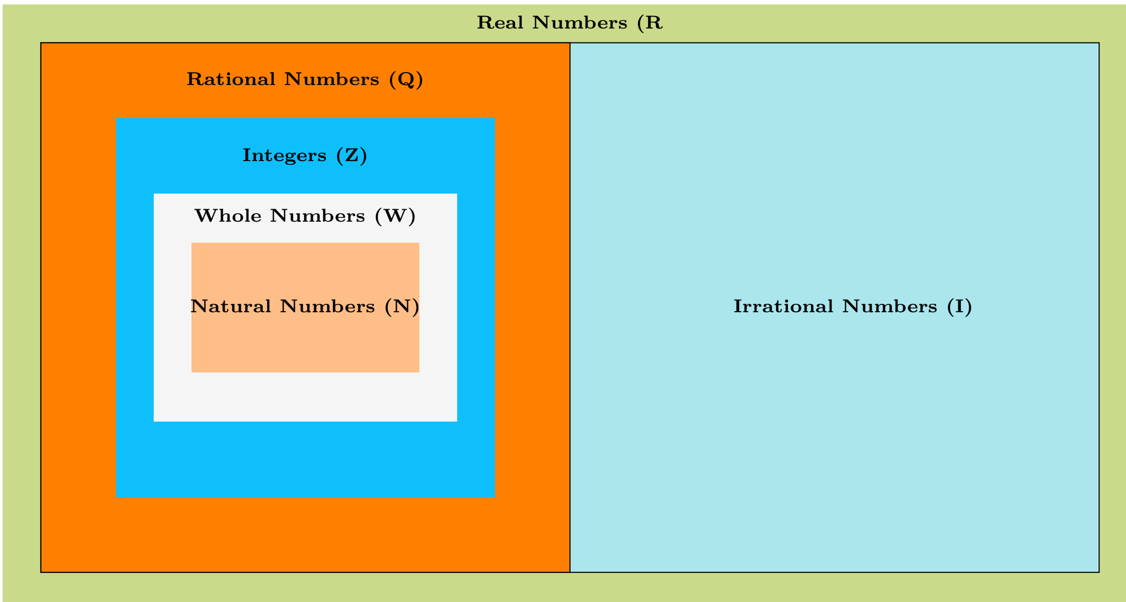


Figure 2

Real Numbers (R)

Real numbers are numbers that are either rational or irrational.

Example 1.3: Determine whether each of the numbers in the following list is a (a) whole number, (b) integer, (c) rational number, (d) irrational number, and (e) real number.

$$-7,\ \frac{14}{5},\ 8,\ \sqrt{5},\ 5.9,\ -\sqrt{64}$$

Solution:

(a) The whole numbers are 0, 1, 2, 3, The number 8 is the only whole number given.

(b) The integers are the whole numbers, their opposites, and 0. From the given numbers, 7 and 8 are integers. Also, notice that 64 is the square of 8 so $-\sqrt{64} = -8$. So the integers are 7, 8, $-\sqrt{64}$.

(c) Since all integers are rational, the numbers 7, 8, and $-\sqrt{64}$ are also rational. Rational numbers also include fractions and decimals that terminate or repeat, so $\frac{14}{5}$ and 5.9 are rational.

(d) The number 5 is not a perfect square, so $\sqrt{5}$ is irrational.

(e) All of the numbers listed are real.

We'll summarize the results in a table.

Number	Whole	Integer	Rational	Irrational	Real
-7		✓	✓		✓
$\frac{14}{5}$			✓		✓
8	✓	✓	✓		✓
$\sqrt{5}$				✓	✓
5.9			✓		✓
$-\sqrt{64}$		✓	✓		✓

1.1 Section Exercises

Rational Numbers

In the following exercises, determine which of the given numbers are rational and which are irrational.

1. $0.75, 0.22\overline{3}, 1.39174\cdots$ **2.** $0.36, 0.94729\cdots, 2.52\overline{8}$ **3.** $0.4\overline{5}, 1.919293\cdots, 3.59$

In the following exercises, identify whether each number is rational or irrational.

4. **a.** $\sqrt{25}$ **b.** $\sqrt{30}$

5. **a.** $\sqrt{44}$ **b.** $\sqrt{49}$

6. **a.** $\sqrt{164}$ **b.** $\sqrt{169}$

Classifying Real Numbers

In the following exercises, determine whether each number is whole, integer, rational, irrational, and real.

7. $-8, 0, 1.95286...., \frac{12}{5}, \sqrt{36}, 9$

8. $-9, -3\frac{4}{9}, -\sqrt{9}, 0.4\overline{09}, \frac{11}{6}, 7$

1.2 Properties of Real Numbers

Addition (+) and Multiplication ($\times$ or $\cdot$) are two important operations defined on the set of all real numbers. In this section, we will take a look at some properties of real numbers with respect to these two operations.

1. Closure Properties

We know the sum of any two real numbers is again a real number and also the product of any two real numbers is also a real number. These properties are given below.

Closure Properties

Closure Property of Addition: If a and b are real numbers, then $a + b$ is also a real number.
Closure Property of Multiplication: If a and b are real numbers, then $a \cdot b$ is also a real number.

2. Commutative Properties

Commutative Properties

Commutative Property of Addition: if a and b are real numbers, then

$$a + b = b + a$$

Commutative Property of Multiplication: if a and b are real numbers, then

$$a \cdot b = b \cdot a$$

Example 1.4: Use the commutative properties to rewrite the following expressions:

a. $-1 + 3 = \_\_\_\_$ **b.** $4 \cdot 9 = \_\_\_\_$

Solution:

a. $-1 + 3 \quad = 3 + (-1)$ Use the commutative property of addition to change the order.

b. $4 \cdot 9 \quad = 9 \cdot 4$ Use the commutative property of multiplication to change the order.

3. Associative Properties

Associative Properties

Associative Property of Addition: if a, b, and c are real numbers, then

$$(a + b) + c = a + (b + c)$$

Associative Property of Multiplication: if a, b, and c are real numbers, then

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Example 1.5: Use the associative properties to rewrite the following expressions:

a. $(3 + 0.6) + 0.4 = \_\_\_\_$ **b.** $\left(-4 \cdot \frac{2}{5}\right) \cdot 15 = \_\_\_\_$

Solution:

a. $(3 + 0.6) + 0.4 = 3+(0.6+0.4)$ Change the grouping.

Notice that 0.6 + 0.4 is 1, so the addition will be easier if we group as shown on the right.

b. $\left(-4 \cdot \frac{2}{5}\right) \cdot 15 = -4 \cdot \left(\frac{2}{5} \cdot 15\right)$ Change the grouping.

Notice that $\frac{2}{5} \cdot 15$ *is 6. The multiplication will be easier if we group as shown on the right.*

4. Distributive Property

In algebra, we use the Distributive Property to remove parentheses as we simplify expressions.

Distributive Property

If a, b, and c are real numbers, then $a(b + c) = ab + ac$

Example 1.6: Simplify: $3(x + 4)$

Solution:

$3(x + 4) = 3 \cdot x + 3 \cdot 4$ Distribute.

$= 3x + 12$ Multiply.

Some students find it helpful to draw in arrows to remind them how to use the Distributive Property. Then the first step in the Example would look like this:

$$3(x + 4)$$
$$3 \cdot x + 3 \cdot 4$$

Example 1.7: Simplify: $-11(4-3a)$

Solution:

$-11(4-3a) = -11 \cdot 4 - (-11) \cdot 3a$ Distribute.

$= -44 - (-33a)$ Multiply.

$= -44 + 33a$ Simplify.

You could also write the result as $33a-44$. Do you know why?

5. Identity Properties of Addition and Multiplication

What happens when we add zero to any number? Adding zero doesnt change the value. For this reason, we call 0 the **additive identity**.

What happens when you multiply any number by one? Multiplying by one doesnt change the value. So we call 1 the **multiplicative identity**.

Identity Properties

The **identity property of addition:** for any real number a,

$$\mathrm{a} + 0 = \mathrm{a} \qquad 0 + \mathrm{a} = \mathrm{a}$$

0 is called the **additive identity**

The **identity property of multiplication**: for any real number a,

$$a \cdot 1 = a \qquad 1 \cdot a = a$$

1 is called the **multiplicative identity**

Example 1.8: Identify whether each equation demonstrates the identity property of addition or multiplication.

a. $7 + 0 = 7$ **b.** $-16(1) = -16$

Solution:

a. We are adding 0. We are using the identity property of addition.

b. We are multiplying by 1. We are using the identity property of multiplication.

6. Inverse Properties of Addition and Multiplication

Well formally state the Inverse Properties here:

Inverse Properties

Inverse Property of Addition for any real number a,

$$a + (-a) = 0$$

$-a$ is the **additive inverse** of a.

Inverse Property of Multiplication for any real number $a \neq 0$,

$$a \cdot \frac{1}{a} = 1$$

$\frac{1}{a}$ is the **multiplicative inverse** of a.

Example 1.9: Find the additive inverse of each expression:

a. 13 **b.** $-\frac{5}{8}$ **c.** 0.6

Solution: To find the additive inverse, we find the opposite.

(a) The additive inverse of 13 is its opposite, -13.

(b) The additive inverse of $-\frac{5}{8}$ is its opposite, $\frac{5}{8}$.

(c) The additive inverse of 0.6 is its opposite, -0.6.

Example 1.10: Find the multiplicative inverse:

a. 9 **b.** $-\frac{1}{9}$ **c.** 0.9

Solution: To find the multiplicative inverse, we find the reciprocal.

(a) The multiplicative inverse of 9 is its reciprocal, $\frac{1}{9}$.

(b) The multiplicative inverse of $-\frac{1}{9}$ is its reciprocal, -9.

(c) To find the multiplicative inverse of 0.9, we first convert 0.9 to a fraction, $\frac{9}{10}$. Then we find the reciprocal, $\frac{10}{9}$.

7. Properties of Zero

We have already learned that zero is the additive identity, since it can be added to any number without changing the numbers identity. But zero also has some special properties when it comes to multiplication and division.

What happens when you multiply a number by 0? Multiplying by 0 makes the product equal zero.

What about dividing with 0? Think about a real example: if there are no cookies in the cookie jar and three people want to share them, how many cookies would each person get? There are 0 cookies to share, so each person gets 0 cookies. $0 \div 3 = 0$.

Now lets think about dividing a number by zero. What is the result of dividing 4 by 0? Is there a number that multiplied by 0 gives 4? There is no real number that can be multiplied by 0 to obtain 4. We can conclude that there is no answer to $4 \div 0$ and is **undefined**.

Properties of Zero

Multiplication by Zero

- For any real number a, $a \cdot 0 = 0 \qquad 0 \cdot a = 0$

Division with Zero

- For any real number $a \neq 0$, $\frac{0}{a} = 0$.
- For any real number a, $\frac{a}{0}$ is undefined.

Example 1.11: **Simplify:** **a.** -25×0 **b.** $\frac{0}{10}$ **c.** $\frac{32}{0}$

Solution: To find the multiplicative inverse, we find the reciprocal.

(a) $-25 \times 0 = 0$ (b) $\frac{0}{10} = 0$ (c) $\frac{32}{0}$ is undefined.

All the properties of real numbers we have used in this chapter are summarized in Table 1.

Property	of Addition	of Multiplication
Closure Property If a and b are real numbers then...	$a+b$ is also a real number	$a \cdot b$ is also a real number
Commutative Property If a and b are real numbers then...	$a+b=b+a$	$a \cdot b=b \cdot a$
Associative Property If a, b and c are real numbers then...	$a+(b+c)=(a+b)+c$	$a \cdot (b \cdot c)=(a \cdot b) \cdot c$
Identity Property For any real number a,	0 is the additive identity $a+0=a$ $0+a=a$	1 is the multiplicative identity $a \cdot 1=a$ $1 \cdot a=a$
Inverse Property For any real number a,	$-a$ is the additive inverse of a $a+(-a)=0$	For $a \neq 0$ $\frac{1}{a}$ is the multiplicative inverse of a $a \cdot \frac{1}{a}=1$
Distributive Property	If a, b, c are real numbers, then $a \cdot (b+c)=a \cdot b+a \cdot c$	
Properties of Zero For any real number a,	$a \cdot 0=0$ and $0 \cdot a=0$	
For any real number a, $a \neq 0$	$\dfrac{0}{a}=0$ $\dfrac{a}{0}$ is undefined.	

Table 1: Properties of Real Numbers

1.2 Section Exercises

Use the Commutative and Associative Properties *In the following exercises, use the commutative properties to rewrite the given expression.*

1. $8+9=$ ____ **2.** $7+6=$ ____ **3.** $8(-12)=$ ____

4. $y+1=$ ____ **5.** $-2a=$ ____ **6.** $-3m=$ ____

In the following exercises, use the associative properties to rewrite the given expression.

7. $(11+9)+14=$ ____ **8.** $(21+14)+9=$ ____ **9.** $(12\cdot 5)\cdot 7=$ ____

10. $(7\cdot 6)\cdot 9=$ ____ **11.** $(-7+9)+8=$ ____ **12.** $3(4x)=$ ____

Simplify Expressions Using the Commutative and Associative Properties

13. $-45a+15+45a$ **14.** $9y+23+(-9y)$ **15.** $\frac{1}{2}+\frac{7}{8}+\left(-\frac{1}{2}\right)$

16. $\left(\frac{5}{6}+\frac{8}{15}\right)+\frac{7}{15}$ **17.** $\left(\frac{1}{12}+\frac{4}{9}\right)+\frac{5}{9}$ **18.** $14x+19y+25x+3y$

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the distributive property.

19. $4(x+8)$ **20.** $3(a+9)$ **21.** $8(4y+9)$

22. $7(3p–8)$ **23.** $5(7u–4)$ **24.** $\frac{1}{2}(n+8)$

25. $(y+4)p$ **26.** $-3(a+11)$ **27.** $-(r+7)$

Recognize the Identity Properties of Addition and Multiplication

In the following exercises, identify whether each example is using the identity property of addition or multiplication.

28. $101+0=101$ **29.** $\frac{3}{5}(1)=\frac{3}{5}$ **30.** $-9\cdot 1=-9$

Use the Inverse Properties of Addition and Multiplication

In the following exercises, find the additive and multiplicative inverse.

31. 8 **32.** 14 **33.** -17

34. 0.5 **35.** $\frac{7}{12}$ **36.** $\frac{8}{13}$

Use the Properties of Zero

In the following exercises, simplify using the properties of zero.

37. $48\cdot 0$ **38.** $0\div\frac{11}{12}$ **39.** $\dfrac{3}{0}$

40. $\dfrac{0}{24}$ **41.** $5.72\div 0$ **42.** $\dfrac{\frac{1}{10}}{0}$

1.3 Fractions, Decimals, Ratios and Percent

In this section, we will learn about numbers that describe parts of a whole. These numbers, called fractions, are very useful both in algebra and in everyday life. Also we will explore decimals, ratios and percentage in this section.

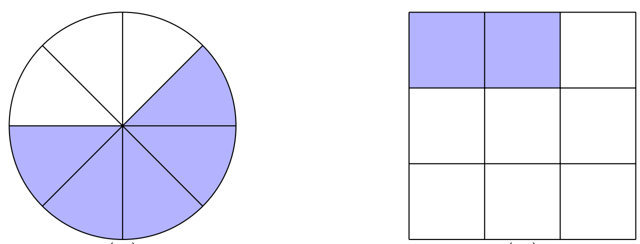
1.3.1 Fractions

A fraction is a way to represent parts of a whole. The denominator b represents the number of equal parts the whole has been divided into, and the numerator a represents how many parts are included. The denominator, b, cannot equal zero because division by zero is undefined.

Fractions

A fraction is written $\frac{a}{b}$, where a and b are integers and $b \neq 0$. In a fraction, a is called the **numerator** and b is called the **denominator**.

Example 1.12: Name the fraction of the shape that is shaded in each of the figures.



(a) (b)

Solution: We need to ask two questions. First, how many equal parts are there? This will be the denominator. Second, of these equal parts, how many are shaded? This will be the numerator.

a. Five out of eight parts are shaded. Therefore, the fraction of the circle shaded is $\frac{5}{8}$.

b. Two out of nine parts are shaded. Therefore, the fraction of the square shaded is $\frac{2}{9}$.

Improper Fractions and Mixed Numbers

Proper and Improper Fractions

The fraction $\frac{a}{b}$ is a proper fraction if $a < b$ and an improper fraction if $a \geq b$.

An improper fraction can be converted into mixed number and vice versa.

The improper fraction $\frac{8}{5}$ is one whole, 1, plus three fifths, $\frac{3}{5}$, or $1\frac{3}{5}$, which is read as *one* *and three-fifths*. The number $a\frac{b}{c}$ is called a **mixed number**. A mixed number consists of a whole number and a fraction.

Convert between Improper Fractions and Mixed Numbers

How To **:: Convert an Improper Fraction to a Mixed number**

How?

Step 1. Divide the denominator by the numerator.

Step 2. Identify the quotient, remainder, and divisor.

Step 3. Write the mixed number as $\mathit{Quotient}\frac{\mathit{remainder}}{\mathit{divisor}}$

Example 1.13: Convert the improper fraction $\frac{33}{8}$ to a mixed number.

Solution: Divide the denominator into the numerator. Identify the quotient, remainder, and divisor.

Quotient $= 4$ and Remainder $= 1$ $\qquad \frac{33}{8}$

$$\frac{33}{8} = 4\frac{1}{8}$$

How To **:: Convert a Mixed number to an Improper Fraction**

How?

Step 1. Multiply the whole number by the denominator.

Step 2. Add the numerator to the product found in Step 1.

Step 3. Write the final sum over the original denominator.

Example 1.14: Convert the mixed number $4\frac{2}{3}$ to an improper fraction.

Solution: $4\frac{2}{3} = \frac{4 \times 3 + 2}{3} = \frac{14}{3}$

Manipulation of fractions

Let us begin with some important concepts which are essential for addition and multiplication of fractions.

Multiples and Factors

A multiple of a number is the product of the number and a counting number. 3, 6, 9, 12, $\cdots$ are the multiples of 3.

Multiple of a Number

A number is a multiple of n if it is the product of a counting number and n.

Another way to say that 20 is a multiple of 4 is to say that 20 is divisible by 4. In fact, $20 \div 4$ is 5, so 20 is 5×4. Notice that 18 is not a multiple of 4. When we divided 18 by 4 we did not get a counting number, so 18 is not divisible by 4.

Divisibility

If a number m is a multiple of n, then we say that m is divisible by n.

Factors of a Number

We know that 72 is the product of 8 and 9, so we can say 72 is a multiple of 8 and 72 is a multiple of 9. Another way to talk about this is to say that 8 and 9 are factors of 72.

Factors

If $a \cdot b = m$, then a and b are factors of m, and m is the product of a and b.

How to find all the factors of any Counting number?

Step 1. Divide the number by each of the counting numbers, in order, until the quotient is smaller than the divisor.

- If the quotient is a counting number, the divisor and quotient are a pair of factors.
- If the quotient is not a counting number, the divisor is not a factor.

Step 2. List all the factor pairs.

Step 3. Write all the factors in order from smallest to largest.

Prime and Composite Numbers

Some numbers, like 72, have many factors. Other numbers, such as 7, have only two factors: 1 and the number. A number with **only two factors** is called a prime number. A number with more than two factors is called a composite number. The number 1 is neither prime nor composite. It has only one factor, itself.

Prime Numbers and Composite Numbers

A prime number is a counting number greater than 1 whose only factors are 1 and itself.

A composite number is a counting number that is not prime.

The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. The composite numbers less than or equal to 25 are 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24 and 25.

Equivalent Fractions

If Ahmed eats $\frac{1}{2}$ of a pizza and Salim eats $\frac{2}{4}$ of the pizza, have they eaten the same amount of pizza? In other words, does $\frac{1}{2} = \frac{2}{4}$?

Equivalent Fractions

Equivalent fractions are fractions that have the same value.

For example, $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{4}{8}$ are all equivalent to $\frac{1}{2}$.
In working with equivalent fractions, you saw that there are many ways to write fractions that have the same value. A fraction is considered simplified if there are no common factors, other than 1, in the numerator and denominator.

Simplified Fraction

A fraction is considered simplified if there are no common factors in the numerator and denominator.

For example, $\frac{2}{3}$ is simplified, but $\frac{10}{15}$ is not simplified. [5 is a common factor of 10 and 15.]

Multiplication of fractions

Fraction Multiplication

If a, b, c and d are numbers where $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Example 1.15: Multiply and write the answer in simplified form:

a. $\frac{3}{4} \cdot \frac{1}{5}$ **b.** $-\frac{5}{8}\left(-\frac{2}{3}\right)$

Solution:

a. $\frac{3}{4} \cdot \frac{1}{5} = \frac{3 \cdot 1}{4 \cdot 5} = \frac{3}{20}$ Multiply the numerators; multiply the denominators.

b. $-\frac{5}{8}\left(-\frac{2}{3}\right) = \frac{5 \cdot 2}{8 \cdot 3}$ The signs are the same, so the product is positive. Multiply the numerators; multiply the denominators.

$= \frac{5 \cdot 2}{2 \cdot 4 \cdot 3}$ Show common factors and then remove them.

$= \frac{5}{12}$ Multiply remaining factors.

Reciprocals

Reciprocal

The reciprocal of the fraction $\frac{a}{b}$ is $\frac{b}{a}$, where $a \neq 0$ and $b \neq 0$.

The product of a number and its reciprocal is 1.

Division of Fractions

Fraction Division

If a, b, c and d are numbers where $b \neq 0$, $c \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

Example 1.16: Divide, and write the answer in simplified form:

a. $\frac{2}{5} \div -\frac{3}{7}$ **b.** $-\frac{3}{4} \div \left(-\frac{7}{8}\right)$

Solution:

a. $\frac{2}{5} \div -\frac{3}{7}$

$= \frac{2}{5}\left(-\frac{7}{3}\right)$ Multiply the first fraction by the reciprocal of the second.

$= -\frac{14}{15}$ Multiply. The product is negative.

b. $-\frac{3}{4} \div \left(-\frac{7}{8}\right)$

$= -\frac{3}{4} \cdot \left(-\frac{8}{7}\right)$ Multiply the first fraction by the reciprocal of the second.

$= \frac{3 \cdot 8}{4 \cdot 7}$ Multiply. Remember to determine the sign first.

$= \frac{3 \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 7}$ Rewrite to show common factors.

$= \frac{6}{7}$ Remove common factors and simplify.

Add and Subtract Fractions with Common Denominators

Fraction Addition and Subtraction

If a, b, and c are numbers where $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

Example 1.17: Simplify:

a. $\frac{3}{5}+\frac{1}{5}$ **b.** $\frac{23}{24}-\frac{14}{24}$

Solution:

a. $\frac{3}{5}+\frac{1}{5} = \frac{3+1}{5}$ Add the numerators and place the sum over the common denominator.

$= \frac{4}{5}$ Simplify.

b. $\frac{23}{24}-\frac{14}{24} = \frac{23-14}{24}$ Subtract the numerators and place the difference over the common denominator.

$= \frac{9}{24}$ Simplify the numerator.

$= \frac{3}{8}$ Simplify the fraction by removing common factors.

Add and Subtract Fractions with Different Denominators

When we add fractions with different denominators we have to convert them to equivalent fractions with a common denominator. Now lets see what you need to do with fractions that have different denominators.

Least Common Denominator

The **least common denominator** (LCD) of two fractions is the **least common multiple** (LCM) of their denominators.

How To **:: Find the Least Common Denominator (LCD) of Two Fractions.**

How?

Step 1. Factor each denominator into its primes.

Step 2. List the primes, matching primes in columns when possible.

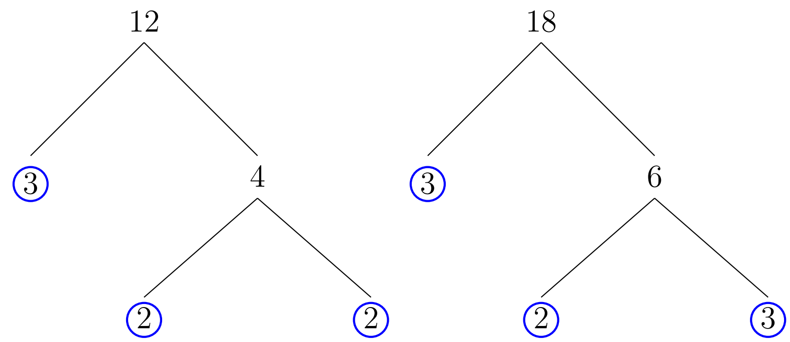
Step 3. Bring down the columns.

Step 4. Multiply the factors. The product is the LCM of the denominators.

Step 5. The LCM of the denominators is the LCD of the fractions.

Example 1.18: Find the LCD for the fractions $\frac{7}{12}$ and $\frac{5}{18}$.

Solution: Factor each denominator into its primes.



List the primes of 12 and the primes of 18 lining them up in columns when possible.

$$12 = 2 \cdot 2 \cdot 3$$
$$18 = 2 \cdot 3 \cdot 3$$

Bring down the columns.

$$\begin{array}{rl} 12 = & 2 \cdot 2 \cdot 3 \\ 18 = & 2 \cdot \, 3 \cdot 3 \\ \hline \textbf{LCD} = & 2 \cdot 2 \cdot 3 \cdot 3 \end{array}$$

The product is the LCM. The LCM of 12 and 18 is 36, so the LCD of $\frac{7}{12}$ and $\frac{5}{18}$ is 36.

Add and Subtract Fractions with Different Denominators

Once we have converted two fractions to equivalent forms with common denominators, we can add or subtract them by adding or subtracting the numerators.

How To **:: Add or Subtract Fractions with Different Denominators.**

Step 1. Find the LCD.
Step 2. Convert each fraction to an equivalent form with the LCD as the denominator.
Step 3. Add or subtract the fractions.
Step 4. Write the result in simplified form.

Example 1.19: Add: $\frac{7}{12} + \frac{5}{18}$.

Solution: First find LCD of $\frac{7}{12}$ $\frac{5}{18}$.

$$\begin{array}{rl} 12 = & 2 \cdot 2 \cdot 3 \\ 18 = & 2 \cdot \, 3 \cdot 3 \\ \hline \textbf{LCD} = & 2 \cdot 2 \cdot 3 \cdot 3 \\ \textbf{LCD} = & 36 \end{array}$$

$\frac{7}{12} + \frac{5}{18} = \frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$ Change into equivalent fractions with the LCD 36.

$= \frac{21}{36} + \frac{10}{36} = \frac{31}{36}$ Simplify the numerators and denominators.

1.3.2 Decimals

Add and Subtract Decimals

> ***How To*** **:: Add and Subtract Decimals.**
>
> Step 1. Write the numbers vertically so the decimal points line up.
> Step 2. Use zeros as place holders, as needed.
> Step 3. Add or subtract the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.

Example 1.20: **a. Add:** $5.7 + 11.9$ **b. Add:** $23.5 + 41.38$

Solution:

a. Write the numbers vertically so the decimal points line up

$$\begin{array}{r} 5.7 \\ \underline{+\ 11.9} \\ 17.6 \end{array}$$

b. Write the numbers vertically so the decimal points line up and Place 0 as a place holder after the 5 in 23.5, so that both numbers have two decimal places

$$\begin{array}{r} 23.50 \\ \underline{+\ 41.38} \\ 64.88 \end{array}$$

Example 1.21: **a. Subtract:** $20{-}14.65$ **b. Subtract:** $2.51{-}7.4$

Solution:

a. Write the numbers vertically so the decimal points line up. Remember 20 is a whole number, so place the decimal point after the 0.

$$\begin{array}{r} 20.00 \\ \underline{-\ 14.65} \\ 5.35 \end{array}$$

b. If we subtract 7.4 from 2.51, the answer will be negative since $7.4 > 2.51$. To subtract easily, we can subtract 2.51 from 7.4. Then we will place the negative sign in the result.

Write the numbers vertically so the decimal points line up. Place 0 as a place holder after the 4 in 7.4, so that both numbers have two decimal places.

$$\begin{array}{r} 7.40 \\ \underline{-\ 2.51} \\ 4.89 \end{array}$$

Remember that we are really subtracting $2.51{-}7.4$ so the answer is negative.

$2.51{-}7.4 = -4.89$

1.3.3 Ratio and Percent

Ratio

Ratios

A ratio compares two numbers or two quantities that are measured with the same unit. The ratio of a to b is written a to b, $\frac{a}{b}$, or a:b.

Because a ratio compares two quantities, we would leave a ratio as $\frac{4}{1}$ instead of simplifying it to 4 so that we can see the two parts of the ratio.

Example 1.22: Write each ratio as a fraction:

a. 15 to 27 **b.** 45 : 18

Solution:

a. 15 to 27 $\frac{15}{27} = \frac{3\cdot 5}{3\cdot 9} = \frac{5}{9}$ Write as a fraction with the first number in the numerator and the second in the denominator.

b. 45 : 18 $\frac{45}{18} = \frac{9\cdot 5}{9\cdot 2} = \frac{5}{2}$

Percent

Percent

A percent is a ratio whose denominator is 100. We use the percent symbol %, to show percent.

Example 1.23: According to a survey, 89 % of college students have a smartphone. Write this percent as a ratio.

Solution: $89\% = \frac{89}{100}$

How To **:: Convert a Percent to a Fraction**

How?

Step 1. Write the percent as a ratio with the denominator 100.

Step 2. Simplify the fraction if possible.

Example 1.24: Convert each percent to a fraction:

a. 36% **b.** 125%

Solution:

a. 36% $= \frac{36}{100} = \frac{9\cdot 4}{25\cdot 4} = \frac{9}{25}$ Write as a ratio with denominator 100 and simplify.

b. 125% $= \frac{125}{100} = \frac{5\cdot 25}{4\cdot 25} = \frac{5}{4}$ Write as a ratio with denominator 100 and simplify.

In Decimals, we learned how to convert fractions to decimals. To convert a percent to a decimal, we first convert it to a fraction and then change the fraction to a decimal.

> ***How To* :: Convert a Pecent to a Decimal**
>
> Step 1. Write the percent as a ratio with the denominator 100.
>
> Step 2. Convert the fraction to a decimal by dividing the numerator by the denominator.

Example 1.25: Convert each percent to a decimal:

a. 135% **b.** 12.5% **c.** 7 %

Solution: Because we want to change to a decimal, we will leave the fractions with denominator 100 instead of removing common factors.

a. $135\% = \dfrac{135}{100} = 1.35$

b. $12.5\% = \dfrac{12.5}{100} = 0.125$

c. $7\% = \dfrac{7}{100} = 0.07$

Do you see the pattern?

To convert a percent number to a decimal number, we move the decimal point two places to the left and remove the % sign. (Sometimes the decimal point does not appear in the percent number, but just like we can think of the integer 6 as 6.0, we can think of 6% as 6.0%.) Notice that we may need to add zeros in front of the number when moving the decimal to the left.

Example 1.26: Convert 77% to: **(a)** a fraction **(b)** a decimal.

Solution:

a. $77\% = \dfrac{77}{100}$ Write as a ratio with denominator 100.

b. $77\% = \dfrac{77}{100} = 0.77$ Write as a ratio with denominator 100 and change the fraction to a decimal

Convert Decimals and Fractions to Percents

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent.

How To **:: Convert a Decimal to a Percent**

Step 1. Write the decimal as a fraction.

Step 2. If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.

Step 3. Write this ratio as a percent.

Example 1.27: Convert each decimal to a percent: **(a.)** 0.05 **(b.)** 0.83

Solution:

a. $0.05 = \frac{5}{100}$ Write as a ratio with denominator 100.

$= 5\%$ Write this ratio as a percent.

b. $0.83 = \frac{83}{100}$ Write as a ratio with denominator 100.

$= 83\%$ Write this ratio as a percent.

Do you see the pattern? To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.

Example 1.28: Convert each fraction or mixed number to a percent:

a. $\frac{3}{4}$ **b.** $\frac{11}{8}$ **c.** $2\frac{1}{5}$

Solution:

a. $\frac{3}{4} = \frac{3}{4} \times 100 = \frac{3 \times 100}{4} = \frac{3 \times 4 \times 25}{4} = 75\%$

b. $\frac{11}{8} = \frac{11}{8} \times 100 = \frac{11 \times 100}{8} = \frac{11 \times 4 \times 25}{2 \times 4} = 137.5\%$

c. $2\frac{1}{5} = \frac{11}{5} \times 100 = \frac{11 \times 100}{5} = \frac{11 \times 5 \times 20}{5} = 220\%$

1.3 Section Exercises

In the following exercises, rewrite the improper fraction as a mixed number.

1. $\frac{3}{2}$ **2.** $\frac{5}{3}$ **3.** $\frac{11}{4}$

In the following exercises, rewrite the mixed number as an improper fraction.

4. $1\frac{2}{3}$ **5.** $2\frac{2}{5}$ **6.** $2\frac{1}{4}$

In the following exercises, simplify each fraction. Do not convert any improper fractions to mixed numbers.

7. $\frac{7}{21}$ **8.** $\frac{8}{24}$ **9.** $\frac{20}{15}$

In the following exercises, multiply, and write the answer in simplified form.

10. $\frac{2}{5} \cdot \frac{1}{3}$

11. $\frac{1}{2} \cdot \frac{3}{8}$

12. $-\frac{5}{9} \cdot \frac{3}{10}$

13. $-\frac{3}{8} \cdot \frac{4}{15}$

14. $\frac{7}{12}\left(-\frac{8}{21}\right)$

15. $\left(-\frac{9}{10}\right)\left(\frac{25}{33}\right)$

In the following exercises, divide, and write the answer in simplified form.

16. $\frac{1}{2} \div \frac{1}{4}$

17. $\frac{3}{4} \div \frac{2}{3}$

18. $-\frac{4}{5} \div \frac{4}{7}$

19. $-\frac{7}{9} \div \left(-\frac{7}{9}\right)$

20. $\frac{3}{4} \div \frac{x}{11}$

21. $\frac{7p}{12} \div \frac{21p}{8}$

In the following exercises, find each sum or difference.

22. $\frac{4}{9} + \frac{1}{9}$

23. $\frac{6}{13} + \left(-\frac{10}{13}\right) + \left(-\frac{12}{13}\right)$

24. $\frac{x}{4} + \frac{3}{4}$

25. $\frac{4}{5} - \frac{1}{5}$

26. $\frac{11}{15} - \frac{7}{15}$

27. $\frac{7}{12} - \frac{2}{12}$

In the following exercises, add or subtract. Write the result in simplified form.

28. $\frac{1}{3} + \frac{1}{5}$

29. $\frac{1}{4} + \frac{1}{10}$

30. $\frac{7}{12} + \frac{5}{8}$

31. $\frac{5}{12} + \frac{3}{18}$

32. $\frac{7}{12} - \frac{9}{16}$

33. $\frac{11}{12} - \frac{3}{8}$

In the following exercises, write each ratio as a fraction.

34. 20 to 36

35. 45 to 54

36. 42 to 48

In the following exercises, write each percent as a ratio.

37. 12%.

38. 35%

39. 2.5%

In the following exercises, convert each percent to a fraction and simplify all fractions.

40. 4%

41. 120%

42. 12.5%

In the following exercises, convert each percent to a decimal.

43. 250%

44. 9%

45. 15%

46. 39.3%

47. 7.5%

48. 100%

In the following exercises, convert each decimal to a percent.

49. 0.01

50. 0.18

51. 1.35

In the following exercises, convert each fraction to a percent.

52. $\frac{1}{4}$

53. $\frac{1}{5}$

54. $5\frac{1}{4}$

Chapter 2

EXPONENTS AND RADICALS

Contents

Learning outcome covered:

c. Demonstrate an understanding of the exponent laws, and apply them to simplify expression and manipulate fractions, ratios, decimals, and percentages.[partial]

e. Simplify rational expressions and rationalize numerators or denominators.

g. Solve linear equations, equations involving radicals, fractional expression and inequalities.

Learning Objectives

At the end of this module, the students will be able to:

- Use the product rule of exponents
- Use the quotient rule of exponents

- ➠ Use the power rule of exponents
- ➠ Use the zero exponent rule
- ➠ Use the rule of negative exponents
- ➠ Find the power of a product and a quotient
- ➠ Simplify Exponential and Radical expressions
- ➠ Add and Subtract Radical Expressions
- ➠ Multiply and Divide Radical Expressions
- ➠ Rationalize denominator
- ➠ Solve Radical Equations

Introduction

Using a calculator, we enter $2048 \times 1536 \times 48 \times 24 \times 3600$ and press ENTER. The calculator displays 1.304596316E13. What does this mean? The E13 portion of the result represents the exponent 13 of ten, so there are a maximum of approximately 1.3×10^{13} bits of data in that one-hour film. In this chapter, we study about integer and rational exponents and review the properties of exponents. Then we define radicals from exponents and explain how to solve radical equations.

2.1 Integer Exponents

Definition of Integer Exponents

When we multiply a number by itself, we **square** it or **raise** it to a power of 2. For example, $4^2 = 4 \cdot 4 = 16$. We can raise any number to any power. In general, the exponential notation a^n means that the number or variable a is used as a factor n times.

$$\overbrace{a \cdot a \cdot a \cdots \cdot a}^{n \text{ times}}$$

In this notation, a^n is read as the n^{th} power of a, where a is called the **base** and n is called the **exponent**.

2.1.1 Rules of Integer Exponents

1. Product Rule of Exponents

Consider the product $x^3 \cdot x^4$. Both terms have the same base x, but they are raised to different exponents.

$$\begin{aligned} x^3 \cdot x^4 &= \overbrace{x \cdot x \cdot x}^{3 \text{ factors}} \cdot \overbrace{x \cdot x \cdot x \cdot x}^{4 \text{ factors}} \\ &= \overbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}^{7 \text{ factors}} \\ &= x^7 \end{aligned}$$

The result is that $x^3 \cdot x^4 = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents.

The product rule of exponents

For any real number a and natural numbers m and n, the product rule of exponents states that

$$a^m \cdot a^n = a^{m+n}$$

Example 2.1: Write each of the following products with a single base. Do not simplify further.

a. $t^5 \cdot t^3$ **b.** $(-3)^5 \cdot (-3)$ **c.** $x^2 \cdot x^5 \cdot x^3$

Solution:

a. $t^5 \cdot t^3 = t^{5+3} = t^8$ [Use the product rule to simplify]

b. $(-3)^5 \cdot (-3) = (-3)^{5+1} = (-3)^6$ [Use the product rule to simplify]

c. $x^2 \cdot x^5 \cdot x^3 = (x^2 \cdot x^5) \cdot x^3 = (x^{2+5}) \cdot x^3 = x^7 \cdot x^3 = x^{7+3} = x^{10}$
Notice we get the same result by adding the three exponents in one step.
$x^2 \cdot x^5 \cdot x^3 = x^{2+5+3} = x^{10}$

2. Quotient Rule of Exponents

The quotient rule of exponents allows us to simplify an expression that divides two numbers with the same base but different exponents.

The quotient rule of exponents

For any real number $a \neq 0$ and positive integers m and n, the power rule of exponents states that

$$\frac{a^m}{a^n} = a^{m-n}$$

Example 2.2: Write each of the following expressions with a single base. Do not simplify further.

a. $\dfrac{(-2)^{14}}{(-2)^9}$ **b.** $\dfrac{t^{23}}{t^{15}}$

Solution:

Use the quotient rule to simplify each expression.

a. $\dfrac{(-2)^{14}}{(-2)^9} = (-2)^{14-9} = (-2)^5$ **b.** $\dfrac{t^{23}}{t^{15}} = t^{23-15} = t^8$

3. Power Rule of Exponents

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power. In this case, you multiply the exponents.

The power rule of exponents

For any real number a and positive integers m and n, the power rule of exponents states that

$$(a^m)^n = a^{mn}$$

Example 2.3: Write each of the following expressions with a single base. Do not simplify further.

a. $\left((x)^2\right)^7$ **b.** $\left((2t)^5\right)^3$

Solution:

Use the quotient rule to simplify each expression.

a. $\left((x)^2\right)^7 = x^{2\cdot 7} = x^{14}$ **b.** $\left((2t)^5\right)^3 = (2t)^{5\cdot 3} = (2t)^{15}$

---

4. Zero Exponent Rule of Exponents

Return to the quotient rule. What would happen if $m = n$? In this case, we would use the zero exponent rule to simplify the expression to 1. To see how this is done, let us begin with an example.

$$\frac{t^8}{t^8} = \frac{\cancel{t^8}}{\cancel{t^8}} = 1$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is $t^0 = 1$. This is true for any non-zero real number, or any variable representing a real number. The sole exception is the expression 0^0.

The zero exponent rule

For any non-zero real number a, the zero exponent rule states that

$$a^0 = 1$$

Example 2.4: Simplify each expression using the zero exponent rule.

a. $\dfrac{c^3}{c^3}$ **b.** $\dfrac{-3x^5}{x^5}$

Solution:

a.
$$\begin{aligned}\frac{c^3}{c^3} &= c^{3-3} && \text{Quotient rule}\\ &= c^0 && \text{Simplify}\\ &= 1 && \text{Zero exponent}\end{aligned}$$

b.
$$\begin{aligned}\frac{-3x^5}{x^5} &= -3\cdot\frac{x^5}{x^5}\\ &= -3\cdot x^{5-5} && \text{Quotient rule}\\ &= -3\cdot x^0 && \text{Simplify}\\ &= -3\cdot 1 && \text{Zero exponent}\\ &= -3\end{aligned}$$

5. Rule of Negative Exponents

When $m < n$ that is, where the difference $m - n$ is negative$(-)$ we can use the rule of negative exponents to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\frac{h^3}{h^5}$.

$$\frac{h^3}{h^5} = \frac{h\cdot h\cdot h}{h\cdot h\cdot h\cdot h\cdot h} = \frac{\cancel{h}\cdot\cancel{h}\cdot\cancel{h}}{\cancel{h}\cdot\cancel{h}\cdot\cancel{h}\cdot h\cdot h} = \frac{1}{h\cdot h} = \frac{1}{h^2}$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{h^3}{h^5} = h^{3-5} = h^{-2}$$

Putting the answers together, we have $h^{-2} = \frac{1}{h^2}$. This is true for any non-zero real number, or any variable representing a non-zero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction barfrom numerator to denominator or vice versa.

$$a^{-n} = \frac{1}{a^n} \text{ and } a^n = \frac{1}{a^{-n}}$$

The rule of negative exponents

For any non-zero real number a and natural number n, the rule of negative exponents states that $a^{-n} = \frac{1}{a^n}$

Example 2.5: Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\frac{\theta^3}{\theta^{10}}$ **b.** $\frac{z^2\cdot z}{z^4}$ **c.** $\frac{(-5t^3)^4}{(-5t^3)^8}$

Solution:

a. $\frac{\theta^3}{\theta^{10}} = \theta^{3-10} = \theta^{-7} = \frac{1}{\theta^7}$

b. $\frac{z^2\cdot z}{z^4} = \frac{z^3}{z^4} = z^{3-4} = z^{-1} = \frac{1}{z}$

c. $\frac{(-5t^3)^4}{(-5t^3)^8} = (-5t^3)^{4-8} = (-5t^3)^{-4} = \frac{1}{(-5t^3)^4}$

6. Power of a Product

To simplify the power of a product of two exponential expressions, we can use the power of a product rule, which breaks up the power of a product of factors into the product of the powers of the factors.

The power of a product rule of exponents

For any non-zero real number a and natural number n, the power of a product rule states that

$$(ab)^n = a^n \cdot b^n$$

Example 2.6: Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $(ab^2)^3$ **b.** $(2t)^{15}$ **c.** $(-2w^3)^3$ **d.** $\dfrac{1}{(-2z)^4}$ **e.** $(e^{-2}f^2)^7$

Solution: Use the product and quotient rules and the new definitions to simplify each expression.

a. $(ab^2)^3 = a^3 \cdot (b^2)^3 = a^{1\cdot 3} \cdot b^{2\cdot 3} = a^3b^6$

b. $(2t)^{15} = (2)^{15} \cdot (t)^{15} = (2)^{15}(t)^{15}$

c. $(-2w^3)^3 = (-2)^3 \cdot (w^3)^3 = -8 \cdot w^{3\cdot 3} = -8w^9$

d. $\dfrac{1}{(-2z)^4} = \dfrac{1}{(-2)^4 \cdot (z)^4} = \dfrac{1}{16z^4}$

e. $(e^{-2}f^2)^7 = (e^{-2})^7 \cdot (f^2)^7 = e^{-2\cdot 7} \cdot f^{2\cdot 7} = e^{-14}f^{14} = \dfrac{f^{14}}{e^{14}}$

7. Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the power of a quotient rule, which states that the power of a quotient of factors is the quotient of the powers of the factors.

The power of a quotient rule of exponents

For any real numbers a and b and any integer $n \neq 0$, the power of a quotient rule of exponents states that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Example 2.7: Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\dfrac{4}{z^{11}}\right)^3$ **b.** $\left(\dfrac{p}{q^3}\right)^6$ **c.** $\left(\dfrac{-1}{t^2}\right)^{27}$

Solution:

a. $\left(\dfrac{4}{z^{11}}\right)^3 = \dfrac{4^3}{(z^{11})^3} = \dfrac{64}{z^{11\cdot 3}} = \dfrac{64}{z^{33}}$

b. $\left(\dfrac{p}{q^3}\right)^6 = \dfrac{p^6}{(q^3)^6} = \dfrac{p^6}{q^{3\cdot 6}} = \dfrac{p^6}{q^{18}}$

c. $\left(\dfrac{-1}{t^2}\right)^{27} = \dfrac{(-1)^{27}}{(t^2)^{27}} = \dfrac{-1}{t^{2\cdot 27}} = \dfrac{-1}{t^{54}}$

Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combining terms or exponents; in other words, to write the expression more simply with fewer terms. The rules of exponents may be combined to simplify expressions.

Example 2.8: Simplify each expression and write the answer with positive exponents only.

a. $(6m^2n^{-1})^3$ **b.** $6^5 \cdot 6^{-4} \cdot 6^{-3}$ **c.** $\left(\dfrac{u^{-1}v}{v^{-1}}\right)^2$

d. $(-2a^3b^{-1})(5a^{-2}b^2)$ **e.** $(x^2y)^4(x^2y)^{-4}$ **f.** $\dfrac{(2w^2)^4}{(6w^{-2})^2}$

Solution:

a. $(6m^2n^{-1})^3$

	$= 6^3(m^2)^3(n^{-1})^3$	The power of a product rule
	$= 6^3m^{2\cdot 3}n^{-1\cdot 3}$	The power rule
	$= 216m^6n^{-3}$	Simplify
	$= \dfrac{216m^6}{n^3}$	The negative exponent rule

b. $6^5 \cdot 6^{-4} \cdot 6^{-3}$

	$= 6^{5-4-3}$	The product rule
	$= 6^{-2}$	Simplify
	$= \dfrac{1}{6^2}$ or $\dfrac{1}{36}$	The negative exponent rule

c. $\left(\dfrac{u^{-1}v}{v^{-1}}\right)^2$

	$= \dfrac{(u^{-1}v)^2}{(v^{-1})^2}$	The power of a quotient rule
	$= \dfrac{u^{-2}v^2}{v^{-2}}$	The power of a product rule
	$= u^{-2}v^{2-(-2)}$	The quotient rule
	$= u^{-2}v^4$	Simplify.
	$= \dfrac{v^4}{u^2}$	The negative exponent rule

d.	$(-2a^3b^{-1})(5a^{-2}b^2)$	$= -2 \cdot 5 \cdot a^3 \cdot a^{-2} \cdot b^{-1} \cdot b^2$	Commutative and associative laws of multiplication
		$= -10 \cdot a^{3-2} \cdot b^{-1+2}$	The product rule
		$= -10ab$	Simplify.
e.	$(x^2y)^4(x^2y)^{-4}$	$= (x^2y)^{4-4}$	The product rule
		$= (x^2y)^0$	Simplify.
		$= 1$	The zero exponent rule
f.	$\dfrac{(2w^2)^4}{(6w^{-2})^2}$	$= \dfrac{2^4(w^2)^4}{6^2(w^{-2})^2}$	The power of a product rule
		$= \dfrac{2^4w^{2\cdot 4}}{6^2w^{-2\cdot 2}}$	The power rule
		$= \dfrac{16w^8}{36w^{-4}}$	Simplify.
		$= \dfrac{4w^{8-(-4)}}{9}$	The quotient rule and reduce fraction
		$= \dfrac{4w^{12}}{9}$	Simplify.

2.1 Section Exercises

Numeric

For the following exercises, simplify the given expression. Write answers with positive exponents.

1. 9^2 **2.** 15^{-2} **3.** $3^2 \cdot 3^3$ **4.** $4^4 \div 4$

5. $(2^2)^{-2}$ **6.** $(5-8)^0$ **7.** $11^3 \div 11^4$ **8.** $6^5 \cdot 6^{-7}$

For the following exercises, write each expression with a single base. Do not simplify further. Write answers with positive exponents.

9. $4^2 \cdot 4^3 \div 4^{-4}$ **10.** $\dfrac{6^{12}}{6^9}$ **11.** $(12^3 \cdot 12)^{10}$ **12.** $10^6 \div (10^{10})^{-2}$

13. $7^6 \cdot 7^{-3}$ **14.** $(3^3 \div 3^4)^5$

Algebraic

For the following exercises, simplify the given expression. Write answers with positive exponents.

15. $\dfrac{a^3a^2}{a}$ **16.** $\dfrac{mn^2}{m^{-2}}$ **17.** $(b^3c^4)^2$ **18.** $\left(\dfrac{x^{-3}}{y^2}\right)^{-5}$

19. $(ab^2) \div d^{-3}$ **20.** $(w^0x^5)^{-1}$ **21.** $\dfrac{m^4}{n^0}$ **22.** $y^{-4}(y^2)^2$

23. $\dfrac{p^{-4}q^2}{p^2q^{-3}}$ **24.** $(l \times w)^2$ **25.** $(y^7)^3 \div x^{14}$ **26.** $\left(\dfrac{a}{2^3}\right)^2$

27. $5^2m \div 5^0m$ **28.** $\dfrac{(16x^2}{y^{-1}}$ **29.** $\dfrac{2^3}{(3a)^{-2}}$ **30.** $(ma^6)^2\dfrac{1}{m^3a^2}$

31. $(b^{-3}c)^3$ **32.** $(x^2y^{13} \div y^0)^2$ **33.** $(9z^3)^{-2}y$

Extensions

For the following exercises, simplify the given expression. Write answers with positive exponents.

34. $\left(\dfrac{3^2}{a^3}\right)^{-2}\left(\dfrac{a^4}{2^2}\right)^2$ **35.** $(6^2 - 24)^2 \div \left(\dfrac{x}{y}\right)^{-5}$ **36.** $\dfrac{m^2n^3}{a^2c^{-3}} \cdot \dfrac{a^{-7}n^{-2}}{m^2c^4}$

37. $\left(\dfrac{x^6y^3}{x^3y^{-3}} \cdot \dfrac{y^{-7}}{x^{-3}}\right)^{10}$ **38.** $\left(\dfrac{(ab^2c)^{-3}}{b^{-3}}\right)^2$

2.2 Rational Exponents

Rational exponents are exponents that are fractions, where the numerator is a power and the denominator is a root. For example, $16^{\frac{1}{2}}$ is another way of writing $\sqrt{16}$, $8^{\frac{1}{3}}$ is another way of writing $\sqrt[3]{8}$. The ability to work with rational exponents is a useful skill, as it is highly applicable in calculus.

Definition

A rational exponent indicates a power in the numerator and a root in the denominator. There are multiple ways of writing an expression, a variable, or a number with a rational exponent:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}}$$

Example 2.9: **Simplify:** **a.** $125^{\frac{2}{3}}$ **b.** $-25^{-\frac{3}{2}}$ **c.** $32^{-\frac{2}{5}}$

Solution:

a.
$$125^{\frac{2}{3}} = (5^3)^{\frac{2}{3}} = 5^{3\cdot\frac{2}{3}} = 5^2 = 25$$

b.
$$-25^{-\frac{3}{2}} = -(5^2)^{-\frac{3}{2}} = -5^{-3} = -\frac{1}{5^3} = -\frac{1}{125}$$

c.
$$32^{-\frac{2}{5}} = (2^5)^{-\frac{2}{5}} = 2^{5\cdot-\frac{2}{5}} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

Properties of Rational Exponents We will list the Properties of Exponenets here to have them for reference as we simplify expressions.

Rational exponents

If a and b are real numbers and m and n are rational numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}$
Power Property	$(a^m)^n = a^{mn}$
Zero Exponent Property	$a^0 = 1, \quad a \neq 0$
Negative Exponent Property	$a^{-n} = \frac{1}{a^n}, \quad a \neq 0$
Product to a Power	$(ab)^m = a^m b^m$
Quotient to a Power	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \quad b \neq 0$

2.2.1 Simplifying Expressions with Rational Exponents

Example 2.10: Simplify:

a. $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$ **b.** $(z^9)^{\frac{2}{3}}$ **c.** $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$

Solution:

a.
$$x^{\frac{1}{2}} \cdot x^{\frac{5}{6}} = x^{\frac{1}{2}+\frac{5}{6}} = x^{\frac{8}{6}} = x^{\frac{4}{3}}$$

b.
$$(z^9)^{\frac{2}{3}} = z^{9\cdot\frac{2}{3}} = z^{\cancel{9}3\cdot\frac{2}{\cancel{3}}} = z^6$$

c.
$$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}} = x^{\frac{1}{3}-\frac{5}{3}} = x^{-\frac{4}{3}} = \frac{1}{x^{\frac{4}{3}}}$$

Example 2.11: Simplify:

a. $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ **b.** $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$

Solution:

a. $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}} = 27^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}} = (3^3)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ The power of a product rule & write 27 as a power of 3.

$= 3^{3\cdot\frac{2}{3}}u^{\frac{1}{2}\cdot\frac{2}{3}}$ The power rule

$= 3^2u^{\frac{1}{3}} = 9u^{\frac{1}{3}}$ Simplify

b. $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}} = \left(m^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(n^{\frac{1}{2}}\right)^{\frac{3}{2}}$ The power of a quotient rule

$= m^{\frac{2}{3}\cdot\frac{3}{2}}n^{\frac{1}{2}\cdot\frac{3}{2}}$ The power rule

$= mn^{\frac{3}{4}}$ Simplify

We will use both the Product Property and the Quotient Property in the next example.

Example 2.12: **Simplify:** **a.** $\dfrac{x^{\frac{3}{4}}\cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ **b.** $\left(\dfrac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$

Solution:

a. $\dfrac{x^{\frac{3}{4}}\cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}} = \dfrac{x^{\frac{3}{4}-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ The product rule

$= \dfrac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$ Simplify

$= x^{\frac{2}{4}+\frac{6}{4}}$ The quotient rule

$= x^{\frac{8}{4}} = x^2$ Simplify

b. $\left(\dfrac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}} = \left(16x^{\frac{4}{3}+\frac{2}{3}}y^{-\frac{5}{6}-\frac{1}{6}}\right)^{\frac{1}{2}}$ The quotient rule

$= \left(4^2x^{\frac{6}{3}}y^{-\frac{6}{6}}\right)^{\frac{1}{2}} = \left(4^2x^2y^{-1}\right)^{\frac{1}{2}}$ Simplify

$= \left(4^2\right)^{\frac{1}{2}}\left(x^2\right)^{\frac{1}{2}}\left(y^{-1}\right)^{\frac{1}{2}}$ The power of a product rule

$= 4^{2\cdot\frac{1}{2}}x^{2\cdot\frac{1}{2}}y^{-1\cdot\frac{1}{2}}$ The power rule

$= 4xy^{-\frac{1}{2}}$ Simplify

$= \dfrac{4x}{y^{\frac{1}{2}}}$ The negative exponent rule

2.2.2 Equations with rational exponents

Sometimes an equation will contain rational exponents. We raise each side of the equation to the power equal to the denominator of the rational exponent.

Example 2.13: **Solve:** $(3x-2)^{\frac{1}{4}}+3=5$

Solution:

$(3x-2)^{\frac{1}{4}}+3$	$=5$	
$(3x-2)^{\frac{1}{4}}$	$=2$	To isolate the term with the rational exponent, subtract 3 from both sides.
$\left[(3x-2)^{\frac{1}{4}}\right]^4$	$=2^4$	Raise each side of the equation to the fourth power.
$3x-2$	$=16$	Simplify.
$3x$	$=18$	Solve the equation.
x	$=6$	

2.2 Section Exercises

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify. Assume all variables are positive.

1. **a.** $c^{\frac{1}{4}}\cdot c^{\frac{5}{8}}$ **b.** $6^{\frac{5}{2}}\cdot 6^{\frac{1}{2}}$ **c.** $y^{\frac{1}{2}}\cdot y^{\frac{3}{4}}$ **d.** $q^{\frac{2}{3}}\cdot q^{\frac{5}{6}}$

2. **a.** $(p^{12})^{\frac{3}{4}}$ **b.** $(b^{15})^{\frac{3}{5}}$ **c.** $(x^{12})^{\frac{2}{3}}$ **d.** $(h^{6})^{\frac{4}{3}}$

3. **a.** $\dfrac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$ **b.** $\dfrac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$ **c.** $\dfrac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$ **d.** $\dfrac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$

4. **a.** $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$ **b.** $\left(m^{\frac{4}{3}}m^{\frac{1}{2}}\right)^{\frac{3}{4}}$ **c.** $\left(4p^{\frac{1}{3}}q^{\frac{1}{2}}\right)^{\frac{3}{2}}$ **d.** $\left(9x^{\frac{2}{5}}y^{\frac{3}{5}}\right)^{\frac{5}{2}}$

5. **a.** $\dfrac{r^{\frac{5}{2}}\cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$ **b.** $\dfrac{a^{\frac{3}{4}}\cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$ **c.** $\dfrac{c^{\frac{5}{3}}\cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$ **d.** $\dfrac{m^{\frac{7}{4}}\cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$

6. **a** $\left(\dfrac{36s^{\frac{1}{5}}t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$ **b** $\left(\dfrac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$ **c** $\left(\dfrac{8x^{\frac{5}{3}}y^{-\frac{1}{2}}}{27x^{-\frac{4}{3}}y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$ **d** $\left(\dfrac{16m^{\frac{1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{2}}$

In the following exercises, simplify.

7. **a.** $81^{\frac{1}{2}}$ **b.** $125^{-\frac{1}{3}}$ **c.** $64^{\frac{1}{2}}$ **d.** $64^{\frac{1}{3}}$ **e.** $32^{\frac{1}{5}}$

8. **a.** $(-32)^{\frac{1}{5}}$ **b.** $(-8)^{\frac{1}{3}}$ **c.** $-49^{\frac{1}{2}}$ **d.** $49^{-\frac{1}{2}}$ **e.** $-16^{\frac{1}{4}}$

In the following exercises, simplify.

9. **a.** $25^{\frac{3}{2}}$ **b.** $(-27)^{\frac{2}{3}}$ **c.** $100^{\frac{3}{2}}$ **d.** $81^{-\frac{3}{2}}$ **e.** $27^{-\frac{2}{3}}$

Solve Equations with Rational Exponents

In the following exercises, solve.

10. **a.** $(6x+1)^{\frac{1}{2}} - 3 = 4$ **b.** $(12x-5)^{\frac{1}{3}} + 8 = 3$ **c.** $(5x-4)^{\frac{1}{4}} + 7 = 9$

11. **a.** $(3x-5)^{\frac{3}{2}} + 1 = 9$ **b.** $(8x+5)^{\frac{1}{3}} + 2 = -1$ **c.** $x^{\frac{2}{3}} - 5 = 4$

2.3 Radicals

n^{th} Root of a number

If $b^n = a$, then b is an n^{th} root of a. The principal n^{th} root of a is written $\sqrt[n]{a}$. n is called the **index** of the radical, a is called the **radicand** and $\sqrt{}$ is called the **radical**.

We use the term **square root** for $\sqrt{a}$ and **cube root** for $\sqrt[3]{a}$.

2.3.1 Converting expressions between radicals and exponents

Rational Exponent $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Example 2.14: Write as a radical expression:

a. $x^{\frac{1}{2}}$ **b.** $y^{\frac{1}{3}}$ **c.** $z^{\frac{1}{4}}$

Solution:

a. $x^{\frac{1}{2}} = \sqrt{x}$ **b.** $y^{\frac{1}{3}} = \sqrt[3]{y}$ **c.** $z^{\frac{1}{4}} = \sqrt[4]{z}$

In the next example, we will write each radical using a rational exponent.

Example 2.15: **Write with a rational exponent:**

a. $\sqrt{5y}$ **b.** $(\sqrt[3]{2x})^4$ **c.** $\sqrt{\left(\dfrac{3a}{4b}\right)^3}$

Solution:

a. $\sqrt{5y} = (5y)^{\frac{1}{2}}$ **b.** $(\sqrt[3]{2x})^4 = (2x)^{\frac{4}{3}}$ **c.** $\sqrt{\left(\dfrac{3a}{4b}\right)^3} = \left(\dfrac{3a}{4b}\right)^{\frac{3}{2}}$

First we define absolute value function before studying the properties of radicals.

Absolute value function

The absolute value function can be defined as a piecewise function $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

2.3.2 Properties of Radicals

1. Definition of $\sqrt[n]{a}$

Properties of $\sqrt[n]{a}$

When n is an **even number** and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an **odd number**, $\sqrt[n]{a}$ is a real number for all values of a.

We will apply these properties in the next two examples. Be alert for the negative signs as well as even and odd powers.

Example 2.16: **Simplify:**

a. $\sqrt[3]{64}$ **b.** $\sqrt[4]{81}$ **c.** $\sqrt[3]{-125}$ **c.** $\sqrt[4]{-16}$

Solution:

a. Since $4^3 = 64$, $\sqrt[3]{64} = 4$.

b. Since $3^4 = 81$, $\sqrt[4]{81} = 3$.

c. Since $(-5)^3 = -125$, $\sqrt[3]{-125} = -5$.

d. Think, $(?)^4 = -16$. No real number raised to the fourth power is negative. Hence $\sqrt[4]{-16}$ is not a real number.

2. Simplifying Odd and Even Roots

Simplifying Odd and Even Roots

For any integer $n \geq 2$,

- when the index n is odd $\sqrt[n]{a^n} = a$.
- when the index n is even $\sqrt[n]{a^n} = |a|$.

`We must use the absolute value signs when we take an even root of an expression with a variable in the radical.`

Example 2.17: Simplify:

a. $\sqrt{x^2}$ **b.** $\sqrt[3]{n^3}$ **c.** $\sqrt{x^6}$ **d.** $\sqrt[4]{z^8}$ **e.** $\sqrt[4]{16q^{12}}$

Solution:

a. We use the absolute value to be sure to get the positive root.Since the index $n = 2$ is even, $\sqrt{x^2} = |x|$.

b. This is an odd indexed root so there is no need for an absolute value sign. Hence $\sqrt[3]{n^3} = n$.

c. $\sqrt{x^6} = \sqrt{(x^3)^2} = |x^3|$. [Since the index is even]

d. $\sqrt[4]{z^8} = \sqrt[4]{(z^2)^4} = |z^2| = z^2$.

[`In this case the absolute value sign is not needed as` z^2 `is positive`].

e. $\sqrt[4]{16q^{12}} = \sqrt[4]{(2q^3)^4} = |2q^3| = 2|q^3|$.[Since the index is even]

3. Simplified Radical Expressions

A radical expression, $\sqrt[n]{a}$, is considered simplified if it has no factors of m^n.

Simplified Radical Expression

For real numbers a and m, and an integer $n \geq 2$,

$$\sqrt[n]{a} \text{ is considered simplified if } a \text{ has no factors of the form } m^n.$$

For example, $\sqrt{6}$ is simplified because there are no perfect square factors in 6. But $\sqrt{24}$ is not simplified because 24 has a perfect square factor of 4. $24 = 6 \cdot 2^2$.

$\sqrt{24} = \sqrt{6 \cdot 2^2} = \sqrt{6}\sqrt{2^2} = |2|\sqrt{6} = 2\sqrt{6}$.

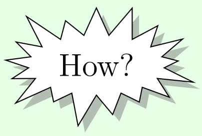
4. Use the Product Property to Simplify Radical Expressions

Product Rule

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and $n \geq 2$ is an integer, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} = \text{ and } \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

How To **:: Simplify a Radical Expression using the Product Property**



Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.

Step 2. Use the product rule to rewrite the radical as the product of two radicals.

Step 3. Simplify the root of the perfect power.

Example 2.18: Simplify:

a. $\sqrt{500}$ **b.** $\sqrt[4]{x^7}$ **c.** $\sqrt[3]{24x^7}$ **d.** $\sqrt{63u^3v^5}$

Solution:

a.	$\sqrt{500}$	$=\sqrt{100 \cdot 5}$	Rewrite the radicand as a product using the largest perfect square factor
		$=\sqrt{100} \cdot \sqrt{5}$	Rewrite the radical as the product of two radicals
		$= 10\sqrt{5}$	Simplify.
b.	$\sqrt[4]{x^7}$	$= \sqrt[4]{x^4 \cdot x^3}$	Rewrite the radicand as a product using the greatest perfect fourth factor
		$= \sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$	Rewrite the radical as the product of two radicals
		$= \lvert x \rvert \sqrt[4]{x^3}$	Simplify.
c.	$\sqrt[3]{24x^7}$	$= \sqrt[3]{8x^6 \cdot 3x}$	Rewrite the radicand as a product using the greatest perfect cube factor
		$= \sqrt[3]{8x^6} \cdot \sqrt[3]{3x}$	Rewrite the radical as the product of two radicals
		$= \sqrt[3]{(2x^2)^3} \cdot \sqrt[3]{3x}$	Rewrite the first radicand as $(2x^2)^3$
		$= 2x^2 \sqrt[3]{3x}$	Simplify.
d.	$\sqrt{63u^3v^5}$	$=\sqrt{9u^2v^4 \cdot 7uv}$	Rewrite the radicand as a product using the largest perfect square factor

$$= \sqrt{9u^2v^4} \cdot \sqrt{7uv} \quad \text{Rewrite the radical as the product of two radicals}$$

$$= \sqrt{(3uv^2)^2} \cdot \sqrt{7uv} \quad \text{Rewrite the first radicand as } (3uv^2)^2$$

$$= |3uv^2|\sqrt{7uv} \quad \text{Simplify.}$$

$$= 3|u|v^2\sqrt{7uv} \quad \text{Simplify.}$$

5. Use the Quotient Property to Simplify Radical Expressions

Quotient Rule

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

***How To* :: Simplify a Radical Expression using the Quotient Property**

How?

Step 1. Simplify the fraction in the radicand, if possible.

Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.

Step 3. Simplify the radicals in the numerator and the denominator.

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them.

Example 2.19: Simplify:

a. $\sqrt[3]{\frac{16}{54}}$ **b.** $\sqrt[4]{\frac{a^{10}}{a^2}}$ **c.** $\sqrt{\frac{18p^5q^7}{32pq^2}}$ **d.** $\frac{\sqrt{48a^7}}{\sqrt{3a}}$

Solution:

a. $\sqrt[3]{\frac{16}{54}}$

$$= \sqrt[3]{\frac{2 \cdot 8}{2 \cdot 27}} \quad \text{Rewrite showing the common factors of the numerator and denominator.}$$

$$= \sqrt[3]{\frac{8}{27}} \quad \text{Simplify the fraction by removing common factors.}$$

$$= \sqrt[3]{\frac{2^3}{3^3}} \quad \text{Rewrite the radicand as a quotient using the greatest perfect cube}$$

$$= \frac{2}{3} \quad \text{Simplify.}$$

b. $\sqrt[4]{\dfrac{a^{10}}{a^2}} = \sqrt[4]{a^8}$ Use Quotient rule of exponent

$= \sqrt[4]{(a^2)^4}$ Rewrite the radicand as a quotient using the greatest perfect fourth power

$= |a^2|$ Since the index n is even

$= a^2$

c. $\sqrt{\dfrac{18p^5q^7}{32pq^2}} = \sqrt{\dfrac{9p^4q^5}{16}}$ Simplify the fraction in the radicand, if possible.

$= \dfrac{\sqrt{9p^4q^5}}{\sqrt{16}}$ Rewrite using the Quotient Property.

$= \dfrac{\sqrt{9p^4q^4 \cdot q}}{4}$ Rewrite each radicand as a product using perfect square factors.

$= \dfrac{\sqrt{9p^4q^4}\sqrt{q}}{4}$ Simplify the radicals in the numerator and the denominator.

$= \dfrac{3p^2q^2\sqrt{q}}{4}$ Simplify.

d. $\dfrac{\sqrt{48a^7}}{\sqrt{3a}} = \sqrt{\dfrac{48a^7}{3a}}$ Use the Quotient Property to write as one radical.

$= \sqrt{16a^6}$ Simplify the fraction under the radical.

$= 4|a^3|$ Simplify.

2.3.3 Add, Subtract, and Multiply Radical Expressions

Add and Subtract Radical Expressions

Like Radicals: Like radicals are radical expressions with the **same index** and the **same radicand**.

We add and subtract like radicals in the same way we add and subtract like terms. When the radicals are not like, you cannot combine the terms.

Example 2.20: Simplify:

a. $2\sqrt{2} - 7\sqrt{2}$ **b.** $5\sqrt[3]{y} + 4\sqrt[3]{y}$ **c.** $\sqrt{20} + 3\sqrt{5}$ **d.** $7\sqrt[4]{x} - 2\sqrt[4]{y}$

Solution

a. $2\sqrt{2} - 7\sqrt{2} = -5\sqrt{2}$.

b. $5\sqrt[3]{y} + 4\sqrt[3]{y} = 9\sqrt[3]{y}$.

c. $\sqrt{20}+3\sqrt{5}=\sqrt{4\cdot 5}+3\sqrt{5}$ Simplify the radicals, when possible.

$=2\sqrt{5}+3\sqrt{5}$ Simplify.

$=5\sqrt{5}$ Combine the like radicals.

d. $7\sqrt[4]{x}-2\sqrt[4]{y}=7\sqrt[4]{x}-2\sqrt[4]{y}$. [*Since the radicals are not like, we cannot subtract them.*]

It is common practice at this point for us to ***assume all variables are greater than or equal to zero so that absolute values are not needed***. We will use this assumption throughout the rest of this chapter.

Example 2.21: Simplify:

a. $9\sqrt{50m^2}-6\sqrt{48m^2}$ **b.** $\sqrt{32x^7}-\sqrt{50x^7}$

Solution:

a. $9\sqrt{50m^2}-6\sqrt{48m^2}=9\sqrt{2\cdot 25m^2}-6\sqrt{3\cdot 16m^2}$ Simplify the radicals.

$=45m\sqrt{2}-24m\sqrt{3}$ The radicals are not like and so cannot be combined.

b. $\sqrt{32x^7}-\sqrt{50x^7}=\sqrt{2\cdot 16x^6\cdot x}-\sqrt{2\cdot 25x^6\cdot x}$ Simplify the radicals, when possible.

$=4x^3\sqrt{2x}-5x^3\sqrt{2x}$ Simplify.

$=-x^3\sqrt{2x}$ Combine the like radicals.

Multiply Radical Expressions

Remember, we assume all variables are greater than or equal to zero. We will use the Distributive Property to multiply expressions with radicals and then simplify the radicals when possible.

Example 2.22: Simplify:

a. $(10\sqrt{6p^3})(4\sqrt{3p})$ **b.** $(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$

Solution:

a. $(10\sqrt{6p^3})(4\sqrt{3p})=40\sqrt{18p^4}$ Multiply using the Product Property.

$=40\sqrt{9p^4}\cdot\sqrt{2}$ Simplify the radical.

$=40\cdot 3p^2\sqrt{2}$ Simplify.

$=120p^2\sqrt{2}$

b. $(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})=3\sqrt{2}\cdot\sqrt{2}+3\sqrt{2}\cdot 4\sqrt{5}-\sqrt{5}\cdot\sqrt{2}-\sqrt{5}\cdot 4\sqrt{5}$ Multiply.

$=3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$ Multiply.

$$= 6 + 12\sqrt{10} - \sqrt{10} - 20 \quad \text{Simplify.}$$
$$= -14 + 11\sqrt{10} \quad \text{Combine like terms.}$$

We will use the special product formulas in the next few examples.
$(a+b)^2 = a^2 + 2ab + b^2$, $(a-b)^2 = a^2 - 2ab + b^2$ and $(a+b)(a-b) = a^2 - b^2$.

Example 2.23: Simplify:

a. $(2+\sqrt{3})^2$ **b.** $(4-2\sqrt{5})^2$ **c.** $(5-2\sqrt{3})(5+2\sqrt{3})$

Solution:

a. $(2+\sqrt{3})^2$

$= 2^2 + 2 \cdot 2 \cdot \sqrt{3} + (\sqrt{3})^2$	Multiply, using the Product of Binomial Squares Pattern
$= 4 + 4\sqrt{3} + 3$	Simplify.
$= 7 + 4\sqrt{3}$	Combine like terms.

b. $(4-2\sqrt{5})^2$

$= 4^2 - 2 \cdot 4 \cdot 2\sqrt{5} + (2\sqrt{5})^2$	Multiply, using the Product of Binomial Squares Pattern
$= 16 - 16\sqrt{5} + 4 \cdot 5$	Simplify.
$= 16 - 16\sqrt{5} + 20$	Simplify.
$= 36 - 16\sqrt{5}$	Combine like terms.

c. $(5-2\sqrt{3})(5+2\sqrt{3})$

$= 5^2 - (2\sqrt{3})^2$	Multiply, using the Product of Conjugates Pattern.
$= 25 - 4 \cdot 3$	Simplify.
$= 25 - 12$	
$= 13$	

2.3.4 Rationalizing the Denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

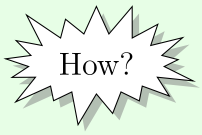
Simplified Radical Expressions

Simplified Radical Expressions

A radical expression is considered simplified if there is

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

1. Rationalize a One Term Denominator

> ***How To*** **:: Rationalize a One Term Denominator**
>
> 
>
>
> - To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.
> - To rationalize a denominator with a higher index radical, we multiply the numerator and denominator by a radical that would give us a radicand that is a perfect power of the index.

To rationalize a denominator with a square root, we use the property that $(\sqrt{a})^2 = a$.

Example 2.24: Rationalize the denominator: a. $\dfrac{4}{\sqrt{3}}$ **b.** $\dfrac{3}{\sqrt{6x}}$

Solution:

a.
$$\frac{4}{\sqrt{3}} = \frac{4\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}$$
Multiply both the numerator and denominator by $\sqrt{3}$.
$$= \frac{4\sqrt{3}}{3}$$
Simplify.

b.
$$\frac{3}{\sqrt{6x}} = \frac{3\cdot\sqrt{6x}}{\sqrt{6x}\cdot\sqrt{6x}}$$
Multiply the numerator and denominator by $\sqrt{6x}$.
$$= \frac{3\sqrt{6x}}{6x}$$
Simplify.
$$= \frac{\sqrt{6x}}{2x}$$
Simplify.

2. Rationalize a Two Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates Pattern to rationalize the denominator.

Example 2.25: Rationalize the denominator of $\dfrac{5}{2-\sqrt{3}}$

Solution: Multiply the numerator and denominator by the conjugate of the denominator.

$$\begin{aligned}\frac{5}{2-\sqrt{3}} &= \frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} \\ &= \frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2} && \text{Multiply the conjugates in the denominator.} \\ &= \frac{5(2+\sqrt{3})}{4-3} && \text{Simplify the denominator.}\end{aligned}$$

$$= \frac{5(2+\sqrt{3})}{1} \qquad \text{Simplify the denominator.}$$

$$= 5(2+\sqrt{3}) \qquad \text{Simplify.}$$

Example 2.26: Rationalize the denominator of $\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$

Solution: Multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}} = \frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$$

$$= \frac{(\sqrt{x}+\sqrt{7})^2}{(\sqrt{x})^2-(\sqrt{7})^2} \qquad \text{Multiply the conjugates in the denominator.}$$

$$= \frac{(\sqrt{x}+\sqrt{7})^2}{x-7} \qquad \text{Simplify the denominator.}$$

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

2.3.5 Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a radical equation.

Radical Equation: An equation in which a variable is in the radicand of a radical expression is called a radical equation.

Example 2.27: Solve: $\sqrt{5n-4}-9=0$

Solution:

Isolate the radical on one side of the equation.	To isolate the radical add 9 to both sides. Simplify	$\sqrt{5n-4}-9=0$ $\sqrt{5n-4}-9+9=0+9$ $\sqrt{5n-4}=9$
Raise both sides of the equation to the power of the index.	Since the index is 2, square both sides.	$(\sqrt{5n-4})^2=9^2$
Solve the new equation.	Remember $(\sqrt{a})^2=a$.	$5n-4=81$ $5n=85$ $n=17$

How To **:: Solve a Radical Equation with One Radical.**

How?

Step 1. Isolate the radical on one side of the equation.

Step 2. Raise both sides of the equation to the power of the index.

Step 3. Solve the new equation.

Step 4. Check the answer in the original equation.

When the index of the radical is 3, we cube both sides to remove the radical.

Example 2.28: Solve: $\sqrt[3]{5x+1}+8=4$

Solution:

$\sqrt[3]{5x+1}=-4$	To isolate the radical, subtract 8 from both sides.
$(\sqrt[3]{5x+1})^3=(-4)^3$	Cube both sides.
$5x+1=-64$	Simplify.
$5x=-65$	Solve the equation.
$x=-13$	

2.3 Section Exercises

Simplify Expressions with Roots

In the following exercises, simplify.

1. **a.** $\sqrt{64}$ **b.** $-\sqrt{81}$ **c.** $\sqrt[5]{-32}$ **d.** $\sqrt{\dfrac{64}{121}}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using ***absolute values*** *as necessary.*

2. **a.** $\sqrt{x^6}$ **b.** $\sqrt[3]{-8c^9}$ **c.** $\sqrt[4]{16x^8}$ **d.** $\sqrt[5]{a^{10}}$

In the following exercises, write as a radical expression.

3. **a.** $x^{\frac{1}{2}}$ **b.** $c^{\frac{1}{9}}$ **c.** $(2x^5y^2)^{\frac{1}{3}}$ **d.** $(2xy^2)^{\frac{1}{5}}$

In the following exercises, write with a rational exponent.

4. **a.** $\sqrt[7]{x}$ **b.** $\sqrt[4]{5x}$ **c.** $\sqrt[4]{\left(\dfrac{2xy}{5z}\right)^2}$ **d.** $\sqrt{x^5}$

Use the Product Property to Simplify Radical Expressions

In the following exercises, simplify using absolute value signs as needed.

5. **a.** $\sqrt{27}$ **b.** $\sqrt[4]{48y^{16}}$ **c.** $\sqrt[4]{32x^5y^4}$ **d.** $\sqrt[3]{81p^7q^8}$

Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify.

6. **a.** $\sqrt{\dfrac{45}{80}}$ **b.** $\sqrt{\dfrac{28p^7}{q^2}}$ **c.** $\sqrt[3]{\dfrac{24x^8y^4}{81x^2y}}$ **d.** $\sqrt[4]{\dfrac{6}{96}}$

Add and Subtract Radical Expressions

In the following exercises, simplify

7. **a.** $\sqrt{48} - \sqrt{27}$ **b.** $\sqrt[3]{24} - \sqrt[3]{81}$ **c.** $8\sqrt[3]{64q^6} - 3\sqrt[3]{125q^6}$

Multiply Radical Expressions

In the following exercises, simplify

8. **a.** $(-2\sqrt{3})(3\sqrt{18})$ **b.** $\sqrt{7}(5+2\sqrt{7})$ **c.** $(3-2\sqrt{7})(5-4\sqrt{7})$

9. **a.** $(3+\sqrt{5})^2$ **b.** $(2-5\sqrt{3})^2$ **c.** $(4+\sqrt{2})(4-\sqrt{2})$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

10. **a.** $\dfrac{10}{\sqrt{6}}$ **b.** $\sqrt{\dfrac{4}{27}}$ **c.** $\dfrac{6}{\sqrt[4]{9x^3}}$

Rationalize a Two Term Denominator

In the following exercises, rationalize the denominator.

11. **a.** $\dfrac{8}{1-\sqrt{5}}$ **b.** $\dfrac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$ **c.** $\dfrac{3\sqrt{2}+2\sqrt{3}}{5\sqrt{2}-3\sqrt{3}}$

Solve Radical Equations

In the following exercises, solve.

12. **a.** $2\sqrt{5x+1} - 8 = 0$ **b.** $\sqrt[3]{4x+5} - 2 = -5$ **c.** $\sqrt{6n+1} + 4 = 8$

Chapter 3

POLYNOMIALS

Contents

Learning outcome covered:

j. `Perform operations on polynomials and manipulate numerical and polynomial expressions and solve first degree equations.`

Learning Objectives

At the end of this chapter, the students will be able to:

- Determine the degree of polynomials
- Add and subtract polynomials
- Multiply a polynomial by another polynomial
- Simplify special products

Introduction

A polynomial is a special algebraic expression and is used widely in algebra. In this chapter you will investigate polynomials and polynomial functions and learn how to perform mathematical operations on them.

3.1 Addition and Subtraction of Polynomials

Let us start with some basic concepts.

Term

A **term** is a constant or the product of a constant and one or more variables.

Examples of terms are 7, y, $5x^2$, $9a$, and b^5.

Coefficient

The **coefficient** of a term is the constant that multiplies the variable in a term.

Think of the coefficient as the number in front of the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x = 1 \cdot x$.

Like Terms

Terms that are either constants or have the same variables raised to the same powers are called **like terms**.

$5x$ and $3x$ are like terms where as $2x^2y$ and $3xy^2$ are not like terms.

If there are like terms in an expression, you can simplify the expression by combining the like terms. We add the coefficients and keep the same variable. The following example shows how to combine like terms.

$$\begin{aligned} 2x^2 + 3x + 7 + x^2 + 4x + 5 &= 2x^2 + x^2 + 3x + 4x + 7 + 5 \\ &= 3x^2 + 7x + 12 \end{aligned}$$

3.1.1 Determine the Degree of Polynomials

We have learned that a term is a constant or the product of a constant and one or more variables. A monomial is an algebraic expression with one term. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a monomial in one variable. Some examples of monomial in one variable are x, $4y^2$, $3a$. Monomials can also have more than one variable such as $-4a^2b^3c^2$.

Monomial

A monomial is an algebraic expression with one term.
A monomial in one variable is a term of the form ax^m, where a is a constant and m is a whole number.

Polynomials

polynomial - A monomial, or two or more monomials, combined by addition or subtraction

monomial - A polynomial with exactly one term

binomial - A polynomial with exactly two terms

trinomial - A polynomial with exactly three terms

Here are some examples of polynomials:

Polynomial	$y+1$	$4a^2-7ab+2b^2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	$-13a^3b^2c$
Binomial	$a+7b$	$4x^2-y^2$	y^2-16	$3p^3q-9p^2q$
Trinomial	$x^2-7x+12$	$9m^2+2mn-8n^2$	$6k^4-k^3+8k$	z^4+3z^2-1

We use the words monomial, binomial, and trinomial when referring to these special polynomials and just call all the rest polynomials.

The **degree of a polynomial** and the degree of its terms are determined by the exponents of the variable.

A monomial that has no variable, just a constant, is a special case. The **degree of a constant term** is 0.

Degree of a Polynomial

The **degree of a term** is the sum of the exponents of its variables.

The **degree of a constant** is 0.

The **degree of a polynomial** is the highest degree of all its terms.

Let's start by looking at a monomial. The monomial $8ab^2$ has two variables a and b. To find the degree we need to find the sum of the exponents. The variable a doesn't have an exponent written, but remember, that means the exponent is 1. The exponent of b is 2. The sum of the exponents, 1 + 2, is 3 so the degree is 3.

Here are some additional examples.

Monomials	14	$8ab^2$	$-9x^3y^5$	$-13a$
Degree	0	3	8	1
Binomials	$h+7$	$7b^2-3b$	x^2y^2-25	$4n^3-8n^2$
Degree of each term	1 0	2 1	4 0	3 2
Degree of polynomial	1	2	4	3
Trinomial	$x^2-12x+7$	$9a^2-6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of each term	2 1 0	2 2 2	4 5 6	4 2 0
Degree of polynomial	2	2	6	4
Polynomial	$y-1$	$3y^2-2y-5$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	1 0	2 1 0	4 3 2 1 0	
Degree of polynomial	1	2	4	

Example 3.1: Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. $7y^2-5y+3$ **b.** $-2a^4b^2$ **c.** $3x^5-4x^3-6x^2+x-8$ **d.** $2y-8xy^3$ **e.** 15

Solution:

	Polynomial	Number of terms	Type	Degree of terms	Degree of polynomial
a.	$7y^2-5y+3$	3	Trinomial	2, 1, 0	2
b.	$-2a^4b^2$	1	Monomial	6	6
c.	$3x^5-4x^3-6x^2+x-8$	5	Polynomial	5, 3, 2, 1, 0	5
d.	$2y-8xy^3$	2	Binomial	1, 4	4
e.	15	1	Monomial	0	0

Working with polynomials is easier when you list the terms in descending order of degrees. When a polynomial is written this way, it is said to be in **standard form of a polynomial**. Get in the habit of writing the term with the highest degree first.

3.1.2 Add and Subtract Polynomials

Example 3.2: Simplify:

a. $a^2 + 7b^2 - 6a^2$ **b.** $u^2v + 5u^2 - 3v^2$

Solution:

a. $a^2 + 7b^2 - 6a^2 = -5a^2 + 7b^2$ Combine like terms.

b. $u^2v + 5u^2 - 3v^2 = u^2v + 5u^2 - 3v^2$ There are no like terms to combine.

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like termsthose with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

Example 3.3: Find the sum: $(7y^2 - 2y + 9) + (4y^2 - 8y - 7)$

Solution:

$$
\begin{aligned}
(7y^2 - 2y + 9) + (4y^2 - 8y - 7) &= (7y^2 - 2y + 9) + (4y^2 - 8y - 7) && \text{Identify like terms.} \\
&= 7y^2 + 4y^2 - 2y - 8y + 9 - 7 && \text{Rewrite without parentheses} \\
&= 11y^2 - 10y + 2 && \text{Combine like terms.}
\end{aligned}
$$

Be careful with the signs as you distribute while subtracting the polynomials in the next example.

Example 3.4: Find the difference: $(9w^2 - 7w + 5) - (2w^2 - 4)$

Solution:

$$
\begin{aligned}
(9w^2 - 7w + 5) - (2w^2 - 4) &= 9w^2 - 7w + 5 - 2w^2 + 4 && \text{Distribute and identify like terms.} \\
&= 9w^2 - 2w^2 - 7w + 5 + 4 && \text{Rearrange the terms.} \\
&= 7w^2 - 7w + 9 && \text{Combine like terms.}
\end{aligned}
$$

To subtract a from b, we write it as $b - a$, placing the b first.

Example 3.5: Subtract $p^2 + 10pq - 2q^2$ from $p^2 + q^2$.

Solution:

$$
\begin{aligned}
(p^2 + q^2) - (p^2 + 10pq - 2q^2) &= p^2 + q^2 - p^2 - 10pq + 2q^2 && \text{Distribute.} \\
&= p^2 - p^2 - 10pq + q^2 + 2q^2 && \text{Rearrange the terms, to put like terms together.} \\
&= -10pq + 3q^2 && \text{Combine like terms.}
\end{aligned}
$$

3.1 Section Exercises

Determine the Type of Polynomials

In the following exercises, determine if the polynomial is a monomial, binomial, trinomial, or other polynomial.

1.

a. $47x^5 - 17x^2y^3 + y^2$

b. $5c^3 + 11c^2 - c - 8$

2.

a. $x^2 - y^2$

b. $-13c^4$

3.

a. $8y - 5x$

b. $y^2 - 5yz - 6z^2$

Add and Subtract Polynomials

In the following exercises, add the polynomials.

4. $(5y^2 + 12y + 4) + (6y^2 - 8y + 7)$

5. $(x^2 + 6x + 8) + (-4x^2 + 11x - 9)$

6. $(8x^2 - 5x + 2) + (3x^2 + 3)$

7. $(5a^2 + 8) + (a^2 - 4a - 9)$

In the following exercises, subtract the polynomials.

8. $(4m^2 - 6m - 3) - (2m^2 + m - 7)$

9. $(3b^2 - 4b + 1) - (5b^2 - b - 2)$

10. $(12s^2 - 15s) - (s - 9)$

11. $(a^2 + 8a + 5) - (a^2 - 3a + 2)$

In the following exercises, subtract the polynomials.

12. Subtract $(9x^2 + 2)$ from $(12x^2 - x + 6)$

13. Subtract $(5y^2 - y + 12)$ from $(5y^2 - 8y - 20)$

14. Subtract $(7w^2 - 4w + 2)$ from $(8w^2 - w + 6)$

15. Subtract $(5x^2 - x + 12)$ from $(9x^2 - 6x - 20)$

In the following exercises, add or subtract the polynomials.

16. $(p^3 - 3p^2q) + (2pq^2 + 4q^3) - (3p^2q + pq^2)$

17. $(a^3 - 2a^2b) + (ab^2 + b^3) - (3a^2b + 4ab^2)$

18. $(x^3 - x^2y) - (4xy^2 - y^3) + (3x^2y - xy^2)$

19. $(x^3 - 2x^2y) - (xy^2 - 3y^3) - (x^2y - 4xy^2)$

20. Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

	Polynomial	Number of terms	Type	Degree of polynomial
a.	$3x^2 - 5xy + 3xy^2$			
b.	$-5xy^2$			
c.	$5x^3y^2z^3 + y^5z^2$			
d.	$5x - 1$			
e.	$x^3 - 3x^2y + 3xy^3 - y^3$			
f.	2			

3.2 Multiplication of Polynomials

3.2.1 Multiplication of Polynomials

Multiply Monomials

We are ready to perform operations on polynomials. Since monomials are algebraic expressions, we can use the properties of exponents to multiply monomials.

Example 3.6: Multiply: **a.** $(3x^2)(-4x^3)$ **b.** $\left(\frac{5}{6}x^3y\right)(2xy^2)$

Solution:

a. $(3x^2)(-4x^3)$

$= 3 \cdot (-4) \cdot x^2 \cdot x^3$ Rearrange the terms.

$= -12x^5$ Multiply.

b. $\left(\frac{5}{6}x^3y\right)(12xy^2)$

$= \frac{5}{6} \cdot 12 \cdot x^3 \cdot x \cdot y \cdot y^2$ Rearrange the terms.

$= 10x^4y^3$ Multiply.

Multiply a Polynomial by a Monomial

Multiplying a polynomial by a monomial is really just applying the Distributive Property.

Example 3.7: Multiply: a. $-2y(4y^2+3y-5)$ **b.** $3x^3y(x^2-8xy+y^2)$

Solution:

a. $-2y(4y^2+3y-5)$

$= -2y\cdot(4y^2)+(-2y)\cdot(3y)-(-2y)\cdot(5)$ Distribute.

$= -8y^3-6y^2+10y$ Multiply.

b. $3x^3y(x^2-8xy+y^2)$

$= 3x^3y\cdot x^2-(3x^3y)\cdot(8xy)+(3x^3y)\cdot y^2$ Distribute.

$= 3x^5y-24x^4y^2+3x^3y^3$ Multiply.

Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial. We will start by using the Distributive Property.

Example 3.8: Multiply: a. $(y+5)(y+8)$ **b.** $(4y+3)(2y-5)$

Solution:

a. $(y+5)(y+8)$

$(y+5)(y+8) = y(y+8)+5(y+8)$ Distribute (y+8).

$= y^2+8y+5y+40$ Distribute again.

$= y^2+13y+40$ Combine like terms.

b. $(4y+3)(2y-5)$

$(4y+3)(2y-5) = 4y(2y-5)+3(2y-5)$ Distribute (y+8).

$= 8y^2-20y+6y-15$ Distribute again.

$= 8y^2-14y-15$ Combine like terms.

F O I L

If you multiply binomials often enough you may notice a pattern.

- Notice that the first term in the result is the product of the first term in each binomial.
- The second and third terms are the product of multiplying the two outer terms and then the two inner terms.
- And the last term results from multiplying the two last terms.
  We abbreviate First, Outer, Inner, Last as ***FOIL****.*

Example 3.9: Multiply: $(y-7)(y+4)$

Solution:

Step 1. Multiply the First terms.	$(y-7)(y+4)$	$=y^2$
Step 2. Multiply the Outer terms.	$(y-7)(y+4)$	$=y^2+4y$
Step 3. Multiply the Inner terms.	$(y-7)(y+4)$	$=y^2+4y-7y$
Step 4. Multiply the Last terms.	$(y-7)(y+4)$	$=y^2+4y-7y-28$
Step 5. Combine like terms.		$=y^2-3y-28$

Multiply a Polynomial by a Polynomial

Now were ready to multiply a polynomial by a polynomial.

Example 3.10: Multiply $(b+3)(2b^2—5b+8)$.

Solution:

$(b+3)(2b^2—5b+8) = b\cdot(2b^2)—b\cdot(5b)+b\cdot 8+3\cdot(2b^2)+3\cdot(-5b)+3\cdot 8$	Distribute.
$= 2b^3—5b^2+8b+6b^2-15b+24$	Multiply.
$= 2b^3+b^2-7b+24$	Combine like terms.

3.2.2 Special Products

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and multiplying them, there is less work to do if you learn to use a pattern.

Binomial Squares Pattern

If a and b are real numbers,

- $(a+b)^2 = a^2+2ab+b^2$
- $(a-b)^2 = a^2-2ab+b^2$

To square a binomial, square the first term, square the last term , double their product.

Example 3.11: **Multiply:** **a.** $(x+5)^2$ **b.** $(2x-3y)^2$

Solution:

a. $(x+5)^2 \quad = x^2 + 2 \cdot x \cdot 5 + 5^2 \quad (a+b)^2 = a^2 + 2ab + b^2$

$= x^2 + 10x + 25 \quad$ Simplify.

b. $(2x-3y)^2 \quad = (2x)^2 - 2 \cdot 2x \cdot 3y + (3y)^2 \quad (a-b)^2 = a^2 - 2ab + b^2$

$= 4x^2 - 12xy + 9y^2 \quad$ Simplify.

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a **conjugate pair** and is of the form $(a–b)$, $(a+b)$.

The product of conjugates is always of the form $a^2–b^2$. This is called a **difference of squares**. This leads to the pattern:

Product of Conjugates Pattern

If a and b are real numbers,

$$(a–b)(a+b) = a^2 - b^2.$$

Example 3.12: **Multiply using the product of conjugates pattern:**

a. $(2x+5)(2x-5)$ **b.** $(5m–9n)(5m+9n)$

Solution:

a. $(2x+5)(2x-5) \quad = (2x)^2 - 5^2 \quad (a–b)(a+b) = a^2 - b^2$

$= 4x^2 - 25 \quad$ Simplify.

b. $(5m–9n)(5m+9n) \quad = (5m)^2 - (9n)^2 \quad (a–b)(a+b) = a^2 - b^2$

$= 25m^2 - 81n^2 \quad$ Simplify.

3.2 Section Exercises

Multiply Monomials

In the following exercises, multiply the monomials.

1. $(6y^7)(-3y^4)$

2. $(-10x^5)(-3x^3)$

3. $(-8u^6)(-9u)$

4. $\left(\frac{4}{7}rs^2\right)(14rs^3)$

5. $\left(\frac{5}{8}x^3y\right)(24x^5y)$

6. $\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)$

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

7. $-8x(x^2+2x–15)$

8. $-5t(t^2+3t–18)$

9. $-8y(y^2+2y–15)$

10. $5pq^3(p^2–2pq+6q^2)$

11. $9r^3s(r^2–3rs+5s^2)$

12. $-4y^2z^2(3y^2+12yz–z^2)$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials.

13. $(w+5)(w+7)$

14. $(2xy+3)(3xy+2)$

15. $(4p+11)(5p-4)$

16. $(7q+4)(3q-8)$

17. $(x^2+3)(x+2)$

18. $(y^2-4)(y+3)$

Multiply a Polynomial by a Polynomial

In the following exercises, multiply the polynomials.

19. $(x+5)(x^2+4x+3)$

20. $(u+4)(u^2+3u+2)$

21. $(y+8)(4y^2+y–7)$

22. $(a+10)(3a^2+a–5)$

23. $(y^2–3y+8)(4y^2+y–7)$

Multiply Special Products

In the following exercises, square each binomial using the Binomial Squares Pattern.

24. $(w+4)^2$

25. $(q+12)^2$

26. $(3x-5)^2$

27. $(2y–3z)^2$

28. $\left(y+\frac{1}{4}\right)^2$

29. $(3x^2+2)^2$

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

30. $(5k+6)(5k-6)$

31. $(9c+5)(9c-5)$

32. $(7w+10x)(7w-10x)$

Chapter 4

FACTORING POLYNOMIALS

Contents

Learning outcome covered:

j. `Perform operations on polynomials and manipulate numerical and polynomial expressions and solve first degree equations.`

Learning Objectives

At the end of this chapter, the students will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial and by grouping
- Factor trinomials of the form $x^2 + bx + c$ and $ax^2 + bx + c$
- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes

Introduction

An epidemic of a disease has broken out. Where did it start? How is it spreading? What can be done to control it? Answers to these and other questions can be found by scientists known as epidemiologists. They collect data and analyze it to study disease and consider possible control measures. Because diseases can spread at alarming rates, these scientists must use their knowledge of mathematics involving **factoring**. In this chapter, you will learn how to factor and apply factoring to real-life situations.

4.1 Greatest Common Factor and Factor by Grouping

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.

Multiply $\longrightarrow$

$$\underbrace{8 \cdot 7}_{factors} = \underbrace{56}_{product} \qquad \underbrace{2x(x+3)}_{factors} = \underbrace{2x^2 + 6x}_{product}$$

$\longleftarrow$ Factor

Greatest Common Factor

The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

We summarize the steps we use to find the greatest common factor.

How To **:: Find the greatest common factor (GCF) of two expressions**

How?

Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.

Step 2. List all factorsmatching common factors in a column. In each column, circle the common factors.

Step 3. Bring down the common factors that all expressions share.

Step 4. Multiply the factors.

The next example will show us the steps to find the greatest common factor of three expressions.

Example 4.1: Find the greatest common factor of $21x^3$, $9x^2$, $15x$.

Solution:

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.

$$\begin{aligned} 21x^3 &= 3 \cdot \quad\quad\quad 7 \cdot x \cdot x \cdot x \\ 9x^2 &= 3 \cdot 3 \cdot \quad\quad\quad x \cdot x \\ 15x &= 3 \cdot \quad\quad 5 \cdot \quad x \end{aligned}$$

Bring down the common factors.

$$\text{GCF} = 3 \cdot \quad\quad\quad x$$

Multiply the factors.

$$\text{GCF} = 3x$$

The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

4.1.1 Factor the Greatest Common Factor from a Polynomial

It is sometimes useful to represent a number as a product of factors, for example, 12 as 2×6 or 3×4. In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such as $3x^2 + 15x$, and end with its factors, $3x(x + 5)$. To do this we apply the Distributive Property in reverse.

***How To* :: Factor the greatest common factor from a polynomial.**

Step 1. Find the GCF of all the terms of the polynomial.
Step 2. Rewrite each term as a product using the GCF.
Step 3. Use the reverse Distributive Property to factor the expression.
Step 4. Check by multiplying the factors.

Example 4.2: Factor: $8m^3 - 12m^2n + 20mn^2$

Solution:

Step 1: Find the GCF of all the terms of the polynomial.	Find the GCF of $8m^3$, $12m^2n$, $20mn^2$	$8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$ $GCF = 2 \cdot 2 \cdot m$ $GCF = 4m$
Step 2: Rewrite each term as a product using the GCF.	Rewrite $8m^3$, $12m^2n$, $20mn^2$ as product of their GCF, $4m$. $8m^3 = 4m \cdot 2m^2$ $12m^2n = 4m \cdot 3mn$ $20mn^2 = 4m \cdot 5n^2$	$8m^3 - 12m^2n + 20mn^2 =$ $4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$

Step 3: Use the reverse Distributive Property to factor the expression.		$4m(2m^2-3mn+5n^2)$
Step 4: Check.		$4m(2m^2-3mn+5n^2)=$ $8m^3-12m^2n+20mn^2$

Example 4.3: Factor: $8x^3y–10x^2y^2+12xy^3$

Solution:

The GCF of $8x^3y$, $-10x^2y^2$, and $12xy^3$ is $2xy$.

$$\begin{array}{rllllllll} 8x^3y & = 2\cdot & 2\cdot 2 & & x\cdot & x\cdot x\cdot & y & & \\ 10x^2y^2 & = 2\cdot & & 5\cdot & x\cdot & x\cdot & y\cdot & y & \\ 12xy^3 & = 2\cdot & 2\cdot 3 & & x\cdot & & y\cdot & y\cdot y & \\ \hline \text{GCF} & = 2\cdot & & & x\cdot & & y & & \\ \text{GCF} & = 2xy & & & & & & & \end{array}$$

$$\begin{aligned} 8x^3y–10x^2y^2+12xy^3 &= 2xy\cdot 4x^2-2xy\cdot 5xy+2xy\cdot 6y^2 \\ &= 2xy(4x^2-5xy+6y^2) \end{aligned}$$

So far our greatest common factors have been monomials. In the next example, the greatest common factor is a binomial.

Example 4.4: Factor: $3y(y+7)-4(y+7)$

Solution:

$3y(y+7)-4(y+7)=3y(y+7)-4(y+7)$ The GCF is the binomial $y+7$.

$=(y+7)(3y-4)$ Factor the GCF, $(y+7)$.

4.1.2 Factor by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the GCF in each part.

***How To* :: Factor by grouping.**

How?

Step 1. Group terms with common factors.

Step 2. Factor out the common factor in each group.

Step 3. Factor the common factor from the expression.

Step 4. Check by multiplying the factors.

Example 4.5: Factor by grouping: $xy + 3y + 2x + 6$

Solution:

Step 1: Group terms with common factors.	Is there a gcf for all four terms? No, so let us separate the first two terms from the second two.	$xy + 3y + 2x + 6 =$ $\underbrace{xy + 3y} + \underbrace{2x + 6}$
Step 2: Factor out the common factor in each group.	Factor the GCF from the first two terms. Factor the GCF from the second two terms.	$y(x + 3) + 2x + 6$ $y(x + 3) + 2(x + 3)$
Step 3: Factor the common factor from the expression.	Notice that each term has a common factor of $(x + 3)$	$y(x + 3) + 2(x + 3) =$ $(x + 3)(y + 2)$
Step 4: Check by multiplying the factors.	Multiply $(x + 3)(y + 2)$. Is the product the original expression?	$(x+3)(y+2) = xy+2x+3y+6$ $= xy + 3y + 2x + 6$

Example 4.6: Factor by grouping:

a. $x^2 - 3x + 2x - 6$ **b.** $6x^2 - 3x - 4x + 2$.

Solution: There is no GCF in all four terms.

a.	$x^2 - 3x + 2x - 6$	$= x^2 - 3x \quad + 2x - 6$	Separate into two parts.
		$= x(x + 3) + 2(x - 3)$	Factor the GCF from both parts.
		$= (x - 3)(x - 2)$	Factor out the common factor.
b.	$6x^2 - 3x - 4x + 2$	$= 6x^2 - 3x \quad - 4x + 2$	Separate into two parts.
		$= 3x(2x - 1) - 2(2x - 1)$	Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.
		$= (2x - 1)(3x - 2)$	Factor out the common factor.

4.1 Section Exercises

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

1. $10p^3q, 12pq^2$

2. $8a^2b^3, 10ab^2$

3. $12m^2n^3, 30m^5n^3$

4. $35x^3y^2, 10x^4y, 5x^5y^3$

5. $27p^2q^3, 45p^3q^4, 9p^4q^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

6. $12x^3 - 10x$

7. $5x^3 - 15x^2 + 20x$

8. $8m^2 - 40m + 16$

9. $24x^3 - 12x^2 + 15x$

10. $24y^3 - 18y^2 - 30y$

11. $12xy^2 + 18x^2y^2 - 30y^3$

12. $-5y^3 + 35y^2 - 15y$

13. $-4p^3q - 12p^2q^2 + 16pq^2$

14. $-6a^3b - 12a^2b^2 + 18ab^2$

15. $5x(x+1) + 3(x+1)$

16. $2x(x-1) + 9(x-1)$

17. $3b(b-2) - 13(b-2)$

Factor by Grouping

In the following exercises, factor by grouping.

18. $ab + 5a + 3b + 15$

19. $pq - 10p + 8q - 80$

20. $uv - 18 - 9u + 2v$

21. $6y^2 + 7y + 24y + 28$

22. $x^2 - x + 4x - 4$

23. $u^2 - u + 6u - 6$

24. $mn - 6m - 4n + 24$

25. $r^2 - 3r - r + 3$

26. $2x^2 - 14x - 5x + 35$

4.2 Factor Trinomials

Splitting a product into factors is called **factoring**.

Multiply

$$\underbrace{(x+2)(x+3)}_{factors} = \underbrace{x^2+5x+6}_{product}$$

Factor

4.2.1 Factor Trinomials of the Form $x^2 + bx + c$

To figure out how we would factor a trinomial of the form $x^2 + bx + c$, such as $x^2 + 5x + 6$ and factor it to $(x+2)(x+3)$, lets start with two general binomials of the form $(x+m)$ and $(x+n)$.

$$\begin{aligned} (x+m)(x+n) &= x^2 + mx + nx + mn \\ &= x^2 + (m+n)x + mn \\ &= \overbrace{x^2 + (m+n)x + mn}^{x^2+\quad b\quad x+\quad c} \end{aligned}$$

This tells us that to factor a trinomial of the form $x^2 + bx + c$, we need two factors $(x+m)$ and $(x+n)$ where the two numbers m **and** n ***multiply to*** c ***and add to*** b.

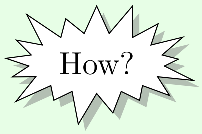
Example 4.7: How to factor a trinomial of the form $x^2 + bx + c$

Factor: $x^2 + 11x + 24$

Solution:

<table>
<tr><td>Step 1: Write the factors as two binomials with first terms x.</td><td>Write two sets of parantheses and put x as the first term.</td><td>$x^2+11x+24 = (x+\quad)(x+\quad)$</td></tr>
<tr><td>Step 2: Find two numbers m and n that
✜ multiply to c, $m \cdot n = c$
✜ add to b, $m + n = b$</td><td>Find two numbers that multiply to 24 and add to 11.
<table>
<tr><th>Factors of 24</th><th>Sum of factors</th></tr>
<tr><td>1, 24</td><td>1+24=25</td></tr>
<tr><td>2, 12</td><td>2+12=14</td></tr>
<tr><td>3, 8</td><td>3+8=11*</td></tr>
<tr><td>4, 6</td><td>4+6=10</td></tr>
</table></td><td></td></tr>
<tr><td>Step 3: Use m and n as the last terms of the factors.</td><td>Use 3 and 8 as the last terms of the factors.</td><td>$x^2+11x+24 = (x+3)(x+8)$</td></tr>
<tr><td>Step 4: Check by multiplying the factors.</td><td>Multiply $(x+3)(x+8)$. Is the product the original expression?</td><td>$(x+3)(x+8) =$
$x^2 + 8x + 3x + 24$
$= x^2 + 11x + 24$</td></tr>
</table>

How To **:: How to factor a trinomial of the form $x^2 + bx + c$**



Step 1. Write the factors as two binomials with first terms x.
$x^2 + bx + c = (x \qquad)(x \qquad)$

Step 2. Find two numbers m and n that

- multiply to c, $m \cdot n = c$
- add to b, $m + n = b$

Step 3. Use m and n as the last terms of the factors. $(x + m)(x + n)$

Step 4. Check by multiplying the factors.

Now, what if the last term in the trinomial is negative? The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too. How do you get a negative product and a positive sum? We use one positive and one negative number. When we factor trinomials, we must have the terms written in descending order-in order from highest degree to lowest degree.

Example 4.8: Factor: $2x + x^2 - 48$

Solution:

First we put the terms in decreasing degree order. $x^2 + 2x - 48$

Factors will be two binomials with first terms x. $x^2 + 2x - 48 = (x \qquad)(x \qquad)$

Find two numbers that: multiply to -48 and add to 2.

Factors of -48	Sum of factors
-1, 48	-1 + 48 = 47
-2, 24	-2 + 24 = 20
-3, 16	-3 + 16 = 13
-4, 12	-4 + 12 = 8
-6, 8	-6 + 8 = 2*

Use 6, 8 as the last terms of the binomials.

$$x^2 + 2x - 48 = (x - 6)(x + 8)$$

Sometimes youll need to factor trinomials of the form $x^2 + bxy + cy^2$ with two variables, such as $x^2 + 12xy + 36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x \cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain y. To get the coefficients b and c, you use the same process summarized in **How To Factor trinomials**4.2.2.

Example 4.9: Factor: $r^2 – 8rs – 9s^2$

Solution: We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

Note that the first terms are r, last terms contain s. $r^2 – 8rs – 9s^2 = (r \quad s)(r \quad s)$

Find two numbers that: multiply to -9 and add to -8.

Factors of -9	Sum of factors
1, -9	1 + (-9) $= -8^*$
-1, 9	-1 + 9 = 8
3, -3	3 + (-3) = 0

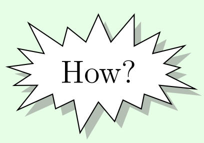
Use 1, -9 as coefficients of the last terms.

$$r^2 – 8rs – 9s^2 = (r + s)(r - 9s)$$

4.2.2 Factor Trinomials of the Form $ax^2 + bx + c$ using the ac Method

The ac method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

How To **::** **How to factor a trinomial of the form $ax^2 + bx + c$ using the 'ac' method**



Step 1. Factor any GCF.

Step 2. Find the product ac.

Step 3. Find two numbers m and n that

- multiply to ac, $m \cdot n = ac$
- add to b, $m + n = b$

Step 4. Split the middle term using m and n. $ax^2 + mx + nx + c$

Step 5. Factor by grouping.

Step 6. Check by multiplying the factors.

Example 4.10: **How to factor a trinomial of the form $ax^2 + bx + c$**

Factor using the ac method: $6x^2 + 7x + 2$

Solution:

Step 1: Factor any GCF.	Is there any greatest common factor? No!	$6x^2 + 7x + 2$

Step 2: Find the product ac.	$a \cdot c = 6 \cdot 2 = 12$	$\overset{ax^2+bx+c}{6x^2+7x+2}$
Step 3: Find the product ac. ✤ multiply to ac, $m \cdot n = ac$ ✤ add to b, $m + n = b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12$ & $3 + 4 = 7$	
Step 4: Split the middle term using m and n. $ax^2 + bx + c$ bx $ax^2 + mx + nx + c$	Rewrite $7x$ as $3x + 4x$. It would also give the same result if we use $4x + 3x$. Notice that $6x^2 + 3x + 4x + 2$ is equal to $6x^2 + 7x + 2$. We just split the middle term to get a more useful form.	$6x^2 + 7x + 2$ $\underbrace{6x^2 + 3x} + \underbrace{4x + 2}$
Step 5: Factor by grouping.		$3x(2x + 1) + 2(2x + 1)$ $(2x + 1)(3x + 2)$
Step 6: Check by multiplying the factors.		$(2x + 1)(3x + 2)$ $= 6x^2 + 4x + 3x + 2$ $= 6x^2 + 7x + 2$✓

Example 4.11: Factor using the ac method: $10y^2 - 55y + 70$

Solution: Is there a greatest common factor? Yes. The GCF is 5. Factor it.

$10y^2 - 55y + 70 = 5(2y^2 - 11y + 14)$ Factor out 5.

$= 5(\overbrace{2y^2 - 11y + 14}^{ax^2+bx+c})$ The trinomial inside the parentheses has a leading coefficient that is not 1.

$ac = 28$ Find the product ac.

$(-4)(-7) = 28$ Find two numbers that multiply to ac

$-4 + (-7) = -11$ and add to b.

$10y^2 - 55y + 70 = 5(2y^2 - 11y + 14)$ Split the middle term.

$= 5(\underbrace{2y^2 - 7y} - \underbrace{4y + 14})$

$= 5[y(2y - 7) - 2(2y - 7)]$ Factor the trinomial by grouping.

$= 5(2y - 7)(y - 2)$

4.2 Section Exercises

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

1. $p^2 + 11p + 30$

2. $w^2 + 10w + 21$

3. $n^2 + 19n + 48$

4. $x^2 – 8x + 12$

5. $q^2 – 13q + 36$

6. $x^2 – 8x + 7$

7. $5p – 6 + p^2$

8. $6n – 7 + n^2$

9. $8 – 6x + x^2$

10. $x^2 – 12 – 11x$

11. $-11 – 10x + x^2$

12. $7x + x^2 + 6$

In the following exercises, factor each trinomial of the form $x^2 + bxy + cy^2$.

13. $x^2 – 2xy – 80y^2$

14. $p^2 – 2pq – 35q^2$

15. $a^2 + 5ab – 24b^2$

16. $r^2 + 3rs – 28s^2$

17. $x^2 – 13xy – 14y^2$

18. $u^2 – 8uv + 12v^2$

Factor Trinomials of the Form $ax^2 + bx + c$ using the ac Method

In the following exercises, factor using the ac method.

19. $3x^2 + 5x + 2$

20. $2w^2 + 3w - 5$

21. $3x^2 + x - 10$

22. $5s^2 – 9s + 4$

23. $6y^2 + y – 15$

24. $5n^2 + 21n + 4$

25. $6n^2 – n – 2$

26. $4x^2 + 11x - 3$

27. $3u^2 + 8u + 5$

4.3 Factor Special Products

We have seen that some binomials and trinomials result from special products-squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

4.3.1 Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.

$$(a+b)^2$$
$$(3x+4)^2$$
$$a^2 \quad + \quad 2ab \quad + \quad b^2$$
$$(3x)^2 \quad + \quad 2 \cdot (3x) \cdot 4 \quad + \quad 4^2$$
$$9x^2 + 24x + 16$$

The trinomial $9x^2 + 24x + 16$ is called a perfect square trinomial. It is the square of the binomial $3x + 4$.

If you recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, you will save yourself a lot of work.

Here is the pattern-the reverse of the binomial squares pattern.

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2 - 2ab + b^2$, which factors to $(a - b)^2$.

The steps are summarized here.

How To Factor Perfect Square Trinomials.

How?

Step 1. Does the trinomial fit the pattern?
Is the first term a perfect square?
Write it as a square.
Is the last term a perfect square?
Write it as a square.
Check the middle term. Is it $2ab$?

Step 2. Write the square of the binomial.

Step 3. Check by multiplying the factors.

$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
$(a)^2$	$(a)^2$
$(a)^2 \quad (b)^2$	$(a)^2 \quad (b)^2$
$2ab$	$2ab$
$(a+b)^2$	$(a-b)^2$

Example 4.12: **Factor:** $9x^2 + 12x + 4$

Solution:

Step 1: Does the trinomial fit the perfect square trinomial pattern, $a^2 + 2ab + b^2$? ✜ Is the first term a perfect square? Write it as a square a^2. ✜ Is the last term a perfect square? Write it as a square b^2. ✜ Check the middle term. Is it $2ab$?	Is $9x^2$ a perfect square? Yes - write it as $(3x)^2$. Is 4 a perfect square? Yes - write it as 2^2. Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial. Yes - write it as $(3x)^2$.	$9x^2 \quad + \quad 12x \quad + \quad 4$ $(3x)^2$ $(3x)^2 \qquad 2^2$ $(3x)^2 \qquad 2^2$ $2(3x)2$ $12x$
Step 2: Write the square of the binomial.		$9x^2 \; + \; 12x \; + \; 4$ $a^2 \; + 2 \cdot a \cdot b \; + b^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $(a+b)^2$ $(3x+2)^2$
Step 3: Check.		$(3x+2)^2$ $= (3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $= 9x^2 + 12x + 4$✓

The next example will be a perfect square trinomial with two variables.

Example 4.13: Factor: $36x^2 - 84xy + 49y^2$

Solution:

	$36x^2 \; - \; 84xy \; + \; 49y^2$
Test each term to verify the pattern.	$a^2 \; - \; 2 \cdot a \cdot b \; + \; b^2$ $(6x)^2 - 2 \cdot (6x) \cdot (7y) + (7y)^2$
Factor.	$(6x - 7y)^2$

Remember the first step in factoring is to look for a greatest common factor.

Example 4.14: **Factor:** $100x^2y - 80xy + 16y$

Solution:

	$100x^2y - 80xy + 16y$
Is there a GCF? Yes, $4y$, so factor it out.	$4y(25x^2 - 20x + 4)$
Is this a perfect square trinomial? Verify the pattern.	$a^2 \; - \; 2 \cdot a \cdot b \; + \; b^2$ $4y[(5x)^2 - 2 \cdot (5x) \cdot (2) + 2^2]$
Factor.	$4y(5x-2)^2$

4.3.2 Factor Differences of Squares

The other special product you saw in the previous chapter was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Heres an example:

$$\overset{(a-b)(a+b)}{(3x-4)(3x+4)} = \overset{a^2-b^2}{(3x)^2-(4)^2} = 9x^2-16$$

A difference of squares factors to a product of conjugates.

Difference of Squares Pattern

$$a^2-b^2=(a-b)(a+b)$$

Example 4.15: Factor: $64y^2-1$

Solution:

$64y^2-1=(8y)^2-1^2$	Is the binomial a difference of squares? Yes.
$=(8y-1)(8y+1)$	Factor as a product of conjugates.

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may disguise the difference of squares and you wont recognize the perfect squares until you factor the GCF. Also, to completely factor the binomial in the next example, well factor a difference of squares twice!

Example 4.16: Factor: $48x^4y^2-243y^2$

Solution:

$48x^4y^2-243y^2=3y^2(16x^4-81)$	Is there a GCF? Yes, $3y^2$ factor it out!
$=3y^2\left[(4x^2)^2-(9)^2\right]$	Is the binomial a difference of squares? Yes.
$=3y^2(4x^2-9)(4x^2+9)$	Factor as a product of conjugates.
$=3y^2\left[(2x)^2-(3)^2\right](4x^2+9)$	Notice the first binomial is also a difference of squares!
$=3y^2(2x-3)(2x+3)(4x^2+9)$	Factor it as the product of conjugates.

4.3.3 Factor sums and differences of cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes.

Sum and Difference of Cubes Pattern

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

The two patterns look very similar, dont they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

Example 4.17: Factor: $x^3 + 64$

Solution:

$$x^3 + 64 = (x)^3 + (4)^3$$ Write the terms as cubes.

$$= \overbrace{(x+4)(x^2 - 4x + 4^2)}^{(a+b)(a^2-ab+b^2)}$$ Use the sum of cubes pattern.

$$= (x+4)(x^2 - 4x + 16)$$ Simplify.

Example 4.18: Factor: $8x^3 - 27y^3$

Solution: This binomial is a difference. The first and last terms are perfect cubes.

$$8x^3 - 27y^3 = (2x)^3 - (3y)^3$$ Write the terms as cubes.

$$= \overbrace{(2x-3y)\left((2x)^2 + (2x)(3y) + (3y)^2\right)}^{(a-b)(a^2+ab+b^2)}$$ Use the difference of cubes pattern.

$$= (2x-3y)\left(4x^2 + 6xy + 9y^2\right)$$ Simplify.

4.3 Section Exercises

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

1. $16y^2 + 24y + 9$

2. $25v^2 + 20v + 4$

3. $36s^2 + 84s + 49$

4. $100x^2–20x + 1$

5. $49x^2 + 28xy + 4y^2$

6. $75u^4–30u^3v + 3u^2v^2$

7. $64x^2y–96xy + 36y$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

8. $25v^2–1$

9. $4–49x^2$

10. $36p^2–49q^2$

11. $x^2–16x + 64–y^2$

12. $a^2 + 6a + 9–9b^2$

13. $6p^2q^2–54p^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

14. $x^3 + 125$

15. $x^3 - 125$

16. $z^6–27$

17. $125–27w^3$

18. $7k^3 + 56$

19. $27y^3 + 8z^3$

20. $(x + 3)^3 + 8x^3$

21. $(y–5)^3–64y^3$

22. $-2x^3y^2–16y^5$

Chapter 5

RATIONAL EXPRESSIONS

Contents

Learning outcome covered:

e. `Simplify rational expressions and rationalize numerators or denominators.`

Learning Objectives

By the end of this chapter, the students will be able to:

- Simplify rational expressions
- Multiply and Divide rational expressions
- Find the least common denominator of rational expressions
- Add and subtract rational expressions
- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Introduction

Rational expressions and rational equations can be useful tools for representing real life situations and for finding answers to real problems. In particular, they are quite good for describing distance-speed-time questions, and modeling multi-person work problems. In this chapter, you will work with rational expressions and perform operations on them.

5.1 Multiplication and Division of Rational Expressions

We previously reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

Rational Expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Here are some examples of rational expressions:

$$-\frac{25}{56} \qquad \frac{5x}{12y} \qquad \frac{4x+1}{x^2-9} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{25}{56}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero. We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

5.1.1 Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1, in its numerator and denominator.

Simplified Rational Expression

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example,

$\frac{x+2}{x+3}$ is simplified because there are no common factors of $x+2$ and $x+3$.

$\frac{2x}{3x}$ is not simplified because x is a common factor of $2x$ and $3x$.

We use the Equivalent Fractions Property to simplify numerical fractions.We restate it here as we will also use it to simplify rational expressions.

Equivalent Fractions Property

If a, b, and c are numbers where $b \neq 0$, $c \neq 0$,

$$\text{then } \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0$, $c \neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\frac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$	$\frac{3x\cancel{(x-9)}}{5\cancel{(x-9)}}$ where $x \neq 9$	$\frac{x+5}{x}$ where $x \neq 0$
$\frac{2}{5}$	$\frac{3x}{5}$ where $x \neq 9$	**No common factor**
We remove the common factors 3 and 7. They are factors of the product.	We remove the common factors $(x-9)$. It is a factors of the product.	While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Example 5.1: How to simplify a rational expression.

Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$

Solution:

Step 1: Factor the numerator and denominator completely.	Factor x^2+5x+6 and $x^2+8x+12$.	$\frac{x^2+5x+6}{x^2+8x+12}$ $\frac{(x+2)(x+3)}{(x+2)(x+6)}$
Step 2: Simplify by dividing out common factors.	Remove the common factor $(x+2)$ from the numerator and denominator.	$\frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$ $\frac{(x+3)}{(x+6)}$ $x \neq -2, \quad x \neq -6$

We now summarize the steps you should follow to simplify rational expressions.

How To **:: Simplify a Rational Expression**

How?

Step 1. Factor the numerator and denominator completely.

Step 2. Simplify by dividing out common factors.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

Example 5.2: Simplify: $\dfrac{3a^2-12ab+12b^2}{6a^2-24b^2}$

Solution:

$\dfrac{3a^2-12ab+12b^2}{6a^2-24b^2} = \dfrac{3(a^2-4ab+4b^2)}{6(a^2-4b^2)}$	Factor the numerator and denominator, first factoring out the GCF.
$= \dfrac{3(a-2b)(a-2b)}{6(a-2b)(a+2b)}$	
$= \dfrac{\cancel{3}\cancel{(a-2b)}(a-2b))}{\cancel{3}\cdot 2\cancel{(a-2b)}(a+2b)}$	Remove the common factors of $a-2b$ and 3.
$= \dfrac{a-2b}{2(a+2b)}$	

Example 5.3: Simplify: $\dfrac{x^2-4x-32}{64-x^2}$

Solution:

$\dfrac{x^2-4x-32}{64-x^2} = \dfrac{(x-8)(x+4)}{(8-x)(8+x)}$	Factor the numerator and the denominator.
$= (-1)\dfrac{\cancel{(x-8)}(x+4)}{\cancel{(x-8)}(8+x)}$	Recognize the factors that are opposites. $8-x=-(x-8)$
$= -\dfrac{x+4}{x+8}$	Simplify.

5.1.2 Multiply Rational Expressions

To multiply rational expressions, we multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Multiplication of Rational Expressions

If p, q, r, and s are polynomials where $q \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded.

Example 5.4: Multiply: $\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$

Solution:

Step 1: Factor each numerator and denominator completely.	Factor x^2-9 and $x^2-7x+12$.	$\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$ $\dfrac{2x}{(x-3)(x-4)} \cdot \dfrac{(x-3)(x+3)}{6x^2}$
Step 2: Multiply the numerators and denominators.	Multiply numerators and denominators. It is helpful to write the monomials first.	$\dfrac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3: Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in the factored form.	$\dfrac{\cancel{2}\cancel{x}\cancel{(x-3)}(x+3)}{\cancel{2}\cdot 3x^{\cancel{2}}\cancel{(x-3)}(x-4)}$ $\dfrac{(x+3)}{3x(x-4)}$

How To :: Multiply Rational Expressions

How?

Step 1. Factor each numerator and denominator completely.

Step 2. Multiply the numerators and denominators.

Step 3. Simplify by dividing out common factors.

Example 5.5: Multiply: $\dfrac{3a^2-8a-3}{a^2-25} \cdot \dfrac{a^2+10a+25}{3a^2-14a-5}$

Solution:

$$\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5} = \frac{(3a+1)(a-3)(a+5)(a+5)}{(a-5)(a+5)(3a+1)(a-5)}$$

Factor the numerators and denominators and then multiply.

$$= \frac{\cancel{(3a+1)}(a-3)\cancel{(a+5)}(a+5)}{(a-5)\cancel{(a+5)}\cancel{(3a+1)}(a-5)}$$

Simplify by dividing out common factors.

$$= \frac{(a-3)(a+5)}{(a-5)(a-5)}$$ Simplify.

$$= \frac{(a-3)(a+5)}{(a-5)^2}$$ Rewrite $(a-5)(a-5)$ using an exponent.

5.1.3 Divide Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

Division of Rational Expressions

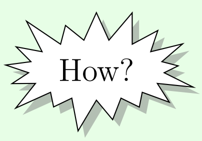
If p, q, r, and s are polynomials where $q \neq 0$, $r \neq 0$ and $s \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions, multiply the first fraction by the reciprocal of the second.

We now summarize the steps you should follow to simplify rational expressions.

How To **:: Divide Rational Expressions**



Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.

Step 2. Factor the numerators and denominators completely.

Step 3. Multiply the numerators and denominators together.

Step 4. Simplify by dividing out common factors.

Example 5.6: How to Divide Rational Expressions

Divide: $\dfrac{p^3+q^3}{2p^2+2pq+2q^2} \div \dfrac{p^2-q^2}{6}$

Solution:

$$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6} = \frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$$ Rewrite the division as multiplication by the reciprocal.

$$= \frac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \frac{2 \cdot 3}{(p+q)(p-q)}$$ Factor the numerators and the denominators.

$$= \frac{(p+q)(p^2-pq+q^2) \cdot 2 \cdot 3}{2(p^2+pq+q^2)(p+q)(p-q)}$$ Multiply the numerators and the denominators.

$$= \frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2} \cdot 3}{\cancel{2}(p^2+pq+q^2)\cancel{(p+q)}(p-q)}$$ Divide out common factors.

$$= \frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$$ Simplify.

5.1 Section Exercises

Simplify Rational Expressions

In the following exercises, simplify each rational expression.

1. $\dfrac{36v^3w^2}{27vw^3}$

2. $\dfrac{x^2+4x-5}{x^2-2x+1}$

3. $\dfrac{8m^3n}{12mn^2}$

4. $\dfrac{a^2-4}{a^2+6a-16}$

5. $\dfrac{x^2-y^2}{x^3-y^3}$

6. $\dfrac{x^3-2x^2-25x+50}{x^2-25}$

Multiply Rational Expressions

In the following exercises, multiply the rational expressions.

7. $\dfrac{5x^2y^4}{12xy^3}\cdot\dfrac{6x^2}{20y^2}$

8. $\dfrac{12a^3b}{b^2}\cdot\dfrac{2ab^2}{9b^3}$

9. $\dfrac{5p^2}{p^2-5p-36}\cdot\dfrac{p^2-16}{10p}$

10. $\dfrac{2y^2-10y}{y^2+10y+25}\cdot\dfrac{y+5}{6y}$

11. $\dfrac{2m^2-3m-2}{2m^2+7m+3}\cdot\dfrac{3m^2-14m+15}{3m^2+17m-20}$

12. $\dfrac{2n^2-3n-14}{25-n^2}\cdot\dfrac{n^2-10n+25}{2n^2-13n+21}$

Divide Rational Expressions

In the following exercises, divide the rational expressions.

13. $\dfrac{v-5}{11-v}\div\dfrac{v^2-25}{v-11}$

14. $\dfrac{10+w}{w-8}\div\dfrac{100-w^2}{8-w}$

15. $\dfrac{x^2+3x-10}{4x}\div(2x^2+20x+50)$

16. $\dfrac{2y^2-10yz-48z^2}{2y-1}\div(4y^2-32yz)$

For the following exercises, perform the indicated operations.

17. $\dfrac{10m^2+80m}{3m-9}\cdot\dfrac{m^2+4m-21}{m^2-9m+20}\div\dfrac{5m^2+10m}{2m-10}$

18. $\dfrac{4n^2+32n}{3n+2}\cdot\dfrac{3n^2-n-2}{n^2+n-30}\div\dfrac{108n^2-24n}{n+6}$

5.2 Addition and Subtraction of Rational Expressions

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

5.2.1 Add and Subtract Rational Expressions with a Common Denominator

Rational Expression Addition and Subtraction

If p, q and r are polynomials where $r \neq 0$, then
$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \text{ and } \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

Example 5.7: Add: $\dfrac{11x+28}{x+4}+\dfrac{x^2}{x+4}$

Solution:

Since the denominator is $x+4$, we must exclude the value $x=-4$.

$\dfrac{11x+28}{x+4}+\dfrac{x^2}{x+4} = \dfrac{11x+28+x^2}{x+4}$	The fractions have a common denominator, so add the numerators and place the sum over the common denominator.
$= \dfrac{x^2+11x+28}{x+4}$	Write the degrees in descending order.
$= \dfrac{(x+4)(x+7)}{x+4}$	Factor the numerator.
$= \dfrac{\cancel{(x+4)}(x+7)}{\cancel{x+4}}$	Simplify by removing common factors.
$= x+7$	Simplify.

Example 5.8: Subtract: $\dfrac{5x^2-7x+3}{x^2-3x-18}-\dfrac{4x^2+x-9}{x^2-3x-18}$

Solution:

$\dfrac{5x^2-7x+3}{x^2-3x-18}-\dfrac{4x^2+x-9}{x^2-3x-18} = \dfrac{5x^2-7x+3-(4x^2+x-9)}{x^2-3x-18}$	Subtract the numerators and place the difference over the common denominator.
$= \dfrac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$	Distribute the sign in the numerator.

$$= \frac{x^2 - 8x + 12}{x^2 - 3x - 18}$$ Combine like terms.

$$= \frac{(x-2)\cancel{(x-6)}}{(x+3)\cancel{(x-6)}}$$ Simplify by removing common factors.

$$= \frac{(x-2)}{(x+3)}$$ Simplify.

Add and Subtract Rational Expressions Whose Denominators are Opposites

Be careful with the signs as you work with the opposites when the fractions are being subtracted.

Example 5.9: Subtract: $\frac{m+3}{m^2-1} + \frac{4}{1-m^2}$

Solution:

$$\frac{m+3}{m^2-1} + \frac{4}{1-m^2} = \frac{m+3}{m^2-1} + \frac{-1(4)}{-1(1-m^2)}$$ The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.

$$= \frac{m+3}{m^2-1} - \frac{4}{m^2-1}$$ Simplify the second fraction.

$$= \frac{m+3-4}{m^2-1}$$ The denominators are the same. Subtract the numerators.

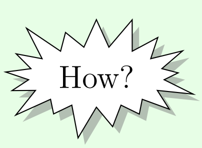
$$= \frac{m-1}{(m-1)(m+1)} = \frac{1}{m+1}$$ Factor the numerator and denominator.

5.2.2 Add and Subtract Rational Expressions with Unlike Denominators

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators.

Find the Least Common Denominator of Rational Expressions

How To **:: Find the Least Common Denominator of Rational Expressions.**



Step 1. Factor each denominator completely.

Step 2. List the factors of each denominator. Match factors vertically when possible.

Step 3. Bring down the columns by including all factors, but do not include common factors twice.

Step 4. Write the LCD as the product of the factors.

Example 5.10: Find the LCD for the expressions $\dfrac{8}{x^2-2x-3}, \dfrac{3x}{x^2+4x+3}$.

Solution: Factor each denominator completely, lining up common factors.

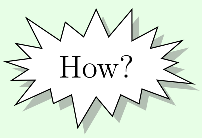
$$\begin{array}{lcll} x^2-2x-3 & = & (x+1) & (x-3) \\ x^2+4x+3 & = & (x+1) & \qquad (x+3) \\ \hline \text{LCD} & = & (x+1)((x-3)(x+3) & \end{array}$$

Bring down the columns.

The LCD is $(x+1)(x-3)(x+3)$.

The steps used to add rational expressions are summarized here.

How To **:: How to add Rational Expressions with Unlike Denominators**



Step 1. Determine if the expressions have a common denominator.

- Yes - goto step 2
- No - Rewrite each rational expression with the LCD
  - Find the LCD
  - Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 2. Add or subtract the rational expressions.

Step 3. Simplify, if possible.

Example 5.11: Add: $\dfrac{3}{x-3} + \dfrac{2}{x-2}$

Solution:

Step 1: Determine if the expressions have a common denominator. ✠ Yes - goto step 2 ✠ No - Rewrite each rational expression with the LCD ✠ Find the LCD ✠ Rewrite each rational expression as an equivalent rational expression with the LCD.	No Find the LCD of $x-3$ and $x-2$ Change into equivalent rational expression with the LCD, $(x-3)(x-2)$ Keep the denominators factored	$\begin{array}{rcll} x-3 & = & (x-3) & \\ x-2 & = & & (x-2) \\ \hline \text{LCD} & = & (x-3)(x-2) & \end{array}$ $\dfrac{3}{x-3} + \dfrac{2}{x-2}$ $\dfrac{3(x-2)}{(x-3)(x-2)} + \dfrac{2(x-3)}{(x-2)(x-3)}$ $\dfrac{3x-6}{(x-3)(x-2)} + \dfrac{2x-6}{(x-2)(x-3)}$

Step 2: Add or subtract the rational expressions.	Add the numerators and place the sum over the common denominator.	$\dfrac{3x-6+2x-6}{(x-3)(x-2)}$
Step 3: Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	$\dfrac{5x-12}{(x-3)(x-2)}$

Example 5.12: Add: $\dfrac{8}{x^2-2x-3}+\dfrac{3x}{x^2+4x+3}$

Solution:

Do the expressions have a common denominator? No.

Rewrite each expression with the LCD.

Find the LCD

$$\begin{aligned} x^2-2x-3 &= (x+1)\ (x-3) \\ x^2+4x+3 &= (x+1)\ \ (x+3) \\ \hline \text{LCD} &= (x+1)((x-3)(x+3) \end{aligned}$$

$$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$$

$$=\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$$ Rewrite each rational expression as an equivalent rational expression with the LCD.

$$=\frac{8x+24}{(x+1)(x-3)(x+3)}+\frac{3x^2-9x}{(x+1)(x+3)(x-3)}$$ Simplify the numerators.

$$=\frac{8x+24+3x^2-9x}{(x+1)(x-3)(x+3)}$$ Add the rational expressions.

$$=\frac{3x^2-x+24}{(x+1)(x-3)(x+3)}$$ Add the rational expressions.

The numerator is a prime polynomial, so there are no common factors.

Example 5.13: Simplify: $\dfrac{2u}{u-1}+\dfrac{1}{u}-\dfrac{2u-1}{u^2-u}$

Solution:

$$\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}=\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u(u-1)}$$ Factor the denominators.

Do the expressions have a common denominator? No.

Rewrite each expression with the LCD.

Find the LCD

$$\begin{array}{lll} u-1 & = & (u-1) \\ u & = u & \\ u^2-u & = u & (u-1) \\ \hline \text{LCD} & = u(u-1) & \end{array}$$

$$\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$$

$$=\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u(u-1)}$$

$$=\frac{2u\cdot u}{(u-1)u}+\frac{1\cdot(u-1)}{u(u-1)}-\frac{2u-1}{u(u-1)}$$

Rewrite each rational expression as an equivalent rational expression with the LCD.

$$=\frac{2u^2}{u(u-1)}+\frac{u-1}{u(u-1)}-\frac{2u-1}{u(u-1)}$$

Simplify the numerators.

$$=\frac{2u^2+u-1-2u+1}{u(u-1)}$$

Write as one rational expression.

$$=\frac{2u^2-u}{u(u-1)}$$

Simplify.

$$=\frac{\cancel{u}(2u-1)}{\cancel{u}(u-1)}$$

Factor the numerator, and remove common factors.

$$=\frac{2u-1}{u-1}$$

Simplify.

5.2 Section Exercises

Add and Subtract Rational Expressions with a Common Denominator

1. $\frac{7m}{2m+n}+\frac{4}{2m+n}$

2. $\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$

3. $\frac{3s^2}{3s-2}+\frac{13s-10}{3s-2}$

4. $\frac{3m^2}{6m-30}-\frac{21m-30}{6m-30}$

5. $\frac{5r^2+7r-33}{r^2-49}-\frac{4r^2+5r+30}{r^2-49}$

Add and Subtract Rational Expressions whose Denominators are Opposites

6. $\frac{10x^2+16x-7}{8x-3}+\frac{2x^2+3x-1}{3-8x}$

7. $\frac{a^2+3a}{a^2-9}-\frac{3a-27}{9-a^2}$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

8. $\frac{7}{10x^2y}+\frac{4}{5xy^2}$

9. $\frac{1}{12a^3b^2}+\frac{5}{9a^2b^3}$

10. $\frac{4}{2x+5}+\frac{2}{x-1}$

11. $\frac{t}{t-6}-\frac{t-2}{t+6}$

12. $\frac{6}{m+6}-\frac{12m}{m^2-36}$

13. $\frac{4}{x^2-6x+5}-\frac{3}{x^2-7x+10}$

14. $\frac{9p-17}{p^2-4p-21}-\frac{p+1}{7-p}$

15. $\frac{2t-30}{t^2+6t-27}-\frac{2}{3-t}$

16. $\frac{5a}{a-2}+\frac{9}{a}-\frac{2a+18}{a^2-2a}$

17. $\frac{6d}{d-5}+\frac{1}{d+4}-\frac{7d-5}{d^2-d-20}$

5.3 Simplification of Complex Rational Expressions

Complex fractions are fractions in which the numerator or denominator contains a fraction. We previously simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

5.3.1 Simplify a Complex Rational Expression by Writing it as Division

Complex Rational Expression

A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are a few complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}} \qquad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}+\frac{y}{x}} \qquad \frac{\frac{2}{x+6}}{\frac{4}{x-6}+\frac{4}{x^2-36}}$$

Fraction bars tell us to divide, so we rewrite $\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$ as the division problem:

$$\frac{6x^2-7x+2}{4x-8} \div \frac{2x^2-7x+3}{x^2-5x+6}$$

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

Example 5.14: Simplify the complex rational expression by writing it as division: $\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$

Solution:

$\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}} = \frac{6}{x-4} \div \frac{3}{x^2-16}$ Rewrite the complex fraction as division.

$= \frac{6}{x-4} \cdot \frac{x^2-16}{3}$ Rewrite as the product of first times the reciprocal of the second.

$= \frac{2 \cdot 3}{x-4} \cdot \frac{(x-4)(x+4)}{3}$ Factor.

$= \frac{2 \cdot 3(x-4)(x+4)}{3(x-4)}$ Multiply.

$= \frac{2 \cdot \cancel{3(x-4)}(x+4)}{\cancel{3(x-4)}}$ Remove common factors.

$= 2(x+4)$ Simplify.

Example 5.15: Simplify the complex rational expression by writing it as division: $\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$

Solution:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}} = \frac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$$

Find the LCD and add the fractions in the numerator. Find the LCD and subtract the fractions in the denominator.

$$= \frac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$$

Simplify the numerator and denominator.

$$= \frac{3}{6} \div \frac{1}{6}$$

Rewrite the complex rational expression as a division problem.

$$= \frac{3}{6}\cdot\frac{6}{1} = 3$$

Multiply the first by the reciprocal of the second.

How To **:: How to Simplify a Complex Rational Expression Using Division**

How?

Step 1. Simplify the numerator and denominator.

Step 2. Rewrite the complex rational expression as a division problem.

Step 3. Divide the expressions.

Example 5.16: Simplify the complex rational expression by writing it as division: $\dfrac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$

Solution:

$$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}} = \frac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$$

Simplify the numerator and denominator. Find common denominators for the numerator and denominator.

$$= \frac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$$

Simplify the numerators.

$$= \frac{\frac{n^2+5n-4n}{n+5}}{\frac{n-5+n+5}{(n+5)(n-5)}}$$

Subtract the rational expressions in the numerator and add in the denominator.

$$= \frac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$$

Simplify. (We now have one rational expression over one rational expression.)

$$
\begin{aligned}
&= \frac{n^2+n}{n+5} \div \frac{2n}{(n+5)(n-5)} && \text{Rewrite as fraction division.} \\
&= \frac{n^2+n}{n+5} \cdot \frac{(n+5)(n-5)}{2n} && \text{Multiply the first times the reciprocal of the second.} \\
&= \frac{n(n+1)(n+5)(n-5)}{2n(n+5)} && \text{Factor any expressions if possible.} \\
&= \frac{\cancel{n}(n+1)\cancel{(n+5)}(n-5)}{2\cancel{n}\cancel{(n+5)}} && \text{Remove common factors.} \\
&= \frac{(n+1)(n-5)}{2} && \text{Simplify.}
\end{aligned}
$$

5.3.2 Simplify a Complex Rational Expression by Using the LCD

***How To* :: How to Simplify a Complex Rational Expression Using the LCD**

Step 1. Find the LCD of all fractions in the complex rational expression.
Step 2. Multiply the numerator and denominator by the LCD.
Step 3. Simplify the expression.

Lets look at the complex rational expression we simplified one way in Example 5.15. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{LCD}{LCD}$ we are multiplying by 1, so the value stays the same.

Example 5.17: Simplify the complex rational expression by using the LCD: $\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$

Solution:

The LCD of all the fractions in the whole expression is 6.

$$
\begin{aligned}
\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}} &= \frac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)} && \text{Clear the fractions by multiplying the numerator and denominator by that LCD.} \\
&= \frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}} && \text{Distribute.} \\
&= \frac{2+1}{3-2} && \text{Simplify.} \\
&= \frac{3}{1} = 3
\end{aligned}
$$

Example 5.18: Simplify the complex rational expression by using the LCD: $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$

Solution:

Find the LCD of all fractions in the complex rational expression.

The LCD is xy.

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} = \frac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$$

Multiply the numerator and denominator by the LCD.

$$= \frac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$$

Simplify the expression.

$$= \frac{y+x}{x^2-y^2}$$

Simplify.

$$= \frac{x+y}{(x+y)(x-y)}$$

Factor.

$$= \frac{\cancel{x+y}}{(x-y)\cancel{(x+y)}}$$

Cancel out common factor.

$$= \frac{1}{(x-y))}$$

Simplify.

5.3 Section Exercises

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify each complex rational expression by writing it as division.

1. $\dfrac{\frac{2a}{a+4}}{\frac{4a^2}{a^2-16}}$

2. $\dfrac{\frac{3b}{b-5}}{\frac{b^2}{b^2-25}}$

3. $\dfrac{\frac{5}{c^2+5c-14}}{\frac{10}{c+7}}$

4. $\dfrac{\frac{8}{d^2+9d+18}}{\frac{12}{d+6}}$

5. $\dfrac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$

6. $\dfrac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

7. $\dfrac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

8. $\dfrac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

9. $\dfrac{\frac{m}{n}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

Chapter 6

SYSTEMS OF MEASUREMENT

Contents

Learning outcome covered:

d. Understand measurements and conversion from one unit to another.

o. Apply knowledge of basic algebra and trigonometry in real life problems.

Learning Objectives

By the end of this chapter, the students will be able to:

- Make unit conversions in the U.S. system
- Use mixed units of measurement in the U.S. system
- Make unit conversions in the Metric system
- Use mixed units of measurement in the Metric system
- Convert between the U.S. and the Metric systems of measurement
- Convert between Fahrenheit and Celsius temperatures

Introduction

There are two systems of measurement commonly used around the world, namely the **Metric System** and the **U.S. System**. Most countries use the metric system. The United States uses a different system of measurement, usually called the U.S. system. In this chapter students will learn how to convert from one unit to another.

6.1 U.S. System

The United States uses a system of measurement, called the U.S. system. We will look at the U.S. system first. The U.S. system of measurement uses units of

- inch, foot, yard, and mile to measure length
- pound and ton to measure weight
- cup, pint, quart and gallons to measure capacity(volume)
- seconds, minutes and hours to measure time

6.1.1 Make Unit Conversions in the U.S. System

The equivalencies among the basic units of the U.S. system of measurement are listed in Table 1. The table also shows, in parentheses, the common abbreviations for each measurement.

U.S. System Units	
Length	**Volume**
1 foot (ft) = 12 inches (in) 1 yard (yd) = 3 feet (ft) 1 mile (mi) = 5280 feet (ft)	3 teaspoons (t) = 1 tablespoon 16 Tablespoons (T) = 1 cup (C) 1 cup (C) = 8 fluid ounces (fl oz) 1 pint (pt) = 2 cups (C) 1 quart (qt) = 2 pints (pt) 1 gallon (gal) = 4 quarts (qt)
Weight	**Time**
pound (lb) = 16 ounces (oz) 1 ton = 2000 pounds (lb)	1 minute (min) = 60 seconds (s) 1 hour (h) = 60 minutes (min) 1 day = 24 hours (h) 1 week (wk) = 7 days 1 year (yr) = 365 days

Table 1

In many real-life applications, we need to convert between units of measurement. We will use the identity property of multiplication to do these conversions. To use the identity property of multiplication, we write 1 in a form that will help us to convert the units.

For example, we want to convert inches to feet. We know that 1 foot is equal to 12 inches, so we can write 1 as the fraction $\frac{1\text{ ft}}{12\text{ in}}$. When we multiply by this fraction, we do not change the value but just change the units.

But $\frac{12\text{ in}}{1\text{ ft}}$ also equals 1. How do we decide whether to multiply by $\frac{1\text{ ft}}{12\text{ in}}$ or $\frac{12\text{ in}}{1\text{ ft}}$? We choose the fraction that will make the units we want to convert ***from*** divide out. For example, suppose we wanted to convert 60 inches to feet. If we choose the fraction that has inches in the denominator, we can eliminate the inches.

$$60\ \cancel{\text{in}} \cdot \frac{1\text{ ft}}{12\ \cancel{\text{in}}} = 5\text{ ft}$$

On the other hand, if we wanted to convert 5 feet to inches, we would choose the fraction that has feet in the denominator.

$$5\ \cancel{\text{ft}} \cdot \frac{12\text{ in}}{1\ \cancel{\text{ft}}} = 60\text{ in}$$

We treat the unit words like factors and divide out common units like we do common factors.

Example 6.1: Asma is 66 inches tall. What is her height in feet?

Solution:

$\textbf{66 inches} = 66\text{ inches} \cdot \dfrac{1\text{ foot}}{12\text{ inches}}$	Write 1 as a fraction relating the units given and the units needed.
$= \dfrac{66\text{ inches} \cdot 1\text{ foot}}{12\text{ inches}}$	Multiply.
$= \dfrac{66\ \cancel{\text{inches}} \cdot 1\text{ foot}}{12\ \cancel{\text{inches}}}$	Simplify the fraction.
$= \dfrac{66\text{ feet}}{12}$	
$= 5.5\text{ feet}$	

Notice that when we simplified the fraction, we first divided out the inches.
Asma is 5.5 feet tall.

Example 6.2: Jumbo, an elephant at the San Diego Safari Park, weighs almost 3.2 tons. Convert her weight to pounds.

Solution:

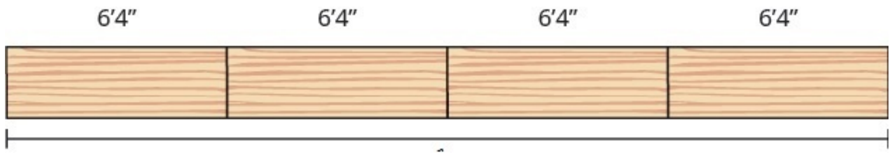
$\textbf{3.2 tons} = 3.2\text{ tons} \cdot \dfrac{2000\text{ lbs}}{1\text{ ton}}$	Write 1 as a fraction relating tons and pounds.
$= \dfrac{3.2\ \cancel{\text{tons}} \cdot 2000\text{ lbs}}{1\ \cancel{\text{ton}}}$	Simplify the fraction.
$= 6400\text{ lbs}$	

Sometimes to convert from one unit to another, we may need to use several other units in between, so we will need to multiply several fractions.

6.1.2 Use Mixed Units of Measurement in the U.S. System

Performing arithmetic operations on measurements with mixed units of measures requires care. Be sure to add or subtract like units.

Example 6.3: Ahmed bought four planks of wood that were each 6 feet 4 inches long. If the four planks are placed end-to-end, what is the total length of the wood?



Solution:

Total length	=	(6 feet 4 inches) × 4	We will multiply the length of one plank by 4 to find the total length.
	=	24 feet + 16 inches	Multiply the inches and then the feet.
	=	24 feet + 1 foot 4 inches	Convert 16 inches to feet.
	=	25 feet + 4 inches	Add the feet.

Ahmed bought 25 feet 4 inches of wood.

6.1 Section Exercises

Make Unit Conversions in the U.S. System

In the following exercises, convert the units.

1. A park bench is 6 feet long. Convert the length to inches.
2. A ribbon is 18 inches long. Convert the length to feet.
3. Salim is 6 feet 4 inches tall. Convert his height to inches.
4. A football field is 160 feet wide. Convert the width to yards.
5. On a baseball diamond, the distance from home plate to first base is 30 yards. Convert the distance to feet.
6. Humaid lives 1.5 miles from school. Convert the distance to feet.
7. Omers dog, Beans, weighs 8 pounds. Convert his weight to ounces.
8. Baby Omer weighed 7 pounds 3 ounces at birth. Convert his weight to ounces.
9. Blue whales can weigh as much as 150 tons. Convert the weight to pounds.

Use Mixed Units of Measurement in the U.S. System

In the following exercises, solve and write your answer in mixed units.

10. Azzan caught three fish. The weights of the fish were 2 pounds 4 ounces, 1 pound 11 ounces, and 4 pounds 14 ounces. What was the total weight of the three fishes?
11. Maryam bought 1 pound 6 ounces of almonds, 2 pounds 3 ounces of walnuts, and 8 ounces of cashews. What was the total weight of the nuts?
12. Lamees attached a 6-foot-6-inch extension cord to her computers 3-foot-8-inch power cord. What was the total length of the cords?

6.2 Metric System

In the metric system, units are related by powers of 10. The root words of their names reflect this relation. For example, the basic unit for measuring length is a meter. One kilometer is 1000 meters; the prefix kilo - means thousand. One centimeter is $\frac{1}{100}$ of a meter, because the prefix centi- means one one-hundredth.

6.2.1 Make Unit Conversions in the Metric System

The equivalencies of measurements in the metric system are shown in Table 2. The common abbreviations for each measurement are given in parentheses. We use m for meter, g for gram and L for liter.

Metric Measurements		
Length	**Mass**	**Volume / Capacity**
1 kilometer (km) = 1000 m 1 hectometer (hm) = 100 m 1 decameter (dam) = 10 m 1 decimeter (dm) = 0.1 m 1 centimeter (cm) = 0.01 m 1 millimeter (mm) = 0.001 m	1 kilogram (kg) = 1000 g 1 hectogram (hg) = 100 g 1 decagram (dag) = 10 g 1 decigram (dg) = 0.1 g 1 centigram (cg) = 0.01 g 1 milligram (mg) = 0.001 g	1 kiloliter (kL) = 1000 L 1 hectoliter (hL) = 100 L 1 decaliter (daL) = 10 L 1 deciliter (dL) = 0.1 L 1 centiliter (cL) = 0.01 L 1 milliliter (mL) = 0.001 L
1 meter = 100 centimeters 1 meter = 1000 millimeters	1 gram = 100 centigrams 1 gram = 1000 milligrams	1 liter = 100 centiliters 1 liter = 1000 milliliters

Table 2

To make conversions in the metric system, we will use the same technique we did in the U.S. system.

Example 6.4: Nasser ran a 10-kilometer race. How many meters did he run?

Solution: We will convert kilometers to meters using the Identity Property of Multiplication and the equivalencies in Table 2.

$$\begin{aligned} \textbf{10 kilometers} &= 10\text{ km} \times 1 && \text{Multiply the measurement to be converted by 1.} \\ &= 10\text{ km} \times \frac{1000\text{ m}}{1\text{ km}} && \text{Write 1 as a fraction relating kilometers and meters.} \\ &= 10\,\cancel{\text{km}} \times \frac{1000\text{ m}}{1\,\cancel{\text{km}}} && \text{Simplify.} \\ &= 10000\text{ m} && \text{Multiply.} \end{aligned}$$

Nasser ran 10,000 meters.

Example 6.5: Majids newborn baby weighed 3200 grams. How many kilograms did the baby weigh?

Solution: We will convert grams to kilograms.

$$\begin{aligned}
\textbf{3200 grams} &= 3200\text{ g} \times 1 && \text{Multiply the measurement to be converted by 1.}\\
&= 3200\text{ g} \times \frac{1\text{ kg}}{1000\text{ g}} && \text{Write 1 as a fraction relating kilograms and grams.}\\
&= 3200\,\cancel{\text{g}} \times \frac{1\text{ kg}}{1000\,\cancel{\text{g}}} && \text{Simplify.}\\
&= 3.2\text{ kg} && \text{Divide - moving the decimal 3 places to the left.}
\end{aligned}$$

The baby weighed 3.2 kilograms.

Example 6.6: Convert: (**a**) 350 liters to kiloliters (**b**) 4.1 liters to milliliters.

Solution:

a. We will convert liters to kiloliters. In Table 2, we see that 1 kiloliter = 1000 liters.

$$\begin{aligned}
\textbf{350 L} &= 350\text{ L} \times 1 && \text{Multiply the measurement to be converted by 1.}\\
&= 350\text{ L} \times \frac{1\text{ kL}}{1000\text{ L}} && \text{Writing 1 as a fraction relating liters to kiloliters.}\\
&= 350\,\cancel{\text{L}} \times \frac{1\text{ kL}}{1000\,\cancel{\text{L}}} && \text{Simplify.}\\
&= 350 \times \frac{1\text{ kL}}{1000}\\
&= 0.35\text{ kL} && \text{Move the decimal 3 units to the left.}
\end{aligned}$$

b. We will convert liters to milliliters. In Table 2, we see that 1 liter = 1000 milliliters.

$$\begin{aligned}
\textbf{4.1 L} &= 4.1\text{ L} \times \frac{1000\text{ mL}}{1\text{ L}} && \text{Multiply by 1, writing 1 as a fraction relating milliliters to liters.}\\
&= 4.1\,\cancel{\text{L}} \times \frac{1000\text{ mL}}{1\,\cancel{\text{L}}} && \text{Simplify.}\\
&= 4.1 \times 1000\text{ mL}\\
&= 4100\text{ mL} && \text{Move the decimal 3 units to the right.}
\end{aligned}$$

6.2.2 Use Mixed Units of Measurement in the Metric System

Performing arithmetic operations on measurements with mixed units of measures in the metric system requires the same care we used in the U.S. system. But it may be easier because of the relation of the units to the powers of 10. We still must *make sure to add or subtract like units.*

Example 6.7: Humaid is 1.6 meters tall. His younger brother is 85 centimeters tall. How much taller is Humaid than his younger brother?
Solution: We will subtract the lengths in meters. Convert 85 centimeters to meters by moving the decimal 2 places to the left; 85 cm is the same as 0.85 m.

Now that both measurements are in meters, subtract to find out how much taller Humaid is than his brother.

$$\begin{array}{r} 1.60 \text{ m} \\ -0.85 \text{ m} \\ \hline 0.75 \text{ m} \end{array}$$

Humaid is 0.75 meters taller than his brother.

Example 6.8: Hibas recipe for tomato soup calls for 150 milliliters of olive oil. Hiba wants to triple the recipe. How many liters of olive oil will she need?
Solution: We will find the amount of olive oil in milliliters then convert to liters.

$$\begin{aligned} \textbf{Olive oil needed} &= 3 \times 150 \text{ mL} && \text{Triple 150 mL} \\ &= 450 \text{ mL} && \text{Multiply.} \\ &= 450 \text{ mL} \times \frac{1 \text{ L}}{1000 \text{ mL}} && \text{Convert to liters.} \\ &= 450 \times \frac{1 \text{ L}}{1000} && \text{Simplify.} \\ &= 0.45 \text{ L} && \text{Divide.} \end{aligned}$$

Hiba needs 0.45 liter of olive oil.

6.2 Section Exercises

Make Unit Conversions in the Metric System

In the following exercises, convert the units.

1. Nabhan ran 5 kilometers. Convert the length to meters.
2. Al Jabal Al Akhdar is 2980 meters tall. Convert the height to kilometers.
3. Bushra is 1.55 meters tall. Convert her height to centimeters.
4. A multivitamin contains 1,500 milligrams of calcium. Convert this to grams.
5. A bottle of medicine contained 300 milliliters. Convert this to liters.
6. One stick of butter contains 91.6 grams of fat. Convert this to milligrams.

Use Mixed Units of Measurement in the Metric System

In the following exercises, solve and write your answer in mixed units.

7. Ahmed mailed 5 packages that weighed 420 grams each. What was the total weight of the packages in kilograms?

8. Khalil is 1.6 meters tall. His sister is 95 centimeters tall. How much taller, in centimeters, is Khalil than his sister?

9. Murshid drinks 200 milliliters of water 8 times a day. How many liters of water does Murshid drink in a day?

6.3 Conversion Between U.S. and Metric Systems

Many measurements in the United States are made in metric units. To work easily in both systems, we need to be able to convert between the two systems.

Conversion Factors Between U.S. and Metric Systems		
Length	**Weight**	**Volume**
1 in = 2.54 cm 1 ft = 0.305 m 1 yd = 0.914 m 1 mi = 1.61 km 1 m = 3.28 ft	1 lb = 0.45 kg 1 oz = 28 g 1 kg = 2.2 lb	1 qt = 0.95 L 1 fl oz = 30 mL 1 gallon = 3.785 L 1 L = 1.06 qt

Table 3

We make conversions between the systems just as we do within the systemsby multiplying by unit conversion factors.

Example 6.9:

a. Mosa is 75 inches tall. Convert his height to centimeters.

b. Mahnad passed a football 24 yards. Convert the pass length to meters.

c. The height of Mount Kilimanjaro is 5,895 meters. Convert the height to feet.

d. The flight distance from New York City to London is 3470 miles. Convert the distance to kilometers.

Solution:

a. **75 inches** $= 75 \text{ in} \times \dfrac{2.54 \text{ cm}}{1 \text{ in}}$ Multiply by a unit conversion factor relating inches and centimeter.

$= 190.5 \text{ cm}$ Simplify.

b.	**24 yards**	$= 24 \text{ yd} \times \dfrac{0.914 \text{ m}}{1 \text{ yd}}$	Multiply by a unit conversion factor relating yard and meter.
		$= 21.936 \text{ m}$	Simplify.
c.	**5895 meters**	$= 5895 \text{ m} \times \dfrac{3.28 \text{ ft}}{1 \text{ m}}$	Multiply by a unit conversion factor relating meter and feet.
		$= 19335.6 \text{ ft}$	Simplify.
d.	**3470 miles**	$= 3470 \text{ mi} \times \dfrac{1.61 \text{ km}}{1 \text{ mi}}$	Multiply by a unit conversion factor relating mile and kilometer.
		$= 5586.7 \text{ km}$	Simplify.

Example 6.10:

a. Saids suitcase weighed 20 kilograms. Convert the weight to pounds.

b. Each American throws out an average of 1,650 pounds of garbage per year. Convert this weight to kilograms.

Solution:

a.	**20 kilograms**	$= 20 \text{ kg} \times \dfrac{2.2 \text{ lb}}{1 \text{ kg}}$	Multiply by a unit conversion factor relating kilogram and pound.
		$= 44 \text{ lb}$	Simplify.
b.	**1650 pounds**	$= 1650 \text{ lb} \times \dfrac{0.45 \text{ kg}}{1 \text{ lb}}$	Multiply by a unit conversion factor relating kilogram and pound.
		$= 742.5 \text{ kg}$	Simplify.

Example 6.11:

a. Dawood bought 8 gallons of paint. Convert the volume to liters.

b. Manals water bottle holds 600 mL of water. How many fluid ounces are in the bottle?

Solution:

a.	**8 gallons**	$= 8 \text{ gallons} \times \dfrac{3.785 \text{ L}}{1 \text{ gallon}}$	Multiply by a unit conversion factor relating gallon and liter.
		$= 30.28 \text{ liters}$	Simplify.
b.	**600 mL**	$= 600 \text{ mL} \times \dfrac{1 \text{ fl oz}}{30 \text{ mL}}$	Multiply by a unit conversion factor relating fluid ounce and mL.
		$= 20 \text{ fl oz}$	Simplify.

6.3 Section Exercises

In the following exercises, convert between U.S. and metric units. Round to the nearest tenth.

1. Majid is 61 inches tall. Convert his height to centimeters.
2. A college basketball court is 84 feet long. Convert this length to meters.
3. Rahma walked 2.5 miles. Convert this distance to kilometers.
4. Khalfan weighs 78 kilograms. Convert his weight to pounds.
5. Steves car holds 30 gallons of gas. Convert this to liters.
6. A box of books weighs 25 pounds. Convert this weight to kilograms.

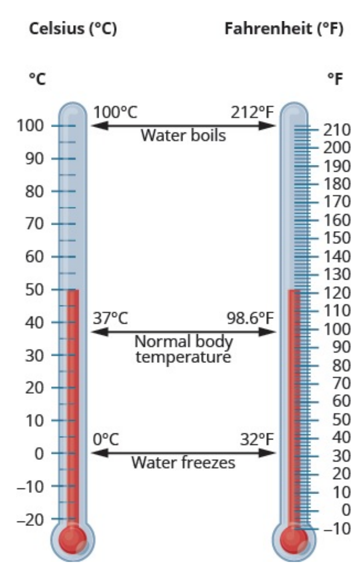
6.4 Measurements of Temperature

The concept of temperature has evolved from the common concepts of hot and cold. Human perception of what feels hot or cold is a relative one. Temperature is operationally defined to be what we measure with a thermometer. The U.S. and metric systems use different scales to measure temperature. The U.S. system uses degrees Fahrenheit, written °F. The metric system uses degrees Celsius, written °C. The figure shows the relationship between the two systems.

The **Celsius scale** (which replaced the slightly different centigrade scale) has the freezing point of water at 0° C and the boiling point at 100° C. Its unit is the degree Celsius ° C.

On the **Fahrenheit scale** (still the most frequently used in the United States), the freezing point of water is at 32° F and the boiling point is 180 degrees higher at 212° F. The unit of temperature on this scale is the degree Fahrenheit ° F.

The **Kelvin scale** is the temperature scale that is commonly used in science. It is an absolute temperature scale defined to have 0 K at the lowest possible temperature, called **absolute zero**. The freezing and boiling points of water are 273 K and 373 K, respectively.



If we know the temperature in one system, we can use a formula to convert it to the other system.

Temperature Conversion

To convert from Fahrenheit temperature, F, to Celsius temperature, C, use the formula

$$C = \frac{5}{9}(F - 32)$$

To convert from Celsius temperature, C, to Fahrenheit temperature, F, use the formula

$$F = \frac{9}{5}C + 32$$

To convert from Celsius C to Kelvin K, use the formula

$$K = C + 273$$

To convert from Kelvin K to Celsius C, use the formula

$$C = K - 273$$

To convert from Fahrenheit F to Kelvin K, use the formula

$$K = \frac{5}{9}(F - 32) + 273$$

To convert from Kelvin K to Fahrenheit F, use the formula

$$F = \frac{9}{5}(K - 273) + 32$$

Example 6.12: Convert 50° F into (a) degrees Celsius and (b) Kelvin.

Solution:

(a) We will substitute 50° F into the formula $C = \frac{5}{9}(F - 32)$ to find C.

$$\begin{aligned} C &= \frac{5}{9}(F - 32) && \text{Use the formula for converting } ^\circ \text{ F to } ^\circ\text{C} \\ &= \frac{5}{9}(50 - 32) && \text{Substitute 50 for F.} \\ &= \frac{5}{9}(18) && \text{Simplify in parentheses.} \\ &= 10 && \text{Multiply.} \end{aligned}$$

We have, 50°F = 10°C.

(b) We will substitute 50° F into the formula $K = \frac{5}{9}(F - 32) + 273$ to find K.

$$\begin{aligned} K &= \frac{5}{9}(F - 32) + 273 && \text{Use the formula for converting } ^\circ \text{ F to } ^\circ\text{C} \\ &= \frac{5}{9}(50 - 32) + 273 && \text{Substitute 50 for F.} \\ &= \frac{5}{9}(18) + 273 && \text{Simplify in parentheses.} \\ &= 10 + 273 && \text{Add.} \\ &= 283 && \text{Simplify.} \end{aligned}$$

We have, 50°F = 283 K.

Also we can do part (b) as given below:

$$\begin{aligned} 50^\circ F &= 10^\circ C && \text{From (a)} \\ &= 10 + 273K = 283K && \text{Convert from Celsius C to Kelvin K.} \end{aligned}$$

Example 6.13: The weather forecast for Paris predicts a high of 20° C. Convert the temperature into (a) degrees Fahrenheit and (b) Kelvin.

Solution:

(a) We will substitute 20° C into the formula $F = \frac{9}{5}C + 32$ to find F.

$$\begin{aligned} F &= \frac{9}{5}C + 32 && \text{Use the formula for converting °C to °F} \\ &= \frac{9}{5}(20) + 32 && \text{Substitute 20 for C.} \\ &= 36 + 32 && \text{Simplify in parentheses.} \\ &= 68 && \text{Multiply.} \end{aligned}$$

So 20° C is equivalent to 68° F.

(b) We will substitute 20° C into the formula $K = C + 273$ to find F.

$$\begin{aligned} 20^\circ C &= 20^\circ C + 273\,K && \text{From (a)} \\ &= 293\,K && \text{Multiply.} \end{aligned}$$

Example 6.14: The surface temperature of the Sun is about 5763 K. What is this temperature on the Fahrenheit scale? What is this on the temperature on the Celsius scale?

Solution:

(a) We will substitute 5763 K into the formula $F = \frac{9}{5}(K - 273) + 32$ to find F.

$$\begin{aligned} F &= \frac{9}{5}(K - 273) + 32 && \text{Use the formula for converting } F = \frac{9}{5}(K-273)+32 \\ &= \frac{9}{5}(5763 - 273) + 32 && \text{Substitute 5763 for K.} \\ &= \frac{9}{5}(5490) + 32 && \text{Simplify.} \\ &= 9 \times 1098 + 32 && \text{Simplify.} \\ &= 9914 && \text{Multiply.} \end{aligned}$$

So 5763 K is equivalent to 9914° F.

(b) We will substitute 5763 K into the formula $C = K - 273$ to find C.

$$\begin{aligned} C &= K - 273 && \text{From (a)} \end{aligned}$$

$= 5763 - 273$ Substitute 5763 for K.

$= 5490$ Simplify.

6.4 Section Exercises

In the following exercises, convert the Fahrenheit temperature to degrees Celsius. Round to the nearest tenth.

1. 86° F

2. 77° F

3. 104° F

4. 14° F

5. 72° F

6. 4° F

7. 0° F

8. 120° F

In the following exercises, convert the Celsius temperatures to degrees Fahrenheit. Round to the nearest tenth.

9. 5° C

10. 25° C

11. -10° C

12. -15° C

13. 22° C

14. 8° C

15. 43° C

16. 16° C

17. To conserve energy, room temperatures are kept at 68.0 ° F in the winter and 77.0 ° F in the summer. What are these temperatures on the Celsius scale?

18. What is the Fahrenheit temperature of a person with a 39° C fever?

19. Frost damage to most plants occurs at temperatures of 28° F or lower. What is this temperature on the Kelvin scale?

20. One of the hottest temperatures ever recorded on the surface of Earth was **134°F** in Death Valley, CA. What is this temperature in Celsius degrees? What is this temperature in Kelvin?

21. At what temperature do the Fahrenheit and Celsius scales have the same numerical value?

22. At what temperature do the Fahrenheit and Kelvin scales have the same numerical value?

Chapter 7

LINEAR EQUATIONS & INEQUALITIES

Contents

Learning outcomes covered:

f. Translate worded problems into mathematical expression and model simple real life problems with equations and inequalities.

g. Solve linear equations, equations involving radicals, fractional expression and inequalities.

o. Apply knowledge of basic algebra and trigonometry in real life problems.

Learning Objectives

By the end of this chapter, the students will be able to:

- Solve linear equations using a general strategy
- Solve word problems
- Solve a formula for a specific variable
- Use formulas to solve geometry applications
- Graph inequalities on the number line
- Solve linear inequalities

Introduction

In this chapter, you will explore linear equations and inequalities, develop a strategy for solving them, and relate them to real-world situations.

7.1 Linear Equations

In this section, we will study about linear equations in one or more variables and develop a strategy for solving them.

7.1.1 Use a General Strategy to Solve Linear Equations

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that make each side of the equation the same so that we end up with a true statement. Any value of the variable that makes the equation true is called a **solution** to the equation. It is the answer to the puzzle!

Solution of an equation

A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

✓ $x = \frac{7}{5}$ is a solution of the equation $5x + 3 = 10x - 4$. [makes a true statement $10 = 10$]

✗ $x = \frac{3}{5}$ is not a solution to the equation $5x + 3 = 10x - 4$. [makes a false statement $6 \neq 2$]

There are many types of equations that we will learn to solve. In this section we will focus on a **linear equation** in one variable.

Linear Equation

A **linear equation** in one variable can be written as $ax + b = 0$, where a and b are real numbers and $a \neq 0$.

In the next example, we will give the steps of a general strategy for solving any linear equation.

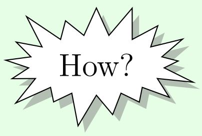
Example 7.1: How to Solve Linear Equations Using the General Strategy

Solve: $7(n–3)–8 = –15$

Solution:

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Note that each side of the equation is simplified as much as possible.	$7(n–3)–8 = –15$ $7n - 21 - 8 = –15$ $7n - 29 = –15$
Step 2. Collect all the variable terms on one side of the equation.	Nothing to do. - All the n's are on the left side.	
Step 3. Collect all the constant terms on the other side of the equation.	To get constants only on the right, add 29 to each side. Simplify.	$7n - 29 + 29 = -15 + 29$ $7n = 14$
Step 4. Make the coefficient of the variable term to equal to 1.	Divide each side by 7 Simplify.	$\frac{7n}{7} = \frac{14}{7}$ $n = 2$

How To **:: General Strategy for Solving Linear Equations.**



Step 1. **Simplify each side of the equation as much as possible.** Use the Distributive Property to remove any parentheses. Combine like terms.

Step 2. **Collect all the variable terms on one side of the equation.** Use the Addition or Subtraction Property of Equality.

Step 3. **Collect all the constant terms on the other side of the equation.** Use the Addition or Subtraction Property of Equality.

Step 4. **Make the coefficient of the variable term to equal to 1.** Use the Multiplication or Division Property of Equality. State the solution to the equation.

Step 5. **Check the solution.** Substitute the solution into the original equation to make sure the result is a true statement.

We can solve equations by getting all the variable terms to either side of the equal sign. By collecting the variable terms on the side where the coefficient of the variable is larger, we avoid working with some negatives. This will be a good strategy when we solve inequalities later in this chapter. It also helps us prevent errors with negatives.

Example 7.2: Solve: $4(x-1)-2 = 5(2x+3)+6$

Solution:

$4(x-1)-2 = 5(2x+3)+6$	
$4x-4-2 = 10x+15+6$	Distribute.
$4x-6 = 10x+21$	Combine like terms.
$4x-6-4x = 10x+21-4x$	Subtract $4x$ from each side to get the variables only on the right since $10 > 4$.
$-6 = 6x+21$	Simplify.
$-6-21 = 6x+21-21$	Subtract 21 from each side to get the constants on left.
$-27 = 6x$	Simplify.
$\dfrac{-27}{6} = \dfrac{6x}{6}$	Divide both sides by 6.
$-\dfrac{9}{2} = x$	Simplify
$x = -\dfrac{9}{2}$	

7.1.2 Solve Equations with Fraction or Decimal Coefficients

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator (LCD) of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions.
Notice in the previous example, once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve.

> ***How To*** **:: Strategy to Solve Equations with Fraction Coefficients.**
>
> Step 1. Find the least common denominator (LCD) of all the fractions and decimals (in fraction form) in the equation.
>
> Step 2. Multiply both sides of the equation by that LCD. This clears the fractions.
>
> Step 3. Solve using the General Strategy for Solving Linear Equations.

Example 7.3: Solve: $\frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y = 5$

Solution:

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the LCD of all fractions in the equation $\frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y = 5$. The LCD is 12.

$$\frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y = 5$$

$$12 \cdot \left(\frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y\right) = (12)5 \qquad \text{Multiply both sides of the equation by 12.}$$

$$12 \cdot \frac{1}{2}y + 12 \cdot \frac{2}{3}y - 12 \cdot \frac{3}{4}y = (12)5 \qquad \text{Distribute.}$$

$$6y + 8y - 9y = 60 \qquad \text{Simplifynotice, no more fractions!}$$

$$5y = 60 \qquad \text{Combine like terms.}$$

$$\frac{5y}{5} = \frac{60}{5} \qquad \text{Divide both sides by 5.}$$

$$y = 12 \qquad \text{Simplify.}$$

7.1.3 Solve a Formula for a Specific Variable

It is often helpful to solve a formula for a specific variable. We isolate that variable on one side of the equals sign with a coefficient of one and all other variables and constants are on the other side of the equal sign.

In the sciences, we often need to change temperature from Fahrenheit to Celsius or vice versa.

Example 7.4: Solve the formula $C = \frac{5}{9}(F - 32)$ for F.

Solution:

$$C = \frac{5}{9}(F - 32) \qquad \text{Write the formula.}$$

$$\frac{9}{5}C = \frac{9}{5} \cdot \frac{5}{9}(F - 32) \qquad \text{Remove the fraction on the right.}$$

$$\frac{9}{5}C = F - 32 \qquad \text{Simplify.}$$

$$\frac{9}{5}C + 32 = F \qquad \text{Add 32 to both sides.}$$

$$F = \frac{9}{5}C + 32$$

Sometimes we might be given an equation that is solved for y and need to solve it for x, or vice versa.

Example 7.5: Solve the formula $8x + 7y = 15$ for y.

Solution:

$$8x + 7y = 15$$

$$8x + 7y - 8x = 15 - 8x \qquad \text{Subtract } 8x \text{ from both sides to isolate the term with } y.$$

$$7y = 15 - 8x \qquad \text{Simplify.}$$

$$\frac{7y}{7} = \frac{15 - 8x}{7} \qquad \text{Divide by 7 to make the coefficient 1.}$$

$$y = \frac{15 - 8x}{7} \qquad \text{Simplify.}$$

7.1 Section Exercises

Solve Equations Using the General Strategy

In the following exercises, solve each linear equation.

1. $3(10–2x) + 54 = 0$

2. $-2(11–7x) + 54 = 4$

3. $\frac{2}{3}(9c - 3) = 22$

4. $-15 + 4(2–5y) = -7(y–4) + 4$

5. $4(p–4)–(p + 7) = 5(p–3)$

6. $4[5–8(4c–3)] = 12(1–13c)–8$

Solve Equations with Fraction or Decimal Coefficients

In the following exercises, solve each equation with fraction coefficients.

7. $\frac{1}{4}x - \frac{1}{2} = -\frac{3}{4}$ **8.** $\frac{3}{4}x - \frac{1}{2} = \frac{1}{4}$ **9.** $\frac{3x+4}{2} + 1 = \frac{5x+10}{8}$

Solve a Formula for a Specific Variable

In the following exercises, solve the formula for the specific variable.

10. $8x + y = 15$ for x **11.** $-4x + y = -6$ for y **12.** $A = \frac{1}{2}bh$ for b

7.2 Applications of Linear Equations

In this section, we will use some common geometry formulas. We will adapt our problem solving strategy to solve some word problems and geometry applications.

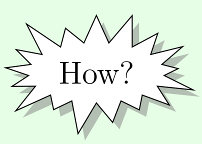
7.2.1 Word Problems

You are now well prepared and you are ready to succeed. If you take control and believe you can be successful, you will be able to master word problems.

Use a Problem Solving Strategy for Word Problems

Now that we can solve equations, we are ready to apply our new skills to word problems. We will develop a strategy we can use to solve any word problem successfully.

How To **:: Solve an Application.**



Step 1. **Read** the problem. Make sure all the words and ideas are understood.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with the important information.

Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

Solve Word Problems

Example 7.6: The sum of seven times a number and eight is thirty-six. Find the number.

Solution:

Step 1: Read the problem.	
Step 2: Identify what you are looking for.	the number
Step 3: Name what we are looking for. Choose a variable to represent that quantity.	Let $n =$ the number.
Step 4: Translate. Restate the problem in one sentence with all the important information.	$\underbrace{\text{The sum of seven times a number and eight}}_{7n+8}\ \underbrace{\text{is}}_{=}\ \underbrace{36}_{36}$
Step 5: Solve the equation. Subtract eight from each side and simplify. Divide each side by seven and simplify.	$7n + 8 = 36$ $7n = 28$ $n = 4$
Step 6: Check: Is the sum of seven times four plus eight equal to 36?	$7n + 8 \stackrel{?}{=} 36$ $28 + 8 \stackrel{?}{=} 36$ $36 = 36$ ✓
Step 7: Answer the question.	The number is 4.

Some number word problems ask us to find **two or more numbers**. It may be tempting to name them all with different variables, but so far, we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable.

Example 7.7: The sum of two numbers is negative fifteen. One number is nine less than the other. Find the numbers.

Solution:

Step 1: Read the problem.	
Step 2: Identify what you are looking for.	two numbers
Step 3: Name what you are looking for by choosing a variable to represent the first number. One number is nine less than the other.	Let $n = 1^{st}$ number. $n{-}9 = 2^{nd}$ number

Step 4: Translate. Write as one sentence. Translate into an equation.	The sum of two numbers is negative fifteen. $\underbrace{1^{st}\text{ number} + 2^{nd}\text{ number}}_{n+n-9}\ \underbrace{\text{was}}_{=}\ \underbrace{\text{-15}}_{-15}$
Step 5: Solve the equation. Combine like terms. Add nine to each side and simplify. Simplify.	$n+n-9=-15$ $2n-9=-15$ $2n=-6$ $n=-3$ $1^{st}\ number = n = -3$ $2^{nd}\ number = n-9 = -12$
Step 6: Answer the question.	The numbers are -3 and -12.

Consecutive Intgers

Some number problems involve **consecutive integers**. Consecutive integers are integers that immediately follow each other. Examples of **consecutive integers** are:

$$\begin{array}{rrrr} 1, & 2, & 3, & 4 \\ 10, & 9, & 8, & 7 \end{array}$$

Notice that each number is one more than the number preceding it. Therefore, if we define the first integer as n, the next consecutive integer is $n+1$. The one after that is one more than $n+1$, so it is $n+1+1$, which is $n+2$.

$$\begin{array}{ll} n & 1^{st}\text{ integer} \\ n+1 & 2^{nd}\text{ consecutive integer} \\ n+2 & 3^{rd}\text{ consecutive integer etc.} \end{array}$$

Example 7.8: Find three consecutive integers whose sum is -54.

Solution:

Step 1: Read the problem.	
Step 2: Identify what you are looking for.	three consecutive integers
Step 3: Name each of the three numbers	Let $n = 1^{st}$ integer $n+1 = 2^{nd}$ consecutive integer $n+2 = 3^{rd}$ consecutive integer
Step 4: Translate. Write as one sentence. Translate into an equation.	The sum of the three integers is -54. $n+n+1+n+2=-54$

Step 5: Solve the equation. Combine like terms. Subtract three from each side. Divide each side by three.	$n+n+1+n+2=-54$ $3n+3=-54$ $3n=-57$ $n=-19$ 1^{st} *integer*, $n=-19$ 2^{nd} *integer*, $n+1=3-19+1=-18$ 3^{rd} *integer*, $n+2==-19+2=-17$
Step 6: Check:	$-19+(-18)+(-17)\stackrel{?}{=}-54$ $-54=-54\checkmark$
Step 7: Answer the question.	The three consecutive integers are -17, -18, and -19.

We will expand our work to include **consecutive even integers** and **consecutive odd integers**. Consecutive even integers are even integers that immediately follow one another. Examples of consecutive even integers are: 24, 26, 28 and -12, -10, -8.

Notice each integer is two more than the number preceding it. If we call the first one n, then the next one is $n+2$. The one after that would be $n+2+2$ or $n+4$.

n 1^{st} even integer

$n+2$ 2^{nd} consecutive even integer

$n+4$ 3^{rd} consecutive even integer etc.

Consecutive odd integers are odd integers that immediately follow one another. Consider the consecutive odd integers 63, 65, and 67.

n 1^{st} odd integer

$n+2$ 2^{nd} consecutive odd integer

$n+4$ 3^{rd} consecutive odd integer etc.

Whether the problem asks for consecutive even numbers or odd numbers, you do not have to do anything different. The pattern is still the same - to get to the next odd or the next even integer, add two.

Example 7.9: Find three consecutive even integers whose sum is 120.

Solution:

Step 1: Read the problem.	
Step 2: Identify what you are looking for.	three consecutive even integers

Step 3: Name each of the three numbers.	Let $n = 1^{st}$ even integer $n + 2 = 2^{nd}$ consecutive even integer $n + 4 = 3^{rd}$ consecutive even integer
Step 4: Translate. Restate as one sentence. Translate into an equation.	The sum of the three integers is 120. $n + n + 2 + n + 4 = 120$
Step 5: Solve the equation. Combine like terms. Subtract 6 from each side. Divide each side by 3.	$n + n + 2 + n + 4 = 120$ $3n + 6 = 120$ $3n = 114$ $n = 38$ 1^{st} *even integer*, $n = 38$ 2^{nd} *even integer*, $n + 2 = 38 + 2 = 40$ 3^{rd} *even integer*, $n + 4 == 38 + 4 = 42$
Step 6: Check:	$38 + 40 + 42 \stackrel{?}{=} 120$ $120 = 120$ ✓
Step 7: Answer the question.	The three consecutive even integers are 38, 40, and 42.

7.2.2 Solving Geometry Applications

How To **:: Solve Geometry Applications.**

Step 1. **Read** the problem. Make sure all the words and ideas are understood.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation. Write the appropriate formula for the situation. Substitute in the given information.

Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

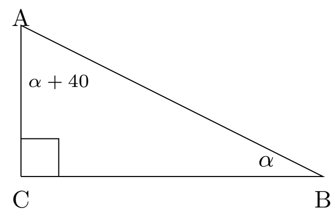
When we solve geometry applications, we often have to use some of the properties of the figures. We will review those properties as needed.

In a triangle $\triangle ABC$,

- the sum of the measures of the angles is 180°. $\angle$ A + $\angle$ B + $\angle$ C = 180.
- the perimeter of a triangle is just the distance around the triangle. We can write this as $P = a + b + c$, where a, b, and c are the lengths of the sides.
- a right triangle has one 90° angle.

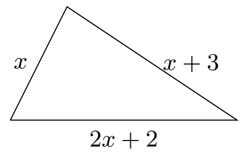
Example 7.10: The measure of one angle of a right triangle is 40 degrees more than the measure of the smallest angle. Find the measures of all three angles.

Solution:

Step 1: Read the problem.	
Step 2: Identify what you are looking for.	the measures of all three angles
Step 3: Name. Choose a variable to represent it. Draw the figure and label it with the given information.	Let $\alpha = 1^{st}$ angle $\alpha + 40 = 2^{nd}$ angle $90 = 3^{rd}$ angle(the right angle) 
Step 4: Translate. Write the appropriate formula. Substitute into the formula.	$m\angle A + m\angle B + m\angle C = 180$ $\alpha + (\alpha + 40) + 90 = 180$
Step 5: Solve the equation.	$2\alpha + 130 = 180$ $2\alpha = 50$ $\alpha = 25$ 1^{st} *angle*, $\alpha = 25°$ 2^{nd} *angle*, $\alpha + 40 = 25 + 40 = 65°$ 3^{rd} *angle*, $=90°$
Step 6: Check:	$25 + 65 + 90 \stackrel{?}{=} 180$ $180 = 180$ ✓
Step 7: Answer the question.	The three angles measure 25°, 65°, and 90°.

Example 7.11: One side of a triangle is three inches more than the first side. The third side is two inches more than twice the first. The perimeter is 29 inches. Find the length of the three sides of the triangle.

Solution:

Step 1: Read the problem.	
Step 2: Identify what you are looking for.	the lengths of the three sides of a triangle
Step 3: Name. Choose a variable to represent the length of the first side. Draw the figure and label it with the given information.	Let $x =$ length of 1^{st} side $x + 3 =$ length of 2^{nd} side $2x + 2 =$ length of 3^{rd} side 
Step 4: Translate. Write the appropriate formula. Substitute into the formula.	$P = a + b + c$ $29 = x + (x + 3) + (2x + 2)$
Step 5: Solve the equation.	$29 = 4x + 5$ $24 = 4x$ $6 = x$ *length of first side*, $x = 6$ *length of second side*, $x + 3 = 6 + 3 = 9$ *length of third side*, $2x + 2 = 2 \cdot 6 + 2 = 14$
Step 6: Check:	$29 \stackrel{?}{=} 6 + 9 + 14$ $29 = 29$ ✓
Step 7: Answer the question.	The lengths of the sides of the triangle are 6, 9, and 14 inches.

7.2 Section Exercises

Use a Problem Solving Strategy for Word Problems

In the following exercises, solve using the problem solving strategy for word problems. Remember to write a complete sentence to answer each question.

1. There are 16 girls in a school club. The number of girls is four more than twice the number of boys. Find the number of boys.

2.The difference of a number and 12 is three. Find the number.

3. The sum of three times a number and eight is 23. Find the number.

4. One number is five more than the other. Their sum is 33. Find the numbers.

5. The sum of two numbers is 20. One number is four less than the other. Find the numbers.

6. Find three consecutive integers whose sum is -3.

7. Find three consecutive integers whose sum is -33.

8. Find three consecutive even integers whose sum is 60.

9. Find four consecutive even integers whose sum is 92.

10. Find three consecutive odd integers whose sum is -99.

11. Find four consecutive odd integers whose sum is 48.

Use Formulas to Solve Geometry Applications

In the following exercises, solve using a geometry formula.

12. The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.

13. The angles in a triangle are such that one angle is twice the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

14. One side of a triangle is seven inches more than the first side. The third side is four inches less than three times the first. The perimeter is 28 inches. Find the length of the three sides of the triangle.

15. One side of a triangle is three feet less than the first side. The third side is five feet less than twice the first. The perimeter is 20 feet. Find the length of the three sides of the triangle.

7.3 Linear Inequalities

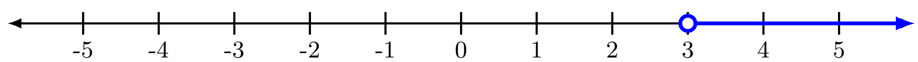
7.3.1 Graph Inequalities on the Number Line

What about the solution of an inequality? What number would make the inequality $x > 3$ true? Are you thinking, x could be 4? Thats correct, but x could be 5 too, or 20, or even 3.001. Any number greater than 3 is a solution to the inequality $x > 3$.

We show the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of 3, to show that all numbers greater than 3 are solutions. Because the number 3 itself is not a solution, we put a circle (hole) at 3.

We can also represent inequalities using **interval notation**. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as **infinity**. It is not an actual number. Figure 7.1 shows both the number line and the interval notation.

$x > 3$



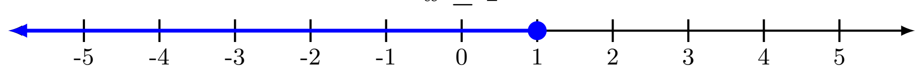
$(3, \infty)$

Figure 7.1: The inequality $x > 3$ is graphed on this number line and written in interval notation.

We use the left parenthesis symbol, (, to show that the endpoint of the inequality is not included. The left bracket symbol, [, shows that the endpoint is included.
The inequality $x \leq 1$ means all numbers less than or equal to 1. Here we need to show that one is a solution, too. We do that by putting a bracket at $x = 1$. We then shade in all the numbers to the left of one, to show that all numbers less than one are solutions. See Figure 7.2.

There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as **negative infinity**. Figure 7.2 shows both the number line and interval notation.

$x \leq 1$

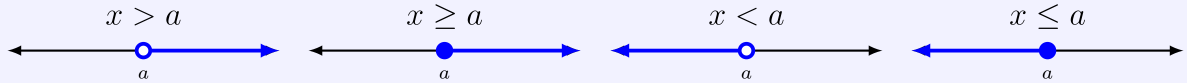


$(-\infty, 1]$

Figure 7.2: The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

Inequalities, Number Lines, and Interval Notation

$x > a$ $\qquad$ $x \geq a$ $\qquad$ $x < a$ $\qquad$ $x \leq a$



(a, ∞) $\qquad$ $[a, \infty)$ $\qquad$ $(-\infty, a)$ $\qquad$ $(-\infty, a]$

Example 7.12: Graph on the number line:

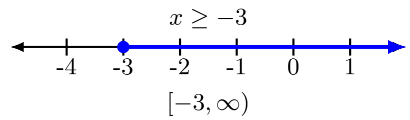
a. $x \geq -3$ **b.** $x < 2.5$ **c.** $x \leq -\frac{1}{2}$

Solution:

a. $x \geq -3$

Shade to the right of 3, and put a bracket at 3.

Write in interval notation.

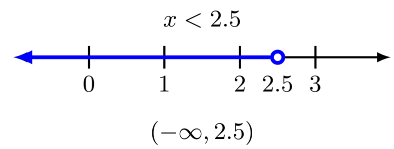


---

b. $x < 2.5$

Shade to the left of 2.5 and put a parenthesis at 2.5.

Write in interval notation.

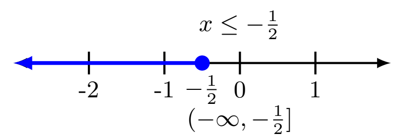


---

c. $x \leq -\frac{1}{2}$

Shade to the left of $-\frac{1}{2}$, and put a bracket at $-\frac{1}{2}$.

Write in interval notation.

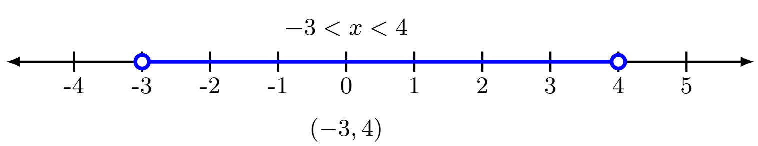


Example 7.13: Graph on the number line:

a. $-3 < x < 4$ **b.** $-6 \leq x < -1$ **c.** $0 \leq x \leq 2.5$

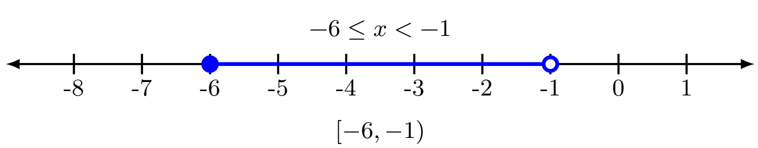
Solution:

a. $-3 < x < 4$



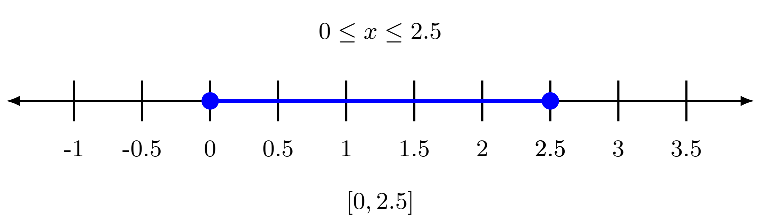
---

b. $-6 \leq x < -1$



---

c. $0 \leq x \leq 2.5$



7.3.2 Solving Linear Inequalities

A linear inequality is much like a linear equation-but the equal sign is replaced with an inequality sign.

Linear Inequality

A linear inequality in one variable is an inequality that can be written in one of the following forms where a, b, and c are real numbers and $a \neq 0$:

$$ax + b < c, \qquad ax + b \leq c, \qquad ax + b > c, \qquad \text{or } ax + b \geq c$$

Addition and Subtraction Property of Inequality

For any numbers a, b, and c, if $a < b$, then

$$\mathrm{a + c < b + c} \qquad \mathrm{a \;\; c < b \;\; c}$$

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality.

Does the inequality stay the same when we divide or multiply by a negative number?

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

Multiplication and Division Property of Inequality

For any numbers a, b, and c,

multiply or divide by a positive

if $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

multiply or divide by a negative

if $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

Example 7.14: Solve each inequality. Graph the solution on the number line, and write the solution in interval notation:

a. $9y < 54$ **b.** $-7x \leq -70$

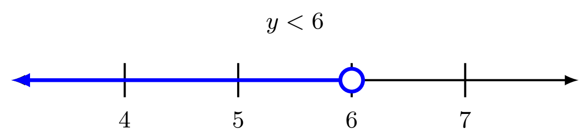
Solution:

a. $9y < 54$

$\dfrac{9y}{9} < \dfrac{54}{9}$ Divide both sides of the inequality by 9; since 9 is positive, the inequality stays the same.

$y < 6$ Simplify.

Graph the solution on the number line.



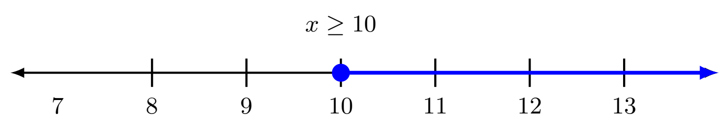
Write the solution in interval notation. $(-\infty, 6)$

b. $-7x \leq -70$

$\dfrac{-7x}{-7} \geq \dfrac{-70}{-7}$ Divide both sides of the inequality by -7. Since -7 is a negative, the inequality reverses.

$x \geq 10$ Simplify.

Graph the solution on the number line.



Write the solution in interval notation. $[10, -\infty)$

Example 7.15: Solve the inequality $6y \leq 11y + 20$, graph the solution on the number line, and write the solution in interval notation.

Solution: $6y \leq 11y + 20$

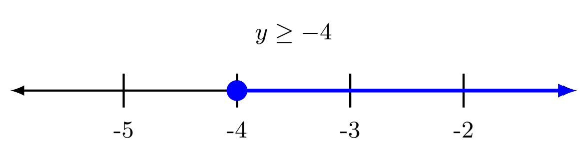
$6y - 11y \leq 11y - 11y + 20$ Subtract $11y$ from both sides to collect the variables on the left.

$-5y \leq 20$ Simplify.

$\dfrac{-5y}{-5} \geq \dfrac{20}{-5}$ Divide both sides of the inequality by 5, and reverse the inequality.

$y \geq -4$ Simplify.

Graph the solution on the number line.



Write the solution in interval notation. $[-4, \infty)$

When solving inequalities, it is usually easiest to collect the variables on the side where the coefficient of the variable is largest. This eliminates negative coefficients and so we dont have to multiply or divide by a negativewhich means we dont have to remember to reverse the inequality sign.

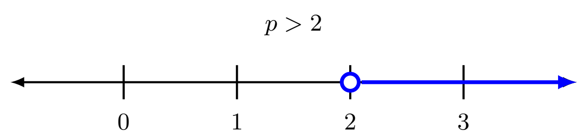
Example 7.16: Solve the inequality $8p + 3(p–12) > 7p–28$, graph the solution on the number line, and write the solution in interval notation.

Solution:

$8p + 3(p–12) > 7p–28$

$8p + 3p–36 > 7p–28$	Simplify each side as much as possible. Distribute
$11p–36 > 7p–28$	Combine like terms.
$11p–36–7p > 7p–28 - 7p$	Subtract $7p$ from both sides to collect the variables on the left, since $11 > 7$..
$4p–36 > -28$	Simplify.
$4p–36 + 36 > -28 + 36$	Add 36 to both sides to collect the constants on the right.
$4p > 8$	Simplify.
$\frac{4p}{4} > \frac{8}{4}$	Divide both sides of the inequality by 4; the inequality stays the same..
$p > 2$	Simplify.

Graph the solution on the number line.



Write the solution in interval notation. $(2, \infty)$

7.3.3 Compound Inequalities

A compound inequality is made up of two inequalities connected by the word and or the word or. For example, the following are compound inequalities.

$$x + 3 > -4 \text{ and } 4x–5 \leq 3$$
$$2(y + 1) < 0 \text{ or } y–5 \geq -2$$

Sometimes we have a compound inequality that can be written more concisely. For example, $a < x$ and $x < b$ can be written simply as $a < x < b$ and then we call it a **double inequality**. These two forms are equivalent.

To solve a double inequality we perform the same operation on all three parts of the double inequality with the goal of isolating the variable in the center.

Example 7.17: Solve $-4 \leq 3x–7 < 8$. Graph the solution and write the solution in interval notation.

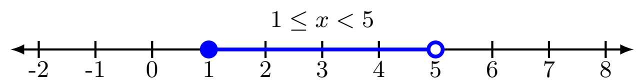
Solution:

-4	$\leq$	$3x–7$	$<$	8	
$-4 + 7$	$\leq$	$3x$	$<$	$8 + 7$	Add 7 to all three parts.
3	$\leq$	$3x$	$<$	15	Simplify.

$$\frac{3}{3} \leq \frac{3x}{3} < \frac{15}{3} \quad \text{Divide each part by three.}$$

$$1 \leq x < 5 \quad \text{Simplify.}$$

Graph the solution:



Write the solution in interval notation: $[1, 5)$

7.3.4 Applications of Linear Inequalities

Translate to an Inequality and Solve - Word Problems

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like more than and less than. But others are not as obvious. The following table shows some common phrases that indicate inequalities.

$>$	$\geq$	$<$	$\leq$
is greater than	is greater than or equal to	is less than	is less than or equal to
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

Example 7.18: Translate and solve. Then graph the solution on the number line, and write the solution in interval notation. Twenty seven less than x is at least 48.

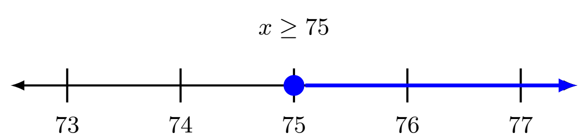
Solution: Twenty seven less than x is **at least** 48.

$$x - 27 \geq 48 \quad \text{Translate}$$

$$x \geq 48 + 27$$

$$x \geq 75$$

Graph on the number line:



Write the solution in interval notation: $[75, \infty)$

Solve Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. Sometimes an application requires the solution to be a whole number, but the algebraic solution to the inequality is not a whole number. In that case, we must round up or round down the algebraic solution to a whole number. The context of the application will determine whether we round up or down.

How to solve application problems?

Step 1. **Read** the problem. Make sure all the words and ideas are understood.
Step 2. **Identify** what we are looking for.
Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.
Step 4. **Translate**. Write a sentence that gives the information to find it. Translate into an inequality.
Step 5. **Solve** the inequality.
Step 6. **Check** the answer in the problem and make sure it makes sense.
Step 7. **Write** a sentence that answers the question.

Example 7.19: Alia has \$25 to spend on juice boxes for her sons preschool picnic. Each pack of juice boxes costs \$2. What is the maximum number of packs she can buy?
Solution: We are looking for the maximum number of packs Alia can buy.
Let $n =$ the number of packs.
Given that: \$2 times the number of packs **is less than or equal to** \$25.

$$\begin{aligned} 2n &\leq 25 && \text{Translate into an inequality.} \\ n &\leq \frac{25}{2} \\ n &\leq 12.5 \\ n &\leq 12 && \text{But } n \text{ must be a whole number of packs, so round to 12.} \end{aligned}$$

Alia can buy a maximum of 12 packs.

Example 7.20: Mohammed is planning a six-day summer vacation trip. He has omr 800 in savings, and he earns omr 10 by selling one painting. The trip will cost him omr 510 for airfare, omr 700 for food and sightseeing, and omr 200 for the hotel. How many paintings must he sell to have enough money to pay for the trip?
Solution: We are looking for the number of painting Mohammed sells
Let $n =$ the number of paintings
The expenses must be less than or equal to the income. **The cost of airfare plus the cost of food and sightseeing plus the hotel bill must be less than or equal to the savings plus the amount earned by selling paintings.**

$$\begin{aligned} 510 + 700 + 200 &\leq 800 + 10n && \text{Translate into an inequality.} \\ 1410 &\leq 800 + 10n \\ 610 &\leq 10n \\ \frac{610}{10} &\leq n \\ n &\geq 61 \\ n &\geq 61 \end{aligned}$$

Mohammed must sell at least 61 paintings.

7.3 Section Exercises

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line and write in interval notation.

1.

a. $x \leq 5$

b. $x \geq -1.5$

2.

a. $-5 < x < 2$

b. $-3 \leq x < -1$

3.

a. $-2 < x < 0$

b. $-5 \leq x < -3$

Solve Linear Inequalities

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

4. $8x > 72$

5. $6y < 48$

6. $20 \geq \frac{2}{5}x$

7. $-5k \leq 20$

8. $3x \geq -15$

9. $12 \leq -4y$

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

10. $4v \geq 9v - 40$

11. $12x + 3(x + 7) > 10x - 24$

12. $8m - 2(3 - m) \geq 2(m + 7) + 3m$

13. $9y + 5(y + 3) < 4y - 35$

14. $6h - 4(h - 1) \leq 7h - 11$

15. $4k - (k - 2) \geq 7k - 26$

16. $\frac{2}{3}b - \frac{3}{4}b < \frac{5}{12}b - \frac{1}{2}$

17. $6n - 12(3 - n) \leq 9(n - 4) + 9n$

18. $18q - 4(10 - 3q) < 5(6q - 8)$

19. $-\frac{9}{4}x \geq -\frac{5}{12}$

Solve Compound Inequalities

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

20. $-5 \leq 4x - 1 < 7$

21. $5 < 4x + 1 < 9$

22. $-6 \leq 4x - 2 < -2$

23. $-1 < 3x + 2 < 8$

Translate to an Inequality and Solve

In the following exercises, translate and solve. Then graph the solution on the number line and write the solution in interval notation.

24. Three more than h is no less than 25.

25. Six more than k exceeds 25.

26. Fifteen less than a is at least 7.

27. Nineteen less than b is at most 22.

Solve Applications with Linear Inequalities

In the following exercises, solve.

28. Ryan charges his neighbors $20 to wash their car. How many cars must he wash next summer if his goal is to earn at least $1500?

29. Maryam got a OMR 20 gift card for the coffee shop. Her favorite iced drink costs OMR 3. What is the maximum number of drinks she can buy with the gift card?

30. Ahmed is a personal chef. He charges OMR 4 per meal. His monthly expenses are OMR 400. How many meals must he sell in order to make a profit of at least OMR 600?

Chapter 8

QUADRATIC EQUATIONS

Contents

Learning outcome covered:

k. Use the quadratic formula to find roots of a second-degree polynomial.

o. Apply knowledge of basic algebra and trigonometry in real life problems.

Learning Objectives

At the end of this chapter, the students will be able to:

- Solve quadratic equations using the quadratic formula
- Use the discriminant to predict the nature of solutions of a quadratic equation
- Solve applications modeled by Quadratic Equations

Introduction

Quadratic equations are equations of the form $ax^2 + bx + c = 0$, where $a \neq 0$. They differ from linear equations by including a term with the variable raised to the second power. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

In this chapter, we solve the quadratic equations using quadratic formula and study the nature of the solutions. Also, we solve some applications modeled by quadratic equation. Quadratic equation can also be solved by factorization.

8.1 Quadratic Formula

8.1.1 Solving Quadratic Equation using Quadratic Formula

Quadratic Formula

The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of a, b, and c into the expression on the right side of the formula. Then, we do all the math to simplify the expression. The result gives the solution(s) to the quadratic equation. **How to solve a Quadratic Equation using the Quadratic Formula**

How To **:: Solve a Quadratic Equation using the Quadratic Formula.**

How?

Step 1. Write the Quadratic Formula in standard form. Identify the a, b and c values.

Step 2. Write the Quadratic Formula. Then substitute the values of a, b and c.

Step 3. Simplify.

Step 4. Check the solutions.

Example 8.1: Solve $2x^2 + 9x - 5 = 0$ by using the Quadratic Formula.

Solution:

Step 1: Write the Quadratic Formula in standard form. Identify the a, b and c values.	The equation is in standard form.	$ax^2 + bx + c = 0$ $2x^2 + 9x - 5 = 0$ $a = 2,\ b = 9,\ c = -5$

Step 2: Write the Quadratic Formula. Then substitute the values of a, b and c.	Substitute $a = 2$, $b = 9$ and $c = -5$.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \dfrac{-9 \pm \sqrt{9^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2}$
Step 3: Simplify the fraction and solve for x.		$x = \dfrac{-9 \pm \sqrt{81 + 40}}{4}$ $x = \dfrac{-9 \pm \sqrt{121}}{4}$ $x = \dfrac{-9 \pm 11}{4}$ $x = \dfrac{-9 + 11}{4}, \quad x = \dfrac{-9 - 11}{4}$ $x = \dfrac{2}{4}, \quad x = \dfrac{-20}{4}$ $x = \dfrac{1}{2}, \quad x = -5$

Example 8.2: Solve $x^2 - 6x + 5 = 0$ by using the Quadratic Formula.

Solution: This equation is in standard form.

$$\overset{ax^2}{x^2} \overset{+\ bx}{-\ 6x} \overset{+\ c}{+\ 5} \overset{=\ 0}{=\ 0}$$

$$a = 1, \quad b = -6, \quad c = 5$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Write the Quadratic Formula.}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} \qquad \text{Substitute the values of } a,\ b,\ c.$$

$$x = \frac{6 \pm \sqrt{36 - 20}}{2}$$

$$x = \frac{6 \pm \sqrt{16}}{2} \qquad \text{Simplify.}$$

$$x = \frac{6 \pm 4}{2}$$

$$x = \frac{6 + 4}{2}, \quad x = \frac{6 - 4}{2} \qquad \text{Rewrite to show two solutions.}$$

$$x = \frac{10}{2}, \quad x = \frac{2}{2}$$

$$x = 5, \quad x = 1$$

Example 8.3: Solve $2x^2 + 10x + 11 = 0$ by using the Quadratic Formula.

Solution:

$$\begin{gathered}\overset{ax^2\,+\,bx\,+\,c\,=\,0}{2x^2 + 10x + 11 = 0} \\ a = 2,\quad b = 10,\ c = 11\end{gathered}$$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Write the Quadratic Formula.
$x = \dfrac{-10 \pm \sqrt{10^2 - 4 \cdot 2 \cdot 11}}{2 \cdot 2}$	Substitute the values of a, b, c.
$x = \dfrac{-10 \pm \sqrt{100 - 88}}{4}$	
$x = \dfrac{-10 \pm \sqrt{12}}{4}$	
$x = \dfrac{-10 \pm 2\sqrt{3}}{4}$	Simplify the radical.
$x = \dfrac{2(-5 \pm \sqrt{3})}{4}$	Factor out the common factor in the numerator.
$x = \dfrac{-5 \pm \sqrt{3}}{2}$	Remove the common factors.
$x = \dfrac{-5 + \sqrt{3}}{2}, \quad x = \dfrac{-5 - \sqrt{3}}{2}$	Rewrite to show two solutions.

Example 8.4: Solve $3p^2 + 2p + 9 = 0$ by using the Quadratic Formula.

Solution:

$$\begin{gathered}\overset{ax^2\,+\,bx\,+\,c\,=\,0}{3p^2 + 2p + 9 = 0} \\ a = 3,\quad b = 2,\ c = 9\end{gathered}$$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Write the Quadratic Formula.
$x = \dfrac{-2 \pm \sqrt{2^2 - 4 \cdot 3 \cdot 9}}{2 \cdot 3}$	Substitute the values of a, b, c.
$x = \dfrac{-2 \pm \sqrt{4 - 108}}{6}$	Simplify.
$x = \dfrac{-2 \pm \sqrt{-104}}{6}$	Simplify the radical.

`We cannot take the square root of a negative number. There is no real solution.`

The quadratic equations we have solved so far in this section were all written in standard form, $ax^2 + bx + c = 0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

Example 8.5: Solve $4x^2 - 20x = -25$ by using the Quadratic Formula.

Solution:

Add 25 to get the equation in standard form.

$$\overset{ax^2+}{4x^2}\overset{bx+}{-20x}+\overset{c}{25}\overset{=}{=}\overset{0}{0}$$

$$a = 4, \quad b = -20, \; c = 25$$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Write the Quadratic Formula.
$x = \dfrac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot 25}}{2 \cdot 4}$	Substitute the values of a, b, c.
$x = \dfrac{20 \pm \sqrt{400 - 400}}{8}$	Simplify.
$x = \dfrac{20 \pm \sqrt{0}}{8}$	Simplify inside radical.
$x = \dfrac{20 \pm 0}{8}$	Simplify the radical.
$x = \dfrac{20 + 0}{8}, \quad x = \dfrac{20 - 0}{8}$	Rewrite to show two solutions.
$x = \dfrac{5}{2}, \quad x = \dfrac{5}{2}$	Simplify the fraction.

`Did you recognize that` $4x^2 - 20x + 25 = 0$ `is a perfect square?`

8.1.2 Use the Discriminant to predict the number of solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two distinct real solutions, sometimes two equal solution, sometimes no real solution. Is there a way to predict the nature and the number of solutions to a quadratic equation without actually solving the equation?

Yes, the quantity inside the radical of the Quadratic Formula makes it easy for us to determine the nature as well as the number of solutions. This quantity is called the discriminant.

Discriminant

In the Quadratic Formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the **discriminant**.

How To **:: Nature of the solutions of Quadratic equation.**

How?

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- ✠ if $b^2 - 4ac > 0$, the equation has two real distinct solutions.
- ✠ if $b^2 - 4ac = 0$, the equation has two equal solutions.
- ✠ if $b^2 - 4ac < 0$, the equation has no real solution.

Example 8.6: Determine the number of solutions to each quadratic equation:

a. $2v^2 - 3v + 6 = 0$ **b.** $3x^2 + 7x - 9 = 0$

Solution:

a. The equation is in standard form, identify a, b, c. $\quad 2v^2 - 3v + 6 = 0$

$$a = 2,\ b = -3,\ c = 6$$

Write the discriminant. $\quad b^2 - 4ac$

Substitute the values of a, b, c. $\quad (-3)^2 - 4 \cdot 2 \cdot 6$

Simplify. $\quad 9 - 48$

$$-39$$

Because the discriminant is negative, there is ***no real solution*** *to the equation.*

b. The equation is in standard form, identify a, b, c. $\quad 3x^2 + 7x - 9 = 0$

$$a = 3,\ b = 7,\ c = -9$$

Write the discriminant. $\quad b^2 - 4ac$

Substitute the values of a, b, c. $\quad (7)^2 - 4 \cdot 3 \cdot (-9)$

Simplify. $\quad 49 + 108$

$$157$$

Because the discriminant is positive, there are two distinct real solutions to the equation.

Example 8.7: Determine the number of solutions to each quadratic equation:

a. $5n^2 + n + 4 = 0$ **b.** $9y^2 + 1 = 6y$

Solution:

a. The equation is in standard form $5n^2 + n + 4 = 0$. $a = 5$, $b = 1$, $c = 4$.

$$\begin{aligned} Discriminant &= b^2 - 4ac \\ &= (1)^2 - 4 \cdot 5 \cdot 4 = 1 - 80 \\ &= -79 \end{aligned}$$

Because the discriminant is negative, there is no real solution to the equation.

b. Write the equation is in standard form $9y^2 – 6y + 1 = 0$. $a = 9$, $b = -6$, $c = 1$.

$$\begin{aligned} \textit{Discreminant} &= b^2 – 4ac \\ &= (-6)^2 – 4 \cdot 9 \cdot 1 = 36 - 36 \\ &= 0 \end{aligned}$$

Because the discriminant is 0, the solutions are equal.

8.1 Section Exercises

Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

1. $4m^2 + m – 3 = 0$ **2.** $4n^2 – 9n + 5 = 0$ **3.** $2p^2 – 7p + 3 = 0$

4. $3q^2 + 8q – 3 = 0$ **5.** $p^2 + 7p + 12 = 0$ **6.** $q^2 + 3q – 18 = 0$

7. $r^2 – 8r – 33 = 0$ **8.** $t^2 + 13t + 40 = 0$ **9.** $3u^2 + 7u – 2 = 0$

10. $6z^2 – 9z + 1 = 0$ **11.** $2a^2 – 6a + 3 = 0$ **12.** $2x^2 + 3x + 9 = 0$

13. $3t(t – 2) = 2$ **14.** $v(v + 5) – 10 = 0$ **15.** $3w(w – 2) – 8 = 0$

16. $6y^2 – 5y + 2 = 0$ **17.** $2a^2 + 12a + 5 = 0$ **18.** $16y^2 + 8y + 1 = 0$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions to each quadratic equation.

19. $4x^2 – 5x + 16 = 0$ **20.** $9v^2 – 15v + 25 = 0$ **21.** $r^2 + 12r + 36 = 0$

22. $6m^2 + 3m – 5 = 0$ **23.** $4u^2 – 12u + 9 = 0$ **24.** $3v^2 – 5v – 1 = 0$

25. $4x^2 – 5x + 16 = 0$ **26.** $5c^2 + 7c – 10 = 0$ **27.** $8t^2 – 11t + 5 = 0$

8.2 Applications of Quadratic Equations

We have solved number applications that involved consecutive even integers and consecutive odd integers by modeling the situation with linear equations. One set of even integers and one set of odd integers are shown below.

Consecutive even integers 64, 66, 68		**Consecutive odd integers 77, 79, 81**	
n	1^{st} even integer	n	1^{st} odd integer
$n+2$	2^{nd} consecutive even integer	$n+2$	2^{nd} consecutive odd integer
$n+4$	3^{rd} consecutive even integer	$n+4$	3^{rd} consecutive odd integer

Some applications of consecutive odd integers or consecutive even integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

Example 8.8: The product of two consecutive odd integers is 195. Find the integers.

Solution:

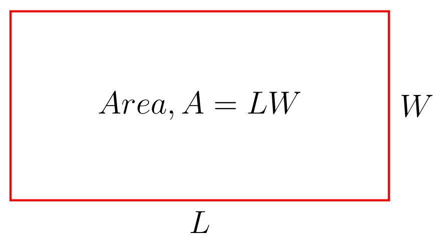
Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive odd integers.
Step 3. Name what we are looking for.	Let $n =$ the first odd integer. $n+2 =$ the next odd integer
Step 4. Translate into an equation. State the problem in one sentence.	"The product of two consecutive odd integers is 195." The product of the first odd integer and the second odd integer is 195.
Translate into an equation	$n(n+2) = 195$
Step 5. Solve the equation. Distribute. Subtract 195 to get the equation in standard form. Identify the a, b, c values. Write the quadratic formula.	$n^2 + 2n = 195$ $\overset{ax^2+}{n^2} \overset{bx+}{+ 2n} \overset{c}{- 195} \overset{=0}{= 0}$ $a = 1, b = 2, c = -195$ $n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c. Simplify. Simplify the radical.	$n = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-195)}}{2 \cdot 1}$ $n = \frac{-2 \pm \sqrt{4 + 780}}{2}$ $n = \frac{-2 \pm \sqrt{784}}{2}$ $n = \frac{-2 \pm 28}{2}$
Rewrite to show two solutions.	$n = \frac{-2 + 28}{2}, n = \frac{-2 - 28}{2}$

Solve each equation.	$n = \dfrac{26}{2} = 13$ and $n = \dfrac{-30}{2} = -15$
There are two values of n that are solutions. This will give us two pairs of consecutive odd integers for our solution.	**when** $n = 13$**:** First odd integer: $n = 13$ next odd integer: $n + 2 = 13 = 2 = 15$ **when** $n = -15$**:** First odd integer: $n = -15$ next odd integer: $n + 2 = -15 + 2 = -13$
Step 6. Answer the question.	The two consecutive odd integers whose product is 195 are 13, 15, and 13, 15.

We will use the formula for the area of a rectangle to solve the next example.

Area of a Triangle:

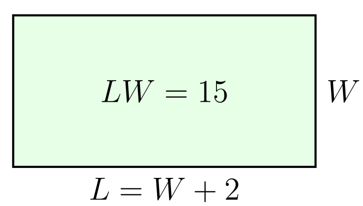
For a recatange with length L and width W , the area, A , is given by the formula $A = LW$.



Example 8.9: A rectangular garden has an area 15 square feet. The length of the garden is two feet more than the width. Find the length and width of the garden.

Solution:

Let L = Length of the garden
W = Width of the garden
A = Area of the garden
$A = LW$
Given: The length of the garden is two feet more than the width.
$L = W + 2$



Given that, Area, $A = 15$ square feet.

$A = 15 \Rightarrow LW = 15$

$\Rightarrow (W + 2)W = 15$

$\Rightarrow W^2 + 2W = 15$ Multiply

$\Rightarrow W^2 + 2W - 15 = 0$ Write the quadratic equation in the standard form

$\Rightarrow W = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Write the quadratic formula

$$\Rightarrow W = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-15)}}{2 \cdot 1} \quad \text{Substitute} a = 1, b = 2, c = -15$$

$$\Rightarrow W = \frac{-2 \pm \sqrt{64}}{2} \quad \text{Simplify}$$

$$\Rightarrow W = \frac{-2 \pm 8}{2} \quad \text{Simplify}$$

$$\Rightarrow W = \frac{-2 + 8}{2}, \frac{-2 - 8}{2} \quad \text{Find the two solutions}$$

$$\Rightarrow W = \frac{6}{2}, \frac{-10}{2} \quad \text{Simplify}$$

$$\Rightarrow W = 3, -5$$

Since W is the width of the garden, it does not make sense for it to be negative. We eliminate that value for W.

Width of the garden, $W = 3$

Length of the garden, $L = W + 2 = 5$.

8.2 Section Exercises

Solve Applications of the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

1. The product of two consecutive integers is 72. Find the numbers.

2. The product of two consecutive odd numbers is 63. Find the numbers.

3. The product of two consecutive even numbers is 80. Find the numbers.

4. The product of two consecutive even numbers is 24. Find the numbers.

5. The product of two consecutive odd numbers is 35. Find the numbers.

6. The product of two consecutive odd numbers is 99. Find the numbers.

7. The product of two consecutive even numbers is 48. Find the numbers.

8. A triangle with area 45 square inches has a height that is two less than four times the width. Find the height and width of the triangle. [Hint: *For a triangle with base b and height h , the area, A , is given by the formula $A = \frac{1}{2}bh$.*]

9. The width of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the height and width of the triangle.

10. A rectangular sign board has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign board.

11. A rectangular lawn has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the lawn.

Chapter 9

COORDINATE GEOMETRY

Contents

Learning outcome covered:

h. Use coordinate plane to solve algebraic and geometric problem, and understand geometric concepts such as equation of circle, perpendicular, parallel, and tangent lines.

i. Use the three types of symmetry of an equation to sketch its graph.

Learning Objectives

By the end of this chapter, the students will be able to:

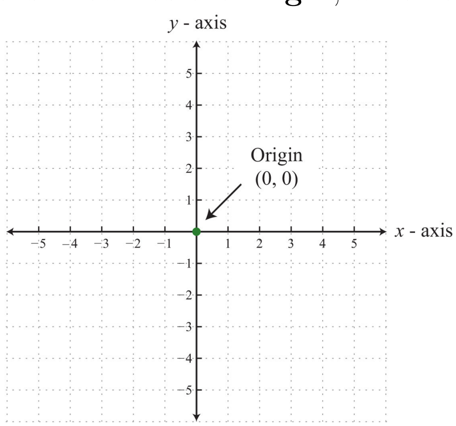
- Plot points using the rectangular coordinate system.
- Calculate the distance between any two points in the rectangular coordinate plane.

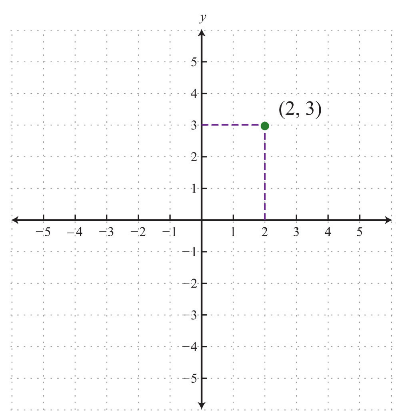
- ➠ Identify and find the slope of a line.
- ➠ Find the equation of the line using the slope and y-intercept.
- ➠ Find the equation of the line using point-slope form.
- ➠ Determine the slopes of parallel and perpendicular lines.
- ➠ Write the equation of a circle in standard form
- ➠ Test Equations for Symmetry

9.1 Rectangular Coordinate System

9.1.1 Rectangular Coordinate System

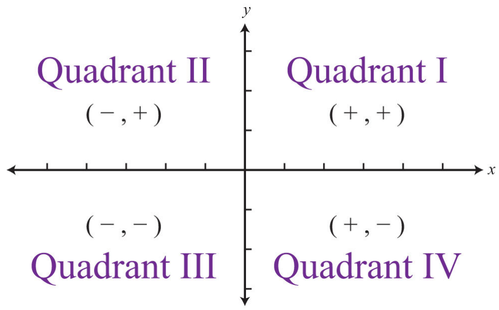
The rectangular coordinate system consists of two real number lines that intersect at a right angle. The horizontal number line is called the x-axis, and the vertical number line is called the y-axis. These two number lines define a flat surface called a plane, and each point on this plane is associated with an ordered pair of real numbers (x, y). The first number is called the x-**coordinate**, and the second number is called the y-**coordinate**. The intersection of the two axes is known as the **origin**, which corresponds to the point (0, 0).





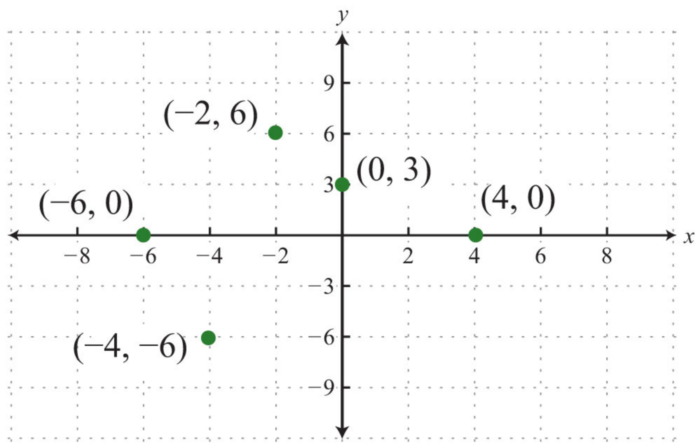
An ordered pair (x, y) represents the position of a point relative to the origin. The x-coordinate represents a position to the right of the origin if it is positive and to the left of the origin if it is negative. The y-coordinate represents a position above the origin if it is positive and below the origin if it is negative. Using this system, every position (point) in the plane is uniquely identified. This system is often called the Cartesian coordinate system.

The x- and y-axes break the plane into four regions called quadrants, named using roman numerals I, II, III, and IV, as pictured. In quadrant I, both coordinates are positive. In quadrant II, the x-coordinate is negative and the y-coordinate is positive. In quadrant III, both coordinates are negative. In quadrant IV, the x-coordinate is positive and the y-coordinate is negative.



Example 9.1: Plot the set of ordered pairs: {(4, 0), (6, 0), (0, 3), (2, 6), (4, 6)}

Solution: Each marked points on the x-axis represents 2 units and each marked points on the y-axis represents 3 units.



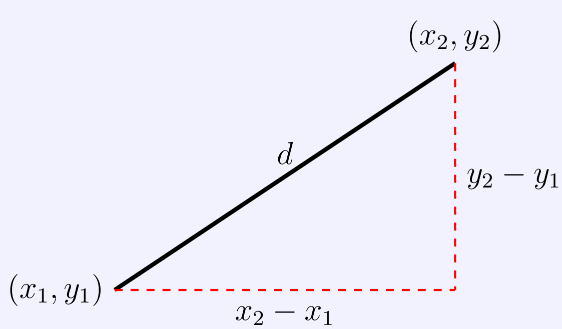
9.1.2 The distance between two points

The distance between two points is the length of the straight line between those two points. The formula to find the distance between two points is given below.

Distance formula

Given two points, (x_1, y_1) and (x_2, y_2), then the distance, d, between them is given by the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example 9.2: Calculate the distance between (3, 1) and (2, 4).

Solution: Use the distance formula.

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-3, -1) & (-2, 4) \end{array}$$

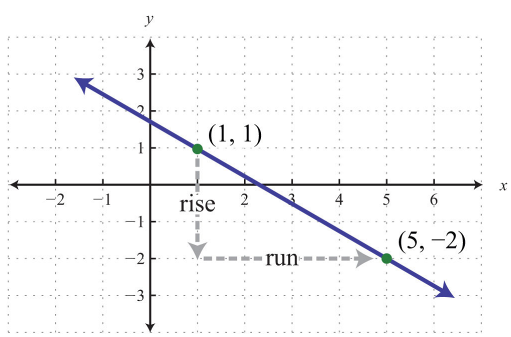
$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Write the distance formula.} \\ &= \sqrt{(-2 - (-3))^2 + (4 - (-1))^2} && \text{Substitute.} \\ &= \sqrt{(-2 + 3)^2 + (4 + 1)^2} && \text{Simplify.} \\ &= \sqrt{1^2 + 5^2} \\ &= \sqrt{1 + 25} \\ &= \sqrt{26} \end{aligned}$$

9.1.3 Slope of a Straight line

The steepness of any incline can be measured as the ratio of the vertical change to the horizontal change. In mathematics, we call the incline of a line the slope and use the letter m to denote it. The vertical change is called the rise and the horizontal change is called the run.

$$\text{Slope, } m = \frac{\text{Vertical Change}}{\text{Horizontal Change}} = \frac{\text{Rise}}{\text{Run}}$$

Example 9.3: Find the slope of the given line:

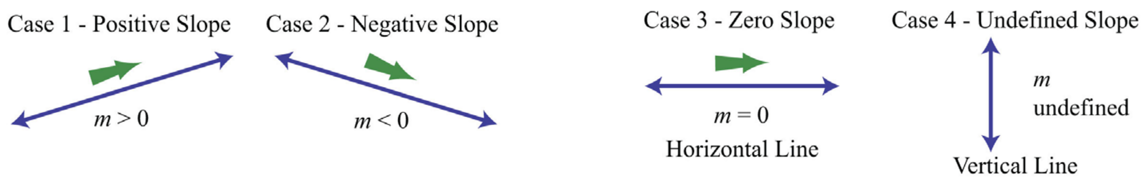


Solution: From the given points on the graph, count 3 units down and 4 units right.

$$\text{Slope, } m = \frac{\text{Rise}}{\text{Run}} = \frac{\text{-3 units}}{\text{4 units}} = -\frac{3}{4}$$

Here we have a negative slope, which means that for every 4 units of movement to the right, the vertical change is 3 units downward.

There are four geometric cases for the value of the slope.Reading the graph from left to right, we see that lines with an upward incline have positive slopes and lines with a downward incline have negative slopes.



Horizontal & Vertical line

❊ For a horizontal line, Rise $= 0$. Hence, slope, $m = \dfrac{Rise}{Run} = 0 = \dfrac{0}{Run} = 0$.

❊ For a vertical line, Run $= 0$. Hence, slope, $m = \dfrac{Rise}{Run} = \dfrac{Rise}{0}$, **undefined**.

Slope of the line joining two points

Given any two points (x_1, y_1) and (x_2, y_2), we can obtain the rise and run by subtracting the corresponding coordinates. This leads us to the slope formula.

Slope of the line joining two points

Given any two points (x_1, y_1) and (x_2, y_2), the slope is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 9.4: Find the slope of the line passing through (3, 5) and (2, 1).

Solution: Given (3, 5) and (2, 1), calculate the difference of the y-values divided by the difference of the x-values. Since subtraction is not commutative, take care to be consistent when subtracting the coordinates.

$$(x_1, y_1) \quad (x_2, y_2)$$

$$(-3, -5) \quad (2, 1)$$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - (-5)}{2 - (-3)} = \frac{1 + 5}{2 + 3} = \frac{6}{5} \end{aligned}$$

9.1 Section Exercises

Ordered pairs

Plot the given set of ordered pairs.

1. $\{(-4, 5), (-1, 1), (-3, -2), (5, -1)\}$

2. $\{(-2, 5), (10, 0), (2, -5), (6, -10)\}$

3. $\{(-8, 3), (-4, 6), (0, -6), (6, 9)\}$

4. $\{(-3.5, 0), (-1.5, 2), (0, 1.5), (2.5, -1.5)\}$

State the quadrant in which the given point lies.

5. $(-3, 2)$

6. $(5, 7)$

7. $(-12, -15)$

8. $(7, -8)$

9. $(-3.8, 4.6)$

10. $(-18, -58)$

Distance Formula

Calculate the distance between the given two points.

11. $(-5, 3)$ and $(-9, 6)$

12. $(6, -2)$ and $(-2, 4)$

13. $(0, 0)$ and $(5, 12)$

14. $(-6, -8)$ and $(0, 0)$

15. $(-7, 8)$ and $(5, -1)$

16. $(1, 2)$ and $(4, 3)$

17. $(2, -4)$ and $(-3, -2)$

18. $(-7, -3)$ and $(2, 6)$

19. $(0, 1)$ and $(1, 0)$

Slope Formula

Find the slope of the line joining the given two points.

20. $(3, 2)$ and $(5, 1)$

21. $(7, 8)$ and $(-3, 5)$

22. $(2, -3)$ and $(-3, 2)$

23. $(-3, 2)$ and $(7, -5)$

24. $(-1, -6)$ and $(3, 2)$

25. $(5, 3)$ and $(4, 12)$

26. $(-3, 1)$ and $(-14, 1)$

27. $(-4, -4)$ and $(5, 5)$

28. $(-2, 3)$ and $(-2, -4)$

29. Find b if the slope of the line passing through $(-2, 3)$ and $(4, b)$ is 12.

30. Find b if the slope of the line passing through $(5, b)$ and $(-4, 2)$ is 0.

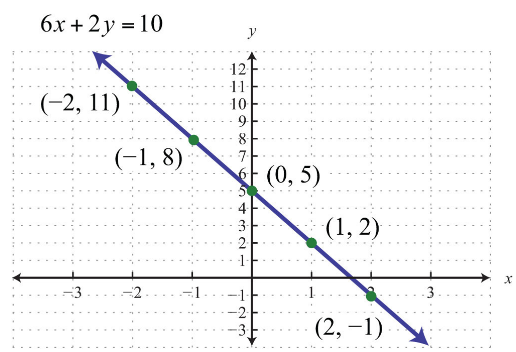
31. Find a if the slope of the line passing through $(-3, 2)$ and $(a, 5)$ is undefined.

9.2 Straight Lines

A linear equation with two variables has standard form $ax + by = c$, where a, b, and c are real numbers and a and b are not both 0. Since the solutions to linear equations are ordered pairs, they can be graphed using the rectangular coordinate system. The set of all solutions to a linear equation can be represented on a rectangular coordinate plane using a **straight line** connecting at least two points.

To illustrate this, plot five ordered pair solutions, (2, 11), (1, 8), (0, 5), (1, 2), (2, 1) to the linear equation $6x + 2y = 10$. Draw a line through the points with a straightedge, and add arrows on either end to indicate that the graph extends indefinitely. The resulting line represents all solutions to $6x + 2y = 10$.

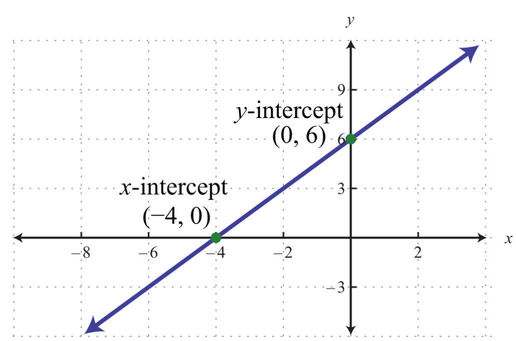
Hence, any linear equation $ax + by = c$ with two variables where a, b, and c are real numbers and a and b are not both 0 always represents a straight line.



9.2.1 Slope - Intercept Form of a Line

Definition of x - and y - Intercepts

The x - intercept is the point where the graph of a line intersects the x - axis. The y - intercept is the point where the graph of a line intersects the y - axis. These points have the form $(x, 0)$ and $(0, y)$, respectively.



To find the x - and y - intercepts:

- To find the x - intercept, set $y = 0$ and determine the corresponding x - value.
- To find the y - intercept, set $x = 0$ and determine the corresponding y - value.

Slope - Intercept Form of a Line

The equation of any non-vertical line can be written in slope - intercept form $y = mx + b$. In this form, we can identify the slope, m, and the $y-$intercept, $(0, b)$.

Example 9.5: Determine the slope and $y-$intercept: $y = -\frac{4}{5}x + 7$

Solution: In this form, the coefficient of x is the slope, and the constant is the y - coordinate of the y - intercept. Therefore, by inspection, we have

$$y = -\frac{4}{5}x + 7$$

$y-$ intercept, $(0,\ 7)$

Slope, $m = -\frac{4}{5}$

The y - intercept is $(0,\ 7)$, and the slope is $m = -\frac{4}{5}$.

Example 9.6: Find the equation of a line with slope $m = -\frac{5}{8}$ and y - intercept $(0,\ 1)$.

Solution: The given y - intercept implies that $b = 1$. Substitute the slope m and the y - value of the y - intercept b into the equation $y = mx + b$.

$$y = \ mx + b$$

$$y = -\frac{5}{8}x + 1$$

Answer: $y = -\frac{5}{8}x + 1$

9.2.2 Point - Slope form of a line

Finding equation of line Using a point and the slope

Given any point on a line and its slope, we can find the equation of that line. Begin by applying the slope formula with a given point (x_1, y_1) and a variable point (x, y).

Point - Slope form of a line

The equation of any non-vertical line with slope m and passing through the point (x_1, y_1) can be written in the point - slope form $y - y_1 = m(x - x_1)$.

Example 9.7: Find the equation of the line with slope $m = \frac{1}{2}$ passing through $(4,\ 1)$.

Solution: Use point - slope form, where $m = \frac{1}{2}$ and $(x_1, y_1) = (4, -1)$.

$$\begin{aligned} y - y_1 \ &= m(x - x_1) \\ y - (-1) \ &= \frac{1}{2}(x - 4) \\ y + 1 \ &= \frac{1}{2}x - 2 \\ y \ &= \frac{1}{2}x - 3 \end{aligned}$$

9.2.3 Standard Equation of Straight Lines

Standard equation of a straight line

The standard equation of a straight line is $ax + by + c = 0$, where a, b and c are real numbers and a and b are not both 0.

For the straight line $ax + by + c = 0$,

- Slope, $m = -\dfrac{a}{b}$
- x- intercept $= \left(-\dfrac{c}{a}, 0\right)$
- y- intercept $= \left(0, -\dfrac{c}{b}\right)$

Example 9.8: Find the slope, x-inercept and y- intercept of the line $2x - 3y + 4 = 0$.

Solution: $a = 2$, $b = -3$ and $c = 4$.

$$\text{Slope, } m = -\frac{a}{b} = -\frac{2}{-3} = \frac{2}{3}$$

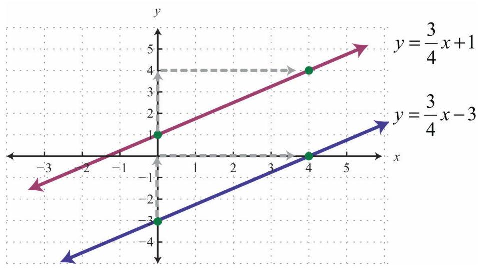
$$x - \text{intercept} = \left(-\frac{c}{a}, 0\right) = \left(-\frac{4}{2}, 0\right) = (-2, 0)$$

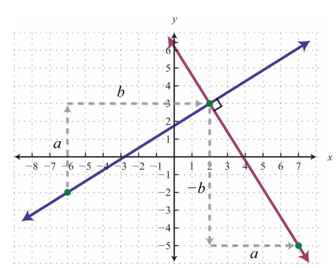
$$y - \text{intercept} = \left(0, -\frac{c}{b}\right) = \left(0, -\frac{4}{-3}\right) = \left(0, \frac{4}{3}\right)$$

9.2.4 Parallel lines and Perpendicular Lines

Parallel lines are lines in the same plane that never intersect. Two lines in the same plane, with slopes m_1 and m_2, are parallel if their slopes are the same, $m_1 = m_2$.

Perpendicular lines are lines in the same plane that intersect at right angles (90 degrees). Two lines in the same plane, with slopes m_1 and m_2, are perpendicular if the product of their slopes is -1: $m_1 \cdot m_2 = -1$. That is, $m_2 = -\dfrac{1}{m_1}$.





If m is the slope of a straight line, then

- the slope of a parallel line is, $m_{\|} = m$.
- the slope of a perpendicular line is, $m_{\perp} = -\dfrac{1}{m}$.

⚽ **If m_1 and m_2 are the slopes of two straight lines L_1 and L_2, then**

- L_1 and L_2 are parallel if $m_1 = m_2$.
- L_1 and L_2 are perpendicular $m_1 \cdot m_2 = -1$.

Example 9.9: Determine the slope of a line parallel to $y = -5x + 3$.

Solution: Since the given line is in slope - intercept form, we can see that its slope is m=5. Thus the slope of any line parallel to the given line must be the same, $m_{\|} = -5$.

Example 9.10: Determine the slopes of the lines that are parallel and perpendicular to $3x-7y = 21$.

Solution: Write the equation in the standard form.

The equation of the straight line is $3x-7y - 21 = 0$. $a = 3,$ $b = -7,$ $c = -21$

Slope, $m = -\dfrac{a}{b} = -\dfrac{3}{-7} = \dfrac{3}{7}$

Slope of the parallel line is, $m_{\|} = m = \dfrac{3}{7}$

Slope of the perpendicular line is, $m_{\perp} = -\dfrac{1}{m} = -\dfrac{7}{3}$

Example 9.11: Determine if the lines $2x - 3y + 1 = 0$ and $9x + 6y - 5 = 0$ are parallel, perpendicular, or neither.

Solution:

Slope of line $L_1 : 2x - 3y + 1 = 0$ is $a = 2,\ b = -3,\ c = 1$

$$m_1 = -\frac{a}{b} = -\frac{2}{-3} = \frac{2}{3}$$

Slope of line $L_2 : 9x + 6y - 5 = 0$ is $a = 9,\ b = 6,\ c = -5$

$$m_2 = -\frac{a}{b} = -\frac{9}{6} = -\frac{3}{2}$$

$m_1 \neq m_2$. The lines are not parallel.

$m_1 \times m_2 = \dfrac{2}{3} \cdot -\dfrac{3}{2} = -1$ The lines are perpendicular.

9.2 Section Exercises

Find the $x-$ intercepts and y - intercepts.

1. $5x-4y=20$ **2.** $-2x+7y=-28$ **3.** $x-y=3$

4. $y=6$ **5.** $x=-1$ **6.** $y=mx+b$

Slope - Intercept Form

Express the given linear equation in slope - intercept form and identify the slope and y - intercept.

7. $6x-5y=30$ **8.** $-2x+7y=28$ **9.** $x-3y=18$

10. $2x-3y=0$ **11.** $-6x+3y=0$ **12.** $9x-y=17$

Given the slope and y - intercept, determine the equation of the line.

13. $m=4;(0,-1)$ **14.** $m=-3;(0,9)$ **15.** $m=0;(0,-1)$

16. $m=\frac{1}{2};(0,5)$ **17.** $m=-\frac{2}{3};(0,-4)$ **18.** $m=5;(0,0)$

Use the point - slope formula to find the equation of the line passing through the two points.

19. $(-4,0),(0,5)$ **20.** $(-1,2),(0,3)$ **21.** $(-3,-2),(3,2)$

22. $(-3,-1),(3,3)$ **23.** $(1,5),(0,5)$ **24.** $(-8,0),(6,0)$

Parallel and Perpendicular Lines

Determine the slopes of the lines that are parallel and perpendicular to the given lines:

25. $y=-34x+8$ **26.** $-2x+7y=28$ **27.** $y=4x+4$

28. $4x+3y=0$ **29.** $-2x+7y=14$ **30.** $-x-y=15$

31. $y=5$ **32.** $x=-12$ **33.** $x-y=0$

Determine if the lines are parallel, perpendicular, or neither.

34. $\begin{cases} y=\frac{2}{3}x+3 \\ y=\frac{2}{3}x-3 \end{cases}$ **35.** $\begin{cases} y=\frac{3}{4}x-1 \\ y=\frac{4}{3}x+3 \end{cases}$ **36.** $\begin{cases} y=-2x+1 \\ y=\frac{1}{2}x+8 \end{cases}$

37. $\begin{cases} x-y=7 \\ 3x+3y=2 \end{cases}$ **38.** $\begin{cases} 2x-6y=4 \\ -x+3y=-2 \end{cases}$ **39.** $\begin{cases} 3x-5y=15 \\ 5x+3y=9 \end{cases}$

9.3 Circle

Circle

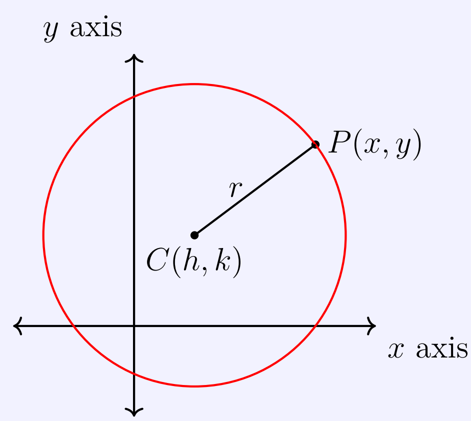
A circle is all points in a plane at a fixed distance from a given point in the plane. The given point is called the center, (h, k), and the fixed distance is called the radius, r, of the circle.

We look at a circle in the rectangular coordinate system. The radius is the distance from the center, (h, k), to a point on the circle, (x, y).

9.3.1 Standard Form of the Equation a Circle

Standard Form of the Equation of a Circle

The standard form of the equation of a circle with center (h, k), and radius r, is $(x - h)^2 + (y - k)^2 = r^2$.



Example 9.12: Write the standard form of the equation of the circle with radius 3 and center (0, 0).

Solution: The standard form of the equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$. Substitute the values $r = 3$, $h = 0$, and $k = 0$.

$$\text{The equation of the circle is } (x - 0)^2 + (y - 0)^2 = 3^2 \Rightarrow \qquad x^2 + y^2 = 9$$

In the last example, the center was (0, 0). Notice what happened to the equation. Whenever the center is (0, 0), the standard form becomes $x^2 + y^2 = r^2$.

Example 9.13: Write the standard form of the equation of the circle with radius 2 and center (1, 3).

Solution: The standard form of the equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$. Substitute the values $r = 2$, $h = -1$, and $k = 3$.

$$\begin{aligned} \text{The equation of the circle is } & (x - (-1))^2 + (y - 3)^2 = 2^2 \\ & (x + 1)^2 + (y - 3)^2 = 4 \end{aligned}$$

In the next example, the radius is not given. To calculate the radius, we use the Distance Formula with the two given points.

Example 9.14: Write the standard form of the equation of the circle with center (2, 4) and passing through the point (2, 1).

Solution: The radius is the distance from the center to any point on the circle so we can use the distance formula to calculate it. We will use the center $(2, 4)$ and point $(-2, 1)$.

Use the Distance Formula to find the radius.

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Distance Formula}$$

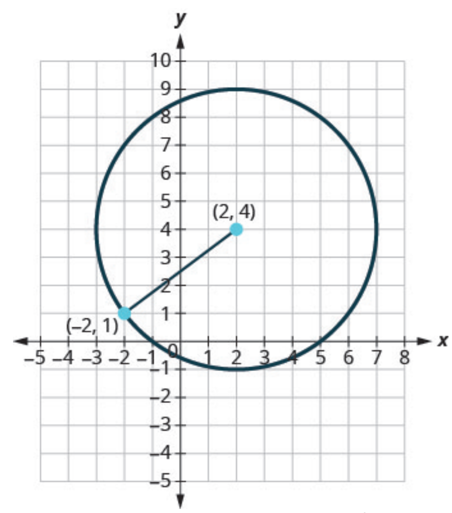
$$= \sqrt{(-2-2)^2 + (1-4)^2} \quad \overset{(x_1,y_1)}{(2,4)} \quad \overset{(x_2,y_2)}{(-2,1)}$$

$$= \sqrt{(-4)^2 + (-3)^2} \quad \text{Simplify}$$

$$= \sqrt{16+9}$$

$$= \sqrt{25} = 5$$

$$= 5$$



Now that we know the radius, r = 5, and the center, (2, 4), we can use the standard form of the equation of a circle to find the equation.

Use the standard form of the equation of a circle. $\quad (x-h)^2 + (y-k)^2 = r^2$

The equation of the circle is, $\quad (x-2)^2 + (y-4)^2 = 5^2$

$$(x-2)^2 + (y-4)^2 = 25$$

Example 9.15: Find the center and radius of the $(x+2)^2 + (y-1)^2 = 9$.

Solution:

$$(x+2)^2 \;+\; (y-1)^2 = 9$$

Use the standard form of the equation of a circle. $(x-h)^2 \;+\; (y-k)^2 = r^2$

Identify the center, (h, k) and radius, r.

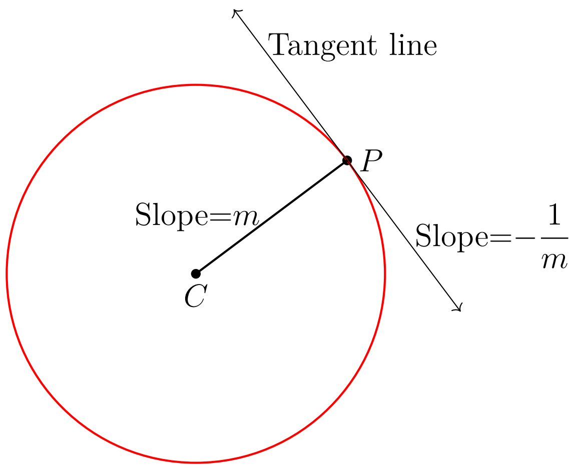
$$(x-(-2))^2 + (y-1)^2 = 3^2$$

Center: $(-2, 1)$ Radius: 3

9.3.2 Tangent lines

Tangent: A tangent to a circle is a line which intersects the circle in exactly one point. A tangent to a circle can also be defined as a straight line which touches the circle at only one point. This point is called the point of tangency or the point of contact.

The tangent to a circle is always perpendicular to the radius at the point of contact.



Finding the equation of the tangent line at a point on the circle.

- Find the center of the circle, $C = (h, k)$.
- Take the point as $P = (x_1, y_1)$.
- Find the slope of the radius PC, $m = \dfrac{y_1 - k}{x_1 - h}$.
- The tangent line at P is perpendicular to the radius PC. Therefore, slope of the tangent is $m' = -\dfrac{1}{m}$.
- The tangent at P is the line passing through (x_1, y_1) and having slope $m' = -\dfrac{1}{m}$.
- Equation of the tangent is $y - y_1 = m'(x - x_1)$.

Example 9.16: Find the equation of the tangent to the circle $(x-2)^2 + (y+1)^2 = 25$ at the point $(5, 3)$.

Solution:

Let C be the center and P be the point. $\quad C\colon (h,k) = (2,-1),\ P\colon (x_1,y_1) = (5,3)$

Slope of the radius CP is, $m = \dfrac{3-(-1)}{5-2} = \dfrac{4}{3}$ $\quad m = \dfrac{y_1 - k}{x_1 - h}$

Slope of the tangent line is, $m' = -\dfrac{1}{m} = -\dfrac{3}{4}$

Equation of the tangent at $(5, 3)$ is

$$\begin{aligned} y - y_1 &= m'(x - x_1) \\ y - 3 &= -\frac{3}{4}(x - 5) \\ 4(y - 3) &= -3(x - 5) \\ 4y - 12 &= -3x + 15 \\ 3x + 4y - 27 &= 0 \end{aligned}$$

9.3 Section Exercises

Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given radius and center (0, 0).

1. Radius: 7

2. Radius: 9

3. Radius: $\sqrt{2}$

4. Radius: $\sqrt{5}$

In the following exercises, write the standard form of the equation of the circle with the given radius and center.

5. Radius: 1, center: $(3, 5)$

6. Radius: 10, center: $(-2, 6)$

7. Radius: $\sqrt{15}$, center: $(0, 0)$

8. Radius: 5, center: $(-1, 0)$

9. Radius: 2.5, center: $(1.5, -3.5)$

10. Radius: $3\sqrt{2}$, center: $(-5.5, -6.5)$

For the following exercises, write the standard form of the equation of the circle with the given center with point on the circle.

11. Center $(3, -2)$ with point $(3, 6)$

12. Center $(6, -6)$ with point $(2, -3)$

13. Center $(0, 0)$ with point $(-6, 8)$

14. Center $(-3, 4)$ with point $(0, 0)$

15. Center $(4, 4)$ with point $(2, 2)$

16. Center $(-5, 6)$ with point $(-2, 3)$

Find the center and radius

In the following exercises, find the center and radius of each circle.

17. $(x + 5)^2 + (y + 3)^2 = 1$

18. $(x - 2)^2 + (y - 3)^2 = 9$

19. $x^2 + (y + 2)^2 = 75$

20. $(x + 2)^2 + (y + 5)^2 = 4$

21. $(x - 4)^2 + (y + 2)^2 = 32$

22. $(x - 1)^2 + y^2 = 36$

Tangent lines

23. Find the equation of the tangent to the circle $(x+9)^2+(y+2)^2=125$ at $(1,3)$.

24. Find the equation of the tangent to the circle $x^2+y^2=29$ at $(-2,5)$.

25. Find the equation of the tangent to the circle $(x-3)^2+y^2-8=0$ at $(1,-2)$.

26. Find the equation of the tangent to the circle $(x+5)^2+(y-7)^2=20$ at $(-7,3)$.

27. Find the slope of the tangent to the circle $(x-1)^2+(y+1)^2=5$ at $(2,-3)$.

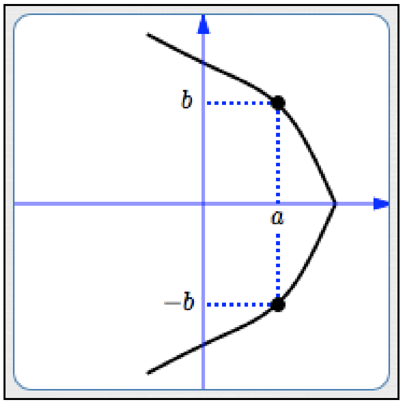
28. Find the slope of the tangent to the circle $(x-2)^2+(y-3)^2=18$ at $(-1,6)$.

9.4 Testing Equations for Symmetry

A knowledge of symmetry can increase your efficiency when working with graphs. This section discusses:

✠ symmetry about the x-axis
✠ symmetry about the y-axis
✠ origin symmetry

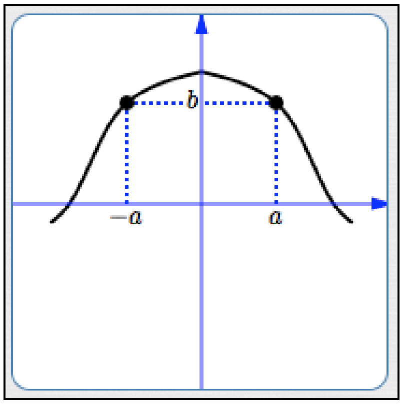
Definition: x-axis symmetry
A graph has symmetry about the x-axis if and only if whenever (a, b) is on the graph, so is $(a, -b)$.



Test for x-axis symmetry

- Replace every y by $-y$ in the equation.
  - ✓ **symmetric with respect to the x- axis** if the equation is the same.
  - ✗ **not symmetric with respect to the x- axis** if the equation is not the same.

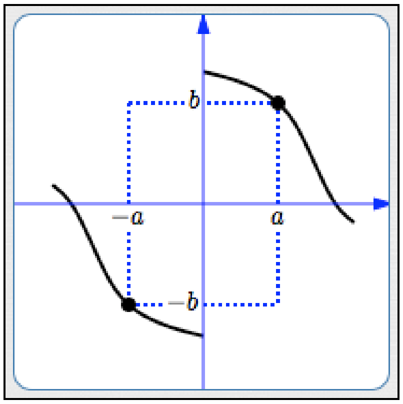
Definition: y-axis symmetry
A graph has symmetry about the y-axis if and only if whenever (a, b) is on the graph, so is $(-a, b)$.



Test for y-axis symmetry

- Replace every x by $-x$ in the equation.
  - ✓ **symmetric with respect to the y- axis** if the equation is the same.
  - ✗ **not symmetric with respect to the y- axis** if the equation is not the same.

Definition: Origin symmetry
A graph has symmetry about the origin if and only if whenever (a, b) is on the graph, so is $(-a, -b)$.



Test for Origin symmetry

- Replace every x by $-x$ and y by $-y$ in the equation.
  - ✓ **symmetric with respect to the origin** if the equation is the same.
  - ✗ **not symmetric with respect to the origin** if the equation is not the same.

Example 9.17: Test $2x - 3y^2 = 10$ for various symmetries.

Solution: The equation is $2x - 3y^2 = 10$ $\cdots(\bigstar)$

x - Axis	➥ Replace y with $-y$. $2x - 3(-y)^2 = 10$ $2x - 3y^2 = 10$ ✜ same equation $(\bigstar)$ ✜ The graph is symmetric with respect to the x - axis.
y - Axis	➥ Replace x with $-x$. $2(-x) - 3y^2 = 10$ $-2x - 3y^2 = 10$ $2x + 3y^2 = -10$ ✜ **not** the same equation $(\bigstar)$ ✜ The graph is **not symmetric** with respect to the y - axis.
Origin	➥ Replace x with $-x$ and y with $-y$. $2(-x) - 3(-y)^2 = 10$ $-2x - 3y^2 = 10$ $2x + 3y^2 = -10$ ✜ **not** the same equation $(\bigstar)$ ✜ The graph is **not symmetric** with respect to the origin.

Example 9.18: Test $5x^2 - 3y^2 = 15$ for various symmetries.

Solution: The equation is $5x^2 - 3y^2 = 15$ $\cdots(\bigstar)$

x - Axis	➥ Replace y with $-y$. $5x^2 - 3(-y)^2 = 15$ $5x^2 - 3y^2 = 15$ ✜ the same equation (★) ✜ The graph is symmetric with respect to the x - axis.
y - Axis	➥ Replace x with $-x$. $5(-x)^2 - 3y^2 = 15$ $5x^2 - 3y^2 = 15$ ✜ the same equation (★) ✜ The graph is symmetric with respect to the y - axis.
Origin	➥ Replace x with $-x$ and y with $-y$. $5(-x)^2 - 3(-y)^2 = 15$ $5x^2 - 3y^2 = 15$ ✜ the same equation (★) ✜ The graph is symmetric with respect to the origin.

9.4 Section Exercises

1. Suppose that the point $(-2, 4)$ lies on a graph that has x-axis symmetry. What other point must lie on the graph?

2. Suppose that the point $(0, 7)$ lies on a graph that has x-axis symmetry. What other point must lie on the graph?

3. Suppose that the point $(-2, 9)$ lies on a graph that has y-axis symmetry. What other point must lie on the graph?

4. Suppose that the point $(4, -3)$ lies on a graph that has y-axis symmetry. What other point must lie on the graph?

5. Suppose that the point $(1, 10)$ lies on a graph that has origin symmetry. What other point must lie on the graph?

6. Suppose that the point $(-9, -1)$ lies on a graph that has origin symmetry. What other point must lie on the graph?

7. Suppose that the point $(6, -5)$ lies on a graph that has origin symmetry. What other point must lie on the graph?

Test the following graphs for various symmetries.

8. $y = 3x + 5$

9. $2y = x^2 - 6$

10. $x + y^2 = 4$

11. $3x^2 - 5xy + 2y^2 = 25$

12. $x^3 - 2xy - xy^2 = 0$

13. $-4x^4y = 8x^6y^7$

14. $x^2 + y^2 = 16$

15. $y = x^3$

Chapter 10

TRIGONOMETRY

Contents

Learning outcome covered:

l. Know the relationship between degree and radian measure of an angle and find the length of a circular arc and the area of a sector.

m. Understand trigonometric and circular functions and use the fundamental trigonometric identities in various problems.

n. Solve a right angle triangle using angle of elevation and depression.

o. Apply knowledge of basic algebra and trigonometry in real life problems.

Learning Objectives

By the end of this chapter, the students will be able to:

- Know the relationship between degree and radian measure of an angle
- Find the length of a circular arc and the area of a sector
- Understand trigonometric and circular functions
- Use the fundamental trigonometric identities
- Understand angle of elevation and depression
- Apply knowledge of trigonometry in real life problems

10.1 Angles and Circle

10.1.1 Angles

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

Drawing Angles in Standard Position

Properly defining an angle first requires that we define a ray. A **ray** consists of one point on a line and all points extending in one direction from that point. The first point is called the **endpoint** of the ray. We can refer to a specific ray by stating its endpoint and any other point on it. The ray in Figure 1 can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

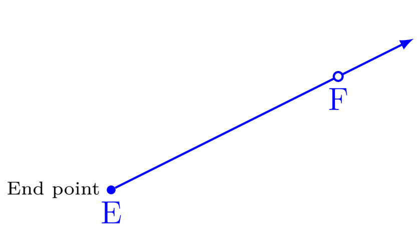


Figure 1: Ray EF

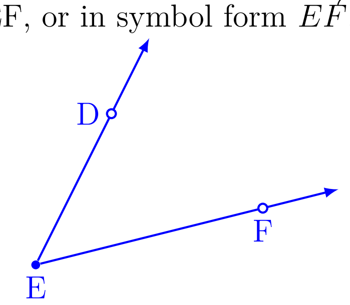


Figure 2: Angle DEF

An angle is the union of two rays having a common endpoint. The endpoint is called the vertex of the angle, and the two rays are the sides of the angle. The angle in Figure 2 is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle DEF, or in symbol form $\angle DEF$. Greek letters are often used as variables for the measure of an angle. Table 1 is a list of Greek letters commonly used to represent angles, and a sample angle is shown in Figure 3.

θ	ϕ or φ	α	β	γ
theta	phi	alpha	beta	gamma

Table 1

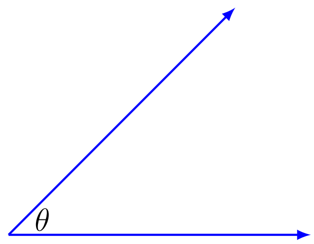


Figure 3: Angle theta, shown as $\angle\theta$

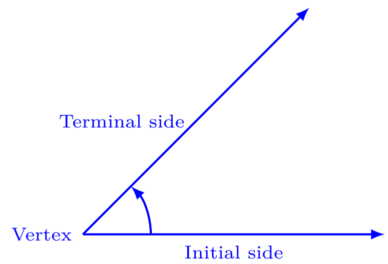


Figure 4

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial** side, and the rotated ray is the **terminal** side. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in Figure 4.

Measure of Angles

1. Degree

The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is $\dfrac{1}{360}$ of a circular rotation, so a complete circular rotation contains **360 degrees**. An angle measured in degrees should always include the unit degrees after the number, or include the degree symbol °. For example, 90 degrees = 90°.

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

2. Radian

One radian is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.

The radian measure of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle.

In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\dfrac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

Note that when an angle is described without a specific unit, it refers to radian measure.

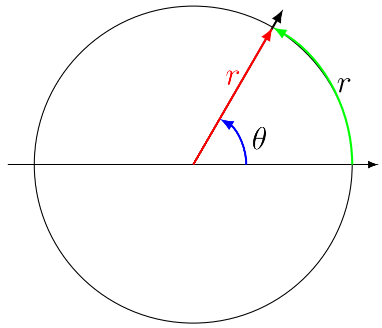


Figure 5: The angle θ sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

10.1.2 Converting Between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion where θ is the measure of the angle in degrees and θ_R is the measure of the angle in radians.

$$\frac{\theta}{180} = \frac{\theta_R}{\pi}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

$$\frac{\text{Degrees}}{180} = \frac{\text{Radians}}{\pi}$$

Example 10.1: Convert each radian measure to degrees.

a. $\frac{\pi}{6}$ **b.** $\frac{2\pi}{5}$

Solution: Because we are given radians and we want degrees, we take $\frac{\text{Degrees}}{180} = \frac{\text{Radians}}{\pi}$.

a. $\theta_R = \frac{\pi}{6}$.

$$\frac{\theta}{180} = \frac{\theta_R}{\pi} \Rightarrow \frac{\theta}{180} = \frac{\frac{\pi}{6}}{\pi}$$

$$\Rightarrow \frac{\theta}{180} = \frac{\cancel{\pi}1}{6\cancel{\pi}}$$

$$\Rightarrow \theta = \frac{180}{6} = 30^\circ$$

b. $\theta_R = \frac{2\pi}{5}$

$$\frac{\theta}{180} = \frac{\theta_R}{\pi} \Rightarrow \frac{\theta}{180} = \frac{\frac{2\pi}{5}}{\pi}$$

$$\Rightarrow \frac{\theta}{180} = \frac{2\cancel{\pi}1}{5\cancel{\pi}}$$

$$\Rightarrow \theta = \frac{2 \times 180}{5} = 72^\circ$$

Example 10.2: Convert each degree measure to radians: **a.** 15 degrees **b.** 420°

Solution: Because we want to convert degree to radian, we take $\frac{\text{Radians}}{\pi} = \frac{\text{Degrees}}{180}$.

a. $\theta = 15$ degrees

$$\frac{\theta_R}{\pi} = \frac{\theta}{180} \Rightarrow \frac{\theta_R}{\pi} = \frac{15}{180}$$

$$\Rightarrow \theta_R = \frac{15\pi}{180}$$

$$\Rightarrow \theta = \frac{\pi}{12}$$

b. $\theta = 420°$

$$\frac{\theta_R}{\pi} = \frac{\theta}{180} \Rightarrow \frac{\theta_R}{\pi} = \frac{420}{180}$$

$$\Rightarrow \theta_R = \frac{420\pi}{180}$$

$$\Rightarrow \theta = \frac{7\pi}{3}$$

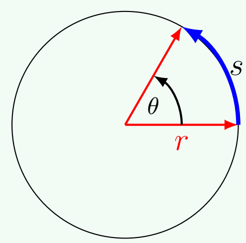
10.1.3 Determining the Length of an Arc and Area of a Sector

Determining the Length of an Arc

Arc length on a circle

In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians is

$$s = r\theta$$



How To

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of θ : $s = r\theta$.

Example 10.3: Find the length of the arc of a circle of radius 12 inches subtended by a central angle of $\frac{\pi}{4}$ radians.

Solution:

$$\text{Central angle, } \theta = \frac{\pi}{4}\text{radians}$$

$$\text{radius, } r = 12 \text{ inches}$$

$$\text{Arc length, } s = r\theta$$

$$= 12 \cdot \frac{\pi}{4} = 3\pi \text{ inches}$$

Example 10.4: Find the arc length along a circle of radius 10 units subtended by an angle of 120°.

Solution: First, we need to convert the angle measure into radians.

$$120° = \frac{120 \cdot \pi}{180} = \frac{2\pi}{3} \text{ radians}$$

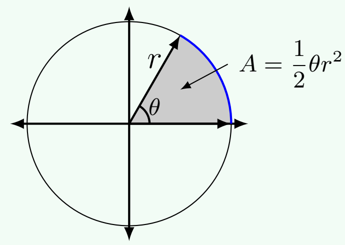
$$\begin{aligned}
\text{Central angle, } \theta &= \frac{2\pi}{3}\text{radians} \\
\text{radius, } r &= 10 \text{ units} \\
\text{Arc length, } s &= r\theta \\
&= 10 \cdot \frac{2\pi}{3} = \frac{20\pi}{3} \text{ units}
\end{aligned}$$

Finding the Area of a Sector of a Circle

Area of a sector

The **area of a sector** of a circle with radius r subtended by an angle θ, measured in **radians**, is

$$A = \frac{1}{2}\theta r^2$$

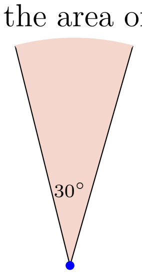


How To . . .

Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r: $A = \frac{1}{2}\theta r^2$.

Example 10.5: An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in the figure. What is the area of the sector of grass the sprinkler waters?



Solution: First, we need to convert the angle measure into radians.

$$30 \text{ degrees} = 30 \cdot \frac{\pi}{180} = \frac{\pi}{6}\text{radians}$$

$$\begin{aligned}
\text{The area of the sector is then } Area &= \frac{1}{2}\theta r^2 \\
&= \frac{1}{2}\left(\frac{\pi}{6}\right)(20)^2 \\
&= \frac{100\pi}{3} \text{ ft}^2
\end{aligned}$$

10.1 Section Exercises

For the following exercises, convert angles in radians to degrees.

1. $\frac{3\pi}{4}$ radians

2. $\frac{\pi}{9}$ radians

3. $-\frac{5\pi}{12}$ radians

4. $\frac{\pi}{3}$ radians

For the following exercises, convert angles in degrees to radians.

5. $90°$

6. $210°$

7. $-540°$

8. $-120°$

For the following exercises, use to given information to find the length of a circular arc. Round to two decimal places.

9. Find the length of the arc of a circle of radius 9 miles subtended by the central angle of $\frac{\pi}{3}$.

10. Find the length of the arc of a circle of diameter 15 meters subtended by the central angle of $\frac{11\pi}{6}$.

11. Find the length of the arc of a circle of radius 10 centimeters subtended by the central angle of $50°$.

12. Find the length of the arc of a circle of radius 5 inches subtended by the central angle of $220°$.

For the following exercises, use the given information to find the ***area*** *of the sector. Round to four decimal places.*

13. A sector of a circle has a central angle of $45°$ and a radius 6 cm.

14. A sector of a circle has a central angle of $30°$ and a radius 20 cm.

15. A sector of a circle with diameter 10 feet and an angle of $\frac{\pi}{2}$ radians.

16. A sector of a circle with radius of 0.7 inches and an angle of π radians.

10.2 Trigonometric Ratios and Identities

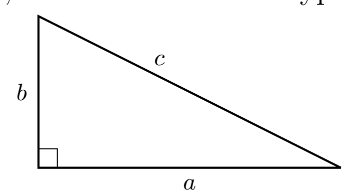
In this section, we will define a new group of functions known as trigonometric functions In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

10.2.1 Using Right Triangles to Evaluate Trigonometric Ratios

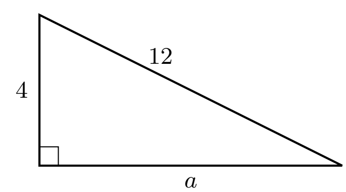
One of the most famous formulas in mathematics is the Pythagorean Theorem. It is based on a right triangle, and states the relationship among the lengths of the sides. It has immeasurable uses in architecture, engineering, the sciences, geometry, trigonometry, and algebra, and in everyday applications.

Pythagorean Theorem

The Pythagorean Theorem is given as $a^2 + b^2 = c^2$ where a and b refer to the legs of a right triangle adjacent to the 90° angle, and c refers to the hypotenuse, as shown in the figure.



Example 10.6: Find the length of the missing side of the right triangle in the figure given below.



Solution: As we have measurements for side b and the hypotenuse, the missing side is a.

$$a^2 + b^2 = c^2$$
$$a^2 + 4^2 = 12^2$$
$$a^2 + 16 = 144$$
$$a^2 = 128$$
$$a = \sqrt{128} = \sqrt{2 \times 64}$$
$$= 8\sqrt{2}$$

Trigonometric Ratios

We can define the trigonometric ratios in terms an angle θ and the lengths of the sides of the triangle. The adjacent side is the side closest to the angle, θ. (Adjacent means next to.) The opposite side is the side across from the angle, θ. The hypotenuse is the side of the triangle opposite the right angle. These sides are labeled in Figure 1.

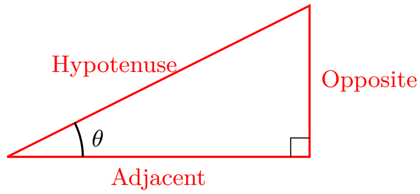


Figure 1: The sides of a right triangle in relation to angle θ.

Given a right triangle with an acute angle of θ, the first three trigonometric ratios are listed.

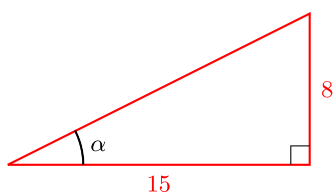
Sine	Cosine	Tangent
$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$	$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$	$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

How To

Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle.

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent as the ratio of the opposite side to the adjacent side.

Example 10.7: Given the triangle shown in the figure, find the values of all trigonometric ratios of the angle α.



Solution: The side adjacent to the angle is 15, and opposite to the angle is 8. Let c be the hypotenuse. By the Pythagorean Theorem,

$$c^2 = 15^2 + 8^2 = 225 + 64 = 289$$
$$c = \sqrt{289} = 17$$

$$\sin\alpha = \frac{\text{opp}}{\text{hyp}} = \frac{8}{17} \qquad \csc\alpha = \frac{17}{8}$$

$$\cos\alpha = \frac{\text{adj}}{\text{hyp}} = \frac{15}{17} \qquad \sec\alpha = \frac{17}{15}$$

$$\tan\alpha = \frac{\text{opp}}{\text{adj}} = \frac{8}{15} \qquad \cot\alpha = \frac{15}{8}$$

Reciprocal Functions

Secant	Cosecant	Cotangent
$\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}}$	$\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}}$	$\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}}$

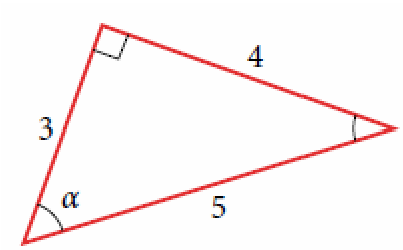
Take another look at these definitions. These functions are the reciprocals of the first three functions.

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Example 10.8: Using the triangle shown in Figure, evaluate $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.



Solution:

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} \qquad \csc\alpha = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{4}$$

$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} \qquad \sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{3}$$

$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3} \qquad \cot\alpha = \frac{\text{adjacent}}{\text{opposite}} = \frac{3}{4}$$

Another approach would have been to find sine, cosine, and tangent first. Then find their reciprocals to determine the other functions.

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

10.2.2 Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

In this section, we will work with the fundamental identities: the Pythagorean identities, the even-odd identities, the reciprocal identities, and the quotient identities.

Pythagorean Identities		
$\sin^2\theta + \cos^2\theta = 1$	$1 + \cot^2\theta = \csc^2\theta$	$1 + \tan^2\theta = \sec^2\theta$

Table 1

The next set of fundamental identities is the set of **even-odd identities**. The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle. (See Table 2).

Even Identity	**Odd Identities**	
$\cos(-\theta) = \cos\theta$	$\sin(-\theta) = -\sin\theta$	$\tan(-\theta) = -\tan\theta$
$\sec(-\theta) = \sec\theta$	$\csc(-\theta) = -\csc\theta$	$\cot(-\theta) = -\cot\theta$

Table 2

The next set of fundamental identities is the set of **reciprocal identities**, which, as their name implies, relate trigonometric functions that are reciprocals of each other. See Table 3.

Reciprocal Identities	
$\sin\theta = \dfrac{1}{\csc\theta}$	$\csc\theta = \dfrac{1}{\sin\theta}$
$\cos\theta = \dfrac{1}{\sec\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$
$\tan\theta = \dfrac{1}{\cot\theta}$	$\cot\theta = \dfrac{1}{\tan\theta}$

Table 3

The final set of identities is the set of **quotient identities**, which define relationships among certain trigonometric functions and can be very helpful in verifying other identities. See Table 4.

Quotient Identities	
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Table 4

Example 10.9: Verify $\tan\theta\cos\theta = \sin\theta$.

Solution: We will start on the left side, as it is the more complicated side:

$$\begin{aligned}\tan\theta\cos\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta\\ &= \left(\frac{\sin\theta}{\cancel{\cos\theta}}\right)\cancel{\cos\theta}\\ &= \sin\theta\end{aligned}$$

Example 10.10: Verify the following equivalency using the even-odd identities:

$(1+\sin x)[1+\sin(-x)] = \cos^2 x$

Solution: Working on the left side of the equation, we have

$$\begin{aligned}(1+\sin x)[1+\sin(-x)] &= (1+\sin x)(1-\sin x) && \text{Since } \sin(-x) = -\sin x\\ &= 1-\sin^2 x && \text{Difference of squares}\\ &= \cos^2 x && \left[\cos^2 x = 1-\sin^2 x\right]\end{aligned}$$

Example 10.11: Verify the identity $\dfrac{\sec^2\theta - 1}{\sec^2} = \sin^2\theta$.

Solution: As the left side is more complicated, lets begin there.

$$\begin{aligned}
\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{(\tan^2\theta + 1) - 1}{\sec^2\theta} && \left[\sec^2\theta = \tan^2\theta + 1\right] \\
&= \frac{\tan^2\theta}{\sec^2\theta} \\
&= \tan^2\theta\left(\frac{1}{\sec^2\theta}\right) \\
&= \tan^2\theta(\cos^2\theta) && \left[\cos^2\theta = \frac{1}{\sec^2\theta}\right] \\
&= \frac{\sin^2\theta}{\cos^2\theta}(\cos^2\theta) && \left[\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}\right] \\
&= \frac{\sin^2\theta}{\cancel{\cos^2\theta}}(\cancel{\cos^2\theta}) \\
&= \sin^2\theta
\end{aligned}$$

Example 10.12: Verify the identity $(1-\cos^2 x)(1+\cot^2 x) = 1$.

Solution: We will work on the left side of the equation

$$\begin{aligned}
(1-\cos^2 x)(1+\cot^2 x) &= (1-\cos^2 x)\left(1 + \frac{\cos^2 x}{\sin^2 x}\right) \\
&= (1-\cos^2 x)\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\right) \\
&= (\sin^2 x)\left(\frac{1}{\sin^2 x}\right) \\
&= 1
\end{aligned}$$

10.2 Section Exercises

Graphical

1. *For the following exercises, use Figure 1 to evaluate each trigonometric ratio of angle θ.*

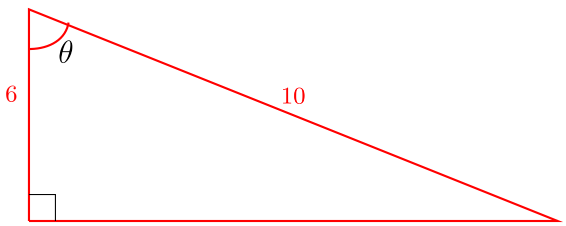


Figure 1

a. $\sin\theta$ **b.** $\cos\theta$ **c.** $\tan\theta$ **d.** $\csc\theta$ **e.** $\sec\theta$ **f.** $\cot\theta$

2. *For the following exercises, use Figure 2 to evaluate each trigonometric ratio of angle α.*

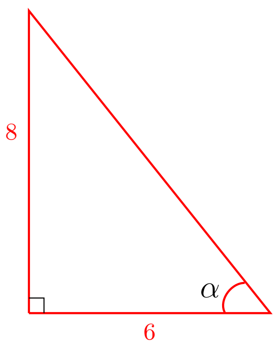


Figure 2

a. $\sin\alpha$ **b.** $\cos\alpha$ **c.** $\tan\alpha$ **d.** $\csc\alpha$ **e.** $\sec\alpha$ **f.** $\cot\alpha$

Verifying Trigonometric Identities

For the following exercises, use the fundamental identities to fully simplify the expression.

3. $\sin x \cos x \sec x$

4. $\sin(-x)\cos(-x)\csc(-x)$

5. $\tan x \sin x + \sec x \cos^2 x$

6. $\csc x + \cos x \cot(-x)$

7. $\dfrac{\cot t + \tan t}{\sec(-t)}$

8. $3\sin^3 t \csc t + \cos^2 t + 2\cos(-t)\cos t$

9. $-\tan(-x)\cot(-x)$

10. $\dfrac{-\sin(-x)\cos x \sec x \csc x \tan x}{\cot x}$

11. $\dfrac{1-\cos^2 x}{\tan^2 x} + 2\sin^2 x$

For the following exercises, verify the identity.

12. $\cos x - \cos^3 x = \cos x \sin^2 x$

13. $\cos x(\tan x - \sec(-x)) = \sin x - 1$

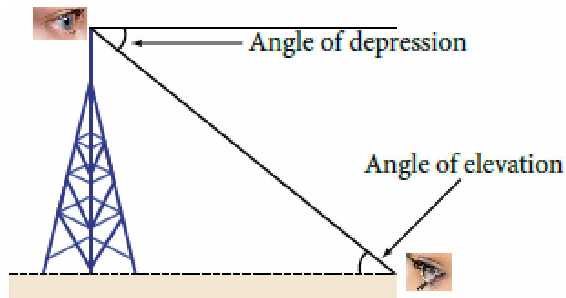
14. $\dfrac{1+\sin^2 x}{\cos^2 x} = 1 + 2\tan^2 x$

15. $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

16. $\cos^2 x - \tan^2 x = 2 - \sin^2 x - \sec^2 x$

10.3 Applications of Trigonometry

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height.



The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye.

The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye.

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 10.13: To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 60° between a line of sight to the top of the tree and the ground, as shown in Figure. Find the height of the tree.

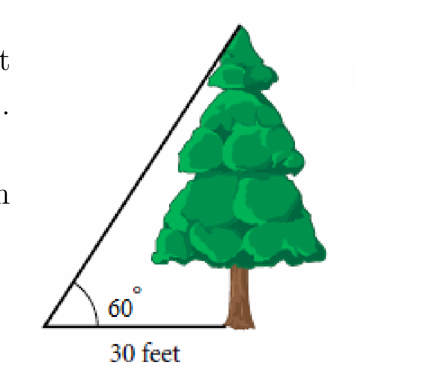
$$\left[\sin 60° = \frac{\sqrt{3}}{2},\ \cos 60° = \frac{1}{2},\ \tan 60° = \sqrt{3}\right]$$

Solution:

We have that the angle of elevation is 60° and the adjacent side is 30 ft long. The opposite side is the unknown height. Let the height $= h$.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

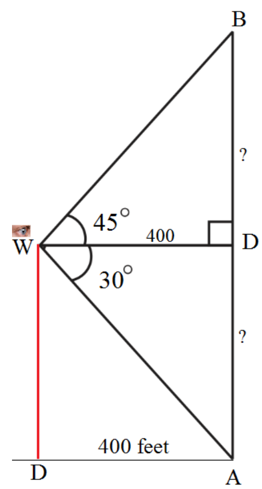


$$\tan(60°) = \frac{h}{30}$$
$$h = 30\tan(60°)$$
$$h = 30\sqrt{3}$$

The tree is $30\sqrt{3}$ feet tall.

Example 10.14: A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 45°, and that the angle of depression to the bottom of the tower is 30°. How tall is the tower?

Solution:
Let AB be the radio tower, where A is the base and B is the top.
Let W be the window in the building.
Draw a line WD perpendicular to AB.



$$AB = AD + DB$$

We have,
$$AD = WD = 400$$

angle of elevation, $\angle DWB = 45°$ and
angle of depression, $\angle DWA = 30°$.

From the triangle, BWD,

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan(45°) = \frac{BD}{400}$$
$$1 = \frac{BD}{400}$$
$$BD = 400 \text{ feet}$$

From the triangle, ADW,

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan(30°) = \frac{AD}{400}$$
$$\frac{1}{\sqrt{3}} = \frac{AD}{400}$$
$$AD = \frac{400}{\sqrt{3}} \text{ feet}$$

$$\text{Height of the tower} = AB = AD + DB$$
$$= \frac{400}{\sqrt{3}} + 400 = 400\left(1 + \frac{1}{\sqrt{3}}\right) \text{ feet}$$

10.3 Section Exercises

Real-World Applications

1. A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 60°. How high does the ladder reach up the side of the building?
2. A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 60ř, and that the angle of depression to the bottom of the tower is 30ř. How tall is the tower?
3. The angle of elevation to the top of a building in New York is found to be 30 degrees from the ground at a distance of 25 meter from the base of the building. Using this information, find the height of the building.
4. From the top of a building of height 200 feet, a person determines that the angle of depression of a stone on the ground is 60 degrees. How far is the stone from the base of the building?

Angle θ		Functions		
Degrees	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	∞
270°	$\frac{3\pi}{2}$	-1	0	∞
360°	2π	0	1	0