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FPMB0001 - BASIC MATHEMATICS

Solution Manual

Prepared by Periyasamy Chellakkani

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Contents

Chapter 1

REAL NUMBERS

1.1 Classification of Real Numbers

Natural Numbers

Natural numbers are the collection of numbers which is used to count. It begins with 1, next number is 2, then 3, and so on. It is denoted by $\boldsymbol{N}$. ie., $N = \{1, 2, 3, 4, 5, \cdots\}$

Example: 1, 30, 120, 3000, 580 and 90 are all Natural numbers

Whole Numbers

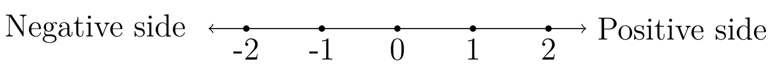
We introduce the number "**0**" (we read as, ZERO) to represent to count the number of chocolates in an empty chocolate box. Whole number is a collection of numbers that includes zero and all counting numbers(natural numbers). It is denoted by $\boldsymbol{W}$. Hence $W = \{0, 1, 2, 3, 4, 5, \cdots\}$

Example: 1, 30, 120, 3000, 580 and 0 are all Whole numbers

Integers

The counting numbers are used to move forward one by one, 1 next one is 2, then 3. Likewise we can count backward or opposite direction next to 0, starts with -1, then -2, and then -3 and so on. It is denoted by $\boldsymbol{Z}$ and $Z = \cdots, -2, -1, 0, 1, 2, \cdots$

Example: -1, -50, 50, 75, -80 are all Integers



Rational Numbers

The collection of following numbers are defined as rational numbers:

- All integers

  Example: $-5, 0, 7$

- Valid fractions, $\frac{p}{q}$, (i.e., p and q are integers and $p \neq 0$)

  Example: $\frac{2}{3}, -\frac{13}{4}, \frac{0}{5} = 0$

- Decimal numbers with finite decimal places

  Example: $13.5785, 10.456, 12.1$

- Decimal numbers with infinite repetition of finite numbers in decimal place

  Example: $5.73636363636\cdots = 5.7\bar{3}6, 81.127127\cdots = 81.1\bar{2}7$

It is denoted by $\boldsymbol{Q}$, here $\frac{1}{10}$, $\frac{5}{1} = 5$, -20.5, $-78.633333\cdots$, $1.535535535\cdots$ ... are all represented as rational numbers

Irrational Numbers

A number which is not rational number, we can call it as an irrational number. The simplest way to define the irrational number is, the decimal number having indefinite(does not stop) and non repeating decimal part. It is represented by the letter $\boldsymbol{I}$

Example:

- $\sqrt{2} = 1.41421356237\cdots$
- $\sqrt{3} = 1.73205080757\cdots$
- $\sqrt{5} = 2.2360679775\cdots$
- $\pi = 3.14159265359\cdots$
- $e = 2.71828182846\cdots$

Hence, we conclude the system of **Real numbers** ($\boldsymbol{R}$) by combining both rational and irrational numbers. Also we can easily verify that natural numbers, whole numbers and integers are part of the rational numbers so as of Real numbers.

Exercise - 1.1

In the following exercises, determine which of the given numbers are rational and which are irrational.

Problem 1: **a.** 0.75, **b.** $0.22\bar{3}$, **c.** $1.39174\cdots$

Solution:

(a) 0.75 - Rational (It has finite decimal part)

(b) $0.22\bar{3}$ - Rational (Finite number of repeated decimal part)

(c) $1.39174\cdots$ - Irrational (Indefinite decimal part without any repetition)

Problem 2: **a.** 0.36, **b.** $0.94729\cdots$, **c.** $2.52\bar{8}$

Solution:

(a) 0.36 - Rational (It has finite decimal part)

(b) $0.94729\cdots$ - Irrational (Indefinite decimal part without any repetition)

(c) $2.52\bar{8}$ - Rational (Finite number of repeated decimal part)

Problem 3: **a.** $0.4\bar{5}$, **b.** $1.919293\cdots$, **c.** 3.59

Solution:

(a) $0.4\bar{5}$ - Rational (Finite number of repeated decimal part)

(b) $1.919293\cdots$ - Irrational (Indefinite decimal part without any repetition)

(c) 3.59 - Rational (It has finite decimal part)

In the following exercises, identify whether each number is rational or irrational

Problem 4: **a.** $\sqrt{25}$, **b.** $\sqrt{30}$

Solution:

(a) $\sqrt{25} = 5$ - It is rational

(b) $\sqrt{30} = \sqrt{2 \times 3 \times 5}$ - It is irrational

Problem 5: **a.** $\sqrt{44}$, **b.** $\sqrt{49}$

Solution:

(a) $\sqrt{44} = 2\sqrt{11}$ - It is irrational

(b) $\sqrt{49} = \sqrt{7}$ - It is rational

Problem 6: **a.** $\sqrt{164}$, **b.** $\sqrt{169}$

Solution:

(a) $\sqrt{164} = 2\sqrt{41}$ - It is irrational

(b) $\sqrt{169} = 13$ - It is rational

In the following exercises, determine whether each number is whole, integer, rational, irrational, and real

Problem 7: **-8, 0, 1.95286$\cdots$, $\frac{12}{5}$, $\sqrt{36}$, 9**

Solution:

	-8	0	$1.95286\cdots$	$\frac{12}{5} = 2.4$	$\sqrt{36} = 6$	9
Whole		✓			✓	✓
Integer	✓	✓			✓	✓
Rational	✓	✓		✓	✓	✓
Irrational			✓			
Real Number	✓	✓	✓	✓	✓	✓

Problem 8: **-9, -3$\frac{4}{9}$, -$\sqrt{9}$, $0.4\overline{09}$, $\frac{11}{6}$, 7**

Solution:

	-9	-3$\frac{4}{9}$	-$\sqrt{9}$=-3	$0.4\overline{09}$	$\frac{11}{6} = 1.8\overline{3}$	7
Whole						✓
Integer	✓		✓			
Rational	✓	✓	✓	✓	✓	✓
Irrational						
Real Number	✓	✓	✓	✓	✓	✓

1.2 Properties of Real Numbers

In this section we discuss about much important operation addition (+) and multiplication (×/·) and its properties.

Closure property

When we add/ multiply two real numbers, the result will give again a real number. This property is known as closure property

$$\begin{aligned} \text{If } a,b \in R &\implies a+b \in R \\ \text{If } a,b \in R &\implies a \times b \in R \end{aligned}$$

Example: Consider two real numbers 5 and 2

- $5+2=7$, the answer 7 is also a real number
- $5 \times 2 = 10$, 10 is also a real number

Commutative property

When we add/multiply any two real numbers, the order of the numbers are doesn't matter. We can change the order and we add/multiply, the resulting answer will be same

$$\begin{aligned} \text{If } a,b \in R &\implies a+b = b+a \\ \text{If } a,b \in R &\implies a \times b = b \times a \end{aligned}$$

Example: Consider two real numbers 8 and 4

- $8+4=4+8=12$
- $8 \times 4 = 4 \times 8 = 32$

Associative property

If a, b and c are real numbers, then

- Associative property for **Addition**: $\boldsymbol{a+(b+c)=(a+b)+c}$
  Example: 2+(4+9)=(2+4)+9
- Associative property for **Multiplication**: $\boldsymbol{a.(b.c)=(a.b).c}$
  Example: $3 \cdot (7 \cdot 5) = (3 \cdot 7) \cdot 5$

Identity number

In real number system, the numbers 0 and 1 are respectively known as identity numbers for addition and multiplication. For any real number a,

- $a + 0 = 0 + a = a$ Example: $5 + 0 = 0 + 5 = 5$; & $9 + 0 = 0 + 9 = 9$
- $a \cdot 1 = 1 \cdot a = a$ Example: $1 \cdot \frac{5}{3} = \frac{5}{3} \cdot 1 = \frac{5}{3}$; & $1 \cdot 7 = 7 \cdot 1 = 7$

Inverse numbers

In real number system, for each number have inverse with respect to addition and for each non-zero number have inverse with respect to multiplication

- Additive inverse for a is defined as $-a$, i.e., $\boldsymbol{a + (-a) = (-a) + a = 0}$, 0 is the additive identity
- For 5, -5 is the inverse with respect to addition, sometimes its called additive inverse
- Multiplicative inverse of a is defined as $\frac{1}{a}$, i.e., $\boldsymbol{a \times \dfrac{1}{a} = \dfrac{1}{a} \times a = 1}$, where 1 is the multiplicative identity
- For 3, $\frac{1}{3}$ is the inverse with respect to multiplication, sometimes its called multiplicative inverse

Distributive Property

Distributive Property states the following

If a, b, c are real numbers, then $\boldsymbol{a \times (b + c) = a \times b + a \times c}$
Example: $4 \times (3 + 5) = (4 \times 3) + (4 \times 5)$

Properties of zero

- For any real number a, $a \cdot 0 = 0 \cdot a = 0$. Example: $100 \times 0 = 0$
- For any non-zero real number a, $\dfrac{0}{a} = 0$. Example: $\dfrac{0}{10} = 0$
- For any real number a, $\dfrac{a}{0}$ is not defined. Example: $\dfrac{5}{0}$ is not defined

Exercise - 1.2

Use the Commutative and Associative Properties

In the following exercises, use the commutative properties to rewrite the given expression.

Problem 1: 8+9

Solution:

Commutative Property: 8+9=9+8

Problem 2: 7+6

Solution:

Commutative Property: 7+6=6+7

Problem 3: 8(-12)

Solution:

Commutative Property: 8(-12)=(-12)8

Problem 4: y+1

Solution:

Commutative Property: y+1=1+y

Problem 5: -2a

Solution:

Commutative Property: -2a=a(-2)

Problem 6: -3m

Solution:

Commutative Property: -3m=m(-3)

In the following exercises, use the associative properties to rewrite the given expression.

Problem 7: (11+9)+14

Solution:

Associative Property: (11+9)+14=11+(9+14)

Problem 8: (21+14)+9

Solution:

Associative Property: (21+14)+9=21+(14+9)

Problem 9: $(12 \cdot 5) \cdot 7$

Solution:

Associative Property: $(12 \cdot 5) \cdot 7 = 12 \cdot (5 \cdot 7)$

Problem 10: $(7 \cdot 6) \cdot 9$

Solution:

Associative Property: $(7 \cdot 6) \cdot 9 = 7 \cdot (6 \cdot 9)$

Problem 11: (-7+9)+8

Solution:

Associative Property: (-7+9)+8=(-7)+(9+8)

Problem 12: $3(4x)$

Solution:

Associative Property: $3(4x) = (3 \cdot 4) \cdot x$

Simplify Expressions Using the Properties of real numbers

Problem 13: $\boldsymbol{-45a + 15 + 45a}$

Solution:

$$\begin{aligned} -45a + 15 + 45a &= -45a + (15 + 45a) & \\ &= -45a + (45a + 15) & \text{[Commutative Property]} \\ &= (-45a + 45a) + 15 & \text{[Associative Property]} \\ &= 0 + 15 & \text{[Additive Inverse Property]} \\ -45a + 15 + 45a &= 15 & \text{[Additive identity Property]} \end{aligned}$$

Problem 14: $\boldsymbol{9y + 23 + (-9y)}$

Solution:

$$\begin{aligned} 9y + 23 + (-9y) &= (9y + 23) + (-9y) & \\ &= (23 + 9y) + (-9y) & \text{[Commutative Property]} \\ &= 23 + [9y + (-9y)] & \text{[Associative Property]} \\ &= 23 + [9y - 9y] & \\ &= 23 + 0 & \text{[Additive Inverse Property]} \\ 9y + 23 + (-9y) &= 23 & \text{[Additive identity Property]} \end{aligned}$$

Problem 15: $\boldsymbol{\frac{1}{2} + \frac{7}{8} + \left(-\frac{1}{2}\right)}$

Solution:

$$\begin{aligned} \frac{1}{2} + \frac{7}{8} + \left(-\frac{1}{2}\right) &= \left(\frac{1}{2} + \frac{7}{8}\right) + \left(-\frac{1}{2}\right) & \\ &= \left(\frac{7}{8} + \frac{1}{2}\right) + \left(-\frac{1}{2}\right) & \text{[Commutative Property]} \\ &= \frac{7}{8} + \left[\frac{1}{2} + (-\frac{1}{2})\right] & \text{[Associative Property]} \\ &= \frac{7}{8} + \left[\frac{1}{2} - \frac{1}{2}\right] & \\ &= \frac{7}{8} + [0] & \text{[Additive Inverse Property]} \\ \frac{1}{2} + \frac{7}{8} + \left(-\frac{1}{2}\right) &= \frac{7}{8} & \text{[Additive identity Property]} \end{aligned}$$

Problem 16: $\left(\frac{5}{6}+\frac{8}{15}\right)+\frac{7}{15}$

Solution:

$$\begin{aligned}
\left(\frac{5}{6}+\frac{8}{15}\right)+\frac{7}{15} &= \frac{5}{6}+\left(\frac{8}{15}+\frac{7}{15}\right) && \text{[Associative Property]}\\
&= \frac{5}{6}+\left(\frac{8+7}{15}\right)\\
&= \frac{5}{6}+\left(\frac{\cancel{15}}{\cancel{15}}\right)\\
\left(\frac{5}{6}+\frac{8}{15}\right)+\frac{7}{15} &= \frac{5}{6}+1
\end{aligned}$$

Problem 17: $\left(\frac{1}{12}+\frac{4}{9}\right)+\frac{5}{9}$

Solution:

$$\begin{aligned}
\left(\frac{1}{12}+\frac{4}{9}\right)+\frac{5}{9} &= \frac{1}{12}+\left(\frac{4}{9}+\frac{5}{9}\right) && \text{[Associative Property]}\\
&= \frac{1}{12}+\left(\frac{4+5}{9}\right)\\
&= \frac{1}{12}+\left(\frac{\cancel{9}}{\cancel{9}}\right)\\
\left(\frac{1}{12}+\frac{4}{9}\right)+\frac{5}{9} &= \frac{1}{12}+1
\end{aligned}$$

Problem 18: $14x+(19y+25x)+3y$

Solution:

$$\begin{aligned}
14x+(19y+25x)+3y &= (14x+25x)+(19y+3y)\\
&= (19y+14x)+25x+3y && \text{[Commutative Property]}\\
&= 19y+(14x+25x)+3y && \text{[Associative Property]}\\
&= \underline{19y+(39x)}+3y\\
&= 39x+19y+3y && \text{[Commutative Property]}\\
14x+19y+25x+3y &= 39x+22y
\end{aligned}$$

In the following exercises, simplify Expressions using the distributive property

Problem 19: $\mathbf{4(x+8)}$

Solution:

Distributive property: $4(x+8) = 4x + (4 \times 8) = 4x + 32$

Problem 20: $\mathbf{3(a+9)}$

Solution:

Distributive property: $3(a+9) = 3a + (3 \times 9) = 3a + 27$

Problem 21: $\mathbf{8(4y+9)}$

Solution:

Distributive property: $8(4y+9) = [8 \times (4y)] + (8 \times 9) = 32y + 72$

Problem 22: $\mathbf{7(3p–8)}$

Solution:

Distributive property: $7(3p–8) = [7 \times (3p)]–(7 \times 8) = 21p - 56$

Problem 23: $\mathbf{5(7u–4)}$

Solution:

Distributive property: $5(7u–4) = [5 \times (7u)] - (5 \times 4) = 35u - 20$

Problem 24: $\mathbf{\frac{1}{2}(n+8)}$

Solution:

Distributive property: $\frac{1}{2}(n+8) = \frac{1}{2}(n) + \left(\frac{1}{2} \times 8\right) = \frac{n}{2} + 4$

Problem 25: $\mathbf{(y+4)p}$

Solution:

Distributive property: $(y+4)p = (y \times p) + (4 \times p) = py + 4p$

Problem 26: $\mathbf{–3(a+11)}$

Solution:

Distributive property: $–3(a+11) = [(–3) \times a] + (3 \times 11) = –3a + 33$

Problem 27: $-(r+7)$

Solution:

Distributive property: $-(r+7) = [(-1)\times r] + [(-1)\times 7] = -r-7$

In the following exercises, identify whether each example is using the identity property of addition or multiplication

Problem 28: $101+0=101$

Solution:

$101+0=101$ $\rightarrow$ **Identity property of addition**

Problem 29: $\frac{3}{5}(1)$

Solution:

$\frac{3}{5}(1) = \frac{3}{5}$ $\rightarrow$ **Identity property of multiplication**

Problem 30: $-9\cdot 1$

Solution:

$-9\cdot 1 = -9$ $\rightarrow$ **Identity property of multiplication**

In the following exercises, find the additive and multiplicative inverse

Problem 31: 8

Solution:

Additive inverse of 8 is -8
Multiplicative inverse of 8 is $\frac{1}{8}$

Problem 32: 32

Solution:

Additive inverse of 14 is -14
Multiplicative inverse of 14 is $\frac{1}{14}$

Problem 33: -17

Solution:

Additive inverse of -17 is 17
Multiplicative inverse of -17 is $\frac{-1}{17}$

Problem 34: $\mathbf{0.5}$

Solution:

Additive inverse of 0.5 is -0.5
Multiplicative inverse of $0.5 = \frac{5}{10}$ is $\frac{10}{5} = 2$

Problem 35: $\frac{\mathbf{7}}{\mathbf{12}}$

Solution:

Additive inverse of $\frac{7}{12}$ is $-\frac{7}{12}$

Multiplicative inverse of $\frac{7}{12}$ is $\frac{12}{7}$

Problem 36: $\frac{\mathbf{8}}{\mathbf{13}}$

Solution:

Additive inverse of $\frac{8}{13}$ is $-\frac{8}{13}$

Multiplicative inverse of $\frac{8}{13}$ is $\frac{13}{8}$

In the following exercises, simplify using the properties of zero.

Problem 37: $\mathbf{48 \cdot 0}$

Solution:

$48 \cdot 0 = 0$ [By using the **property of zero**, $\boldsymbol{a \cdot 0 = 0}$]

Problem 38: $\mathbf{0 \div 11}$

Solution:

$0 \div 11 = 0$ [By using the **property of zero**, $\boldsymbol{0 \div a = 0}$]

Problem 39: $\frac{\mathbf{3}}{\mathbf{0}}$

Solution:

$\frac{3}{0}$ is not defined [By using the **property of zero**, $\boldsymbol{a \div 0}$ **is not defined**]

Problem 40: $\mathbf{\dfrac{0}{24}}$

Solution:

$\dfrac{0}{24} = 0$ [By using the **property of zero**, $\boldsymbol{\dfrac{0}{a} = 0}$]

Problem 41: $\mathbf{5.72 \div 0}$

Solution:

$5.72 \div 0$ is not defined [By using the **property of zero**, $\boldsymbol{a \div 0}$ **is not defined**]

Problem 42: $\mathbf{\dfrac{\frac{1}{10}}{0}}$

Solution:

$\dfrac{\frac{1}{10}}{0}$ is not defined [By using the **property of zero**, $\boldsymbol{a \div 0}$ **is not defined**]

1.3 Fractions, Decimals, Ratios and Percent

Fraction

Fraction is a kind of real number which is written in the form $\frac{a}{b}$, where a and b are integers with $b \neq 0$ (to maintain the validness of the number). Here a can be called as **numerator** and b as **denominator**

Types of fractions

Fractions can be classified into two types. Each of these are defined and explained follows:

- **Proper Fraction:** The fraction $\frac{a}{b}$ is called a proper fraction if $a < b$
- **Improper Fraction:** The fraction $\frac{a}{b}$ is called an improper fraction if $a \geq b$

Mixed number

Improper fractions can be written/converted as a form of mixed numbers. If $\frac{a}{b}$ is an improper fraction and r is the remainder and q is the quotient when a is divided by b, then $q\frac{r}{b}$ is defined as the mixed number form for the improper fraction

- **Conversion of improper fraction into mixed number**
  * $\frac{a}{b} = q\frac{r}{b}$ or $\left(\text{Quotient}\frac{\text{Remainder}}{\text{Divisor}}\right)$
- **Conversion of mixed number into improper fraction**
  * $q\frac{r}{b} = \frac{(b \times q) + r}{b}$

Equivalent Fraction

Equivalent fractions are fractions that have the same value. For example, $\frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{15}{10}$ and $\frac{18}{12}$ are all known to be equivalent fractions since each of them have the same value.

Reciprocal

Reciprocal of a fraction is defined as its multiplicative inverse. For example reciprocal of $\frac{2}{7}$ is defined as $\frac{7}{2}$

Decimals

- Decimals are defined as the simplified fractions
- Decimals are represented by as its 10^{th} position as numerals have
- Example:

$$\begin{aligned} 187.125 &= 100 + 80 + 7 + 0.1 + 0.02 + 0.005 \\ &= (1 \times 10^2) + (8 \times 10^1) + (7 \times 10^0) + (1 \times 10^{-1}) + (2 \times 10^{-2}) + (5 \times 10^{-3}) \end{aligned}$$

Ratio

A ratio is used to represent the number with respect to another where the unit of both numbers are same. The ratio of two numbers a and b is written a to b or $a : b$. The meaning of $a : b$ is $\frac{a}{b}$

Percent

A percent is a ratio whose denominator is 100. This symbol % is used to represent the percent of a number.

Exercise - 1.3

In the following exercises, rewrite the improper fraction as a mixed number.

Problem 1: $\frac{3}{2}$

Solution:

$\frac{3}{2}$: When 3 is divided by 2, the remainder is 1 and the quotient is also 1. Hence $\boxed{\frac{3}{2} = 1\frac{1}{2}}$

Problem 2: $\frac{5}{3}$

Solution:

$\frac{5}{3}$: When 5 is divided by 3, the remainder is 2 and the quotient is 1. Hence $\boxed{\frac{5}{3} = 1\frac{2}{3}}$

Problem 3: $\frac{11}{4}$

Solution:

$\frac{11}{4}$: When 11 is divided by 4, the remainder is 3 and the quotient is 2. Hence $\boxed{\frac{11}{4} = 2\frac{3}{4}}$

In the following exercises, rewrite the mixed number as an improper fraction.

Problem 4: $1\frac{2}{3}$

Solution:

$1\frac{2}{3} = \frac{x}{3}$, Where the number x is divided by 3, the quotient is 1 and the remainder is 3. Hence $x = 1 \times 3 + 2 = 5$. Hence $\boxed{1\frac{2}{3} = \frac{5}{3}}$

Problem 5: $\mathbf{2\frac{2}{5}}$

Solution:

$2\frac{2}{5} = \frac{x}{5}$, Where the number x is divided by 5, the quotient is 2 and the remainder is 2. Hence $x = 2 \times 5 + 2 = 12$. Hence $\boxed{2\frac{2}{5} = \frac{12}{5}}$

Problem 6: $\mathbf{2\frac{1}{4}}$

Solution:

$2\frac{1}{4} = \frac{x}{4}$, Where the number x is divided by 4, the quotient is 2 and the remainder is 1. Hence $x = 2 \times 4 + 1 = 9$. Hence $\boxed{2\frac{1}{4} = \frac{9}{4}}$

In the following exercises, simplify each fraction. Do not convert any improper fractions to mixed numbers

Problem 7: $\mathbf{\frac{7}{21}}$

Solution:

$$\frac{7}{21} = \frac{1 \times \cancel{7}}{3 \times \cancel{7}} = \frac{1}{7} \implies \boxed{\frac{7}{21} = \frac{1}{3}}$$

Problem 8: $\mathbf{\frac{8}{24}}$

Solution:

$$\frac{8}{24} = \frac{1 \times \cancel{8}}{3 \times \cancel{8}} = \frac{1}{3} \implies \boxed{\frac{8}{24} = \frac{1}{3}}$$

Problem 9: $\mathbf{\frac{20}{15}}$

Solution:

$$\frac{20}{15} = \frac{4 \times \cancel{5}}{3 \times \cancel{5}} = \frac{4}{3} \implies \boxed{\frac{20}{15} = \frac{4}{3}}$$

In the following exercises, multiply, and write the answer in simplified form

Problem 10: $\frac{2}{5} \cdot \frac{1}{3}$

Solution:

$$\frac{2}{5} \cdot \frac{1}{3} = \frac{2 \times 1}{5 \times 3} = \frac{2}{15} \implies \boxed{\frac{2}{5} \cdot \frac{1}{3} = \frac{2}{15}}$$

Problem 11: $\frac{1}{2} \cdot \frac{3}{8}$

Solution:

$$\frac{1}{2} \cdot \frac{3}{8} = \frac{1 \times 3}{2 \times 8} = \frac{3}{16} \implies \boxed{\frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16}}$$

Problem 12: $-\frac{5}{9} \cdot \frac{3}{10}$

Solution:

$$-\frac{5}{9} \cdot \frac{3}{10} = \frac{-5 \times 3}{9 \times 10} = \frac{-\cancel{5} \times \cancel{3}}{\cancel{3} \times 3 \times 2 \times \cancel{5}} = \frac{-1}{6} \implies \boxed{-\frac{5}{9} \cdot \frac{3}{10} = \frac{-1}{6}}$$

Problem 13: $-\frac{3}{8} \cdot \frac{4}{15}$

Solution:

$$-\frac{3}{8} \cdot \frac{4}{15} = \frac{-3 \times 4}{8 \times 15} = \frac{-\cancel{3} \times \cancel{4}}{\cancel{2} \times 2 \times 5 \times \cancel{3}} = \frac{-1}{10} \implies \boxed{-\frac{3}{8} \cdot \frac{4}{15} = \frac{-1}{10}}$$

Problem 14: $\left(\frac{7}{12}\right)\left(\frac{-8}{21}\right)$

Solution:

$$\left(\frac{7}{12}\right)\left(\frac{-8}{21}\right) = \frac{-7 \times 8}{12 \times 21} = \frac{-\cancel{7} \times 2 \times \cancel{4}}{\cancel{4} \times 3 \times 3 \times \cancel{7}} = \frac{-2}{9} \implies \boxed{\left(\frac{7}{12}\right)\left(\frac{-8}{21}\right) = \frac{-2}{9}}$$

Problem 15: $\left(\frac{-9}{10}\right)\left(\frac{25}{33}\right)$

Solution:

$$\left(\frac{-9}{10}\right)\left(\frac{25}{33}\right) = \frac{-9\times 25}{10\times 33} = \frac{-\cancel{3}\times 3\times 5\times \cancel{5}}{\cancel{5}\times 2\times 11\times \cancel{3}} = \frac{-15}{22} \implies \boxed{\left(\frac{-9}{10}\right)\left(\frac{25}{33}\right) = \frac{-15}{22}}$$

In the following exercises, divide, and write the answer in simplified form.

Problem 16: $\frac{1}{2}\div\frac{1}{4}$

Solution:

$$\frac{1}{2}\div\frac{1}{4} = \frac{1}{2}\times\frac{4}{1} = \frac{1\times 4}{2\times 1} = \frac{2\times\cancel{2}}{\cancel{2}} = 2 \implies \boxed{\frac{1}{2}\div\frac{1}{4} = 2}$$

Problem 17: $\frac{3}{4}\div\frac{2}{3}$

Solution:

$$\frac{3}{4}\div\frac{2}{3} = \frac{3}{4}\times\frac{3}{2} = \frac{3\times 3}{4\times 2} = \frac{9}{8} \implies \boxed{\frac{3}{4}\div\frac{2}{3} = \frac{9}{8}}$$

Problem 18: $-\frac{4}{5}\div\frac{4}{7}$

Solution:

$$-\frac{4}{5}\div\frac{4}{7} = -\frac{4}{5}\times\frac{7}{4} = \frac{-4\times 7}{5\times 4} = \frac{-7\times\cancel{4}}{7\times\cancel{4}} = \frac{-7}{5} \implies \boxed{-\frac{4}{5}\div\frac{4}{7} = \frac{-7}{5}}$$

Problem 19: $-\frac{7}{9}\div\left(-\frac{7}{9}\right)$

Solution:

$$-\frac{7}{9}\div\left(-\frac{7}{9}\right) = \frac{-7}{9}\times\frac{-9}{7} = \frac{\cancel{7}\times\cancel{9}}{\cancel{9}\times\cancel{7}} = 1 \implies \boxed{-\frac{7}{9}\div\left(-\frac{7}{9}\right) = 1}$$

Problem 20: $\frac{3}{4}\div\frac{x}{11}$

Solution:

$$\frac{3}{4}\div\frac{x}{11} = \frac{3}{4}\times\frac{11}{x} = \frac{3\times 11}{4\times x} = \frac{33}{4x} \implies \boxed{\frac{3}{4}\div\frac{x}{11} = \frac{33}{4x}}$$

Problem 21: $\frac{7p}{12} \div \frac{21p}{8}$

Solution:

$$\frac{7p}{12} \div \frac{21p}{8} = \frac{7p}{12} \times \frac{8}{21p} = \frac{7p \times 8}{12 \times 21p} = \frac{p \times 7 \times 2 \times 4}{p \times 3 \times 4 \times 3 \times 7} = \frac{\cancel{p} \times \cancel{7} \times 2 \times \cancel{4}}{\cancel{p} \times 3 \times \cancel{4} \times 3 \times \cancel{7}} = \frac{2}{9}$$

$$\Longrightarrow \boxed{\frac{7p}{12} \div \frac{21p}{8} = \frac{2}{9}}$$

In the following exercises, find each sum or difference.

Problem 22: $\frac{4}{9} + \frac{1}{9}$

Solution:

Here, the denominators are same. Hence $\frac{4}{9} + \frac{1}{9} = \frac{4+1}{9} = \frac{5}{9}$

Problem 23: $\frac{6}{13} + \left(-\frac{10}{13}\right) + \left(-\frac{12}{13}\right)$

Solution:

Here, the denominators are same. Hence $\frac{6}{13} + \left(-\frac{10}{13}\right) + \left(-\frac{12}{13}\right) = \frac{6-10-12}{13} = \frac{-16}{13}$

Problem 24: $\frac{x}{4} + \frac{3}{4}$

Solution:

Here, the denominators are same. Hence $\frac{x}{4} + \frac{3}{4} = \frac{x+3}{4}$

Problem 25: $\frac{4}{5} - \frac{1}{5}$

Solution:

Here, the denominators are same. Hence $\frac{4}{5} - \frac{1}{5} = \frac{4-1}{5} = \frac{3}{5}$

Problem 26: $\frac{11}{15} - \frac{7}{15}$

Solution:

Here, the denominators are same. Hence $\frac{11}{15} - \frac{7}{15} = \frac{11-7}{15} = \frac{4}{15}$

Problem 27: $\mathbf{\frac{7}{12} - \frac{2}{12}}$

Solution:

Here, the denominators are same. Hence $\frac{7}{12} - \frac{2}{12} = \frac{7-2}{12} = \frac{5}{12}$

In the following exercises, add or subtract. Write the result in simplified form.

Problem 28: $\mathbf{\frac{1}{3} + \frac{1}{5}}$

Solution:

Since the denominators are not same, we need to calculate the LCM for 3 and 5
LCM(3,5)=15, Hence $\frac{1}{3} + \frac{1}{5} = \frac{1 \times 5}{3 \times 5} + \frac{1 \times 3}{5 \times 3} = \frac{5}{15} + \frac{3}{15} = \frac{5+3}{15} = \frac{8}{15}$

$\Longrightarrow \boxed{\frac{1}{3} + \frac{1}{5} = \frac{8}{15}}$

Problem 29: $\mathbf{\frac{1}{4} + \frac{1}{10}}$

Solution:

Since the denominators are not same, we need to calculate the LCM for 4 and 10
LCM(4,10)=20, Hence $\frac{1}{4} + \frac{1}{10} = \frac{1 \times 5}{4 \times 5} + \frac{1 \times 2}{10 \times 2} = \frac{5}{20} + \frac{2}{20} = \frac{5+2}{20} = \frac{7}{20}$

$\Longrightarrow \boxed{\frac{1}{4} + \frac{1}{10} = \frac{7}{20}}$

Problem 30: $\mathbf{\frac{7}{12} + \frac{5}{8}}$

Solution:

Since the denominators are not same, we need to calculate the LCM for 12 and 8
LCM(12,8)=24, Hence $\frac{7}{12} + \frac{5}{8} = \frac{7 \times 2}{12 \times 2} + \frac{5 \times 3}{8 \times 3} = \frac{14}{24} + \frac{15}{24} = \frac{14+15}{24} = \frac{29}{24}$

$\Longrightarrow \boxed{\frac{7}{12} + \frac{5}{8} = \frac{29}{24}}$

Problem 31: $\frac{5}{12}+\frac{3}{18}$

Solution:

Since the denominators are not same, we need to calculate the LCM for 12 and 18
LCM(12,18)=36, Hence $\frac{5}{12}+\frac{3}{18}=\frac{5\times 3}{12\times 3}+\frac{3\times 2}{18\times 2}=\frac{15}{36}+\frac{6}{36}=\frac{15+6}{36}=\frac{21}{36}$

$\Longrightarrow \boxed{\frac{5}{12}+\frac{3}{18}=\frac{21}{36}}$

Problem 32: $\frac{7}{12}-\frac{9}{16}$

Solution:

Since the denominators are not same, we need to calculate the LCM for 12 and 16
LCM(12,16)=48, Hence $\frac{7}{12}-\frac{9}{16}=\frac{7\times 4}{12\times 4}-\frac{9\times 3}{16\times 3}=\frac{28}{48}-\frac{27}{48}=\frac{28-27}{48}=\frac{1}{48}$

$\Longrightarrow \boxed{\frac{7}{12}-\frac{9}{16}=\frac{1}{48}}$

Problem 33: $\frac{11}{12}-\frac{3}{8}$

Solution:

Since the denominators are not same, we need to calculate the LCM for 12 and 8
LCM(12,8)=24, Hence $\frac{11}{12}-\frac{3}{8}=\frac{11\times 2}{12\times 2}-\frac{3\times 3}{8\times 3}=\frac{22}{24}-\frac{9}{24}=\frac{22-9}{24}=\frac{13}{9}$

$\Longrightarrow \boxed{\frac{11}{12}-\frac{3}{8}=\frac{13}{9}}$

In the following exercises, write each ratio as a fraction

Problem 34: **20 to 36**

Solution:

Required fraction for the ratio 20 to 36 is $\boxed{\frac{20}{36}}$

Problem 35: **45 to 54**

Solution:

Required fraction for the ratio 45 to 54 is $\boxed{\frac{45}{54}}$

Problem 36: 42 to 48

Solution:

Required fraction for the ratio 42 to 48 is $\boxed{\dfrac{42}{48}}$

In the following exercises, write each percent as a ratio

Problem 37: 12%

Solution:

$12\% = \dfrac{12}{100} \implies$ Ratio of 12% is **12 to 100**

Problem 38: 35%

Solution:

$35\% = \dfrac{35}{100} \implies$ Ratio of 35% is **35 to 100**

Problem 39: 2.5%

Solution:

$2.5\% = \dfrac{2.5}{100} \implies$ Ratio of 2.5% is **2.5 to 100**

In the following exercises, convert each percent to a fraction and simplify all fractions

Problem 40: 4%

Solution:

$$4\% = \frac{4}{100} = \frac{4}{25 \times 4} = \frac{4}{25 \times \cancel{4}} = \frac{1}{25} \implies \boxed{4\% = \frac{1}{25}}$$

Problem 41: 120%

Solution:

$$120\% = \frac{120}{100} = \frac{20 \times 6}{20 \times 5} = \frac{\cancel{20} \times 6}{\cancel{20} \times 5} = \frac{6}{5} \implies \boxed{120\% = \frac{6}{5}}$$

Problem 42: 12.5%

Solution:

$$12.5\% = \frac{12.5}{100} = \frac{12.5 \times 10}{100 \times 10} = \frac{125}{1000} = \frac{125}{8 \times \cancel{125}} = \frac{1}{8} \implies \boxed{12.5\% = \frac{1}{8}}$$

In the following exercises, convert each percent to a decimal

Problem 43: 250%

Solution:

$$250\% = \frac{250}{100} = 2.5 \implies \boxed{250\% = 2.5}$$

Problem 44: 9%

Solution:

$$9\% = \frac{9}{100} = 0.09 \implies \boxed{9\% = 0.09}$$

Problem 45: 15%

Solution:

$$15\% = \frac{15}{100} = 0.15 \implies \boxed{15\% = 0.15}$$

Problem 46: 39.3%

Solution:

$$39.3\% = \frac{39.3}{100} = 0.393 \implies \boxed{39.3\% = 0.393}$$

Problem 47: 7.5%

Solution:

$$7.5\% = \frac{7.5}{100} = 0.075 \implies \boxed{7.5\% = 0.075}$$

Problem 48: 100%

Solution:

$$100\% = \frac{100}{100} = 1 \implies \boxed{100\% = 1}$$

In the following exercises, convert each decimal to a percent

Problem 49: 0.01

Solution:

$0.01 = \frac{1}{100} = 1\% \implies \boxed{0.01 = 1\%}$

Problem 50: 0.18

Solution:

$0.18 = \frac{18}{100} = 18\% \implies \boxed{0.18 = 18\%}$

Problem 51: 1.35

Solution:

$1.35 = \frac{135}{100} = 135\% \implies \boxed{1.35 = 135\%}$

In the following exercises, convert each fraction to a percent

Problem 52: $\mathbf{\frac{1}{4}}$

Solution:

For percentage, we need 100 in the denominator, so we need to convert 4 as 100 by multiplying 25

$\frac{1}{4} = \frac{1 \times 25}{4 \times 25} = \frac{25}{100} = 25\% \implies \boxed{\frac{1}{4} = 25\%}$

Problem 53: $\mathbf{\frac{1}{5}}$

Solution:

For percentage, we need 100 in the denominator, so we need to convert 5 as 100 by multiplying 20

$\frac{1}{5} = \frac{1 \times 20}{5 \times 20} = \frac{20}{100} = 20\% \implies \boxed{\frac{1}{5} = 20\%}$

Problem 54: $\mathbf{5\frac{1}{4}}$

Solution:

For percentage, we need 100 in the denominator, so we need to convert 4 as 100 by multiplying 25

$$5\frac{1}{4} = \frac{4 \times 5 + 1}{4} = \frac{21}{4} = \frac{21 \times 25}{4 \times 25} = \frac{525}{100} = 525\% \implies \boxed{5\frac{1}{5} = 525\%}$$

Chapter 2

EXPONENTS AND RADICALS

In Mathematics, we use the notation a^n, to represent a is multiplied by itself n times. We read this as n^{th} power of a, here a is the base and n is the exponent.

2.1 Integer Exponents

- **Product rule of Exponents:** $a^m \times a^n = a^{m+n}$
- **Quotient rule of Exponents:** $\dfrac{a^m}{a^n} = a^{m-n}$
- **Power rule of Exponents:** $(a^m)^n = a^{mn}$
- **Zero Exponent:** $a^0 = 1$
- **Rule of Negative Exponent:** $a^{-n} = \dfrac{1}{a^n}$
- **Exponent rule of Power of Product:** $(ab)^n = a^n \times b^n$
- **Exponent rule of Power of Quotient:** $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

Exercise - 2.1

For the following exercises, simplify the given expression. Write answers with positive exponents.

Problem 1: $\mathbf{9^2}$

Solution:

$\mathbf{9^2}$, It is itself in the simpler form and it has the positive exponent

Problem 2: $\mathbf{15^{-2}}$

Solution:

$15^{-2} = \dfrac{\mathbf{1}}{\mathbf{15^2}}$ [Rule of Negative Exponents]

Problem 3: $\mathbf{3^2 \cdot 3^3}$

Solution:

$3^2 \cdot 3^3 = 3^{2+3} = \mathbf{3^5}$ [Product rule of Exponents]

Problem 4: $\mathbf{4^4 \div 4}$

Solution:

$4^4 \div 4 = 4^4 \times \dfrac{1}{4} = \dfrac{4^4}{4^1} = 4^{4-1} = \mathbf{4^3}$ [Quotient rule of Exponents]

Problem 5: $\mathbf{(2^2)^{-2}}$

Solution:

$(2^2)^{-2} = 2^{2\times(-2)} = 2^{-4} = \dfrac{\mathbf{1}}{\mathbf{2^4}}$ [Power rule of Exponents]

Problem 6: $\mathbf{(5-8)^0}$

Solution:

$(5-8)^0 = (-3)^0 = \mathbf{1}$ [Zero Exponent]

Problem 7: $\mathbf{11^3 \div 11^4}$

Solution:

$11^3 \div 11^4 = 11^3 \times \dfrac{1}{11^4} = \dfrac{11^3}{11^4} = \dfrac{1}{11^{4-3}} = \dfrac{\mathbf{1}}{\mathbf{11}}$ [Quotient rule of Exponents]

Problem 8: $\mathbf{6^5 \cdot 6^{-7}}$

Solution:

$6^5 \cdot 6^{-7} = 6^{5-7} = 6^{-2} = \dfrac{\mathbf{1}}{\mathbf{6^2}}$ [Product rule and Quotient rule of Exponents]

For the following exercises, write each expression with a single base. Do not simplify further. Write answers with positive exponents

Problem 9: $\mathbf{4^2 \cdot 4^3 \div 4^{-4}}$

Solution:

$$\begin{aligned}
4^2 \cdot 4^3 \div 4^{-4} &= \frac{4^2 \times 4^3}{4^{-4}} \\
&= 4^2 \times 4^3 \times 4^4 && \text{[Division Rule of Exponent]} \\
&= 4^{2+3+4} && \text{[Product Rule of Exponent]} \\
\mathbf{4^2 \cdot 4^3 \div 4^{-4}} &= \mathbf{4^9}
\end{aligned}$$

Problem 10: $\mathbf{\dfrac{6^{12}}{6^9}}$

Solution:

$$\frac{6^{12}}{6^9} = 6^{12-9} = \mathbf{6^3} \text{ [Quotient rule of Exponents]}$$

Problem 11: $\mathbf{(12^3 \cdot 12)^{10}}$

Solution:

$$\begin{aligned}
(12^3 \cdot 12)^{10} &= (12^{3+1})^{10} && \text{[Product Rule of Exponent]} \\
&= (12^4)^{10} \\
&= 12^{4\times 10} && \text{[Power Rule of Exponent]} \\
\mathbf{(12^3 \cdot 12)^{10}} &= \mathbf{12^{40}}
\end{aligned}$$

Problem 12: $\mathbf{10^6 \div (10^{10})^{-2}}$

Solution:

$$\begin{aligned}
10^6 \div (10^{10})^{-2} &= 10^6 \div 10^{10\times(-2)} && \text{[Power Rule of Exponent]} \\
&= 10^6 \div 10^{-20} \\
&= 10^6 \times \frac{1}{10^{-20}} \\
&= 10^6 \times 10^{20} && \text{[Rule Negative of Exponent]} \\
&= 10^{20+6} && \text{[Product Rule of Exponent]} \\
\mathbf{10^6 \div (10^{10})^{-2}} &= \mathbf{10^{26}}
\end{aligned}$$

Problem 13: $\mathbf{7^6 \cdot 7^{-3}}$

Solution:

$7^6 \cdot 7^{-3} = 7^{6-3} = \mathbf{7^3}$ [Product Rule of Exponent]

Problem 14: $\mathbf{(3^3 \div 3^4)^5}$

Solution:

$$\begin{aligned}
(3^3 \div 3^4)^5 &= (3^3 \times \frac{1}{3^4})^5 \\
&= \left(\frac{1}{3^{-3} \times 3^4}\right)^5 && \text{[Rule of Negative Exponents]} \\
&= \left(\frac{1}{3^{4-3}}\right)^5 && \text{[Product Rule of Exponent]} \\
&= \left(\frac{1}{3}\right)^5 \\
\mathbf{(3^3 \div 3^4)^5} &= \mathbf{\frac{1}{3^5}}
\end{aligned}$$

For the following exercises, simplify the given expression. Write answers with positive exponents.

Problem 15: $\boldsymbol{\frac{a^3a^2}{a}}$

Solution:

$$\begin{aligned}
\frac{a^3a^2}{a} &= \frac{a^{3+2}}{a} && \text{[Product rule of Exponents]} \\
&= \frac{a^5}{a} \\
&= a^{5-1} && \text{[Division rule of Exponents]} \\
\boldsymbol{\frac{a^3a^2}{a}} &= \boldsymbol{a^4}
\end{aligned}$$

Problem 16: $\boldsymbol{\dfrac{mn^2}{m^{-2}}}$

Solution:

$$\begin{aligned}
\frac{mn^2}{m^{-2}} &= m^1m^2n^2 \quad \text{[Rule of Negative Exponent]}\\
&= m^{(1+2)}n^2 \quad \text{[Product rule of Exponents]}\\
\boldsymbol{\frac{mn^2}{m^{-2}}} &\boldsymbol{= m^3n^2}
\end{aligned}$$

Problem 17: $\boldsymbol{(b^3c^4)^2}$

Solution:

$$\begin{aligned}
(b^3c^4)^2 &= (b^3)^2(c^4)^2 \quad \text{[Exponent rule of power of product]}\\
&= b^{(3\times2)}c^{(4\times2)} \quad \text{[Power rule of Exponent]}\\
\boldsymbol{(b^3c^4)^2} &\boldsymbol{= b^6c^8}
\end{aligned}$$

Problem 18: $\boldsymbol{\left(\dfrac{x^{-3}}{y^2}\right)^{-5}}$

Solution:

$$\begin{aligned}
\left(\frac{x^{-3}}{y^2}\right)^{-5} &= \frac{(x^{-3})^{-5}}{(y^2)^{-5}} \quad \text{[Exponent rule of Power of quotient]}\\
&= \frac{x^{(-3)\times(-5)}}{y^{2\times(-5)}} \quad \text{[Power rule of Exponent]}\\
\boldsymbol{\left(\frac{x^{-3}}{y^2}\right)^{-5}} &\boldsymbol{= x^{15}y^{10}}
\end{aligned}$$

Problem 19: $\boldsymbol{(ab^2) \div d^{-3}}$

Solution:

$$\begin{aligned}
(ab^2) \div d^{-3} &= \frac{ab^2}{d^{-3}}\\
&= (ab^2)(d^3) \quad \text{[Rule of Negative Exponent]}\\
\boldsymbol{(ab^2) \div d^{-3}} &\boldsymbol{= ab^2d^3}
\end{aligned}$$

Problem 20: $\left(w^0x^5\right)^{-1}$

Solution:

$$\begin{aligned}
\left(w^0x^5\right)^{-1} &= \left(1\times x^5\right)^{-1} && \text{[Property of Zero Exponent]}\\
&= \left(x^5\right)^{-1}\\
&= x^{-5} && \text{[Power rule of Exponents]}\\
\boldsymbol{\left(w^0x^5\right)^{-1}} &\boldsymbol{= \frac{1}{x^5}}
\end{aligned}$$

Problem 21: $\dfrac{m^4}{n^0}$

Solution:

$$\begin{aligned}
\frac{m^4}{n^0} &= \frac{m^4}{1} && \text{[Property of Zero Exponent]}\\
\boldsymbol{\frac{m^4}{n^0}} &\boldsymbol{= m^4}
\end{aligned}$$

Problem 22: $y^{-4}\left(y^2\right)^2$

Solution:

$$\begin{aligned}
y^{-4}\left(y^2\right)^2 &= y^{-4}\left(y^{2\times 2}\right) && \text{[Power rule of Exponents]}\\
&= y^{-4}\left(y^4\right)\\
&= y^{-4+4} && \text{[Product rule of Exponents]}\\
&= y^0\\
\boldsymbol{y^{-4}\left(y^2\right)^2} &\boldsymbol{= 1} && \text{[Property of zero Exponent]}
\end{aligned}$$

Problem 23: $\dfrac{p^{-4}q^2}{p^2q^{-3}}$

Solution:

$$\begin{aligned}
\frac{p^{-4}q^2}{p^2q^{-3}} &= \left(\frac{p^{-4}}{p^2}\right)\cdot\left(\frac{q^2}{q^{-3}}\right) && \text{[Quotient rule of Exponents]}\\
&= \left(p^{-4-2}\right)\left(q^{2+3}\right)\\
&= p^{-6}q^5\\
\boldsymbol{\frac{p^{-4}q^2}{p^2q^{-3}}} &\boldsymbol{= \frac{q^5}{p^6}} && \text{[Rule of negative exponents]}
\end{aligned}$$

Problem 24: $\boldsymbol{(l \times w)^2}$

Solution:

$$
\begin{aligned}
(l \times w)^2 &= (l)^2 \times (w)^2 \quad \text{[Exponent rule of power of product]}\\
\boldsymbol{(l \times w)^2} &= \boldsymbol{l^2 w^2}
\end{aligned}
$$

Problem 25: $\boldsymbol{\left(y^7\right)^3 \div x^{14}}$

Solution:

$$
\begin{aligned}
\left(y^7\right)^3 \div x^{14} &= \left(y^7\right)^3 \div x^{14}\\
&= \left(y^{7\times 3}\right) \div x^{14} \quad \text{[Power rule of Exponent]}\\
\boldsymbol{\left(y^7\right)^3 \div x^{14}} &= \boldsymbol{\frac{y^{21}}{x^{14}}}
\end{aligned}
$$

Problem 26: $\boldsymbol{\left(\frac{a}{2^3}\right)^2}$

Solution:

$$
\begin{aligned}
\left(\frac{a}{2^3}\right)^2 &= \frac{a^2}{\left(2^3\right)^2} \quad \text{[Exponent rule of power of quotient]}\\
&= \frac{a^2}{2^{3\times 2}} \quad [\because (a^m)^n = a^{mn}]\\
\boldsymbol{\left(\frac{a}{2^3}\right)^2} &= \boldsymbol{\frac{a^2}{2^6}}
\end{aligned}
$$

Problem 27: $\boldsymbol{5^2 m \div 5^0 m}$

Solution:

$$
\begin{aligned}
5^2 m \div 5^0 m &= 5^2 m \div (1) m \quad \text{[Property of zero Exponent]}\\
&= \frac{5^2 \cdot m}{m}\\
&= 5^2 \cdot m^{1-1} \quad \text{[Quotient rule of Exponent]}\\
&= 5^2 \cdot m^0\\
&= 5^2 \cdot 1 \quad \text{[Property of zero Exponent]}\\
\boldsymbol{5^2 m \div 5^0 m} &= \boldsymbol{5^2}
\end{aligned}
$$

Problem 28: $\boldsymbol{\frac{16x^2}{y^{-1}}}$

Solution:

$$
\boldsymbol{\frac{16x^2}{y^{-1}}} = 16x^2y^1 = \boldsymbol{16x^2y} \quad \text{[Rule of negative exponent]}
$$

Problem 29: $\dfrac{\mathbf{2^3}}{\mathbf{(3a)^{-2}}}$

Solution:

$$\begin{aligned}
\frac{2^3}{(3a)^{-2}} &= \frac{2^3}{(3)^{-2}(a)^{-2}} && \text{[Exponent rule of power of product]}\\
\boldsymbol{\frac{2^3}{(3a)^{-2}}} &\boldsymbol{= 2^3 \cdot 3^2 \cdot a^2} && \text{[Rule of Negative exponent]}
\end{aligned}$$

Problem 30: $\boldsymbol{(ma^6)^2 \dfrac{1}{m^3a^2}}$

Solution:

$$\begin{aligned}
(ma^6)^2 \frac{1}{m^3a^2} &= m^2 \left(a^6\right)^2 \frac{1}{m^3a^2} && \text{[Exponent rule of Power of Product]}\\
&= m^2 \left(a^{6\times 2}\right) \frac{1}{m^3a^2} && \text{[Power rule of Exponents]}\\
&= m^2a^{12}\frac{1}{m^3a^2}\\
&= m^{2-3} \cdot a^{12-2} && \text{[Quotient rule of Exponent]}\\
&= m^{-1} \cdot a^{10}\\
\boldsymbol{(ma^6)^2 \frac{1}{m^3a^2}} &\boldsymbol{= \frac{a^{10}}{m}} && \text{[Rule of Negative Exponents]}
\end{aligned}$$

Problem 31: $\boldsymbol{(b^{-3}c)^3}$

Solution:

$$\begin{aligned}
(b^{-3}c)^3 &= \left(b^{-3}\right)^3 (c)^3 && \text{[Exponent rule of power of product]}\\
&= b^{-3\times 3} \cdot c^3 && \text{[Power rule of Exponent]}\\
&= b^{-9} \cdot c^3\\
\boldsymbol{(b^{-3}c)^3} &\boldsymbol{= \frac{c^3}{b^9}} && \text{[Rule of Negative Exponent]}
\end{aligned}$$

Problem 32: $\boldsymbol{(x^2y^{13} \div y^0)^2}$

Solution:

$$\begin{aligned}
(x^2y^{13} \div y^0)^2 &= (x^2y^{13} \div 1)^2 && \text{[Property of Zero Exponent]}\\
&= (x^2y^{13})^2\\
&= (x^2)^2\,(y^{13})^2 && \text{[Exponent rule of power of Product]}\\
&= x^{2\times 2} \cdot y^{13\times 2}\\
\boldsymbol{(x^2y^{13} \div y^0)^2} &\boldsymbol{= x^4y^{26}}
\end{aligned}$$

Problem 33: $\boldsymbol{(9z^3)^{-2}\,y}$

Solution:

$$\begin{aligned}
(9z^3)^{-2}\,y &= \frac{y}{(9z^3)^2} && \text{[Rule of Negative Exponent]}\\
&= \frac{y}{9^2 \cdot (z^3)^2} && \text{[Exponent rule of power of Product]}\\
&= \frac{y}{9^2 \cdot z^{(3\times 2)}} && \text{[Power rule of Exponent]}\\
\boldsymbol{(9z^3)^{-2}\,y} &\boldsymbol{= \frac{y}{9^2 \cdot z^6}}
\end{aligned}$$

For the following exercises, simplify the given expression. Write answers with positive exponents.

Problem 34: $\boldsymbol{\left(\frac{3^2}{a^3}\right)^{-2}\left(\frac{a^4}{2^2}\right)^2}$

Solution:

$$\begin{aligned}
\left(\frac{3^2}{a^3}\right)^{-2}\left(\frac{a^4}{2^2}\right)^2 &= \frac{(3^2)^{-2}}{(a^3)^{-2}} \cdot \frac{(a^4)^2}{(2^2)^2} && \text{[Exponent rule of Power of Quotient]}\\
&= \frac{3^{2\times(-2)}}{a^{3\times(-2)}} \cdot \frac{a^{4\times 2}}{2^{2\times 2}} && \text{[Power rule of Exponents]}\\
&= \frac{3^{-4} \cdot a^8}{a^{-6} \cdot 2^4}\\
&= \frac{a^{8+6}}{2^4 \cdot 3^4} && \text{[Rule of Negative Exponents]}\\
&= \frac{a^{14}}{(2 \times 3)^4} && \text{[Exponent rule of power of product]}\\
\boldsymbol{\left(\frac{3^2}{a^3}\right)^{-2}\left(\frac{a^4}{2^2}\right)^2} &\boldsymbol{= \frac{a^{14}}{6^4}}
\end{aligned}$$

Problem 35: $\boldsymbol{(6^2-24)^2 \div \left(\frac{x}{y}\right)^{-5}}$

Solution:

$$
\begin{aligned}
(6^2-24)^2 \div \left(\frac{x}{y}\right)^{-5} &= (36-24)^2 \div \left(\frac{x}{y}\right)^{-5} \\
&= 12^2 \div \left(\frac{x}{y}\right)^{-5} \\
&= 12^2 \div \left(\frac{x^{-5}}{y^{-5}}\right) && \text{[Exponent rule of Power of Quotient]} \\
&= 12^2 \div \left(\frac{y^5}{x^5}\right) && \text{[Rule of Negative Exponent]} \\
&= 12^2 \times \left(\frac{x^5}{y^5}\right) \\
\boldsymbol{(6^2-24)^2 \div \left(\frac{x}{y}\right)^{-5}} &= \boldsymbol{\frac{12^2 \cdot x^5}{y^5}}
\end{aligned}
$$

Problem 36: $\boldsymbol{\frac{m^2n^3}{a^2c^{-3}} \cdot \frac{a^{-7}n^{-2}}{m^2c^4}}$

Solution:

$$
\begin{aligned}
\frac{m^2n^3}{a^2c^{-3}} \cdot \frac{a^{-7}n^{-2}}{m^2c^4} &= \frac{m^2}{m^2} \cdot \frac{a^{-7}}{a^2} \cdot \frac{n^3n^{-2}}{c^4c^{-3}} && \text{[Separating the variables]} \\
&= m^{2-2} \cdot \frac{1}{a^2 \cdot a^7} \cdot \frac{n^{3-2}}{c^{4-3}} && \text{[Quotient, Negative Exponent and Product rules]} \\
&= m^0 \cdot \frac{1}{a^{2+7}} \cdot \frac{n}{c} && \text{[Product rule of Exponent]} \\
&= 1 \cdot \frac{1}{a^9} \cdot \frac{n}{c} && \text{[Zero Exponent rule]} \\
\boldsymbol{\frac{m^2n^3}{a^2c^{-3}} \cdot \frac{a^{-7}n^{-2}}{m^2c^4}} &= \boldsymbol{\frac{n}{ca^9}}
\end{aligned}
$$

Problem 37: $\boldsymbol{\left(\frac{x^6y^3}{x^3y^{-3}} \cdot \frac{y^{-7}}{x^{-3}}\right)^{10}}$

Solution:

$$\left(\frac{x^6y^3}{x^3y^{-3}}\cdot\frac{y^{-7}}{x^{-3}}\right)^{10} = \left(\frac{x^6}{x^3x^{-3}}\cdot\frac{y^3y^{-7}}{y^{-3}}\right)^{10} \quad \text{[Separating the variables]}$$

$$= \left(\frac{x^6}{x^{3-3}}\cdot\frac{y^{-7+3}}{y^{-3}}\right)^{10} \quad \text{[Product rule of Exponent]}$$

$$= \left(\frac{x^6}{x^0}\cdot\frac{y^{-4}}{y^{-3}}\right)^{10}$$

$$= \left(\frac{x^6}{1}\cdot\frac{1}{y^{-3}y^4}\right)^{10} \quad \text{[Zero Exponent Rule, Negative Exponent Rule]}$$

$$= \left(\frac{x^6}{y^{4-3}}\right)^{10}$$

$$= \left(\frac{x^6}{y}\right)^{10}$$

$$= \frac{(x^6)^{10}}{y^{10}} \quad \text{[Quotient rule of power of Exponent]}$$

$$= \frac{x^{(6\times10)}}{y^{10}} \quad \text{[Power rule of Exponent]}$$

$$\boldsymbol{\left(\frac{x^6y^3}{x^3y^{-3}}\cdot\frac{y^{-7}}{x^{-3}}\right)^{10} = \frac{x^{60}}{y^{10}}}$$

Problem 38: $\boldsymbol{\left(\frac{(ab^2c)^{-3}}{b^{-3}}\right)^2}$

Solution:

$$\left(\frac{(ab^2c)^{-3}}{b^{-3}}\right)^2 = \left(\frac{a^{-3}\,(b^2)^{-3}\,c^{-3}}{b^{-3}}\right)^2 \quad \text{[Power rule of Exponent]}$$

$$= (a^{-3})^2\,(c^{-3})^2\left(\frac{b^{2\times(-3)}}{b^{-3}}\right)^2 \quad \text{[Power rule of Exponent]}$$

$$= (a^{-3\times2})\,(c^{-3\times2})\left(\frac{b^{-6}}{b^{-3}}\right)^2 \quad \text{[Power rule of Exponent]}$$

$$= (a^{-6})\,(c^{-6})\,(b^{-6+3})^2 \quad \text{[Quotient rule of Exponent]}$$

$$= (a^{-6})\,(c^{-6})\,(b^{-3\times2}) \quad \text{[Power rule of Exponent]}$$

$$\boldsymbol{\left(\frac{(ab^2c)^{-3}}{b^{-3}}\right)^2 = \frac{1}{a^6b^6c^6}}$$

2.2 Rational Exponents

In a rational exponent, numerator indicates a power, denominator indicates a root.

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}}$$

All the rules that are applicable for integer exponents can also be applicable for rational exponents.

- **Product rule of Exponents:** $a^m \times a^n = a^{m+n}$
- **Quotient rule of Exponents:** $\dfrac{a^m}{a^n} = a^{m-n}$
- **Power rule of Exponents:** $(a^m)^n = a^{mn}$
- **Zero Exponent:** $a^0 = 1$
- **Rule of Negative Exponent:** $a^{-n} = \dfrac{1}{a^n}$
- **Exponent rule of Power of Product:** $(ab)^n = a^n \times b^n$
- **Exponent rule of Power of Quotient:** $\left(\dfrac{a}{b}\right)^m = \frac{a^m}{b^m}$

Exercise - 2.2

In the following exercises, simplify. Assume all variables are positive

Problem 1:

a. $\boldsymbol{c^{\frac{1}{4}}c^{\frac{5}{8}}}$

Solution:

$$\begin{aligned}
c^{\frac{1}{4}}c^{\frac{5}{8}} &= c^{\frac{1}{4}+\frac{5}{8}} && \text{[Product rule of Exponents]}\\
&= c^{\frac{2}{8}+\frac{5}{8}} && \text{[L.C.M(4,8)=8]}\\
&= c^{\frac{2+5}{8}}\\
\boldsymbol{c^{\frac{1}{4}}c^{\frac{5}{8}}} &= \boldsymbol{c^{\frac{7}{8}}}
\end{aligned}$$

b. $\mathbf{6^{\frac{5}{2}}6^{\frac{1}{2}}}$

Solution:

$$\begin{aligned}
6^{\frac{5}{2}}6^{\frac{1}{2}} &= 6^{\frac{5}{2}+\frac{1}{2}} \quad \text{[Product rule of Exponents]}\\
&= 6^{\frac{5+1}{2}}\\
&= 6^{\frac{6}{2}}\\
\mathbf{6^{\frac{5}{2}}6^{\frac{1}{2}}} &= \mathbf{6^{3}}
\end{aligned}$$

c. $\boldsymbol{y^{\frac{1}{2}}y^{\frac{3}{4}}}$

Solution:

$$\begin{aligned}
y^{\frac{1}{2}}y^{\frac{3}{4}} &= y^{\frac{1}{2}+\frac{3}{4}} \quad \text{[Product rule of Exponents]}\\
&= y^{\frac{2}{4}+\frac{3}{4}} \quad \text{[L.C.M(2,4)=4]}\\
&= y^{\frac{2+3}{4}}\\
\boldsymbol{y^{\frac{1}{2}}y^{\frac{3}{4}}} &= \boldsymbol{y^{\frac{5}{4}}}
\end{aligned}$$

d. $\boldsymbol{q^{\frac{2}{3}}q^{\frac{5}{6}}}$

Solution:

$$\begin{aligned}
q^{\frac{2}{3}}q^{\frac{5}{6}} &= q^{\frac{2}{3}+\frac{5}{6}} \quad \text{[Product rule of Exponents]}\\
&= q^{\frac{4}{6}+\frac{5}{6}} \quad \text{[LCM(3,6)=6]}\\
&= q^{\frac{4+5}{6}}\\
&= q^{\frac{9}{6}}\\
\boldsymbol{q^{\frac{2}{3}}q^{\frac{5}{6}}} &= \boldsymbol{q^{\frac{3}{2}}}
\end{aligned}$$

Problem 2:

a. $\boldsymbol{(p^{12})^{\frac{3}{4}}}$

Solution:

$$\begin{aligned}
(p^{12})^{\frac{3}{4}} &= p^{12\times\frac{3}{4}} \quad \text{[Power rule of Exponents]}\\
&= p^{3\times 3}\\
\boldsymbol{(p^{12})^{\frac{3}{4}}} &= \boldsymbol{p^{9}}
\end{aligned}$$

b. $\boldsymbol{(b^{15})^{\frac{3}{5}}}$

Solution:

$$\begin{aligned} (b^{15})^{\frac{3}{5}} &= b^{15\times\frac{3}{5}} \quad \text{[Power rule of Exponents]} \\ &= b^{3\times 3} \\ \boldsymbol{(b^{15})^{\frac{3}{5}}} &\boldsymbol{= b^{9}} \end{aligned}$$

c. $\boldsymbol{(x^{12})^{\frac{2}{3}}}$

Solution:

$$\begin{aligned} (x^{12})^{\frac{2}{3}} &= x^{12\times\frac{2}{3}} \quad \text{[Power rule of Exponents]} \\ &= x^{4\times 2} \\ \boldsymbol{(x^{12})^{\frac{2}{3}}} &\boldsymbol{= x^{8}} \end{aligned}$$

d. $\boldsymbol{(h^{6})^{\frac{4}{3}}}$

Solution:

$$\begin{aligned} (h^{6})^{\frac{4}{3}} &= h^{6\times\frac{4}{3}} \quad \text{[Power rule of Exponents]} \\ &= h^{2\times 4} \\ \boldsymbol{(h^{6})^{\frac{4}{3}}} &\boldsymbol{= h^{8}} \end{aligned}$$

Problem 3:

a. $\boldsymbol{\frac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}}$

Solution:

$$\begin{aligned} \frac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}} &= r^{\frac{4}{5}-\frac{9}{5}} \quad \text{[Quotient rule of Exponents]} \\ &= r^{\frac{4-9}{5}} \\ &= r^{\frac{-5}{5}} \\ &= r^{-1} \\ \boldsymbol{\frac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}} &\boldsymbol{= \frac{1}{r}} \quad \text{[Rule of Negative Exponents]} \end{aligned}$$

b. $\dfrac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

Solution:

$$\begin{aligned}\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}} &= w^{\frac{2}{7}-\frac{9}{7}} && \text{[Quotient rule of Exponents]}\\ &= w^{\frac{2-9}{7}}\\ &= w^{\frac{-7}{7}}\\ &= w^{-1}\\ \boldsymbol{\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}} &\boldsymbol{= \frac{1}{w}} && \text{[Rule of Negative Exponents]}\end{aligned}$$

c. $\dfrac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$

Solution:

$$\begin{aligned}\frac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}} &= m^{\frac{5}{8}-\frac{13}{8}} && \text{[Quotient rule of Exponents]}\\ &= m^{\frac{5-13}{8}}\\ &= m^{\frac{-8}{8}}\\ &= m^{-1}\\ \boldsymbol{\frac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}} &\boldsymbol{= \frac{1}{m}} && \text{[Rule of Negative Exponents]}\end{aligned}$$

d. $\dfrac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$

Solution:

$$\begin{aligned}\frac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}} &= n^{\frac{3}{5}-\frac{8}{5}} && \text{[Quotient rule of Exponents]}\\ &= n^{\frac{3-8}{5}}\\ &= n^{\frac{-5}{5}}\\ &= n^{-1}\\ \boldsymbol{\frac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}} &\boldsymbol{= \frac{1}{n}} && \text{[Rule of Negative Exponents]}\end{aligned}$$

Problem 4:

a. $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$

Solution:

$$\begin{aligned}
\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}} &= (27)^{\frac{4}{3}}\left(q^{\frac{3}{2}}\right)^{\frac{4}{3}} && \text{[Power rule of Exponents]}\\
&= \left(3^{\cancel{3}}\right)^{\frac{4}{\cancel{3}}}\left(q^{\frac{\cancel{3}}{\cancel{2}}\times\frac{\cancel{4}^{2}}{\cancel{3}}}\right)\\
\boldsymbol{\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}} &\boldsymbol{= 3^4q^2}
\end{aligned}$$

b. $\left(m^{\frac{4}{3}}m^{\frac{1}{2}}\right)^{\frac{3}{4}}$

Solution:

$$\begin{aligned}
\left(m^{\frac{4}{3}}m^{\frac{1}{2}}\right)^{\frac{3}{4}} &= \left(m^{\frac{4}{3}+\frac{1}{2}}\right)^{\frac{3}{4}} && \text{[Product rule of Exponents]}\\
&= \left(m^{\frac{8}{6}+\frac{3}{6}}\right)^{\frac{3}{4}} && \text{[LCM(2,3)=6]}\\
&= \left(m^{\frac{8+3}{6}}\right)^{\frac{3}{4}}\\
&= \left(m^{\frac{11}{6}}\right)^{\frac{3}{4}}\\
&= \left(m^{\frac{11}{2\cancel{6}}\times\frac{\cancel{3}}{4}}\right) && \text{[Power rule of Exponents]}\\
\boldsymbol{\left(m^{\frac{4}{3}}m^{\frac{1}{2}}\right)^{\frac{3}{4}}} &\boldsymbol{= m^{\frac{11}{8}}}
\end{aligned}$$

c. $\left(4p^{\frac{1}{3}}p^{\frac{1}{2}}\right)^{\frac{3}{2}}$

Solution:

$$\begin{aligned}
\left(4p^{\frac{1}{3}}p^{\frac{1}{2}}\right)^{\frac{3}{2}} &= \left(4p^{\frac{1}{3}+\frac{1}{2}}\right)^{\frac{3}{2}} && \text{[Product rule of Exponents]}\\
&= \left(4p^{\frac{2}{6}+\frac{3}{6}}\right)^{\frac{3}{2}} && \text{[LCM(2,3)=6]}\\
&= \left(2^2p^{\frac{2+3}{6}}\right)^{\frac{3}{2}}\\
&= \left(2^2\right)^{\frac{3}{2}}\left(p^{\frac{5}{6}}\right)^{\frac{3}{2}}\\
&= \left(2^{\cancel{2}\times\frac{3}{\cancel{2}}}\right)\left(p^{\frac{5}{2\cancel{6}}\times\frac{\cancel{3}}{2}}\right) && \text{[Power rule of Exponents]}\\
\boldsymbol{\left(4p^{\frac{1}{3}}p^{\frac{1}{2}}\right)^{\frac{3}{2}}} &\boldsymbol{= 2^3p^{\frac{5}{4}}}
\end{aligned}$$

d. $\left(9x^{\frac{2}{5}}y^{\frac{3}{5}}\right)^{\frac{5}{2}}$

Solution:

$$\begin{aligned}
\left(9x^{\frac{2}{5}}y^{\frac{3}{5}}\right)^{\frac{5}{2}} &= (9)^{\frac{5}{2}}\left(x^{\frac{2}{5}}\right)^{\frac{5}{2}}\left(y^{\frac{3}{5}}\right)^{\frac{5}{2}} && \text{[Exponent rule of power of product]}\\
&= \left(3^{\cancel{2}\times\frac{5}{\cancel{2}}}\right)\left(x^{\frac{\cancel{2}}{\cancel{5}}\times\frac{\cancel{5}}{\cancel{2}}}\right)\left(y^{\frac{3}{\cancel{5}}\times\frac{\cancel{5}}{2}}\right) && \text{[Power rule of Exponents]}\\
\boldsymbol{\left(9x^{\frac{2}{5}}y^{\frac{3}{5}}\right)^{\frac{5}{2}}} &\boldsymbol{= 3^5xy^{\frac{3}{2}}}
\end{aligned}$$

Problem 5:

a. $\frac{r^{\frac{5}{2}}\cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$

Solution:

$$\begin{aligned}
\frac{r^{\frac{5}{2}}\cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}} &= \frac{r^{\frac{5}{2}-\frac{1}{2}}}{r^{-\frac{3}{2}}} && \text{[Product rule of Exponents]}\\
&= \frac{r^{\frac{5-1}{2}}}{r^{-\frac{3}{2}}}\\
&= \frac{r^{\frac{4}{2}}}{r^{-\frac{3}{2}}}\\
&= r^{\frac{4}{2}+\frac{3}{2}} && \text{[Quotient rule of Exponents]}\\
&= r^{\frac{4+3}{2}}\\
\boldsymbol{\frac{r^{\frac{5}{2}}\cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}} &\boldsymbol{= r^{\frac{7}{2}}}
\end{aligned}$$

b. $\frac{a^{\frac{3}{4}}\cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$

Solution:

$$\begin{aligned}
\frac{a^{\frac{3}{4}}\cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}} &= \frac{a^{\frac{3}{4}-\frac{1}{4}}}{a^{-\frac{10}{4}}} && \text{[Product rule of Exponents]}\\
&= \frac{a^{\frac{3-1}{4}}}{a^{-\frac{10}{4}}} = \frac{a^{\frac{2}{4}}}{a^{-\frac{10}{4}}}\\
&= a^{\frac{2}{4}+\frac{10}{4}} && \text{[Quotient rule of Exponents]}\\
&= a^{\frac{2+10}{4}} = a^{\frac{\cancel{12}3}{\cancel{4}}}\\
\boldsymbol{\frac{a^{\frac{3}{4}}\cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}} &\boldsymbol{= a^3}
\end{aligned}$$

c. $\dfrac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$

Solution:

$$\begin{aligned}
\frac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}} &= \frac{c^{\frac{5}{3}-\frac{1}{3}}}{c^{-\frac{2}{3}}} \quad \text{[Product rule of Exponents]} \\
&= \frac{c^{\frac{5-1}{3}}}{c^{-\frac{2}{3}}} \\
&= \frac{c^{\frac{4}{3}}}{c^{-\frac{2}{3}}} \\
&= c^{\frac{4}{3}+\frac{2}{3}} \quad \text{[Quotient rule of Exponents]} \\
&= c^{\frac{4+2}{3}} \\
&= c^{\frac{\cancel{6}2}{\cancel{3}}} \\
\frac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}} &= \boldsymbol{c^2}
\end{aligned}$$

d. $\dfrac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$

Solution:

$$\begin{aligned}
\frac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}} &= \frac{m^{\frac{7}{4}-\frac{5}{4}}}{m^{-\frac{2}{4}}} \quad \text{[Product rule of Exponents]} \\
&= \frac{m^{\frac{7-5}{4}}}{m^{-\frac{2}{4}}} \\
&= \frac{m^{\frac{2}{4}}}{m^{-\frac{2}{4}}} \\
&= m^{\frac{2}{4}+\frac{2}{4}} \quad \text{[Quotient rule of Exponents]} \\
&= m^{\frac{2+2}{4}} \\
&= m^{\frac{\cancel{4}}{\cancel{4}}} \\
\frac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}} &= \boldsymbol{m}
\end{aligned}$$

Problem 6:

a. $\left(\frac{36s^{\frac{1}{5}}t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$

Solution:

$$\begin{aligned}
\left(\frac{36s^{\frac{1}{5}}t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}} &= (36)^{\frac{1}{2}}\left(\frac{s^{\frac{1}{5}}}{s^{-\frac{9}{5}}}\right)^{\frac{1}{2}}\left(\frac{t^{-\frac{3}{2}}}{t^{\frac{1}{2}}}\right)^{\frac{1}{2}} && \text{[Exponent rule of power of product]}\\
&= (6^2)^{\frac{1}{2}}\left(s^{\frac{1}{5}+\frac{9}{5}}\right)^{\frac{1}{2}}\left(t^{-\frac{3}{2}-\frac{1}{2}}\right)^{\frac{1}{2}} && \text{[Quotient rule of Exponents]}\\
&= (6^2)^{\frac{1}{2}}\left(s^{\frac{9+1}{5}}\right)^{\frac{1}{2}}\left(t^{\frac{-3-1}{2}}\right)^{\frac{1}{2}}\\
&= (6^2)^{\frac{1}{2}}\left(s^{\frac{\cancel{10}2}{\cancel{5}}}\right)^{\frac{1}{2}}\left(t^{\frac{-\cancel{4}2}{\cancel{2}}}\right)^{\frac{1}{2}}\\
&= \left(6^{2\times\frac{1}{2}}\right)\left(s^{2\times\frac{1}{2}}\right)\left(t^{-2\times\frac{1}{2}}\right) && \text{[Power rule of Exponent]}\\
&= 6s\left(t^{-1}\right) && \text{[Power rule of Exponent]}\\
\left(\frac{\mathbf{36}s^{\frac{1}{5}}\cdot t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}} &= \frac{\mathbf{6}s}{t}
\end{aligned}$$

b. $\left(\frac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

Solution:

$$\begin{aligned}
\left(\frac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}} &= (27)^{\frac{1}{3}}\left(\frac{b^{\frac{2}{3}}}{b^{-\frac{7}{3}}}\right)^{\frac{1}{3}}\left(\frac{c^{-\frac{5}{2}}}{c^{\frac{1}{2}}}\right)^{\frac{1}{3}} && \text{[Exponent rule of power of product]}\\
&= (3^3)^{\frac{1}{3}}\left(b^{\frac{2}{3}+\frac{7}{3}}\right)^{\frac{1}{3}}\left(c^{-\frac{5}{2}-\frac{1}{2}}\right)^{\frac{1}{3}} && \text{[Quotient rule of Exponents]}\\
&= (3^3)^{\frac{1}{3}}\left(b^{\frac{2+7}{3}}\right)^{\frac{1}{3}}\left(c^{\frac{-5-1}{2}}\right)^{\frac{1}{3}}\\
&= (3^3)^{\frac{1}{3}}\left(b^{\frac{\cancel{9}3}{\cancel{3}}}\right)^{\frac{1}{3}}\left(c^{\frac{-\cancel{6}3}{\cancel{2}}}\right)^{\frac{1}{3}}\\
&= \left(3^{3\times\frac{1}{3}}\right)\left(b^{3\times\frac{1}{3}}\right)\left(c^{-3\times\frac{1}{3}}\right) && \text{[Power rule of Exponent]}\\
\left(\frac{\mathbf{27}b^{\frac{2}{3}}\cdot c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}} &= \frac{\mathbf{3}b}{c}
\end{aligned}$$

c. $\left(\dfrac{8x^{\frac{5}{3}}y^{-\frac{1}{2}}}{27x^{-\frac{4}{3}}y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$

Solution:

$$\begin{aligned}
\left(\frac{8x^{\frac{5}{3}}y^{-\frac{1}{2}}}{27x^{-\frac{4}{3}}y^{\frac{5}{2}}}\right)^{\frac{1}{3}} &= \left(\frac{8}{27}\right)^{\frac{1}{3}}\left(\frac{x^{\frac{5}{3}}}{x^{\frac{-4}{3}}}\right)^{\frac{1}{3}}\left(\frac{y^{-\frac{1}{2}}}{y^{\frac{5}{2}}}\right)^{\frac{1}{3}} && \text{[Exponent rule of power of product]}\\
&= \left(\frac{2^3}{3^3}\right)^{\frac{1}{3}}\left(x^{\frac{5}{3}+\frac{4}{3}}\right)^{\frac{1}{3}}\left(y^{-\frac{1}{2}-\frac{5}{2}}\right)^{\frac{1}{3}} && \text{[Quotient rule of Exponents]}\\
&= \frac{(2^3)^{\frac{1}{3}}}{(3^3)^{\frac{1}{3}}}\left(x^{\frac{5+4}{3}}\right)^{\frac{1}{3}}\left(y^{\frac{-1-5}{2}}\right)^{\frac{1}{3}} && \text{[Exponent rule of power of product]}\\
&= \frac{\left(2^{3\times\frac{1}{3}}\right)}{\left(3^{3\times\frac{1}{3}}\right)}\left(x^{\frac{9}{3}\times\frac{1}{3}}\right)\left(y^{\frac{-6}{2}\times\frac{1}{3}}\right) && \text{[Power rule of Exponent]}\\
&= \frac{\left(2^{\cancel{3}\times\frac{1}{\cancel{3}}}\right)}{\left(3^{\cancel{3}\times\frac{1}{\cancel{3}}}\right)}\left(x^{\frac{\cancel{9}\cancel{3}}{\cancel{3}}\times\frac{1}{\cancel{3}}}\right)\left(y^{\frac{-\cancel{6}\cancel{2}}{\cancel{2}}\times\frac{1}{\cancel{3}}}\right)\\
&= \frac{2}{3}xy^{-1}\\
\left(\frac{8x^{\frac{5}{3}}y^{-\frac{1}{2}}}{27x^{-\frac{4}{3}}y^{\frac{5}{2}}}\right)^{\frac{1}{3}} &= \frac{2x}{3y}
\end{aligned}$$

d. $\left(\dfrac{16m^{\frac{1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{2}}$

Solution:

$$\begin{aligned}
\left(\frac{16m^{\frac{1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{2}} &= \left(\frac{16}{81}\right)^{\frac{1}{2}}\left(\frac{m^{\frac{1}{5}}}{m^{\frac{9}{5}}}\right)^{\frac{1}{2}}\left(\frac{n^{\frac{3}{2}}}{n^{\frac{-1}{2}}}\right)^{\frac{1}{2}} && \text{[Exponent rule of power of product]}\\
&= \left(\frac{4^2}{9^2}\right)^{\frac{1}{2}}\left(m^{\frac{1}{5}-\frac{9}{5}}\right)^{\frac{1}{2}}\left(n^{\frac{3}{2}+\frac{1}{2}}\right)^{\frac{1}{2}} && \text{[Quotient rule of Exponents]}\\
&= \frac{(4^2)^{\frac{1}{2}}}{(9^2)^{\frac{1}{2}}}\left(m^{\frac{1-9}{5}}\right)^{\frac{1}{2}}\left(n^{\frac{3+1}{2}}\right)^{\frac{1}{2}} && \text{[Exponent rule of power of product]}\\
&= \frac{\left(4^{2\times\frac{1}{2}}\right)}{\left(9^{2\times\frac{1}{2}}\right)}\left(m^{\frac{-8}{5}\times\frac{1}{2}}\right)\left(n^{\frac{4}{2}\times\frac{1}{2}}\right) && \text{[Power rule of Exponent]}\\
&= \frac{\left(4^{\cancel{2}\times\frac{1}{\cancel{2}}}\right)}{\left(9^{\cancel{2}\times\frac{1}{\cancel{2}}}\right)}\left(m^{\frac{-\cancel{8}4}{5}\times\frac{1}{\cancel{2}}}\right)\left(n^{\frac{\cancel{4}\cancel{2}}{\cancel{2}}\times\frac{1}{\cancel{2}}}\right)\\
&= \frac{4}{9}\left(m^{-\frac{4}{5}}\right)n\\
\left(\frac{16m^{\frac{1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{2}} &= \frac{4n}{9m^{\frac{4}{5}}}
\end{aligned}$$

e. $\left(\frac{16m^{\frac{-1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{2}}$

Solution:

$$
\begin{aligned}
\left(\frac{16m^{\frac{-1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{2}} &= \left(\frac{16}{81}\right)^{\frac{1}{2}}\left(\frac{m^{\frac{-1}{5}}}{m^{\frac{9}{5}}}\right)^{\frac{1}{2}}\left(\frac{n^{\frac{3}{2}}}{n^{\frac{-1}{2}}}\right)^{\frac{1}{2}} && \text{[Exponent rule of power of product]}\\
&= \left(\frac{4^2}{9^2}\right)^{\frac{1}{2}}\left(m^{-\frac{1}{5}-\frac{9}{5}}\right)^{\frac{1}{2}}\left(n^{\frac{3}{2}+\frac{1}{2}}\right)^{\frac{1}{2}} && \text{[Quotient rule of Exponents]}\\
&= \frac{(4^2)^{\frac{1}{2}}}{(9^2)^{\frac{1}{2}}}\left(m^{\frac{-1-9}{5}}\right)^{\frac{1}{2}}\left(n^{\frac{3+1}{2}}\right)^{\frac{1}{2}} && \text{[Exponent rule of power of product]}\\
&= \frac{\left(4^{2\times\frac{1}{2}}\right)}{\left(9^{2\times\frac{1}{2}}\right)}\left(m^{\frac{-10}{5}\times\frac{1}{2}}\right)\left(n^{\frac{4}{2}\times\frac{1}{2}}\right) && \text{[Power rule of Exponent]}\\
&= \frac{\left(4^{\cancel{2}\times\frac{1}{\cancel{2}}}\right)}{\left(9^{\cancel{2}\times\frac{1}{\cancel{2}}}\right)}\left(m^{\frac{\cancel{-10}^{2}}{\cancel{5}}\times\frac{1}{\cancel{2}}}\right)\left(n^{\frac{\cancel{4}^{2}}{\cancel{2}}\times\frac{1}{\cancel{2}}}\right)\\
&= \frac{4}{9}\left(m^{-1}\right)n\\
\boldsymbol{\left(\frac{16m^{\frac{-1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{2}}} &= \boldsymbol{\frac{4n}{9m}}
\end{aligned}
$$

Simplify the following exercises

Problem 7:

a. $81^{\frac{1}{2}}$

Solution:

$$
\begin{aligned}
81^{\frac{1}{2}} &= (9^2)^{\frac{1}{2}}\\
&= 9^{2\times\frac{1}{2}} && \text{[Power rule of exponents]}\\
&= 9^{\cancel{2}\times\frac{1}{\cancel{2}}}\\
\boldsymbol{81^{\frac{1}{2}}} &= \boldsymbol{9}
\end{aligned}
$$

b. $\mathbf{125^{\frac{-1}{3}}}$

Solution:

$$\begin{aligned}
125^{\frac{-1}{3}} &= (5^3)^{\frac{-1}{3}} \\
&= 5^{3\times\left(\frac{-1}{3}\right)} \quad \text{[Power rule of exponents]} \\
&= 5^{\cancel{3}\times\left(\frac{-1}{\cancel{3}}\right)} \\
&= 5^{-1} \\
\mathbf{125^{\frac{-1}{3}}} &\mathbf{= \frac{1}{5}}
\end{aligned}$$

c. $\mathbf{64^{\frac{1}{2}}}$

Solution:

$$\begin{aligned}
64^{\frac{1}{2}} &= (8^2)^{\frac{1}{2}} \\
&= 8^{2\times\frac{1}{2}} \quad \text{[Power rule of exponents]} \\
&= 8^{\cancel{2}\times\frac{1}{\cancel{2}}} \\
\mathbf{64^{\frac{1}{2}}} &\mathbf{= 8}
\end{aligned}$$

d. $\mathbf{64^{\frac{1}{3}}}$

Solution:

$$\begin{aligned}
64^{\frac{1}{3}} &= (4^3)^{\frac{1}{3}} \\
&= 4^{3\times\frac{1}{3}} \quad \text{[Power rule of exponents]} \\
&= 4^{\cancel{3}\times\frac{1}{\cancel{3}}} \\
\mathbf{64^{\frac{1}{3}}} &\mathbf{= 4}
\end{aligned}$$

e. $\mathbf{32^{\frac{1}{5}}}$

Solution:

$$\begin{aligned}
32^{\frac{1}{5}} &= (2^5)^{\frac{1}{5}} \\
&= 2^{5\times\frac{1}{5}} \quad \text{[Power rule of exponents]} \\
&= 2^{\cancel{5}\times\frac{1}{\cancel{5}}} \\
\mathbf{32^{\frac{1}{5}}} &\mathbf{= 2}
\end{aligned}$$

Problem 8:

a. $\mathbf{(-32)^{\frac{1}{5}}}$

Solution:

$$\begin{aligned}
(-32)^{\frac{1}{5}} &= ((-2)^5)^{\frac{1}{5}} \\
&= (-2)^{5\times\frac{1}{5}} \quad \text{[Power rule of exponents]} \\
&= (-2)^{\cancel{5}\times\frac{1}{\cancel{5}}} \\
\mathbf{(-32)^{\frac{1}{5}}} &= \mathbf{-2}
\end{aligned}$$

b. $\mathbf{(-8)^{\frac{1}{3}}}$

Solution:

$$\begin{aligned}
(-8)^{\frac{1}{3}} &= ((-2)^3)^{\frac{1}{3}} \\
&= (-2)^{3\times\frac{1}{3}} \quad \text{[Power rule of exponents]} \\
&= (-2)^{\cancel{3}\times\frac{1}{\cancel{3}}} \\
\mathbf{(-8)^{\frac{1}{3}}} &= \mathbf{-2}
\end{aligned}$$

c. $\mathbf{-49^{\frac{1}{2}}}$

Solution:

$$\begin{aligned}
-49^{\frac{1}{2}} &= -\left(7^2\right)^{\frac{1}{2}} \\
&= -(7)^{2\times\frac{1}{2}} \quad \text{[Power rule of exponents]} \\
&= -(7)^{\cancel{2}\times\frac{1}{\cancel{2}}} \\
\mathbf{-49^{\frac{1}{2}}} &= \mathbf{-7}
\end{aligned}$$

d. $\mathbf{49^{\frac{-1}{2}}}$

Solution:

$$\begin{aligned}
49^{\frac{-1}{2}} &= (7^2)^{\frac{-1}{2}} \\
&= 7^{2\times\left(-\frac{1}{2}\right)} \quad \text{[Power rule of exponents]} \\
&= 7^{\cancel{2}\times\left(-\frac{1}{\cancel{2}}\right)} \\
&= 7^{-1} \\
\mathbf{49^{-\frac{1}{2}}} &= \mathbf{\frac{1}{7}}
\end{aligned}$$

e. $\mathbf{-16^{\frac{1}{4}}}$

Solution:

$$\begin{aligned}
-16^{\frac{1}{4}} &= -\left(2^4\right)^{\frac{1}{4}} \\
&= -(2)^{4\times\frac{1}{4}} \quad \text{[Power rule of exponents]} \\
&= -(2)^{\cancel{4}\times\frac{1}{\cancel{4}}} \\
\mathbf{-16^{\frac{1}{4}}} &= \mathbf{-2}
\end{aligned}$$

In the following exercises, simplify.

Problem 9:

a. $\mathbf{25^{\frac{3}{2}}}$

Solution:

$$\begin{aligned}
25^{\frac{3}{2}} &= \left(5^2\right)^{\frac{3}{2}} \\
&= 5^{2\times\frac{3}{2}} \quad \text{[Power rule of exponents]} \\
&= 5^{\cancel{2}\times\frac{3}{\cancel{2}}} \\
&= 5^3 \\
\mathbf{25^{\frac{3}{2}}} &= \mathbf{125}
\end{aligned}$$

b. $\mathbf{(-27)^{\frac{2}{3}}}$

Solution:

$$\begin{aligned}
(-27)^{\frac{2}{3}} &= \left((-3)^3\right)^{\frac{2}{3}} \\
&= (-3)^{3\times\left(\frac{2}{3}\right)} \quad \text{[Power rule of exponents]} \\
&= (-3)^{\cancel{3}\times\left(\frac{2}{\cancel{3}}\right)} \\
&= (-3)^2 \\
\mathbf{(-27)^{\frac{2}{3}}} &= \mathbf{9}
\end{aligned}$$

c. $\mathbf{100^{\frac{3}{2}}}$

Solution:

$$\begin{aligned}
100^{\frac{3}{2}} &= (10^2)^{\frac{3}{2}} \\
&= 10^{2\times\frac{3}{2}} \quad \text{[Power rule of exponents]} \\
&= 10^{\cancel{2}\times\frac{3}{\cancel{2}}} \\
&= 10^3 \\
\mathbf{100^{\frac{3}{2}}} &= \mathbf{1000}
\end{aligned}$$

d. $\mathbf{81^{\frac{-3}{2}}}$

Solution:

$$\begin{aligned}
81^{\frac{-3}{2}} &= (9^2)^{\frac{-3}{2}} \\
&= 9^{2\times\left(\frac{-3}{2}\right)} \quad \text{[Power rule of exponents]} \\
&= 9^{\cancel{2}\times\left(\frac{-3}{\cancel{2}}\right)} \\
&= 9^{-3} \\
\mathbf{81^{\frac{-3}{2}}} &= \mathbf{\frac{1}{9^3}}
\end{aligned}$$

e. $\mathbf{27^{\frac{-2}{3}}}$

Solution:

$$\begin{aligned}
27^{\frac{-2}{3}} &= (3^3)^{\frac{-2}{3}} \\
&= 3^{3\times\left(\frac{-2}{3}\right)} \quad \text{[Power rule of exponents]} \\
&= 3^{\cancel{3}\times\left(\frac{-2}{\cancel{3}}\right)} \\
&= 3^{-2} \\
\mathbf{27^{\frac{-2}{3}}} &= \mathbf{\frac{1}{9}}
\end{aligned}$$

Problem 10:

a. $\mathbf{(6x+1)^{\frac{1}{2}}-3=4}$

Solution:

$$
\begin{aligned}
 & (6x+1)^{\frac{1}{2}}-3 &&= 4 \\
\Rightarrow\quad & (6x+1)^{\frac{1}{2}} &&= 4+3 \\
\Rightarrow\quad & (6x+1)^{\frac{1}{2}} &&= 7 \\
\Rightarrow\quad & 6x+1 &&= 7^2 = 49, \quad \text{[Squaring on both sides]} \\
\Rightarrow\quad & 6x &&= 49-1 = 48 \\
\Rightarrow\quad & x &&= \frac{\cancel{48}^{8}}{\cancel{6}} = 8
\end{aligned}
$$

$\Longrightarrow \boxed{x=8}$

b. $\mathbf{(12x-5)^{\frac{1}{3}}+8=3}$

Solution:

$$
\begin{aligned}
 & (12x-5)^{\frac{1}{3}}+8 &&= 3 \\
\Rightarrow\quad & (12x-5)^{\frac{1}{3}} &&= 3-8 \\
\Rightarrow\quad & (12x-5)^{\frac{1}{3}} &&= -5 \\
\Rightarrow\quad & 12x-5 &&= (-5)^3 = -125, \quad \text{[Raising to the power to 3 for both sides]} \\
\Rightarrow\quad & 12x &&= -125+5 = -120 \\
\Rightarrow\quad & x &&= \frac{\cancel{-120}^{10}}{\cancel{12}} = -10
\end{aligned}
$$

$\Longrightarrow \boxed{x=-10}$

c. $\mathbf{(5x-4)^{\frac{1}{4}}+7=9}$

Solution:

$$
\begin{aligned}
 & (5x-4)^{\frac{1}{4}}+7 &&= 9 \\
\Rightarrow\quad & (5x-4)^{\frac{1}{4}} &&= 9-7 \\
\Rightarrow\quad & (5x-4)^{\frac{1}{4}} &&= 2 \\
\Rightarrow\quad & 5x-4 &&= 2^4 = 16, \quad \text{[Raising to the power to 4 for both sides]} \\
\Rightarrow\quad & 5x &&= 16+4 = 20 \\
\Rightarrow\quad & x &&= \frac{\cancel{20}^{4}}{\cancel{5}} = 4
\end{aligned}
$$

$\Longrightarrow \boxed{x=4}$

Problem 11:

a. $\mathbf{(3x-5)^{\frac{3}{2}}+1=9}$

Solution:

$$\begin{aligned}
& (3x-5)^{\frac{3}{2}}+1 &&= 9 \\
\Rightarrow\quad & (3x-5)^{\frac{3}{2}} &&= 9-1 \\
\Rightarrow\quad & (3x-5)^{\frac{3}{2}} &&= 8 \\
\Rightarrow\quad & (3x-5)^{3} = && 8^2, \quad \text{[Squaring on both sides]} \\
\Rightarrow\quad & (3x-5)^{3} &&= 64 = 4^3 \\
\Rightarrow\quad & 3x-5 &&= 4 \\
\Rightarrow\quad & 3x &&= 4+5=9 \\
\Rightarrow\quad & x &&= \frac{\cancel{9}^3}{\cancel{3}} = 3
\end{aligned}$$

$\Longrightarrow\ \boxed{x=3}$

b. $\mathbf{(8x+5)^{\frac{1}{3}}+2=-1}$

Solution:

$$\begin{aligned}
& (8x+5)^{\frac{1}{3}}+2 &&= -1 \\
\Rightarrow\quad & (8x+5)^{\frac{1}{3}} &&= -1-2 \\
\Rightarrow\quad & (8x+5)^{\frac{1}{3}} &&= -3 \\
\Rightarrow\quad & 8x+5 &&= (-3)^3=-27, \quad \text{[Raising to the power to 3 for both sides]} \\
\Rightarrow\quad & 8x+5 &&= -27, \\
\Rightarrow\quad & 8x &&= -27-5=-32 \\
\Rightarrow\quad & x &&= \frac{-\cancel{32}^4}{\cancel{8}} = -4
\end{aligned}$$

$\Longrightarrow\ \boxed{x=-4}$

c. $\mathbf{x^{\frac{2}{3}}-5=4}$

Solution:

$$\begin{aligned}
& x^{\frac{2}{3}}-5 &&= 4 \\
\Rightarrow\quad & x^{\frac{2}{3}} &&= 4+5 \\
\Rightarrow\quad & x^{\frac{2}{3}} &&= 9 \\
\Rightarrow\quad & x^2 &&= 9^3, \quad \text{[Raising to the power to 3 for both sides]} \\
\Rightarrow\quad & x^2 &&= 729 \\
\Rightarrow\quad & x\times x &&= 729
\end{aligned}$$

$\Longrightarrow\ \boxed{x=27 \text{ or } \ -27}$

2.3 Radicals

If $x = y^2$, then $y = x^{\frac{1}{2}}$

Here $x^{\frac{1}{2}}$, sometimes denoted by using root notation, it can be written as, $\sqrt{x}$. In general, $x^{\frac{1}{n}}$ can be denoted by $\sqrt[n]{x}$, here n is called as a index of the radical($\sqrt{\ }$), and x is known to be radicand.

Properties of Radicals

1. Properties of $\sqrt[n]{a}$
   (a) When n is an even number
      i. $\sqrt[n]{a}$ is a real number if $a \geq 0$
      ii. $\sqrt[n]{a}$ is not a real number if $a < 0$
   (b) When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a
2. Absolute value property: For any $n \geq 2$,
   (a) If n is odd, $\sqrt[n]{a^n} = a$
   (b) If n is even, $\sqrt[n]{a^n} = |a|$
3. Simplified Radical Expression:
   - $\sqrt[n]{a}$ is considered as simplified one if a should not have the factors, which is of the form m^n
4. Add and Subtract Radical Expressions:
   - We can add and subtract like radicals, the way we used to add/ subtract like terms. (**Like Radicals:** Like radicals are radical expressions with the same index and the same radicand)
5. Product rule
   - $\boldsymbol{\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}}$ and $\boldsymbol{\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}}$, if $\sqrt[n]{ab}$, $\sqrt[n]{a}$ & $\sqrt[n]{b}$ are defined
6. Quotient rule
   - $\boldsymbol{\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}}$ and $\boldsymbol{\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}}$, if $\sqrt[n]{\frac{a}{b}}$, $\sqrt[n]{a}$ & $\sqrt[n]{b}$ are defined and $b \neq 0$
7. Conjugate of a Radical

- The conjugate of the radical $a+\sqrt{b}$ is defined as $a-\sqrt{b}$. It is used to rationalize the denominators.

Exercise - 2.3

Simplify Expressions with Roots

In the following exercises, simplify

Problem 1:

a. $\sqrt{64}$

Solution:

Since $8^2=64$, hence $\boxed{\sqrt{64}=8}$

b. $-\sqrt{81}$

Solution:

Since $-9^2=-81$, hence $\boxed{-\sqrt{81}=-9}$

c. $\sqrt[5]{-32}$

Solution:

Since $(-2)^5=-32$, hence $\boxed{\sqrt[5]{-32}=-2}$

d. $\sqrt{\dfrac{64}{121}}$

Solution:

Since $8^2=64$ & $11^2=121$, hence $\boxed{\sqrt{\dfrac{64}{121}}=\dfrac{8}{11}}$

Simplify Variable Expressions with Roots

Problem 2:

a. $\boldsymbol{\sqrt{x^6}}$

Solution:

$$\sqrt{x^6} = \sqrt{(x^3)^2} = |x^3| \Rightarrow \boxed{\sqrt{x^6} = |x^3|}$$

b. $\boldsymbol{\sqrt[3]{-8c^9}}$

Solution:

$$\sqrt[3]{-8c^9} = \sqrt[3]{(-2)^3\,(c^3)^3} = -2c^3 \Rightarrow \boxed{\sqrt[3]{-8c^9} = -2c^3}$$

c. $\boldsymbol{\sqrt[4]{16x^8}}$

Solution:

$$\sqrt[4]{16x^8} = \sqrt[4]{(2x^2)^4} = |2x^2| = 2x^2 \Rightarrow \boxed{\sqrt[4]{16x^8} = 2x^2}$$

d. $\boldsymbol{\sqrt[5]{a^{10}}}$

Solution:

$$\sqrt[5]{a^{10}} = \sqrt[5]{(a^2)^5} = a^2 \Rightarrow \boxed{\sqrt[5]{a^{10}} = a^2}$$

In the following exercises, write as a radical expression

Problem 3:

a. $\boldsymbol{x^{\frac{1}{2}}}$

Solution:

$$\boxed{x^{\frac{1}{2}} = \sqrt{x}}$$

b. $\boldsymbol{c^{\frac{1}{9}}}$

Solution:

$\boxed{c^{\frac{1}{9}} = \sqrt[9]{c}}$

c. $\boldsymbol{(2x^5y^2)^{\frac{1}{3}}}$

Solution:

$\boxed{(2x^5y^2)^{\frac{1}{3}} = \sqrt[3]{2x^5y^2}}$

d. $\boldsymbol{(2xy^2)^{\frac{1}{5}}}$

Solution:

$\boxed{(2xy^2)^{\frac{1}{5}} = \sqrt[5]{2xy^2}}$

In the following exercises, write with a rational exponent

Problem 4:

a. $\boldsymbol{\sqrt[7]{x}}$

Solution:

$\boxed{\sqrt[7]{x} = x^{\frac{1}{7}}}$

b. $\boldsymbol{\sqrt[4]{5x}}$

Solution:

$\boxed{\sqrt[4]{5x} = (5x)^{\frac{1}{4}}}$

c. $\sqrt[4]{\left(\dfrac{2xy}{5z}\right)^2}$

Solution:

$$\sqrt[4]{\left(\frac{2xy}{5z}\right)^2} = \left(\left(\frac{2xy}{5z}\right)^{\cancel{2}}\right)^{\frac{1}{\cancel{4}2}} = \left(\frac{2xy}{5z}\right)^{\frac{1}{2}} \Rightarrow \boxed{\sqrt[4]{\left(\frac{2xy}{5z}\right)^2} = \left(\frac{2xy}{5z}\right)^{\frac{1}{2}}}$$

d. $\sqrt{x^5}$

Solution:

$$\sqrt{x^5} = (x^5)^{\frac{1}{2}} \Rightarrow \boxed{\sqrt{x^5} = x^{\frac{5}{2}}}$$

Use the Product Property to Simplify Radical Expressions

In the following exercises, simplify using absolute value signs as needed

Problem 5:

a. $\sqrt{27}$

Solution:

$$\sqrt{27} = \sqrt{3 \times 3 \times 3} \Rightarrow \boxed{\sqrt{27} = 3\sqrt{3}}$$

b. $\sqrt[4]{48y^{16}}$

Solution:

$$\sqrt[4]{48y^{16}} = \sqrt[4]{(2^4 \times 3)\,(y^4)^4} = |2y^4\sqrt[4]{3}| = 2 \cdot \sqrt[4]{3} \cdot y^4 \Rightarrow \boxed{\sqrt[4]{48y^{16}} = 2 \cdot \sqrt[4]{3} \cdot y^4}$$

c. $\sqrt[4]{16x^8}$

Solution:

$$\sqrt[4]{16x^8} = \sqrt[4]{(2x^2)^4} = |2x^2| = 2x^2 \Rightarrow \boxed{\sqrt[4]{16x^8} = 2x^2}$$

d. $\sqrt[3]{81p^7q^8}$

Solution:

$$\sqrt[3]{81p^7q^8} = \sqrt[3]{(3^3 \times 3)\left((p^2)^3 \times p\right)\left((q^2)^3 \times q^2\right)} \Rightarrow \boxed{\sqrt[3]{81p^7q^8} = 3p^2q^2\sqrt[3]{3pq^2}}$$

Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify.

Problem 6:

a. $\sqrt{\dfrac{45}{80}}$

Solution:

$$\sqrt{\frac{45}{80}} = \sqrt{\frac{9 \times \cancel{5}}{16 \times \cancel{5}}} = \sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} \Rightarrow \boxed{\sqrt{\frac{45}{80}} = \frac{3}{4}}$$

b. $\sqrt{\dfrac{28p^7}{q^2}}$

Solution:

$$\sqrt{\frac{28p^7}{q^2}} = \frac{\sqrt{28p^7}}{\sqrt{q^2}} = \frac{\sqrt{4 \times 7 \times p^6 \times p}}{\sqrt{q^2}} = \frac{2\sqrt{7 \times (p^3)^2 \times p}}{q} \Rightarrow \boxed{\sqrt{\frac{28p^7}{q^2}} = 2\sqrt{7p} \cdot \left|\frac{p^3}{q}\right|}$$

c. $\sqrt[3]{\dfrac{24x^8y^4}{81x^2y}}$

Solution:

$$\sqrt[3]{\frac{24x^8y^4}{81x^2y}} = \sqrt[3]{\frac{8 \times \cancel{3} \times \cancel{x^2} \times x^6 \times y^3 \times \cancel{y}}{\cancel{3} \times 27 \times \cancel{x^2} \times \cancel{y}}} = \sqrt[3]{\frac{2^3 \times (x^2)^3 \times y^3}{3^3}} = \frac{\sqrt[3]{2^3 \times (x^2)^3 \times y^3}}{\sqrt[3]{3^3}}$$
$$\Rightarrow \boxed{\sqrt[3]{\frac{24x^8y^4}{81x^2y}} = \frac{2x^2y}{3}}$$

d. $\sqrt[4]{\dfrac{6}{96}}$

Solution:

$$\sqrt[4]{\frac{6}{96}} = \sqrt[4]{\frac{6}{16\times 6}} = \sqrt[4]{\frac{\cancel{6}}{16\times \cancel{6}}} = \sqrt[4]{\frac{1}{2^4}} = \frac{\sqrt[4]{1}}{\sqrt[4]{2^4}} \Rightarrow \boxed{\sqrt[4]{\frac{6}{96}} = \frac{1}{2}}$$

Add and Subtract Radical Expressions

Simplify the following exercises

Problem 7:

a. $\sqrt{48} - \sqrt{27}$

Solution:

$$\begin{aligned}\sqrt{48} - \sqrt{27} &= \sqrt{2^4\times 3} - \sqrt{3^2\times 3}\\ &= \sqrt{4^2\times 3} - 3\sqrt{3}\\ &= 4\sqrt{3} - 3\sqrt{3}\\ &= (4-3)\sqrt{3}\end{aligned}$$

$$\Longrightarrow \boxed{\sqrt{48} - \sqrt{27} = \sqrt{3}}$$

b. $\sqrt[3]{24} - \sqrt[3]{81}$

Solution:

$$\begin{aligned}\sqrt[3]{24} - \sqrt[3]{81} &= \sqrt[3]{2^3\times 3} - \sqrt[3]{3^4}\\ &= \sqrt[3]{2^3\times 3} - \sqrt[3]{3^3\times 3}\\ &= 2\sqrt[3]{3} - 3\sqrt[3]{3}\\ &= (2-3)\sqrt[3]{3}\end{aligned}$$

$$\Longrightarrow \boxed{\sqrt[3]{24} - \sqrt[3]{81} = -\sqrt[3]{3}}$$

c. $\mathbf{8\sqrt[3]{64q^6} - 3\sqrt[3]{125q^6}}$

Solution:

$$\begin{aligned} 8\sqrt[3]{64q^6} - 3\sqrt[3]{125q^6} &= 8\sqrt[3]{2^6q^6} - 3\sqrt[3]{5^3q^6} \\ &= 8\sqrt[3]{4^3\left(q^2\right)^3} - 3\sqrt[3]{5^3\left(q^2\right)^3} \\ &= 8 \times 4 \times q^2 - 3 \times 5 \times q^2 \\ &= 32q^2 - 15q^2 \end{aligned}$$

$$\Longrightarrow \boxed{8\sqrt[3]{64q^6} - 3\sqrt[3]{125q^6} = 17q^2}$$

In the following exercises, Multiply Radical Expressions

Problem 8:

a. $\mathbf{(-2\sqrt{3})(3\sqrt{18})}$

Solution:

$$\begin{aligned} (-2\sqrt{3})(3\sqrt{18}) &= (-2 \times 3)\sqrt{3}\sqrt{2 \times 3^2} \\ &= -6\sqrt{3 \times 2 \times 3^2} \\ &= -6(3)\sqrt{3 \times 2} \\ &= -18\sqrt{6} \end{aligned}$$

$$\Longrightarrow \boxed{(-2\sqrt{3})(3\sqrt{18}) = -18\sqrt{6}}$$

b. $\mathbf{\sqrt{7}(5 + 2\sqrt{7})}$

Solution:

$$\begin{aligned} \sqrt{7}(5 + 2\sqrt{7}) &= 5\sqrt{7} + 2(\sqrt{7} \times \sqrt{7}) \\ &= 5\sqrt{7} + 2(7) \\ &= 14 + 5\sqrt{7} \end{aligned}$$

$$\Longrightarrow \boxed{\sqrt{7}(5 + 2\sqrt{7}) = 14 + 5\sqrt{7}}$$

c. $\mathbf{(3-2\sqrt{7})(5-4\sqrt{7})}$

Solution:

$$\begin{aligned}(3-2\sqrt{7})(5-4\sqrt{7}) &= (3)(5-4\sqrt{7})-2\sqrt{7}(5-4\sqrt{7})\\ &= (3\times 5)-(3\times 4)\sqrt{7}-2\sqrt{7}(5)+(2)(4)(\sqrt{7})(\sqrt{7})\\ &= 15-12\sqrt{7}-10\sqrt{7}+(8\times 7)\\ &= 15+56-(12+10)\sqrt{7}\end{aligned}$$

$$\Longrightarrow \boxed{(3-2\sqrt{7})(5-4\sqrt{7}) = 71-22\sqrt{7}}$$

Problem 9:

a. $\mathbf{(3+\sqrt{5})^2}$

Solution:

$$\begin{aligned}(a+b)^2 &= a^2+2ab+b^2,\ \text{here, } a=3, b=\sqrt{5}\\ (3+\sqrt{5})^2 &= 3^2+2(3)(\sqrt{5})+(\sqrt{5})^2\\ &= 9+6\sqrt{5}+5\end{aligned}$$

$$\Longrightarrow \boxed{(3+\sqrt{5})^2 = 14+6\sqrt{5}}$$

b. $\mathbf{(2-5\sqrt{3})^2}$

Solution:

$$\begin{aligned}(a-b)^2 &= a^2-2ab+b^2,\ \text{here, } a=2, b=5\sqrt{3}\\ (2-5\sqrt{3})^2 &= 2^2-2(2)(5\sqrt{3})+(5\sqrt{3})^2\\ &= 4-20\sqrt{3}+75\end{aligned}$$

$$\Longrightarrow \boxed{(2-5\sqrt{3})^2 = 79-20\sqrt{3}}$$

c. $\mathbf{(4+\sqrt{2})(4-\sqrt{2})}$

Solution:

$$\begin{aligned}(a+b)(a-b) &= a^2-b^2,\ \text{here, } a=4, b=\sqrt{2}\\ (4+\sqrt{2})(4-\sqrt{2}) &= 4^2-(\sqrt{2})^2\\ &= 16-2\end{aligned}$$

$$\Longrightarrow \boxed{(4+\sqrt{2})(4-\sqrt{2})=14}$$

In the following exercises, rationalize the denominator

Problem 10:

a. $\mathbf{\dfrac{10}{\sqrt{6}}}$

Solution:

$$\begin{aligned}\frac{10}{\sqrt{6}} &= \frac{10}{\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}, && \text{[Multiply and divide by } \sqrt{6}]\\ &= \frac{\cancel{2}\times 5\times\sqrt{6}}{\cancel{6}^{3}}\end{aligned}$$

$$\Longrightarrow \boxed{\frac{10}{\sqrt{6}}=\frac{5\sqrt{6}}{3}}$$

b. $\mathbf{\sqrt{\dfrac{4}{27}}}$

Solution:

$$\begin{aligned}\sqrt{\frac{4}{27}} &= \sqrt{\frac{2\times 2}{3\times 3\times 3}}\\ &= \frac{2}{3\sqrt{3}}\\ &= \frac{2}{3\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}, && \text{[Multiply and divide by } \sqrt{3}]\\ &= \frac{2\sqrt{3}}{3(3)} \Longrightarrow \boxed{\sqrt{\frac{4}{27}}=\frac{2\sqrt{3}}{9}}\end{aligned}$$

c. $\dfrac{\mathbf{6}}{\sqrt[4]{\mathbf{9x^3}}}$

Solution:

$$\begin{aligned}
\frac{6}{\sqrt[4]{9x^3}} &= \frac{6}{(9x^3)^{\frac{1}{4}}} \times \frac{(9x^3)^{\frac{3}{4}}}{(9x^3)^{\frac{3}{4}}}, \qquad \text{[Multiply and divide by } (9x^3)\frac{3}{4}\text{]} \\
&= \frac{6 \times \left((3^2x^3)^3\right)^{\frac{1}{4}}}{(9x^3)^{\frac{1+3}{4}}} \\
&= \frac{6 \times (3^6x^9)^{\frac{1}{4}}}{(9x^3)^{\frac{\cancel{4}}{\cancel{4}}}} \\
&= \frac{6 \times (3^4 \times 3^2 \times x^8 \times x)^{\frac{1}{4}}}{9x^3} \\
&= \frac{6 \times (3^4)^{\frac{1}{4}} \times (x^8)^{\frac{1}{4}} \times \sqrt[4]{3^2x}}{9x^3} \\
&= \frac{\cancel{6}^2 \times \cancel{3} \times \cancel{x^2}\sqrt[4]{9x}}{\cancel{9}^{\cancel{3}}x^{\cancel{3}}}
\end{aligned}$$

$$\implies \boxed{\frac{6}{\sqrt[4]{9x^3}} = \frac{2\sqrt[4]{9x}}{x}}$$

In the following exercises, rationalize the denominator.

Problem 11:

a. $\dfrac{\mathbf{8}}{\mathbf{1-\sqrt{5}}}$

Solution:

$$\begin{aligned}
\frac{8}{1-\sqrt{5}} &= \frac{8}{1-\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}}, \\
&\qquad \text{[Multiply and divide by the conjugate of } (1-\sqrt{5})\text{]} \\
&= \frac{8(1+\sqrt{5})}{1^2-(\sqrt{5})^2} \qquad [(a+b)(a-b) = a^2-b^2] \\
&= \frac{8(1+\sqrt{5})}{1-5} \\
&= \frac{(2\times\cancel{4})(1+\sqrt{5})}{-\cancel{4}} \implies \boxed{\frac{8}{1-\sqrt{5}} = -(2+2\sqrt{5})}
\end{aligned}$$

b. $\boldsymbol{\dfrac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}}$

Solution:

$$\begin{aligned}
\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}} &= \frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}} \times \frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}+\sqrt{8}}, \\
&\quad\text{[Multiply and divided by the conjugate of } (\sqrt{x}-\sqrt{8})] \\
&= \frac{(\sqrt{x}+\sqrt{8})^2}{(\sqrt{x})^2-(\sqrt{8})^2} \qquad [(a+b)(a-b)=a^2-b^2] \\
&= \frac{(\sqrt{x})^2+2\sqrt{x}\sqrt{8}+(\sqrt{8})^2}{x-8} \qquad [(a+b)^2=a^2+2ab+b^2] \\
&= \frac{x+8+2(2)\sqrt{2x}}{x-8}
\end{aligned}$$

$$\Longrightarrow \boxed{\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}} = \frac{(x+8)+4\sqrt{2x}}{x-8}}$$

c. $\boldsymbol{\dfrac{3\sqrt{2}+2\sqrt{3}}{5\sqrt{2}-3\sqrt{3}}}$

Solution:

$$\begin{aligned}
\frac{3\sqrt{2}+2\sqrt{3}}{5\sqrt{2}-3\sqrt{3}} &= \frac{3\sqrt{2}+2\sqrt{3}}{5\sqrt{2}-3\sqrt{3}} \times \frac{5\sqrt{2}+3\sqrt{3}}{5\sqrt{2}+3\sqrt{3}}, \quad \text{[Multiply and divided by } (5\sqrt{2}+3\sqrt{3})] \\
&= \frac{3\sqrt{2}(5\sqrt{2}+3\sqrt{3})+2\sqrt{3}(5\sqrt{2}+3\sqrt{3})}{(5\sqrt{2})^2-(3\sqrt{3})^2} \qquad [(a+b)(a-b)=a^2-b^2] \\
&= \frac{15(\sqrt{2}\times\sqrt{2})+9(\sqrt{2}\times\sqrt{3})+10(\sqrt{2}\times\sqrt{3})+6(\sqrt{3}\times\sqrt{3})}{25(\sqrt{2})^2-9(\sqrt{3})^2} \\
&= \frac{15(2)+9\sqrt{6}+10\sqrt{6}+6(3)}{25(2)-9(3)} \\
&= \frac{30+18+19\sqrt{6}}{50-27}
\end{aligned}$$

$$\Longrightarrow \boxed{\frac{3\sqrt{2}+2\sqrt{3}}{5\sqrt{2}-3\sqrt{3}} = \frac{48+19\sqrt{6}}{23}}$$

Solve the following Radical Equations

Problem 12:

a. $\mathbf{2\sqrt{5x+1} - 8 = 0}$

Solution:

$$\begin{aligned}
& 2\sqrt{5x+1} - 8 &&= 0 \\
\Rightarrow\quad & 2\sqrt{5x+1} &&= 8 \\
\Rightarrow\quad & \sqrt{5x+1} &&= \frac{\cancel{8}^4}{\cancel{2}} = 4 \\
\Rightarrow\quad & (\sqrt{5x+1})^2 &&= 4^2, && \text{[Squaring on both sides]} \\
\Rightarrow\quad & 5x+1 &&= 16 \\
\Rightarrow\quad & 5x &&= 16 - 1 = 15 \\
\Rightarrow\quad & x &&= \frac{15}{5} \Longrightarrow \boxed{x = 3}
\end{aligned}$$

b. $\mathbf{\sqrt[3]{4x+5} - 2 = -5}$

Solution:

$$\begin{aligned}
& \sqrt[3]{4x+5} - 2 &&= -5 \\
\Rightarrow\quad & \sqrt[3]{4x+5} &&= -5 + 2 = -3 \\
\Rightarrow\quad & (\sqrt[3]{4x+5})^3 &&= (-3)^3, && \text{[Raising to the power 3 for both sides]} \\
\Rightarrow\quad & 4x+5 &&= -27 \\
\Rightarrow\quad & 4x &&= -27 - 5 = -32 \\
\Rightarrow\quad & x &&= \frac{-32}{4} \Longrightarrow \boxed{x = -8}
\end{aligned}$$

c. $\mathbf{\sqrt{6n+1} + 4 = 8}$

Solution:

$$\begin{aligned}
& \sqrt{6n+1} + 4 &&= 8 \\
\Rightarrow\quad & \sqrt{6n+1} &&= 8 - 4 = 4 \\
\Rightarrow\quad & (\sqrt{6n+1})^2 &&= 4^2, && \text{[Squaring on both sides]} \\
\Rightarrow\quad & 6n+1 &&= 16 \\
\Rightarrow\quad & 6n &&= 16 - 1 = 15 \\
\Rightarrow\quad & n &&= \frac{15}{6} \Longrightarrow \boxed{n = \frac{5}{2}}
\end{aligned}$$

Chapter 3

POLYNOMIAL

A polynomial is a mathematical expression which is the combination of variables, constants and exponents, that are combined by using the mathematical operations. Ex: $7x^4 - 5x^3 + 2x^2 + 4$

3.1 Basic Concept

1. ***Terms:*** A term is a constant or the product of constant and variables.
   Example: $7, y, 9a^2, 5ab$ are all called terms

2. ***Co-efficient of a variable:*** A coefficient of a variable is defined as the the constant which is multiplied with the variable.
   Example: Co-efficient of x^2 in the polynomail $3x^4 - x^2 + 3x - 1$ is -1

3. ***Degree:*** The degree of the polynomial is the highest exponent of a variable occurs in the expression, if a polynomial having more than one variable, the highest power of the term including the product of two or more variables will be considered as a degree.
   Example:

   - Degree of $\boldsymbol{9x^4} + 2x^3 - 14x - 25$ is 4
   - Degree of $\boldsymbol{x^2y} + xy + 5x + 3y$ is 3
   - Degree of $\boldsymbol{4xyz} + 3xz + 4yz - 5z + 7$ is 3
   - Degree of $\boldsymbol{3x^2} - 0x^3 + 4x + 8$ is 2

4. ***Like Terms:*** Terms that are either constants or have the same variables raised to the same powers are called like terms. **Example:** $5x$ and $3x$ are like terms where as $2x^2y$ and $3xy^2$ are not like terms.

5. ***Types of Polynomials:*** Using the term, we can classify the polynomials.
   - ***Monomial:*** A monomial is an algebraic expression with exactly one term
   - ***Binomial:*** A binomial is an algebraic expression with exactly two term
   - ***Trinomial:*** A trinomial is an algebraic expression with exactly three term
6. ***Add and Subtract Polynomials:*** To add /subtract the polynomial, we have to combine the like terms with respect to the sign/ operation involved in the terms.

Exercise - 3.1

Determine the Type of Polynomials

In the following exercises, determine whether the polynomial is a monomial, binomial, trinomial, or other polynomial?

Problem 1:

a. $\boldsymbol{47x^5 - 17x^2y^3 + y^2}$

Solution:

$47x^5 - 17x^2y^3 + y^2$: **It is trinomial, since it has three terms**

b. $\boldsymbol{5c^3 + 11c^2 - c - 8}$

Solution:

$5c^3+11c^2-c-8$: **It is a polynomial, since it has more than three terms**

Problem 2:

a. $\boldsymbol{x^2 - y^2}$

Solution:

$x^2 - y^2$: **It is binomial, since it has exactly two terms**

b. $\boldsymbol{-13c^4}$

Solution:

$-13c^4$: **It is a monomial, since it has exactly one term**

Problem 3:

a. $\boldsymbol{y^2 - 5yz - 6z^2}$

Solution:

$y^2 - 5yz - 6z^2$: **It is trinomial, since it has three terms**

b. $\boldsymbol{8y - 5x}$

Solution:

$8y - 5x$: **It is binomial, since it has exactly two terms**

Add and Subtract Polynomials

In the following exercises, add the polynomials

Problem 4: $\boldsymbol{(5y^2 + 12y + 4) + (6y^2 - 8y + 7)}$

Solution:

Like Terms	Addition
$5y^2$ & $6y^2$	$11y^2$
$12y$ & $-8y$	$4y$
4 & 7	11

Hence, $\boldsymbol{(5y^2 + 12y + 4) + (6y^2 - 8y + 7) = 11y^2 + 4y + 11}$

Problem 5: $(\boldsymbol{x^2+6x+8)+(-4x^2+11x-9)}$

Solution:

Like Terms	Addition
x^2 & $-4x^2$	$-3x^2$
$6x$ & $11x$	$17x$
8 & -9	-1

Hence, $\boldsymbol{(x^2+6x+8)+(-4x^2+11x-9)=-3x^2+17x-1}$

Problem 6: $\boldsymbol{(8x^2-5x+2)+(3x^2+3x)}$

Solution:

Like Terms	Addition
$8x^2$ & $3x^2$	$11x^2$
$-5x$ & $3x$	$-2x$

Hence, $\boldsymbol{(8x^2-5x+2)+(3x^2+3x)=11x^2-2x+2}$

Problem 7: $\boldsymbol{(5a^2+8)+(a^2-4a-9)}$

Solution:

Like Terms	Addition
$5a^2$ & a^2	$6a^2$
8 & -9	-1

Hence, $\boldsymbol{(5a^2+8)+(a^2-4a-9)=6a^2-4a-1}$

In the following exercises, subtract the polynomials

Problem 8: $\mathbf{(4m^2 - 6m - 3) - (2m^2 + m - 7)}$

Solution:

Like Terms	Subtraction
$4m^2$ & $2m^2$	$4m^2 - 2m^2 = 2m^2$
$-6m$ & m	$-6m - m = -7m$
-3 & -7	$-3 - (-7) = -3 + 7 = 4$

Hence, $\mathbf{(4m^2 - 6m - 3) - (2m^2 + m - 7) = 2m^2 - 7m + 4}$

Problem 9: $\mathbf{(3b^2 - 4b + 1) - (5b^2 - b - 2)}$

Solution:

Like Terms	Subtraction
$3b^2$ & $5b^2$	$3b^2 - 5b^2 = -2b^2$
$-4b$ & $-b$	$-4b - (-b) = -4b + b = -3b$
1 & -2	$1 - (-2) = 1 + 2 = 3$

Hence, $\mathbf{(3b^2 - 4b + 1) - (5b^2 - b - 2) = -2b^2 - 3b + 3}$

Problem 10: $\mathbf{(12s^2 - 15s) - (s - 9)}$

Solution:

Like Terms	Subtraction
$-15s$ & s	$-15s - s = -16s$

Hence, $\mathbf{(12s^2 - 15s) - (s - 9) = 12s^2 - 16s + 9}$

Problem 11: $(a^2 + 8a + 5) - (a^2 - 3a + 2)$

Solution:

Like Terms	Subtraction
a^2 & a^2	$a^2 - a^2 = 0$
$8a$ & $-3a$	$8a - (-3a) = 8a + 3a = 11a$
5 & 2	$5 - 2 = 3$

Hence, $\mathbf{(a^2 + 8a + 5) - (a^2 - 3a + 2) = 11a + 3}$

In the following exercises, subtract the polynomials

Problem 12: **Subtract** $(9x^2 + 2)$ **from** $(12x^2 - x + 6)$

Solution:

Like Terms	Subtraction
$12x^2$ & $9x^2$	$12x^2 - 9x^2 = 3x^2$
6 & 2	$6 - 2 = 4$

Hence, $\mathbf{(12x^2 - x + 6) - (9x^2 + 2) = 3x^2 - x + 4}$

Problem 13: **Subtract** $(5y^2 - y + 12)$ **from** $(5y^2 - 8y - 20)$

Solution:

Like Terms	Subtraction
$5y^2$ & $5y^2$	$5y^2 - 5y^2 = 0$
$-8y$ & $-y$	$-8y - (-y) = -8y + y = -7y$
-20 & 12	$-20 - 12 = -32$

Hence, $\mathbf{(5y^2 - 8y - 20) - (5y^2 - y + 12) = -7y - 32}$

Problem 14: **Subtract $(7w^2 - 4w + 2)$ from $(8w^2 - w + 6)$**

Solution:

Like Terms	Subtraction
$8w^2$ & $7w^2$	$8w^2 - 7w^2 = w^2$
$-w$ & $-4w$	$-w - (-4w) = -w + 4w = 3w$
6 & 2	$6 - 2 = 4$

Hence, $\mathbf{(8w^2 - w + 6) - (7w^2 - 4w + 2) = w^2 + 3w + 4}$

Problem 15: **Subtract $(5x^2 - x + 12)$ from $(9x^2 - 6x - 20)$**

Solution:

Like Terms	Subtraction
$9x^2$ & $5x^2$	$9x^2 - 5x^2 = 4x^2$
$-6x$ & $-x$	$-6x - (-x) = -6x + x = -5x$
-20 & 12	$-20 - 12 = -32$

Hence, $\mathbf{(9x^2 - 6x - 20) - (5x^2 - x + 12) = 4x^2 - 5x - 32}$

In the following exercises, add or subtract the polynomials

Problem 16: $\mathbf{(p^3 - 3p^2q) + (2pq^2 + 4q^3) - (3p^2q + pq^2)}$

Solution:

$(p^3 - 3p^2q) + (2pq^2 + 4q^3) - (3p^2q + pq^2) = p^3 - 3p^2q + 2pq^2 + 4q^3 - 3p^2q - pq^2$

Like Terms	Simplification
p^3	p^3
$-3p^2q$ & $-3p^2q$	$-6p^2q$
$2pq^2$ & $-pq^2$	pq^2
$4q^3$	$4q^3$

Hence, $\mathbf{(p^3 - 3p^2q) + (2pq^2 + 4q^3) - (3p^2q + pq^2) = p^3 - 6p^2q + pq^2 + 4q^3}$

Problem 17: $(a^3 - 2a^2b) + (ab^2 + b^3) - (3a^2b + 4ab^2)$

Solution:

$(a^3 - 2a^2b) + (ab^2 + b^3) - (3a^2b + 4ab^2) = a^3 - 2a^2b + ab^2 + b^3 - 3a^2b - 4ab^2$

Like Terms	Simplification
a^3	a^3
$-2a^2b$ & $-3a^2b$	$-5a^2b$
ab^2 & $-4ab^2$	$-3ab^2$
b^3	b^3

Hence, $\boldsymbol{(a^3 - 2a^2b) + (ab^2 + b^3) - (3a^2b + 4ab^2) = a^3 - 5a^2b - 3ab^2 + b^3}$

Problem 18: $(x^3 - x^2y) - (4xy^2 - y^3) + (3x^2y - xy^2)$

Solution:

$(x^3 - x^2y) - (4xy^2 - y^3) + (3x^2y - xy^2) = x^3 - x^2y - 4xy^2 + y^3 + 3x^2y - xy^2$

Like Terms	Simplification
x^3	x^3
$-x^2y$ & $3x^2y$	$2x^2y$
$-4xy^2$ & $-xy^2$	$-5xy^2$
y^3	y^3

Hence, $\boldsymbol{(x^3 - x^2y) - (4xy^2 - y^3) + (3x^2y - xy^2) = x^3 + 2x^2y - 5xy^2 + y^3}$

Problem 19: $(x^3 - 2x^2y) - (xy^2 - 3y^3) - (x^2y - 4xy^2)$

Solution:

$(x^3 - 2x^2y) - (xy^2 - 3y^3) - (x^2y - 4xy^2) = x^3 - 2x^2y - xy^2 + 3y^3 - x^2y + 4xy^2$

Like Terms	Simplification
x^3	x^3
$-2x^2y$ & $-x^2y$	$-3x^2y$
$-xy^2$ & $4xy^2$	$3xy^2$
$3y^3$	$3y^3$

Hence,
$\boldsymbol{(x^3 - 2x^2y) - (xy^2 - 3y^3) - (x^2y - 4xy^2) = x^3 - 3x^2y + 3xy^2 + 3y^3}$

Problem 20: Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Also find the degree of each polynomial

Solution:

	Polynomial	**Number of terms**	**Type**	**Degree**
a	$3x^2 - 5xy + 3xy^2$	3	Trinomial	3
b	$-5xy^2$	1	Monomial	3
c	$5x^3y^2z^3 + y^5z^2$	2	Binomial	8
d	$5x - 1$	2	Binomial	1
e	$x^3 - 3x^2 + 3xy^2 - y^3$	4	Polynomial	3
f	2	1	Monomial	0

3.2 Multiplication of a Polynomial

Special Products

Let a and b be any two real numbers

1. **Binomial Squares Pattern:**
   - $(a + b)^2 = a^2 + 2ab + b^2$
   - $(a - b)^2 = a^2 - 2ab + b^2$
2. **Products of conjugate pattern**
   - $(a + b)(a - b) = a^2 - b^2$

Exercise - 3.2

In the following exercises, multiply the monomials.

Problem 1: $(6y^7)(-3y^4)$

Solution:

$$\begin{aligned}
(6y^7)(-3y^4) &= (6)(y^7)(-3)(y^4)\\
&= (6)(-3)(y^7)(y^4)\\
&= [6\times(-3)]y^{4+7}\\
\boldsymbol{(6y^7)(-3y^4)} &= \boldsymbol{-18y^{11}}
\end{aligned}$$

Problem 2: $(-10x^5)(-3x^3)$

Solution:

$$\begin{aligned}
(-10x^5)(-3x^3) &= (-10)(x^5)(-3)(x^3)\\
&= (-10)(-3)(x^5)(x^3)\\
&= [(-10)\times(-3)]x^{5+3}\\
\boldsymbol{(-10x^5)(-3x^3)} &= \boldsymbol{30x^8}
\end{aligned}$$

Problem 3: $(-8u^6)(-9u)$

Solution:

$$\begin{aligned}
(-8u^6)(-9u) &= (-8)(u^6)(-9)(u)\\
&= (-8)(-9)(u^6)(u^1)\\
&= [(-8)\times(-9)]u^{6+1}\\
\boldsymbol{(-8u^6)(-9u)} &= \boldsymbol{72u^7}
\end{aligned}$$

Problem 4: $\left(\frac{4}{7}rs^2\right)(14rs^3)$

Solution:

$$\begin{aligned}
\left(\frac{4}{7}rs^2\right)(14rs^3) &= \left(\frac{4}{7}\right)(14)(r \times r)(s^2 \times s^3) \\
&= \left(\frac{4}{\cancel{7}} \times \cancel{14}^2\right)(r^{1+1})(s^{2+3}) \\
&= (4 \times 2)r^2s^5 \\
\boldsymbol{\left(\frac{4}{7}rs^2\right)(14rs^3)} &= \boldsymbol{8r^2s^5}
\end{aligned}$$

Problem 5: $\left(\frac{5}{8}x^3y\right)(24x^5y)$

Solution:

$$\begin{aligned}
\left(\frac{5}{8}x^3y\right)(24x^5y) &= \left(\frac{5}{8}\right)(24)(x^3 \times x^5)(y \times y) \\
&= \left(\frac{5}{\cancel{8}} \times \cancel{24}^3\right)(x^{3+5})(y^{1+1}) \\
&= (5 \times 3)x^8y^2 \\
\boldsymbol{\left(\frac{5}{8}x^3y\right)(24x^5y)} &= \boldsymbol{15x^8y^2}
\end{aligned}$$

Problem 6: $\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)$

Solution:

$$\begin{aligned}
\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right) &= \left(\frac{2}{3} \times \frac{3}{4}\right)(x^2 \cdot x)(y \cdot y^2) \\
&= \left(\frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{4}^2}\right)(x^{2+1})(y^{1+2}) \\
&= \left(\frac{1}{2}\right)x^3y^3 \\
\boldsymbol{\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)} &= \boldsymbol{\frac{x^3y^3}{2}}
\end{aligned}$$

In the following exercises, multiply a Polynomial by a Monomial

Problem 7: $\boldsymbol{-8x(x^2+2x-15)}$

Solution:

$$\begin{aligned}
-8x(x^2+2x-15) &= (-8x)x^2+(-8x)2x-(-8x)15\\
&= (-8)(x\cdot x^2)+(-8\cdot 2)(x\cdot x)-(-8\cdot 15)x\\
\boldsymbol{-8x(x^2+2x-15)} &= \boldsymbol{-8x^3-16x^2+120x}
\end{aligned}$$

Problem 8: $\boldsymbol{-5t(t^2+3t-18)}$

Solution:

$$\begin{aligned}
-5t(t^2+3t-18) &= (-5t)t^2+(-5t)3t-(-5t)18\\
&= (-5)(t\cdot t^2)+(-5\cdot 3)(t\cdot t)-(-5\cdot 18)t\\
\boldsymbol{-5t(t^2+3t-18)} &= \boldsymbol{-5t^3-15t^2+90t}
\end{aligned}$$

Problem 9: $\boldsymbol{-8y(y^2+2y-15)}$

Solution:

$$\begin{aligned}
-8y(y^2+2y-15) &= (-8y)y^2+(-8y)2y-(-8y)15\\
&= (-8)(y\cdot y^2)+(-8\cdot 2)(y\cdot y)-(-8\cdot 15)y\\
\boldsymbol{-8y(y^2+2y-15)} &= \boldsymbol{-8y^3-16y^2+120y}
\end{aligned}$$

Problem 10: $\boldsymbol{5pq^3(p^2-2pq+6q^2)}$

Solution:

$$\begin{aligned}
5pq^3(p^2-2pq+6q^2) &= (5pq^3)p^2-(5pq^3)(2pq)+(5pq^3)(6q^2)\\
&= 5(p\cdot p^2)q^3-(5\cdot 2)(p\cdot p)(q^3\cdot q)+(5\cdot 6)p(q^2\cdot q^3)\\
\boldsymbol{5pq^3(p^2-2pq+6q^2)} &= \boldsymbol{5p^3q^3-10p^2q^4+30pq^5}
\end{aligned}$$

Problem 11: $\boldsymbol{9r^3s(r^2 - 3rs + 5s^2)}$

Solution:

$$\begin{aligned}
9r^3s(r^2 - 3rs + 5s^2) &= (9r^3s)r^2 - (9r^3s)(3rs) + (9r^3s)(5s^2) \\
&= 9(r^3 \cdot r^2)s - (9 \cdot 3)(r^3 \cdot r)(s \cdot s) + (9 \cdot 5)r^3(s \cdot s^2) \\
\mathbf{9r^3s(r^2 - 3rs + 5s^2)} &= \mathbf{9r^5s - 27r^4s^2 + 45r^3s^3}
\end{aligned}$$

Problem 12: $\boldsymbol{-4y^2z^2(3y^2 + 12yz - z^2)}$

Solution:

$$\begin{aligned}
-4y^2z^2(3y^2 + 12yz - z^2) &= (-4y^2z^2)(3y^2) + (-4y^2z^2)(12yz) - (-4y^2z^2)z^2 \\
&= (-4 \cdot 3)(y^2 \cdot y^2)z^2 + (-4 \cdot 12)(y^2 \cdot y)(z^2 \cdot z) + 4y^2(z^2 \cdot z^2) \\
\mathbf{-4y^2z^2(3y^2 + 12yz - z^2)} &= \mathbf{-12y^4z^2 - 48y^3z^3 + 4y^2z^4}
\end{aligned}$$

In the following exercises, multiply the binomials

Problem 13: $\boldsymbol{(w + 5)(w + 7)}$

Solution:

$$\begin{aligned}
(w + 5)(w + 7) &= w \cdot (w + 7) + 5 \cdot (w + 7) \\
&= (w \cdot w) + (w \cdot 7) + (5 \cdot w) + (5 \cdot 7) \\
&= w^2 + 7w + 5w + 35 \\
\mathbf{(w + 5)(w + 7)} &= \mathbf{w^2 + 12w + 35}
\end{aligned}$$

Problem 14: $\boldsymbol{(2xy + 3)(3xy + 2)}$

Solution:

$$\begin{aligned}
(2xy + 3)(3xy + 2) &= 2xy \cdot (3xy + 2) + 3 \cdot (3xy + 2) \\
&= (2xy \cdot 3xy) + (2xy \cdot 2) + (3 \cdot 3xy) + (3 \cdot 2) \\
&= 6x^2y^2 + 4xy + 9xy + 6 \\
\mathbf{(2xy + 3)(3xy + 2)} &= \mathbf{6x^2y^2 + 13xy + 6}
\end{aligned}$$

Problem 15: $(4p+11)(5p-4)$

Solution:

$$\begin{aligned}(4p+11)(5p-4) &= 4p\cdot(5p-4)+11\cdot(5p-4)\\ &= (4p\cdot 5p)+(4p\cdot(-4))+(11\cdot 5p)+(11\cdot(-4))\\ &= 20p^2-16p+55p-44\\ \mathbf{(4p+11)(5p-4)} &= \mathbf{20p^2+39p-44}\end{aligned}$$

Problem 16: $(7q+4)(3q-8)$

Solution:

$$\begin{aligned}(7q+4)(3q-8) &= 7q\cdot(3q-8)+4\cdot(3q-8)\\ &= (7q\cdot 3q)+(7q\cdot(-8))+(4\cdot 3q)+(4\cdot(-8))\\ &= 21q^2-56q+12q-32\\ \mathbf{(7q+4)(3q-8)} &= \mathbf{21q^2-44q-32}\end{aligned}$$

Problem 17: $(x^2+3)(x+2)$

Solution:

$$\begin{aligned}(x^2+3)(x+2) &= x^2\cdot(x+2)+3\cdot(x+2)\\ &= (x^2\cdot x)+(x^2\cdot 2)+(3\cdot x)+(3\cdot 2)\\ \mathbf{(x^2+3)(x+2)} &= \mathbf{x^3+2x^2+3x+6}\end{aligned}$$

Problem 18: $(y^2-4)(y+3)$

Solution:

$$\begin{aligned}(y^2-4)(y+3) &= y^2\cdot(y+3)-4\cdot(y+3)\\ &= (y^2\cdot y)+(y^2\cdot 3)-(4\cdot y)-(4\cdot 3)\\ \mathbf{(y^2-4)(y+3)} &= \mathbf{y^3+3y^2-4y-12}\end{aligned}$$

In the following exercises, multiply the polynomials

Problem 19: $\boldsymbol{(x+5)(x^2+4x+3)}$

Solution:

$$\begin{aligned}(x+5)(x^2+4x+3) &= (x+5)x^2+(x+5)4x+(x+5)(3)\\ &= (x)x^2+5x^2+(x)4x+5(4x)+3x+3(5)\\ &= x^3+5x^2+4x^2+20x+3x+15\\ \boldsymbol{(x+5)(x^2+4x+3)} &= \boldsymbol{x^3+9x^2+23x+15}\end{aligned}$$

Problem 20: $\boldsymbol{(u+4)(u^2+3u+2)}$

Solution:

$$\begin{aligned}(u+4)(u^2+3u+2) &= (u+4)u^2+(u+4)3u+(u+4)(2)\\ &= (u)u^2+4u^2+(u)3u+4(3u)+u(2)+4(2)\\ &= u^3+4u^2+3u^2+12u+2u+8\\ \boldsymbol{(u+4)(u^2+3u+2)} &= \boldsymbol{u^3+7u^2+14u+8}\end{aligned}$$

Problem 21: $\boldsymbol{(y+8)(4y^2+y-7)}$

Solution:

$$\begin{aligned}(y+8)(4y^2+y-7) &= (y+8)(4y^2)+(y+8)y+(y+8)(-7)\\ &= 4(y)y^2+8(4)y^2+(y)(y)+8(y)+(-7)y+(-7)8\\ &= 4y^3+32y^2+y^2+8y-7y-56\\ \boldsymbol{(y+8)(4y^2+y-7)} &= \boldsymbol{4y^3+33y^2+y-56}\end{aligned}$$

Problem 22: $(a+10)(3a^2+a-5)$

Solution:

$$\begin{aligned}(a+10)(3a^2+a-5) &= (a+10)(3a^2)+(a+10)a+(a+10)(-5)\\ &= 3(a)(a^2)+(10\cdot 3)a^2+(a)(a)+10a-5a-5(10)\\ &= 3a^3+30a^2+a^2+10a-5a-50\\ \mathbf{(a+10)(3a^2+a-5)} &= \mathbf{3a^3+31a^2+5a-50}\end{aligned}$$

Problem 23: $(y^2-3y+8)(4y^2+y-7)$

Solution:

$$\begin{aligned}&(y^2-3y+8)(4y^2+y-7)\\ &= (y^2-3y+8)(4y^2)+(y^2-3y+8)(y)-(y^2-3y+8)(7)\\ &= (y^2)(4y^2)-3y(4y^2)+8(4y^2)+y^3-3y^2+8y-7y^2+21y-56\\ &= 4y^4-12y^3+32y^2+y^3-3y^2+8y-7y^2+21y-56\\ &= 4y^4+(-12y^3+y^3)+(32y^2-3y^2-7y^2)+(8y+21y)-56\end{aligned}$$

$$\mathbf{(y^2-3y+8)(4y^2+y-7)=4y^4-11y^3+22y^2+29y-56}$$

In the following exercises, square each binomial using the Binomial Squares Pattern

Problem 24: $(w+4)^2$

Solution:

$$\begin{aligned}(a+b)^2 &= a^2+2ab+b^2 \qquad \text{(Formula) Here } a=w,\ b=4\\ (w+4)^2 &= w^2+2(w)(4)+4^2\\ \mathbf{(w+4)^2} &= \mathbf{w^2+8w+16}\end{aligned}$$

Problem 25: $(q+12)^2$

Solution:

$$\begin{aligned}(a+b)^2 &= a^2+2ab+b^2 \qquad \text{(Formula) Here } a=q,\ b=12\\ (q+12)^2 &= q^2+2(q)(12)+12^2\\ \mathbf{(q+12)^2} &= \mathbf{q^2+24q+144}\end{aligned}$$

Problem 26: $\mathbf{(3x-5)^2}$

Solution:

$$
\begin{aligned}
(a-b)^2 &= a^2-2ab+b^2 \qquad \text{(Formula) Here } a=3x,\ b=5\\
(3x-5)^2 &= (3x)^2-2(3x)(5)+5^2\\
\mathbf{(3x-5)^2} &= \mathbf{9x^2-30x+25}
\end{aligned}
$$

Problem 27: $\mathbf{(2y-3z)^2}$

Solution:

$$
\begin{aligned}
(a-b)^2 &= a^2-2ab+b^2 \qquad \text{(Formula) Here } a=2y,\ b=3z\\
(2y-3z)^2 &= (2y)^2-2(2y)(3z)+(3z)^2\\
\mathbf{(2y-3z)^2} &= \mathbf{4y^2-12yz+9z^2}
\end{aligned}
$$

Problem 28: $\mathbf{\left(y+\frac{1}{4}\right)^2}$

Solution:

$$
\begin{aligned}
(a+b)^2 &= a^2+2ab+b^2 \qquad \text{(Formula) Here } a=y,\ b=\tfrac{1}{4}\\
\left(y+\frac{1}{4}\right)^2 &= (y)^2+\cancel{2}(y)\left(\frac{1}{\cancel{4}^{2}}\right)+\left(\frac{1}{4}\right)^2\\
\mathbf{\left(y+\frac{1}{4}\right)^2} &= \mathbf{y^2+\frac{y}{2}+\frac{1}{16}}
\end{aligned}
$$

Problem 29: $\mathbf{(3x^2+2)^2}$

Solution:

$$
\begin{aligned}
(a+b)^2 &= a^2+2ab+b^2 \qquad \text{(Formula) Here } a=3x^2,\ b=2\\
(3x^2+2)^2 &= (3x^2)^2+2(3x^2)(2)+2^2\\
\mathbf{(3x^2+2)^2} &= \mathbf{9x^4+12x^2+4}
\end{aligned}
$$

In the following exercise, multiply each pair of conjugates using the Product of Conjugates Pattern.

Problem 30: $(5k+6)(5k-6)$

Solution:

$$\begin{aligned}(a+b)(a-b) &= a^2-b^2 \qquad \text{(Formula) Here } a=5k,\ b=6\\ (5k+6)(5k-6) &= (5k)^2-(6)^2\\ \mathbf{(5k+6)(5k-6)} &= \mathbf{25k^2-36}\end{aligned}$$

Problem 31: $(9c+5)(9c-5)$

Solution:

$$\begin{aligned}(a+b)(a-b) &= a^2-b^2 \qquad \text{(Formula) Here } a=9c,\ b=5\\ (9c+5)(9c-5) &= (9c)^2-(5)^2\\ \mathbf{(9c+5)(9c-5)} &= \mathbf{81c^2-25}\end{aligned}$$

Problem 32: $(7w+10x)(7w-10x)$

Solution:

$$\begin{aligned}(a+b)(a-b) &= a^2-b^2 \qquad \text{(Formula) Here } a=7w,\ b=10x\\ (7w+10x)(7w-10x) &= (7w)^2-(10x)^2\\ \mathbf{(7w+10x)(7w-10x)} &= \mathbf{49w^2-100x^2}\end{aligned}$$

Chapter 4

FACTORIZING POLYNOMIALS

4.1 Factorizing

Earlier we multiplied factors together to get a product. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called factoring.

Greatest Common Factor

The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

Steps to find the greatest common factor (GCF) of two expressions:

- Factor each coefficient into prime factors and write all variables with exponents in expanded form.
- Circle the common factors
- Bring down all the common factors and multiply

Factor by Grouping

Steps to find Factor by grouping:

- Group terms with common factors
- Take out the common factor in each group
- Take out the common factor from the expression
- Check by multiplying the factors if it is necessary

Exercise - 4.1

In the following exercises, find the greatest common factor.

Problem 1: $\mathbf{10p^3q, 12pq^2}$

Solution:

$$\begin{array}{l|l} 10p^3q & 2 \times 5 \times p \times p \times p \times q \\ 12pq^2 & 2 \times 2 \times 3 \times p \times q \times q \end{array}$$

Hence $\text{GCF}(10p^3q, 12pq^2) = 2 \times p \times q \implies \boxed{\text{GCF}(10p^3q, 12pq^2) = 2pq}$

Problem 2: $\mathbf{8a^2b^3, 10ab^2}$

Solution:

$$\begin{array}{l|l} 8a^2b^3 & 2 \times 2 \times 2 \times a \times a \times b \times b \times b \\ 10ab^2 & 2 \times 5 \times a \times b \times b^2 \end{array}$$

Hence $\text{GCF}(8a^2b^3, 10ab^2) = 2 \times a \times b \implies \boxed{\text{GCF}(8a^2b^3, 10ab^2) = 2ab^2}$

Problem 3: $\mathbf{12m^2n^3, 30m^5n^3}$

Solution:

$$\begin{array}{l|l} 12m^2n^3 & 2 \times 2 \times 3 \times m \times m \times n \times n \times n \\ 30m^5n^3 & 2 \times 3 \times 5 \times m \times m \times m \times m \times m \times n \times n \times n \end{array}$$

Hence $\text{GCF}(12m^2n^3, 30m^5n^3) = 2 \times 3 \times m \times m \times n \times n \times n$
$\implies \boxed{\text{GCF}(12m^2n^3, 30m^5n^3) = 6m^2n^3}$

Problem 4: $\mathbf{35x^3y^2, 10x^4y, 5x^5y^3}$

Solution:

$$\begin{array}{l|l} 35x^3y^2 & 5 \times 7 \times x \times x \times x \times y \times y \\ 10x^4y & 2 \times 5 \times x \times x \times x \times x \times y \\ 5x^5y^3 & 5 \times x \times x \times x \times x \times x \times y \times y \times y \end{array}$$

Hence $\text{GCF}(35x^3y^2, 10x^4y, 5x^5y^3) = 5 \times x \times x \times x \times y$
$\implies \boxed{\text{GCF}(35x^3y^2, 10x^4y, 5x^5y^3) = 5x^3y}$

Problem 5: $\mathbf{27p^2q^3, 45p^3q^4, 9p^4q^3}$

Solution:

$$\begin{array}{l|l} 27p^2q^3 & 3 \times 3 \times 3 \times p \times p \times q \times q \times q \\ 45p^3q^4 & 3 \times 3 \times 5 \times p \times p \times p \times q \times q \times q \times q \\ 9p^4q^3 & 3 \times 3 \times p \times p \times p \times p \times q \times q \times q \end{array}$$

Hence $\text{GCF}(27p^2q^3, 45p^3q^4, 9p^4q^3) = 3 \times 3 \times p \times p \times q \times q \times q$

$\implies \boxed{\text{GCF}(27p^2q^3, 45p^3q^4, 9p^4q^3) = 9p^2q^3}$

In the following exercises, factor the greatest common factor from each polynomial

Problem 6: $\mathbf{12x^3 - 10x}$

Solution:

$$\begin{array}{l|l} 12x^3 & 2 \times 2 \times 3 \times x \times x \times x \\ 10x & 2 \times 5 \times x \end{array}$$

Hence $\text{GCF}(12x^3, 10x) = 2x$

$12x^3 - 10x = 2x(6x^2) - 2x(5)$

$\implies \boxed{12x^3 - 10x = 2x(6x^2 - 5)}$

Problem 7: $\mathbf{5x^3 - 15x^2 + 20x}$

Solution:

$$\begin{array}{l|l} 5x^3 & 5 \times x \times x \times x \\ 15x^2 & 3 \times 5 \times x \times x \\ 20x & 2 \times 2 \times 5 \times x \end{array}$$

Hence $\text{GCF}(5x^3, 15x^2, 20x) = 5x$

$5x^3 - 15x^2 + 20x = 5x(x^2) - 5x(3x) + 5x(4)$

$\implies \boxed{5x^3 - 15x^2 + 20x = 5x(x^2 - 3x + 4)}$

Problem 8: $\boldsymbol{8m^2 - 40m + 16}$

Solution:

$$\begin{array}{l|l} 8m^2 & 2 \times 2 \times 2 \times m \times m \\ 40m & 2 \times 2 \times 2 \times 5 \times m \\ 16 & 2 \times 2 \times 2 \times 2 \end{array}$$

Hence $\text{GCF}(8m^2, 40m, 16) = 8$
$8m^2 - 40m + 16 = 8(m^2) - 8(5m) + 8(2)$
$\Longrightarrow \boxed{8m^2 - 40m + 16 = 8(m^2 - 5m + 2)}$

Problem 9: $\boldsymbol{24x^3 - 12x^2 + 15x}$

Solution:

$$\begin{array}{l|l} 24x^3 & 2 \times 2 \times 2 \times 3 \times x \times x \times x \\ 12x^2 & 2 \times 2 \times 3 \times x \times \\ 15x & 3 \times 5 \times x \end{array}$$

Hence $\text{GCF}(24x^3, 12x^2, 15x) = 3x$
$24x^3 - 12x^2 + 15x = 3x(8x^2) - 3x(4x) + 3x(5)$
$\Longrightarrow \boxed{24x^3 - 12x^2 + 15x = 3x(8x^2 - 4x + 5)}$

Problem 10: $\boldsymbol{24y^3 - 18y^2 - 30y}$

Solution:

$$\begin{array}{l|l} 24y^3 & 2 \times 2 \times 2 \times 3 \times y \times y \times y \\ 18y^2 & 2 \times 3 \times 3 \times y \times y \\ 30y & 2 \times 3 \times 5 \times y \end{array}$$

Hence $\text{GCF}(24y^3, 18y^2, 30y) = 6y$
$24y^3 - 18y^2 - 30y = 6y(4y^2) - 6y(3y) - 6y(5)$
$\Longrightarrow \boxed{24y^3 - 18y^2 - 30y = 6y(4y^2 - 3y - 5)}$

Problem 11: $\boldsymbol{12xy^2 + 18x^2y^2 - 30y^3}$

Solution:

$$\begin{array}{l|l} 12xy^2 & 2 \times 2 \times 3 \times x \times y \times y \\ 18x^2y^2 & 2 \times 3 \times 3 \times x \times x \times y \times y \\ 30y^3 & 2 \times 3 \times 5 \times y \times y \times y \end{array}$$

Hence $\text{GCF}(12xy^2, 18x^2y^2, 30y^3) = 6y^2$

$12xy^2 + 18x^2y^2 - 30y^3 = 6y^2(2x) + 6y^2(3x^2) - 6y^2(5y)$

$\Longrightarrow \boxed{12xy^2 + 18x^2y^2 - 30y^3 = 6y^2(2x + 3x^2 - 5y)}$

Problem 12: $\boldsymbol{-5y^3 + 35y^2 - 15y}$

Solution:

$$\begin{array}{l|l} 5y^3 & 5 \times y \times y \times y \\ 35y^2 & 5 \times 7 \times y \times y \\ 15y & 3 \times 5 \times y \end{array}$$

Hence $\text{GCF}(5y^3, 35y^2, 15y) = 5y$

$-5y^3 + 35y^2 - 15y = 5y(-y^2) + 5y(7y) - 5y(3) = 5y(-y^2 + 7y - 3)$

$\Longrightarrow \boxed{-5y^3 + 35y^2 - 15y = -5y(y^2 - 7y + 3)}$

Problem 13: $\boldsymbol{-4p^3q - 12p^2q^2 + 16pq^2}$

Solution:

$$\begin{array}{l|l} 4p^3q & 2 \times 2 \times p \times p \times p \times q \\ 12p^2q^2 & 2 \times 2 \times 3 \times p \times p \times q \times q \\ 16pq^2 & 2 \times 2 \times 2 \times 2 \times p \times q \times q \end{array}$$

Hence $\text{GCF}(4p^3q, 12p^2q^2, 16pq^2) = 4pq$

$-4p^3q - 12p^2q^2 + 16pq^2 = -p^2(4pq) - 3pq(4pq) + 4q(4pq)$

$\Longrightarrow \boxed{-4p^3q - 12p^2q^2 + 16pq^2 = -4pq(p^2 + 3pq - 4q)}$

Problem 14: $\boldsymbol{-6a^3b - 12a^2b^2 + 18ab^2}$

Solution:

$$\begin{array}{l|l} 6a^3b & 2 \times 3 \times a \times a \times a \times b \\ 12a^2b^2 & 2 \times 2 \times 3 \times a \times a \times b \times b \\ 18ab^2 & 2 \times 3 \times 3 \times a \times b \times b \end{array}$$

Hence $\text{GCF}(6a^3b, 12a^2b^2, 18ab^2) = 6ab$
$-6a^3b - 12a^2b^2 + 18ab^2 = -a^2(6ab) - 2ab(6ab) + 3b(6ab)$
$\implies \boxed{-6a^3b - 12a^2b^2 + 18ab^2 = -6ab(a^2 + 2ab - 3b)}$

Problem 15: $\boldsymbol{5x(x + 1) + 3(x + 1)}$

Solution:

$$\begin{array}{l|l} 5x(x+1) & 5 \times x \times \underbrace{(x+1)} \\ 3(x+1) & 3 \times \underbrace{(x+1)} \end{array}$$

Hence $\text{GCF}(5x(x + 1), 3(x + 1)) = x + 1$
$\implies \boxed{5x(x + 1) + 3(x + 1) = (x + 1)(5x + 3)}$

Problem 16: $\boldsymbol{2x(x - 1) + 9(x - 1)}$

Solution:

$$\begin{array}{l|l} 2x(x-1) & 2 \times x \times \underbrace{(x-1)} \\ 9(x-1) & 3 \times 3 \times \underbrace{(x-1)} \end{array}$$

Hence $\text{GCF}(2x(x - 1), 9(x - 1)) = x - 1$
$\implies \boxed{2x(x - 1) + 9(x - 1) = (x - 1)(2x + 9)}$

Problem 17: $\boldsymbol{3b(b - 2) - 13(b - 2)}$

Solution:

$$\begin{array}{l|l} 3b(b-2) & 3 \times b \times \underbrace{(b-2)} \\ 13(b-2) & 13 \times \underbrace{(b-2)} \end{array}$$

Hence $\text{GCF}(3b(b - 2) - 13(b - 2)) = b - 2$
$\implies \boxed{3b(b - 2) - 13(b - 2) = (b - 2)(3b - 13)}$

In the following exercises, factor by grouping

Problem 18: $\boldsymbol{ab + 5a + 3b + 15}$

Solution:

$$\begin{aligned} ab + 5a + 3b + 15 &= (ab + 5a) + (3b + 15) \\ &= a(b+5) + 3(b+5) \\ \boldsymbol{ab + 5a + 3b + 15} &= \boldsymbol{(b+5)(a+3)} \end{aligned}$$

Problem 19: $\boldsymbol{pq - 10p + 8q - 80}$

Solution:

$$\begin{aligned} pq - 10p + 8q - 80 &= (pq - 10p) + (8q - 80) \\ &= p(q-10) + 8(q-10) \\ \boldsymbol{pq - 10p + 8q - 80} &= \boldsymbol{(q-10)(p+8)} \end{aligned}$$

Problem 20: $\boldsymbol{uv - 18 - 9u + 2v}$

Solution:

$$\begin{aligned} uv - 18 - 9u + 2v &= uv - 9u + 2v - 18 \\ &= (uv - 9u) + (2v - 18) \\ &= u(v-9) + 2(v-9) \\ \boldsymbol{uv - 18 - 9u + 2v} &= \boldsymbol{(v-9)(u+2)} \end{aligned}$$

Problem 21: $\boldsymbol{6y^2 + 7y + 24y + 28}$

Solution:

$$\begin{aligned} 6y^2 + 7y + 24y + 28 &= (6y^2 + 7y) + (24y + 28) \\ &= y(6y+7) + 4(6y+7) \\ \boldsymbol{6y^2 + 7y + 24y + 28} &= \boldsymbol{(6y+7)(y+4)} \end{aligned}$$

Problem 22: $\boldsymbol{x^2 - x + 4x - 4}$

Solution:

$$\begin{aligned}
x^2 - x + 4x - 4 &= (x^2 - x) + (4x - 4) \\
&= x(x-1) + 4(x-1) \\
\boldsymbol{x^2 - x + 4x - 4} &= \boldsymbol{(x-1)(x+4)}
\end{aligned}$$

Problem 23: $\boldsymbol{u^2 - u + 6u - 6}$

Solution:

$$\begin{aligned}
u^2 - u + 6u - 6 &= (u^2 - u) + (6u - 6) \\
&= u(u-1) + 6(u-1) \\
\boldsymbol{u^2 - u + 6u - 6} &= \boldsymbol{(u-1)(u+6)}
\end{aligned}$$

Problem 24: $\boldsymbol{mn - 6m - 4n + 24}$

Solution:

$$\begin{aligned}
mn - 6m - 4n + 24 &= (mn - 6m) - (4n - 24) \\
&= m(n-6) - 4(n-6) \\
\boldsymbol{mn - 6m - 4n + 24} &= \boldsymbol{(n-6)(m-4)}
\end{aligned}$$

Problem 25: $\boldsymbol{r^2 - 3r - r + 3}$

Solution:

$$\begin{aligned}
r^2 - 3r - r + 3 &= (r^2 - 3r) - (r - 3) \\
&= r(r-3) - 1(r-3) \\
\boldsymbol{r^2 - 3r - r + 3} &= \boldsymbol{(r-3)(r-1)}
\end{aligned}$$

Problem 26: $\mathbf{2x^2 - 14x - 5x + 35}$

Solution:

$$\begin{aligned} 2x^2 - 14x - 5x + 35 &= (2x^2 - 14x) - (5x - 35) \\ &= 2x(x-7) - 5(x-7) \\ \mathbf{2x^2 - 14x - 5x + 35} &= \mathbf{(x-7)(2x-5)} \end{aligned}$$

4.2 Factor Trinomials

How to factor a trinomial of the form $x^2 + bx + c$

- Write the factors as two binomials with first terms x, $x^2 + bx + c = (x \quad)(x \quad)$
- Find two numbers m and n such that $m \cdot n = c$ and $m + n = b$
- Use m and n as the last terms of the factors $(x+m)(x+n)$
- If it is necessary, verify $x^2 + bx + c = (x+m)(x+n)$

Exercise - 4.2

In the following exercises, factor each trinomial of the form $\mathbf{x^2 + bx + c}$

Problem 1: $\mathbf{p^2 + 11p + 30}$

Solution:

$$\begin{aligned} p^2 + 11p + 30 &= p^2 + 5p + 6p + 30 \qquad [30 = 5 \times 6, 5 + 6 = 11] \\ &= p(p+5) + 6(p+5) \\ \mathbf{p^2 + 11p + 30} &= \mathbf{(p+5)(p+6)} \end{aligned}$$

Problem 2: $\mathbf{w^2 + 10w + 21}$

Solution:

$$\begin{aligned} w^2 + 10w + 21 &= w^2 + 7w + 3w + 21 \qquad [21 = 7 \times 3; 7 + 3 = 10] \\ &= w(w+7) + 3(w+7) \\ \mathbf{w^2 + 10w + 21} &= \mathbf{(w+7)(w+3)} \end{aligned}$$

Problem 3: $\boldsymbol{n^2 + 19n + 48}$

Solution:

$$\begin{aligned} n^2+19n+48 &= n^2 + 3n + 16n+48 \qquad [48 = 3 \times 16; 3 + 16 = 19] \\ &= n(n+3) + 16(n+3) \\ \boldsymbol{n^2 + 19n + 48} &= \boldsymbol{(n+3)(n+16)} \end{aligned}$$

Problem 4: $\boldsymbol{x^2 - 8x + 12}$

Solution:

$$\begin{aligned} x^2-8x+12 &= x^2-2x - 6x+12 \qquad [12 = (-2) \times (-6); (-2) + (-6) = -8] \\ &= x(x-2) - 6(x-2) \\ \boldsymbol{x^2 - 8x + 12} &= \boldsymbol{(x-2)(x-6)} \end{aligned}$$

Problem 5: $\boldsymbol{q^2 - 13q + 36}$

Solution:

$$\begin{aligned} q^2-13q+36 &= q^2-4q - 9q+36 \qquad [36 = (-4) \times (-9); (-4) + (-9) = -13] \\ &= q(q-4) - 9(q-4) \\ \boldsymbol{q^2 - 13q + 36} &= \boldsymbol{(q-4)(q-9)} \end{aligned}$$

Problem 6: $\boldsymbol{x^2 - 8x + 7}$

Solution:

$$\begin{aligned} x^2-8x+7 &= x^2-x - 7x+7 \qquad [7 = (-1) \times (-7); (-1) + (-7) = -8] \\ &= x(x-1) - 7(x-1) \\ \boldsymbol{x^2 - 8x + 7} &= \boldsymbol{(x-7)(x-1)} \end{aligned}$$

Problem 7: $\mathbf{5p - 6 + p^2}$

Solution:

$$\begin{aligned} 5p - 6 + p^2 &= p^2 + 5p - 6 & [-6 = (-1) \times (6); (-1) + (6) = 5] \\ &= p^2 - p + 6p - 6 \\ &= p(p-1) + 6(p-1) \\ \mathbf{5p - 6 + p^2} &= \mathbf{(p-1)(p+6)} \end{aligned}$$

Problem 8: $\mathbf{6n - 7 + n^2}$

Solution:

$$\begin{aligned} 6n - 7 + n^2 &= n^2 + 6n - 7 & [-7 = (-1) \times (7); (-1) + (7) = 6] \\ &= n^2 - n + 7n - 7 \\ &= n(n-1) + 7(n-1) \\ \mathbf{6n - 7 + n^2} &= \mathbf{(n-1)(n+7)} \end{aligned}$$

Problem 9: $\mathbf{8 - 6x + x^2}$

Solution:

$$\begin{aligned} 8 - 6x + x^2 &= x^2 - 6x + 8 & [8 = (-2) \times (-4); (-2) + (-4) = -6] \\ &= x^2 - 2x - 4x + 8 \\ &= x(x-2) - 4(x-2) \\ \mathbf{8 - 6x + x^2} &= \mathbf{(x-2)(x-4)} \end{aligned}$$

Problem 10: $\mathbf{x^2 - 12 - 11x}$

Solution:

$$\begin{aligned} x^2 - 12 - 11x &= x^2 - 11x - 12 & [-12 = (1) \times (-12); (1) + (-12) = -11] \\ &= x^2 + x - 12x - 12 \\ &= x(x+1) - 12(x+1) \\ \mathbf{x^2 - 12 - 11x} &= \mathbf{(x+1)(x-12)} \end{aligned}$$

Problem 11: $\boldsymbol{-11 - 10x + x^2}$

Solution:

$$\begin{aligned} -11 - 10x + x^2 &= x^2 - 10x - 11 && [-11 = (1) \times (-11); (1) + (-11) = -10] \\ &= x^2 + x - 11x - 11 \\ &= x(x+1) - 11(x+1) \\ \boldsymbol{-11 - 10x + x^2} &= \boldsymbol{(x+1)(x-11)} \end{aligned}$$

Problem 12: $\boldsymbol{7x + x^2 + 6}$

Solution:

$$\begin{aligned} 7x + x^2 + 6 &= x^2 + 7x + 6 && [6 = (1) \times (6); (1) + (6) = 7] \\ &= x^2 + x + 6x + 6 \\ &= x(x+1) + 6(x+1) \\ \boldsymbol{7x + x^2 + 6} &= \boldsymbol{(x+1)(x+6)} \end{aligned}$$

In the following exercises, factor each trinomial of the form $\boldsymbol{x^2 + bxy + cy^2}$

Problem 13: $\boldsymbol{x^2 - 2xy - 80y^2}$

Solution:

$$\begin{aligned} x^2 - 2xy - 80y^2 &= x^2 - 8xy + 10xy - 80y^2 && [\text{-80=(-8)(10); -8+10=-2}] \\ &= x(x-8y) + 10y(x-8y) \\ \boldsymbol{x^2 - 2xy - 80y^2} &= \boldsymbol{(x-8y)(x+10y)} \end{aligned}$$

Problem 14: $\boldsymbol{p^2 - 2pq - 35q^2}$

Solution:

$$\begin{aligned} p^2 - 2pq - 35q^2 &= p^2 - 7pq + 5pq - 35q^2 && [\text{-35=(5)(-7); -7+5=-2}] \\ &= p(p-7q) + 5q(p-7q) \\ \boldsymbol{p^2 - 2pq - 35q^2} &= \boldsymbol{(p-7q)(p+5q)} \end{aligned}$$

Problem 15: $a^2 + 5ab - 24b^2$

Solution:

$$\begin{aligned} a^2+5ab-24b^2 &= a^2-3ab+8ab-24b^2 \\ &= a(a-3b)+8b(a-3b) \\ \boldsymbol{a^2+5ab-24b^2} &= \boldsymbol{(a-3b)(a+8b)} \end{aligned}$$

[-24=(-3)(8); -3+8=5]

Problem 16: $r^2 + 3rs - 28s^2$

Solution:

$$\begin{aligned} r^2+3rs-28s^2 &= r^2-4rs+7rs-28s^2 \\ &= r(r-4s)+7s(r-4s) \\ \boldsymbol{r^2+3rs-28s^2} &= \boldsymbol{(r-4s)(r+7s)} \end{aligned}$$

[-28=(-4)(7); -4+7=3]

Problem 17: $x^2 - 13xy - 14y^2$

Solution:

$$\begin{aligned} x^2-13xy-14y^2 &= x^2+xy-14xy-14y^2 \\ &= x(x+y)-14y(x+y) \\ \boldsymbol{x^2-13xy-14y^2} &= \boldsymbol{(x+y)(x-14y)} \end{aligned}$$

[-14=(1)(-14); -14+1=-13]

Problem 18: $u^2 - 8uv + 12v^2$

Solution:

$$\begin{aligned} u^2-8uv+12v^2 &= u^2-2uv-6uv+12v^2 \\ &= u(u-2v)-6v(u-2v) \\ \boldsymbol{u^2-8uv+12v^2} &= \boldsymbol{(u-2v)(u-6v)} \end{aligned}$$

[12=(-2)(-6); -2+(-6)=-8]

In the following exercises, Factor Trinomials of the Form $ax^2 + bx + c$ using the ac Method

Problem 19: $3x^2 + 5x + 2$

Solution:

$$\begin{aligned} 3x^2+5x+2 &= 3x^2+3x + 2x+2 \qquad [3 \times 2 = 6;\ 6=(2)(3);\ 2+3=5] \\ &= 3x(x+1) + 2(x+1) \\ \mathbf{3x^2 + 5x + 2} &= \mathbf{(x+1)(3x+2)} \end{aligned}$$

Problem 20: $2w^2 + 3w - 5$

Solution:

$$\begin{aligned} 2w^2+3w-5 &= 2w^2-2w + 5w-5 \qquad [2 \times (-5) = -10;\ -10=(-2)(5);\ -2+5=3] \\ &= 2w(w-1) + 5(w-1) \\ \mathbf{2w^2 + 3w - 5} &= \mathbf{(w-1)(2w+5)} \end{aligned}$$

Problem 21: $3x^2 + x - 10$

Solution:

$$\begin{aligned} 3x^2+x-10 &= 3x^2+6x - 5x-10 \qquad [3 \times (-10) = -30;\ -30=(-5)(6);\ -5+6=1] \\ &= 3x(x+2) - 5(x+2) \\ \mathbf{3x^2 + x - 10} &= \mathbf{(x+2)(3x-5)} \end{aligned}$$

Problem 22: $5s^2 - 9s + 4$

Solution:

$$\begin{aligned} 5s^2-9s+4 &= 5s^2-5s - 4s+4 \qquad [5 \times 4 = 20;\ 20=(-5)(-4);\ (-5)+(-4)=-9] \\ &= 5s(s-1) - 4(s-1) \\ \mathbf{5s^2 - 9s + 4} &= \mathbf{(s-1)(5s-4)} \end{aligned}$$

Problem 23: $\boldsymbol{6y^2 + y - 15}$

Solution:

$$\begin{aligned}
6y^2+y-15 &= 6y^2-9y+10y-15 \qquad [6\times(-15)=-90;\ \text{-90=(-9)(10); -9+10=1}] \\
&= 3y(2y-3)+5(2y-3) \\
\boldsymbol{6y^2+y-15} &= \boldsymbol{(2y-3)(3y+5)}
\end{aligned}$$

Problem 24: $\boldsymbol{5n^2 + 21n + 4}$

Solution:

$$\begin{aligned}
5n^2+21n+4 &= 5n^2+n+20n+4 \qquad [5\times 4=20;\ \text{20=(1)(20); 1+20=21}] \\
&= n(5n+1)+4(5n+1) \\
\boldsymbol{5n^2+21n+4} &= \boldsymbol{(5n+1)(n+4)}
\end{aligned}$$

Problem 25: $\boldsymbol{6n^2 - n - 2}$

Solution:

$$\begin{aligned}
6n^2-n-2 &= 6n^2-4n+3n-2 \qquad [6\times(-2)=-12;\ \text{-12=(-4)(3); -4+3=-1}] \\
&= 2n(3n-2)+1(3n-2) \\
\boldsymbol{6n^2-n-2} &= \boldsymbol{(3n-2)(2n+1)}
\end{aligned}$$

Problem 26: $\boldsymbol{4x^2 + 11x - 3}$

Solution:

$$\begin{aligned}
4x^2+11x-3 &= 4x^2+12x-x-3 \qquad [4\times(-3)=-12;\ \text{-12=(12)(-1); -1+12=11}] \\
&= 4x(x+3)-(x+3) \\
\boldsymbol{4x^2+11x-3} &= \boldsymbol{(x+3)(4x-1)}
\end{aligned}$$

Problem 27: $\mathbf{3u^2 + 8u + 5}$

Solution:

$$\begin{aligned} 3u^2+8u+5 &= 3u^2+3u+5u+5 && [3\times 5 = 15;\ 15=(5)(3);\ 5+3=8] \\ &= 3u(u+1)+5(u+1) \\ \mathbf{3u^2+8u+5} &= \mathbf{(u+1)(3u+5)} \end{aligned}$$

4.3 Factor Special Products

Binomial Identities

1. $(a+b)^2 = a^2+2ab+b^2$
2. $(a-b)^2 = a^2-2ab+b^2$
3. $a^2-b^2 = (a+b)(a-b)$
4. $a^3+b^3 = (a+b)(a^2-ab+b^2)$
5. $a^3-b^3 = (a-b)(a^2+ab+b^2)$

Exercise - 4.3

In the following exercises, factor completely using the perfect square trinomials pattern

Problem 1: $\mathbf{16y^2 + 24y + 9}$

Solution:

Terms	Verification	Result
First term	$16y^2 = (4y)^2$	Perfect square
Last term	$9 = 3^2$	Perfect square
Middle term	$2(4y)(3) = 24y$	Fit

$\boxed{\text{Formula: } a^2+2ab+b^2 = (a+b)^2}$

Hence, $16y^2+24y+9 = (4y)^2+2(4y)(3)+3^2$

$\implies \boxed{16y^2+24y+9 = (4y+3)^2}$

Problem 2: $\mathbf{25v^2 + 20v + 4}$

Solution:

Terms	Verification	Result
First term	$25v^2 = (5v)^2$	Perfect square
Last term	$4 = 2^2$	Perfect square
Middle term	$2(5v)(2) = 20v$	Fit

$\boxed{\text{Formula: } a^2 + 2ab + b^2 = (a+b)^2}$

Hence, $25v^2 + 20v + 4 = (5v)^2 + 2(5v)(2) + 2^2$

$\Longrightarrow \boxed{25v^2 + 20v + 4 = (5v+2)^2}$

Problem 3: $\mathbf{36s^2 + 84s + 49}$

Solution:

Terms	Verification	Result
First term	$36s^2 = (6s)^2$	Perfect square
Last term	$49 = 7^2$	Perfect square
Middle term	$2(6s)(7) = 84s$	Fit

$\boxed{\text{Formula: } a^2 + 2ab + b^2 = (a+b)^2}$

Hence, $36s^2 + 84s + 49 = (6s)^2 + 2(6s)(7) + 7^2$

$\Longrightarrow \boxed{36s^2 + 84s + 49 = (6s+7)^2}$

Problem 4: $\mathbf{100x^2 - 20x + 1}$

Solution:

Terms	Verification	Result
First term	$100x^2 = (10x)^2$	Perfect square
Last term	$1 = 1^2$	Perfect square
Middle term	$2(10x)(1) = 20x$	Fit

$\boxed{\text{Formula: } a^2 - 2ab + b^2 = (a-b)^2}$

Hence, $100x^2 - 20x + 1 = (10x)^2 - 2(10x)(1) + 1^2$

$\Longrightarrow \boxed{100x^2 - 20x + 1 = (10x-1)^2}$

Problem 5: $\boldsymbol{49x^2 + 28xy + 4y^2}$

Solution:

Terms	Verification	Result
First term	$49x^2 = (7x)^2$	Perfect square
Last term	$4y^2 = (2y)^2$	Perfect square
Middle term	$2(7x)(2y) = 28xy$	Fit

$\boxed{\text{Formula: } a^2 + 2ab + b^2 = (a+b)^2}$

Hence, $49x^2 + 28xy + 4y^2 = (7x)^2 + 2(7x)(2y) + (2y)^2$

$\implies \boxed{49x^2 + 28xy + 4y^2 = (7x+2y)^2}$

Problem 6: $\boldsymbol{75u^4 - 30u^3v + 3u^2v^2}$

Solution:

$75u^4 - 30u^3v + 3u^2v^2 = 3u^2(25u^2 - 10uv + v^2)$, Since GCF is $3u^2$

Terms	Verification	Result
First term	$25u^2 = (5u)^2$	Perfect square
Last term	$v^2 = (v)^2$	Perfect square
Middle term	$2\,(5u)\,(v) = 10uv$	Fit

$\boxed{\text{Formula: } a^2 - 2ab + b^2 = (a-b)^2}$

Hence, $25u^2 - 10uv + v^2 = (5u)^2 - 2\,(5u)\,(v) + (v)^2$

$\implies 25u^2 - 10uv + v^2 = (5u - v)^2$

$\implies \boxed{75u^4 - 30u^3v + 3u^2v^2 = 3u^2(5u - v)^2}$

Problem 7: $\boldsymbol{64x^2y - 96xy + 36y}$

Solution:

$64x^2y - 96xy + 36y = 4y(16x^2 - 24x + 9)$, Since GCF is $4y$

Terms	Verification	Result
First term	$16x^2 = (4x)^2$	Perfect square
Last term	$9 = 3^2$	Perfect square
Middle term	$2\,(4x)\,(3) = 24x$	Fit

$\boxed{\text{Formula: } a^2 - 2ab + b^2 = (a-b)^2}$

Hence, $16x^2 - 24x + 9 = (4x)^2 - 2\,(4x)\,(3) + (3)^2$

$\implies 16x^2 - 24x + 9 = (4x-3)^2$

$\implies \boxed{64x^2y - 96xy + 36y = 4y(4x-3)^2}$

In the following exercises, factor completely using the difference of squares pattern, if possible

Problem 8: $\mathbf{25v^2 - 1}$

Solution:

$$\begin{aligned} a^2 - b^2 &= (a+b)(a-b) && \text{[Formula]} \\ 25v^2 - 1 &= (5v)^2 - 1^2 \\ 25v^2 - 1 &= (5v+1)(5v-1) && \text{[Using Formula]} \end{aligned}$$

$$\boxed{25v^2 - 1 = (5v+1)(5v-1)}$$

Problem 9: $\mathbf{4 - 49x^2}$

Solution:

$$\begin{aligned} a^2 - b^2 &= (a+b)(a-b) && \text{[Formula]} \\ 4 - 49x^2 &= 2^2 - (7x)^2 \\ 4 - 49x^2 &= (2+7x)(2-7x) && \text{[Using Formula]} \end{aligned}$$

$$\boxed{4 - 49x^2 = (2+7x)(2-7x)}$$

Problem 10: $\mathbf{36p^2 - 49q^2}$

Solution:

$$\begin{aligned} a^2 - b^2 &= (a+b)(a-b) && \text{[Formula]} \\ 36p^2 - 49q^2 &= (6p)^2 - (7q)^2 \\ 36p^2 - 49q^2 &= (6p+7q)(6p-7q) && \text{[Using Formula]} \end{aligned}$$

$$\boxed{36p^2 - 49q^2 = (6p+7q)(6p-7q)}$$

Problem 11: $\boldsymbol{x^2 - 16x + 64 - y^2}$

Solution:

$$\begin{aligned}
x^2 - 16x + 64 - y^2 &= (x^2 - 16x + 64) - y^2 && \text{[Grouping the terms]} \\
&= [x^2 - 2(x)(8) + 8^2] - y^2 && \text{[Arranging the terms]} \\
&= (x - 8)^2 - y^2 && [(a+b)^2 = a^2 + 2ab + b^2] \\
x^2 - 16x + 64 - y^2 &= (x - 8 + y)(x - 8 - y) && [a^2 - b^2 = (a+b)(a-b)]
\end{aligned}$$

$$\boxed{x^2 - 16x + 64 - y^2 = (x + y - 8)(x - y - 8)}$$

Problem 12: $\boldsymbol{a^2 + 6a + 9 - 9b^2}$

Solution:

$$\begin{aligned}
a^2 + 6a + 9 - 9b^2 &= (a^2 + 6a + 9) - 9b^2 && \text{[Grouping the terms]} \\
&= [a^2 + 2(a)(3) + 3^2] - (3b)^2 && \text{[Arranging the terms]} \\
&= (a + 3)^2 - (3b)^2 && [(a+b)^2 = a^2 + 2ab + b^2] \\
a^2 + 6a + 9 - 9b^2 &= (a + 3 + 3b)(a + 3 - 3b) && [a^2 - b^2 = (a+b)(a-b)]
\end{aligned}$$

$$\boxed{a^2 + 6a + 9 - 9b^2 = (a + 3b + 3)(a - 3b + 3)}$$

Problem 13: $\boldsymbol{6p^2q^2 - 54p^2}$

Solution:

$6p^2q^2 - 54p^2 = 6p^2(q^2 - 9)$, since HCF is $6p^2$

$$\begin{aligned}
a^2 - b^2 &= (a + b)(a - b) && \text{[Formula]} \\
q^2 - 9 &= q^2 - 3^2 && \\
q^2 - 9 &= (q + 3)(q - 3) && \text{[Using Formula]}
\end{aligned}$$

$$\boxed{36p^2 - 49q^2 = 6p^2(q + 3)(q - 3)}$$

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

Problem 14: $\boldsymbol{x^3 + 125}$

Solution:

$$\begin{aligned}
a^3 + b^3 &= (a + b)(a^2 - ab + b^2) && \text{[Formula]} \\
x^3 + 125 &= x^3 + 5^3 && \\
x^3 + 125 &= (x + 5)(x^2 - (x)(5) + 5^2) && \text{[Using Formula]}
\end{aligned}$$

$$\boxed{x^3 + 125 = (x + 5)(x^2 - 5x + 25)}$$

Problem 15: $x^3 - 125$

Solution:

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2+ab+b^2) && \text{[Formula]} \\ x^3 - 125 &= x^3 - 5^3 \\ x^3 - 125 &= (x-5)(x^2+(x)(5)+5^2) && \text{[Using Formula]} \end{aligned}$$

$$\boxed{x^3 - 125 = (x-5)(x^2+5x+25)}$$

Problem 16: $z^6 - 27$

Solution:

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2+ab+b^2) && \text{[Formula]} \\ z^6 - 27 &= (z^2)^3 - 3^3 \\ z^6 - 27 &= (z^2-3)((z^2)^2+(z^2)(3)+3^2) && \text{[Using Formula]} \end{aligned}$$

$$\boxed{z^6 - 27 = (z^2-3)(z^4+3z^2+9)}$$

Problem 17: $125 - 27w^3$

Solution:

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2+ab+b^2) && \text{[Formula]} \\ 125 - 27w^3 &= 5^3 - (3w)^3 \\ 125 - 27w^3 &= (5-3w)(5^2+(5)(3w)+(3w)^2) && \text{[Using Formula]} \end{aligned}$$

$$\boxed{125 - 27w^3 = (5-3w)(9w^2+15w+25)}$$

Problem 18: $7k^3 + 56$

Solution:

$7k^3 + 56 = 7(k^3+8)$

$$\begin{aligned} a^3 + b^3 &= (a+b)(a^2-ab+b^2) && \text{[Formula]} \\ k^3 + 8 &= k^3 + 2^3 \\ k^3 + 8 &= (k+2)(k^2-(k)(2)+2^2) && \text{[Using Formula]} \end{aligned}$$

$$\boxed{7k^3 + 56 = 7(k+2)(k^2-2k+4)}$$

Problem 19: $\mathbf{27y^3 + 8z^3}$

Solution:

$$\begin{aligned} a^3+b^3 &= (a+b)(a^2-ab+b^2) && \text{[Formula]} \\ 27y^3+8z^3 &= (3y)^3+(2z)^3 \\ 27y^3+8z^3 &= (3y+2z)[(3y)^2-(3y)(2z)+(2z)^2] && \text{[Using Formula]} \end{aligned}$$

$$\boxed{27y^3+8z^3 = (3y+2z)(9y^2-6yz+4z^2)}$$

Problem 20: $\mathbf{(x+3)^3 + 8x^3}$

Solution:

$$\begin{aligned} a^3+b^3 &= (a+b)(a^2-ab+b^2) && \text{[Formula]} \\ (x+3)^3+8x^3 &= (x+3)^3+(2x)^3 \\ &= (x+3+2x)[(x+3)^2-(x+3)(2x)+(2x)^2] && \text{[Using Formula]} \\ &= (3x+3)([x^2+6x+9]-[2x^2+6x]+4x^2) \\ &= 3(x+1)([x^2-2x^2+4x^2]+[6x-6x]+9) \\ (x+3)^3+8x^3 &= 3(x+1)(3x^2+9) \end{aligned}$$

$$\boxed{(x+3)^3+8x^3 = 9(x+1)(x^2+3)}$$

Problem 21: $\mathbf{(y-5)^3 - 64y^3}$

Solution:

$$\begin{aligned} a^3-b^3 &= (a-b)(a^2+ab+b^2) && \text{[Formula]} \\ (y-5)^3-64y^3 &= (y-5)^3-(4y)^3 \\ &= (y-5-4y)[(y-5)^2+(y-5)(4y)+(4y)^2] && \text{[Using Formula]} \\ &= (-3y-5)([y^2-10y+25]+[4y^2-20y]+16y^2) \\ &= -(3y+5)([y^2+4y^2+16y^2]+[-10y-20y]+25) \\ (y-5)^3-64y^3 &= -(3y+5)(21y^2-30y+25) \end{aligned}$$

$$\boxed{(y-5)^3-64y^3 = -(3y+5)(21y^2-30y+25)}$$

Problem 22: $\mathbf{-2x^3y^2 - 16y^5}$

Solution:

$$-2x^3y^2-16y^5 = -2y^2(x^3+8y^3)$$

$$\begin{aligned} a^3+b^3 &= (a+b)(a^2-ab+b^2) && \text{[Formula]} \\ x^3+8y^3 &= x^3+(2y)^3 \\ x^3+8y^3 &= (x+2y)(x^2-(x)(2y)+(2y)^2) && \text{[Using Formula]} \end{aligned}$$

$$\boxed{-2x^3y^2-16y^5 = -2y^2(x+2y)(x^2-2xy+4y^2)}$$

Chapter 5

RATIONAL EXPRESSIONS

5.1 Multiplication and Division of Rational Expressions

Rational Expression

A rational expression is an expression of the form $\dfrac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Simplified Rational Expression

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

Equivalent Fractions Property

If a, b and c are numbers, where $b \neq 0$ and $c \neq 0$, then

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

Multiplication of Rational Expressions

If p, q, r and s are polynomials, where $q \neq 0$ and $s \neq 0$, then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

Division of Rational Expressions

If p, q, r and s are polynomials where $q \neq 0$, $r \neq 0$ and $s \neq 0$ then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r} = \frac{ps}{qr}$$

Exercise - 5.1

In the following exercises, simplify each rational expression.

Problem 1: $\boldsymbol{\frac{36v^3w^2}{27vw^3}}$

Solution:

$$\frac{36v^3w^2}{27vw^3} = \frac{(\cancel{9} \times 4) \cdot (\cancel{v} \times v^2) \cdot \cancel{w^2}}{(\cancel{9} \times 3) \cdot \cancel{v} \cdot (w \times \cancel{w^2})}$$

$$\boldsymbol{\frac{36v^3w^2}{27vw^3} = \frac{4v^2}{3w}} \quad \text{[No common factor in the numerator and denominator]}$$

Problem 2: $\boldsymbol{\frac{x^2+4x-5}{x^2-2x+1}}$

Solution:

Expression	Product	Sum	Factors
x^2+4x-5	$-5=-1\times 5$	$4=-1+5$	$(x-1)(x+5)$
x^2-2x+1	$1=(-1)\times(-1)$	$-2=(-1)+(-1)$	$(x-1)(x-1)$

$$\frac{x^2+4x-5}{x^2-2x+1} = \frac{\cancel{(x-1)}(x+5)}{\cancel{(x-1)}(x-1)}$$

$$\boldsymbol{\frac{x^2+4x-5}{x^2-2x+1} = \frac{x+5}{x-1}} \quad \text{[No common factor in the numerator and denominator]}$$

Problem 3: $\boldsymbol{\dfrac{8m^3n}{12mn^2}}$

Solution:

$$\frac{8m^3n}{12mn^2} = \frac{(2\times\cancel{4})\cdot(\cancel{m}\times m^2)\cdot\cancel{n}}{(3\times\cancel{4})\cdot\cancel{m}\cdot(n\times\cancel{n})}$$

$$\boldsymbol{\frac{8m^3n}{12mn^2} = \frac{2m^2}{3n}}$$ [No common factor in numerator and denominator]

Problem 4: $\boldsymbol{\dfrac{a^2-4}{a^2+6a-16}}$

Solution:

Expression	Factors
$a^2+6a-16$	$(a-2)(a+8)$
a^2-4	$(a-2)(a+2)$

$$\frac{a^2-4}{a^2+6a-16} = \frac{\cancel{(a-2)}(a+2)}{\cancel{(a-2)}(a+8)}$$

$$\boldsymbol{\frac{a^2-4}{a^2+6a-16} = \frac{a+2}{a+8}}$$ [No common factor in numerator and denominator]

Problem 5: $\boldsymbol{\dfrac{x^2-y^2}{x^3-y^3}}$

Solution:

Expression	Factors
x^2-y^2	$(x-y)(x+y)$
x^3-y^3	$(x-y)(x^2+xy+y^2)$

$$\frac{x^2-y^2}{x^3-y^3} = \frac{\cancel{(x-y)}(x+y)}{\cancel{(x-y)}(x^2+xy+y^2)}$$

$$\boldsymbol{\frac{x^2-y^2}{x^3-y^3} = \frac{x+y}{x^2+xy+y^2}}$$ [No common factor in numerator and denominator]

Problem 6: $\dfrac{x^3 - 2x^2 - 25x + 50}{x^2 - 25}$

Solution:

$$\begin{aligned}
\frac{x^3 - 2x^2 - 25x + 50}{x^2 - 25} &= \frac{x^2(x-2) - 25(x-2)}{x^2 - 25} \\
&= \frac{(x-2)\cancel{(x^2-25)}}{\cancel{x^2-25}}
\end{aligned}$$

$$\boldsymbol{\frac{x^3 - 2x^2 - 25x + 50}{x^2 - 25} = x - 2} \quad \text{[No common factor in numerator and denominator]}$$

In the following exercises, multiply the rational expressions.

Problem 7: $\dfrac{5x^2y^4}{12xy^3} \cdot \dfrac{6x^2}{20y^2}$

Solution:

$$\begin{aligned}
\frac{5x^2y^4}{12xy^3} \cdot \frac{6x^2}{20y^2} &= \frac{(5 \times 6)x^{2+2}y^4}{(12 \times 20)xy^{3+2}} \\
&= \frac{\cancel{30}x^{\cancel{4}^3}\cancel{y^4}}{\cancel{240}^8\cancel{x}y^{\cancel{5}}}
\end{aligned}$$

$$\boldsymbol{\frac{5x^2y^4}{12xy^3} \cdot \frac{6x^2}{20y^2} = \frac{x^3}{8y}}$$

Problem 8: $\dfrac{12a^3b}{b^2} \cdot \dfrac{2ab^2}{9b^3}$

Solution:

$$\begin{aligned}
\frac{12a^3b}{b^2} \cdot \frac{2ab^2}{9b^3} &= \frac{(12 \times 2)a^{3+1}b^{2+1}}{9b^{2+3}} \\
&= \frac{\cancel{24}^8a^4\cancel{b^3}}{\cancel{9}^3b^{\cancel{5}^2}}
\end{aligned}$$

$$\boldsymbol{\frac{12a^3b}{b^2} \cdot \frac{2ab^2}{9b^3} = \frac{8a^4}{3b^2}}$$

Problem 9: $\dfrac{5p^2}{p^2-5p-36}\cdot\dfrac{p^2-16}{10p}$

Solution:

$$\begin{aligned}\frac{5p^2}{p^2-5p-36}\cdot\frac{p^2-16}{10p} &= \frac{\cancel{5}p^{\cancel{2}}}{(p+4)(p-9)}\cdot\frac{(p^2-4^2)}{\cancel{10}^2\cancel{p}}\\ &= \frac{p(p-4)\cancel{(p+4)}}{2(p-9)\cancel{(p+4)}}\end{aligned}$$

$$\mathbf{\frac{5p^2}{p^2-5p-36}\cdot\frac{p^2-16}{10p} = \frac{p(p-4)}{2(p-9)}}$$

Problem 10: $\dfrac{2y^2-10y}{y^2+10y+25}\cdot\dfrac{y+5}{6y}$

Solution:

$$\frac{2y^2-10y}{y^2+10y+25}\cdot\frac{y+5}{6y} = \frac{\cancel{2}\cancel{y}(y-5)}{(y+5)^{\cancel{2}}}\cdot\frac{\cancel{(y+5)}}{\cancel{6}^3\cancel{y}}$$

$$\mathbf{\frac{2y^2-10y}{y^2+10y+25}\cdot\frac{y+5}{6y} = \frac{y-5}{3(y+5)}}$$

Problem 11: $\dfrac{2m^2-3m-2}{2m^2+7m+3}\cdot\dfrac{3m^2-14m+15}{3m^2+17m-20}$

Solution:

$2m^2-3m-2=(2m+1)(m-2)$
$2m^2+7m+3=(2m+1)(m+3)$
$3m^2-14m+15=(m-3)(3m-5)$
$3m^2+17m-20=(m-1)(3m+20)$

$$\frac{2m^2-3m-2}{2m^2+7m+3}\cdot\frac{3m^2-14m+15}{3m^2+17m-20} = \frac{\cancel{(2m+1)}(m-2)}{\cancel{(2m+1)}(m+3)}\cdot\frac{(m-3)(3m-5)}{(m-1)(3m+20)}$$

$$\mathbf{\frac{2m^2-3m-2}{2m^2+7m+3}\cdot\frac{3m^2-14m+15}{3m^2+17m-20} = \frac{(m-2)(m-3)(3m-5)}{(m+3)(m-1)(3m+20)}}$$

Problem 12: $\dfrac{2n^2-3n-14}{25-n^2} \cdot \dfrac{n^2-10n+25}{2n^2-13n+21}$

Solution:

$$\begin{aligned}
\frac{2n^2-3n-14}{25-n^2} \cdot \frac{n^2-10n+25}{2n^2-13n+21} &= \frac{(n+2)\cancel{(2n-7)}}{(5+n)(5-n)} \cdot \frac{(n-5)^2}{(n-3)\cancel{(2n-7)}} \\
&= \frac{(n+2)\cancel{(2n-7)}}{-(5+n)\cancel{(n-5)}} \cdot \frac{(n-5)^{\cancel{2}}}{(n-3)\cancel{(2n-7)}} \\
&= \frac{-(n+2)(n-5)}{(5+n)(n-3)}
\end{aligned}$$

$$\boldsymbol{\frac{2n^2-3n-14}{25-n^2} \cdot \frac{n^2-10n+25}{2n^2-13n+21} = \frac{-(n+2)(n-5)}{(5+n)(n-3)}}$$

In the following exercises, divide the rational expressions

Problem 13: $\dfrac{v-5}{11-v} \div \dfrac{v^2-25}{v-11}$

Solution:

$$\begin{aligned}
\frac{v-5}{11-v} \div \frac{v^2-25}{v-11} &= \frac{v-5}{11-v} \times \frac{v-11}{v^2-25} \\
&= \frac{\cancel{(v-5)}}{-\cancel{(v-11)}} \times \frac{\cancel{(v-11)}}{(v+5)\cancel{(v-5)}}
\end{aligned}$$

$$\boldsymbol{\frac{v-5}{11-v} \div \frac{v^2-25}{v-11} = \frac{-1}{v+5}}$$

Problem 14: $\dfrac{10+w}{w-8} \div \dfrac{100-w^2}{8-w}$

Solution:

$$\begin{aligned}
\frac{10+w}{w-8} \div \frac{100-w^2}{8-w} &= \frac{10+w}{w-8} \times \frac{8-w}{100-w^2} \\
&= \frac{\cancel{(10+w)}}{-\cancel{(8-w)}} \times \frac{\cancel{(8-w)}}{(10-w)\cancel{(10+w)}} \\
&= \frac{1}{-1(10-w)}
\end{aligned}$$

$$\boldsymbol{\frac{10+w}{w-8} \div \frac{100-w^2}{8-w} = \frac{1}{w-10}}$$

Problem 15: $\boldsymbol{\dfrac{x^2+3x-10}{4x} \div (2x^2+20x+50)}$

Solution:

$$\begin{aligned}
\frac{x^2+3x-10}{4x} \div (2x^2+20x+50) &= \frac{x^2+3x-10}{4x} \times \frac{1}{2x^2+20x+50} \\
&= \frac{(x+5)(x-2)}{4x} \times \frac{1}{2(x^2+10x+25)} \\
&= \frac{\cancel{(x+5)}(x-2)}{4x} \times \frac{1}{2(x+5)^{\cancel{2}}} \\
\boldsymbol{\frac{x^2+3x-10}{4x} \div (2x^2+20x+50)} &= \boldsymbol{\frac{x-2}{8x(x+5)}}
\end{aligned}$$

Problem 16: $\boldsymbol{\dfrac{2y^2-10yz-48z^2}{2y-1} \div (4y^2-32yz)}$

Solution:

$$\begin{aligned}
\frac{2y^2-10yz-48z^2}{(2y-1)} \div (4y^2-32yz) &= \frac{2y^2-10yz-48z^2}{2y-1} \times \frac{1}{(4y^2-32yz)} \\
&= \frac{2(y^2-5yz-24z^2)}{(2y-1)} \times \frac{1}{4y(y-8z)} \\
&= \frac{2(y^2+3yz-8yz-24z^2)}{(2y-1)} \times \frac{1}{4y(y-8z)} \\
&= \frac{2\left[y(y+3z)-8z(y+3z)\right]}{(2y-1)} \times \frac{1}{4y(y-8z)} \\
&= \frac{\cancel{2}(y+3z)\cancel{(y-8z)}}{(2y-1)} \times \frac{1}{\cancel{4}^{2}y\cancel{(y-8z)}} \\
\boldsymbol{\frac{2y^2-10yz-48z^2}{(2y-1)} \div (4y^2-32yz)} &= \boldsymbol{\frac{y+3z}{2y(2y-1)}}
\end{aligned}$$

For the following exercises, perform the indicated operations

Problem 17: $\dfrac{10m^2+80m}{3m-9}\cdot\dfrac{m^2+4m-21}{m^2-9m+20}\div\dfrac{5m^2+10m}{2m-10}$

Solution:

$$\begin{aligned}
&\frac{10m^2+80m}{3m-9}\cdot\frac{m^2+4m-21}{m^2-9m+20}\div\frac{5m^2+10m}{2m-10}\\
&\qquad=\frac{10m^2+80m}{3m-9}\times\frac{m^2+4m-21}{m^2-9m+20}\times\frac{2m-10}{5m^2+10m}\\
&\qquad=\frac{\cancel{10}^2\cancel{m}(m+8)}{3\cancel{(m-3)}}\times\frac{(m+7)\cancel{(m-3)}}{(m-4)\cancel{(m-5)}}\times\frac{2\cancel{(m-5)}}{\cancel{5}\cancel{m}(m+2)}
\end{aligned}$$

$$\boldsymbol{\frac{10m^2+80m}{3m-9}\cdot\frac{m^2+4m-21}{m^2-9m+20}\div\frac{5m^2+10m}{2m-10}=\frac{4(m+7)(m+8)}{3(m-4)(m+2)}}$$

Problem 18: $\dfrac{4n^2+32n}{3n+2}\cdot\dfrac{3n^2-n-2}{n^2+n-30}\div\dfrac{108n^2-24n}{n+6}$

Solution:

$$\begin{aligned}
&\frac{4n^2+32n}{3n+2}\cdot\frac{3n^2-n-2}{n^2+n-30}\div\frac{108n^2-24n}{n+6}\\
&\qquad=\frac{4n(n+8)}{3n+2}\cdot\frac{(n-1)(3n+2)}{(n+6)(n-5)}\div\frac{12n(9n-2)}{n+6}\\
&\qquad=\frac{\cancel{4}\cancel{n}(n+8)}{\cancel{3n+2}}\cdot\frac{(n-1)\cancel{(3n+2)}}{\cancel{(n+6)}(n-5)}\times\frac{\cancel{n+6}}{\cancel{12}^3\cancel{n}(9n-2)}
\end{aligned}$$

$$\boldsymbol{\frac{4n^2+32n}{3n+2}\cdot\frac{3n^2-n-2}{n^2+n-30}\div\frac{108n^2-24n}{n+6}=\frac{(n-1)(n+8)}{3(n-5)(9n-2)}}$$

5.2 Addition and Subtraction of Rational Expressions

Add and Subtract Rational Expressions with a Common Denominator

If p, q and r are polynomials where $r\neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r}\text{ and }\frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

Exercise - 5.2

Add and Subtract Rational Expressions with a Common Denominator

Problem 1: $\frac{7m}{2m+n} + \frac{4}{2m+n}$

Solution:

$$\frac{7m}{2m+n} + \frac{4}{2m+n} = \frac{7m+4}{2m+n}$$

Problem 2: $\frac{2r^2}{2r-1} + \frac{15r-8}{2r-1}$

Solution:

$$\begin{aligned}\frac{2r^2}{2r-1} + \frac{15r-8}{2r-1} &= \frac{2r^2+15r-8}{2r-1} \\ &= \frac{\cancel{(2r-1)}(r+8)}{\cancel{2r-1}}\end{aligned}$$

$$\frac{2r^2}{2r-1} + \frac{15r-8}{2r-1} = r+8$$

Problem 3: $\frac{3s^2}{3s-2} + \frac{13s-10}{3s-2}$

Solution:

$$\begin{aligned}\frac{3s^2}{3s-2} + \frac{13s-10}{3s-2} &= \frac{3s^2+13s-10}{3s-2} \\ &= \frac{\cancel{(3s-2)}(s+5)}{\cancel{3s-2}}\end{aligned}$$

$$\frac{3s^2}{3s-2} + \frac{13s-10}{3s-2} = s+5$$

Problem 4: $\dfrac{3m^2}{6m-30} - \dfrac{21m-30}{6m-30}$

Solution:

$$\begin{aligned}
\frac{3m^2}{6m-30} - \frac{21m-30}{6m-30} &= \frac{3m^2-(21m-30)}{6m-30} \\
&= \frac{3m^2-21m+30}{6m-30} \\
&= \frac{3(m^2-7m+10)}{6(m-5)} \\
&= \frac{\cancel{3}(m-2)\cancel{(m-5)}}{\cancel{6}2\cancel{(m-5)}} \\
\frac{3m^2}{6m-30} - \frac{21m-30}{6m-30} &= \frac{m-2}{2}
\end{aligned}$$

Problem 5: $\dfrac{5r^2+7r-33}{r^2-49} - \dfrac{4r^2+5r+30}{r^2-49}$

Solution:

$$\begin{aligned}
\frac{5r^2+7r-33}{r^2-49} - \frac{4r^2+5r+30}{r^2-49} &= \frac{(5r^2+7r-33)-(4r^2+5r+30)}{r^2-49} \\
&= \frac{5r^2+7r-33-4r^2-5r-30}{r^2-49} \\
&= \frac{r^2+2r-63}{r^2-7^2} \\
&= \frac{(r+9)\cancel{(r-7)}}{(r+7)\cancel{(r-7)}} \\
\frac{5r^2+7r-33}{r^2-49} - \frac{4r^2+5r+30}{r^2-49} &= \frac{r+9}{r+7}
\end{aligned}$$

Add and Subtract Rational Expressions whose Denominators are Opposites

Problem 6: $\dfrac{10x^2+16x-7}{8x-3}+\dfrac{2x^2+3x-1}{3-8x}$

Solution:

$$\begin{aligned}
\frac{10x^2+16x-7}{8x-3}+\frac{2x^2+3x-1}{3-8x} &= \frac{10x^2+16x-7}{8x-3}+\frac{2x^2+3x-1}{[-(8x-3)]}\\
&= \frac{10x^2+16x-7}{8x-3}-\frac{2x^2+3x-1}{8x-3}\\
&= \frac{(10x^2+16x-7)-(2x^2+3x-1)}{8x-3}\\
&= \frac{8x^2+13x-6}{8x-3}\\
&= \frac{(x+2)\cancel{(8x-3)}}{\cancel{(8x-3)}}\\
\boldsymbol{\frac{10x^2+16x-7}{8x-3}+\frac{2x^2+3x-1}{3-8x}} &\boldsymbol{= x+2}
\end{aligned}$$

Problem 7: $\dfrac{a^2+3a}{a^2-9}-\dfrac{3a-27}{9-a^2}$

Solution:

$$\begin{aligned}
\frac{a^2+3a}{a^2-9}-\frac{3a-27}{9-a^2} &= \frac{a^2+3a}{a^2-9}-\frac{3a-27}{-(a^2-9)}\\
&= \frac{a^2+3a}{a^2-9}+\frac{3a-27}{a^2-9}\\
&= \frac{(a^2+3a)+(3a-27)}{a^2-9}\\
&= \frac{a^2+6a-27}{a^2-3^2}\\
&= \frac{(a+9)\cancel{(a-3)}}{(a+3)\cancel{(a-3)}}\\
\boldsymbol{\frac{a^2+3a}{a^2-9}-\frac{3a-27}{9-a^2}} &\boldsymbol{= \frac{a+9}{a+3}}
\end{aligned}$$

In the following exercises, perform the indicated operations.

Problem 8: $\dfrac{7}{10x^2y}+\dfrac{4}{5xy^2}$

Solution:

$$\begin{aligned}
\frac{7}{10x^2y}+\frac{4}{5xy^2} &= \frac{7}{10x^2y}\cdot\left(\frac{y}{y}\right)+\frac{4}{5xy^2}\cdot\left(\frac{2x}{2x}\right)\\
&\qquad [\text{LCD}(10x^2y,5xy^2)=10x^2y^2]\\
&= \frac{7y}{10x^2y^2}+\frac{8x}{10x^2y^2}\\
&= \frac{7y+8x}{10x^2y^2}\\
\mathbf{\frac{7}{10x^2y}+\frac{4}{5xy^2}} &= \mathbf{\frac{8x+7y}{10x^2y^2}}
\end{aligned}$$

Problem 9: $\dfrac{1}{12a^3b^2}+\dfrac{5}{9a^2b^3}$

Solution:

$$\begin{aligned}
\frac{1}{12a^3b^2}+\frac{5}{9a^2b^3} &= \frac{1}{12a^3b^2}\cdot\left(\frac{3b}{3b}\right)+\frac{5}{9a^2b^3}\cdot\left(\frac{4a}{4a}\right)\\
&\qquad [\text{LCD}(12a^3b^2,9a^2b^3)=36a^3b^3]\\
&= \frac{3b}{36a^3b^3}+\frac{20a}{36a^3b^3}\\
&= \frac{3b+20a}{36a^3b^3}\\
\mathbf{\frac{1}{12a^3b^2}+\frac{5}{9a^2b^3}} &= \mathbf{\frac{20a+3b}{36a^3b^3}}
\end{aligned}$$

Problem 10: $\boldsymbol{\dfrac{4}{2x+5}+\dfrac{2}{x-1}}$

Solution:

$$\begin{aligned}
\frac{4}{2x+5}+\frac{2}{x-1} &= \frac{4(x-1)}{(2x+5)(x-1)}+\frac{2(2x+5)}{(x-1)(2x+5)} \\
&\qquad [\text{LCD}(2x+5, x-1) = (2x+5)(x-1)] \\
&= \frac{4x-4}{(x-1)(2x+5)}+\frac{4x+10}{(x-1)(2x+5)} \\
&= \frac{4x-4+4x+10}{(x-1)(2x+5)} \\
&= \frac{8x+6}{(x-1)(2x+5)} \\
\boldsymbol{\frac{4}{2x+5}+\frac{2}{x-1}} &\boldsymbol{= \frac{2(4x+3)}{(x-1)(2x+5)}}
\end{aligned}$$

Problem 11: $\boldsymbol{\dfrac{t}{t-6}-\dfrac{t-2}{t+6}}$

Solution:

$$\begin{aligned}
\frac{t}{t-6}-\frac{t-2}{t+6} &= \frac{t(t+6)}{(t-6)(t+6)}-\frac{(t-2)(t-6)}{(t-6)(t+6)} \\
&\qquad [\text{LCD}(t-6, t+6) = (t-6)(t+6)] \\
&= \frac{t^2+6t}{(t-6)(t+6)}-\frac{t^2-2t-6t+12}{(t-6)(t+6)} \\
&= \frac{t^2+6t}{(t-6)(t+6)}-\frac{t^2-8t+12}{(t-6)(t+6)} \\
&= \frac{(t^2+6t)-(t^2-8t+12)}{(t-6)(t+6)} \\
&= \frac{t^2+6t-t^2+8t-12}{t^2-6^2} \\
&= \frac{14t-12}{t^2-36} \\
\boldsymbol{\frac{t}{t-6}-\frac{t-2}{t+6}} &\boldsymbol{= \frac{2(7t-6)}{t^2-36}}
\end{aligned}$$

Problem 12: $\dfrac{6}{m+6} - \dfrac{12m}{m^2-36}$

Solution:

$$\begin{aligned}
\frac{6}{m+6} - \frac{12m}{m^2-36} &= \frac{6}{m+6} - \frac{12m}{(m+6)(m-6)} \\
&\quad [\text{LCD}(m+6, (m+6)(m-6)) = (m+6)(m-6)] \\
&= \frac{6(m-6)}{(m+6)(m-6)} - \frac{12m}{(m+6)(m-6)} \\
&= \frac{6m-36}{(m+6)(m-6)} - \frac{12m}{(m+6)(m-6)} \\
&= \frac{6m-36-12m}{(m+6)(m-6)} \\
&= \frac{-6m-36}{(m+6)(m-6)} \\
&= \frac{-6\cancel{(m+6)}}{\cancel{(m+6)}(m-6)}
\end{aligned}$$

$$\boldsymbol{\frac{6}{m+6} - \frac{12m}{m^2-36} = \frac{-6}{m-6}}$$

Problem 13: $\dfrac{4}{x^2-6x+5} - \dfrac{3}{x^2-7x+10}$

Solution:

$$\begin{aligned}
&\frac{4}{x^2-6x+5} - \frac{3}{x^2-7x+10} \\
&= \frac{4}{(x-1)(x-5)} - \frac{3}{(x-2)(x-5)} \\
&\qquad [\text{LCD}((x-1)(x-5), (x-2)(x-5)) = (x-1)(x-2)(x-5)] \\
&= \frac{4(x-2)}{(x-1)(x-2)(x-5)} - \frac{3(x-1)}{(x-1)(x-2)(x-5)} \\
&= \frac{4x-8}{(x-1)(x-2)(x-5)} - \frac{3x-3}{(x-1)(x-2)(x-5)}
\end{aligned}$$

$$= \frac{(4x-8)-(3x-3)}{(x-1)(x-2)(x-5)}$$
$$= \frac{4x-8-3x+3}{(x-1)(x-2)(x-5)}$$
$$= \frac{\cancel{x-5}}{(x-1)(x-2)\cancel{(x-5)}}$$
$$\boxed{\frac{4}{x^2-6x+5} - \frac{3}{x^2-7x+10} = \frac{1}{(x-1)(x-2)}}$$

Problem 14: $\dfrac{9p-17}{p^2-4p-21} - \dfrac{p+1}{7-p}$

Solution:

$$\frac{9p-17}{p^2-4p-21} - \frac{p+1}{7-p}$$
$$= \frac{9p-17}{(p-7)(p+3)} - \frac{p+1}{-(p-7)}$$
$$= \frac{9p-17}{(p-7)(p+3)} + \frac{p+1}{p-7}$$
$$[\text{LCD}((p-7)(p+3),(p-7)) = (p-7)(p+3)]$$
$$= \frac{9p-17}{(p-7)(p+3)} + \frac{(p+1)(p+3)}{(p-7)(p+3)}$$
$$= \frac{9p-17}{(p-7)(p+3)} + \frac{p^2+4p+3}{(p-7)(p+3)}$$
$$= \frac{(9p-17)+(p^2+4p+3)}{(p-7)(p+3)}$$
$$= \frac{9p-17+p^2+4p+3}{(p-7)(p+3)}$$
$$= \frac{p^2+13p-14}{(p-7)(p+3)}$$
$$= \frac{(p+14)(p-1)}{(p-7)(p+3)}$$
$$\boxed{\frac{9p-17}{p^2-4p-21} - \frac{p+1}{7-p} = \frac{(p+14)(p-1)}{(p-7)(p+3)}}$$

Problem 15: $\dfrac{2t-30}{t^2+6t-27} - \dfrac{2}{3-t}$

Solution:

$$\begin{aligned}
&\frac{2t-30}{t^2+6t-27} - \frac{2}{3-t} \\
&= \frac{2t-30}{(t+9)(t-3)} - \frac{2}{-(t-3)} \\
&= \frac{2t-30}{(t+9)(t-3)} + \frac{2}{t-3} \\
&\qquad [\text{LCD}((t+9)(t-3),(t-3)) = (t+9)(t-3)] \\
&= \frac{2t-30}{(t+9)(t-3)} + \frac{2(t+9)}{(t+9)(t-3)} \\
&= \frac{2t-30}{(t+9)(t-3)} + \frac{2t+18}{(t+9)(t-3)} \\
&= \frac{2t-30+2t+18}{(t+9)(t-3)} \\
&= \frac{4t-12}{(t+9)(t-3)} \\
&= \frac{4\cancel{(t-3)}}{(t+9)\cancel{(t-3)}} \\
&\boxed{\frac{2t-30}{t^2+6t-27} - \frac{2}{3-t} = \frac{4}{t+9}}
\end{aligned}$$

Problem 16: $\dfrac{5a}{a-2} + \dfrac{9}{a} - \dfrac{2a+18}{a^2-2a}$

Solution:

$$\begin{aligned}
&\frac{5a}{a-2} + \frac{9}{a} - \frac{2a+18}{a^2-2a} \\
&= \frac{5a}{a-2} + \frac{9}{a} - \frac{2a+18}{a(a-2)} \\
&\qquad [\text{LCD}((a-2),a,a(a-2)) = a(a-2)] \\
&= \frac{5a(a)}{a(a-2)} + \frac{9(a-2)}{a(a-2)} - \frac{2a+18}{a(a-2)}
\end{aligned}$$

$$= \frac{5a^2}{a(a-2)} + \frac{9a-18}{a(a-2)} - \frac{2a+18}{a(a-2)}$$
$$= \frac{5a^2+9a-18-(2a+18)}{a(a-2)}$$
$$= \frac{5a^2+9a-18-2a-18}{a(a-2)}$$
$$= \frac{5a^2+7a-36}{a(a-2)}$$
$$\boxed{\frac{5a}{a-2} + \frac{9}{a} - \frac{2a+18}{a^2-2a} = \frac{5a^2+7a-36}{a(a-2)}}$$

Problem 17: $\frac{6d}{d-5} + \frac{1}{d+4} - \frac{7d-5}{d^2-d-20}$

Solution:

$$\frac{6d}{d-5} + \frac{1}{d+4} - \frac{7d-5}{d^2-d-20}$$
$$= \frac{6d}{d-5} + \frac{1}{d+4} - \frac{7d-5}{(d-5)(d+4)}$$
$$[\text{LCD}((d-5),(d+4),(d-5)(d+4)) = (d-5)(d+4)]$$
$$= \frac{6d(d+4)}{(d-5)(d+4)} + \frac{(d-5)}{(d-5)(d+4)} - \frac{7d-5}{(d-5)(d+4)}$$
$$= \frac{6d^2+24d}{(d-5)(d+4)} + \frac{(d-5)}{(d-5)(d+4)} - \frac{7d-5}{(d-5)(d+4)}$$
$$= \frac{6d^2+24d+d-5-(7d-5)}{(d-5)(d+4)}$$
$$= \frac{6d^2+24d+d-5-7d+5}{(d-5)(d+4)}$$
$$= \frac{6d^2+18d}{(d-5)(d+4)}$$
$$\boxed{\frac{6d}{d-5} + \frac{1}{d+4} - \frac{7d-5}{d^2-d-20} = \frac{6d(d+3)}{(d-5)(d+4)}}$$

5.3 Simplification of Complex Rational Expressions

Complex Rational Expression

A complex rational expression is a rational expression in which the numerator and/or the denominator contains a rational expression.

Exercise - 5.3

In the following exercises, simplify each complex rational expression by writing it as division

Problem 1: $\dfrac{\dfrac{2a}{a+4}}{\dfrac{4a^2}{a^2-16}}$

Solution:

$$\begin{aligned}
\dfrac{\dfrac{2a}{a+4}}{\dfrac{4a^2}{a^2-16}} &= \frac{2a}{a+4} \div \frac{4a^2}{a^2-16} \\
&= \frac{2a}{a+4} \times \frac{a^2-16}{4a^2} \\
&= \frac{\cancel{2}\cancel{a}}{a+4} \times \frac{a^2-4^2}{\cancel{4}^2 a^{\cancel{2}}} \\
&= \frac{\cancel{(a+4)}(a-4)}{2a\cancel{(a+4)}} \\
&= \frac{a-4}{2a}
\end{aligned}$$

$$\boldsymbol{\dfrac{\dfrac{2a}{a+4}}{\dfrac{4a^2}{a^2-16}} = \frac{a-4}{2a}}$$

Problem 2: $\dfrac{\dfrac{3b}{b-5}}{\dfrac{b^2}{b^2-25}}$

Solution:

$$\begin{aligned}
\frac{\dfrac{3b}{b-5}}{\dfrac{b^2}{b^2-25}} &= \frac{3b}{b-5} \div \frac{b^2}{b^2-25} \\
&= \frac{3b}{b-5} \times \frac{b^2-25}{b^2} \\
&= \frac{3\cancel{b}}{b-5} \times \frac{b^2-5^2}{b^{\cancel{2}}} \\
&= \frac{3\cancel{(b-5)}(b+5)}{b\cancel{(b-5)}}
\end{aligned}$$

$$\frac{\dfrac{3b}{b-5}}{\dfrac{b^2}{b^2-25}} = \frac{3(b+5)}{b}$$

Problem 3: $\dfrac{\dfrac{5}{c^2+5c-14}}{\dfrac{10}{c+7}}$

Solution:

$$\begin{aligned}
\frac{\dfrac{5}{c^2+5c-14}}{\dfrac{10}{c+7}} &= \frac{5}{c^2+5c-14} \div \frac{10}{c+7} \\
&= \frac{\cancel{5}}{c^2+5c-14} \times \frac{c+7}{\cancel{10}2} \\
&= \frac{\cancel{(c+7)}}{2\cancel{(c+7)}(c-2)}
\end{aligned}$$

$$\frac{\dfrac{5}{c^2+5c-14}}{\dfrac{10}{(c+7)}} = \frac{1}{2(c-2)}$$

Problem 4: $\dfrac{\dfrac{\mathbf{8}}{\boldsymbol{d^2+9d+18}}}{\dfrac{\mathbf{12}}{\boldsymbol{d+6}}}$

Solution:

$$\begin{aligned}\frac{\frac{8}{d^2+9d+18}}{\frac{12}{d+6}} &= \frac{\cancel{8}2}{d^2+9d+18}\times\frac{d+6}{\cancel{12}3}\\ &= \frac{2\cancel{(d+6)}}{3(d+3)\cancel{(d+6)}}\end{aligned}$$

$$\boldsymbol{\frac{\frac{8}{d^2+9d+18}}{\frac{12}{d+6}} = \frac{2}{3(d+3)}}$$

Problem 5: $\dfrac{\frac{\mathbf{1}}{\mathbf{2}}+\frac{\mathbf{5}}{\mathbf{6}}}{\frac{\mathbf{2}}{\mathbf{3}}+\frac{\mathbf{7}}{\mathbf{9}}}$

Solution:

$$\begin{aligned}\frac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}} &= \left(\frac{1}{2}+\frac{5}{6}\right)\div\left(\frac{2}{3}+\frac{7}{9}\right)\\ &= \left(\frac{1\times3}{2\times3}+\frac{5}{6}\right)\div\left(\frac{2\times3}{3\times3}+\frac{7}{9}\right) \qquad \text{[LCD(2,6)=6; LCD(3,9)=9]}\\ &= \left(\frac{3}{6}+\frac{5}{6}\right)\div\left(\frac{6}{9}+\frac{7}{9}\right)\\ &= \left(\frac{3+5}{6}\right)\div\left(\frac{6+7}{9}\right) = \frac{8}{6}\div\frac{13}{9}\\ &= \frac{\cancel{8}4}{\cancel{6}3}\times\frac{\cancel{9}3}{13}\end{aligned}$$

$$\boldsymbol{\frac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}} = \frac{12}{13}}$$

Problem 6: $\dfrac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

Solution:

$$\begin{aligned}
\frac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}} &= \left(\frac{2}{3}-\frac{1}{9}\right)\div\left(\frac{3}{4}+\frac{5}{6}\right)\\
&= \left(\frac{2\times 3}{3\times 3}-\frac{1}{9}\right)\div\left(\frac{3\times 3}{4\times 3}+\frac{5\times 2}{6\times 2}\right) \qquad [\text{LCD(3,9)=9; LCD(4,6)=12}]\\
&= \left(\frac{6}{9}-\frac{1}{9}\right)\div\left(\frac{9}{12}+\frac{10}{12}\right) = \left(\frac{6-1}{9}\right)\div\left(\frac{9+10}{12}\right) = \frac{5}{9}\div\frac{19}{12} = \frac{5}{\cancel{9}^{3}}\times\frac{\cancel{12}^{4}}{19}
\end{aligned}$$

$$\frac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}} = \mathbf{\frac{20}{57}}$$

Problem 7: $\dfrac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

Solution:

$$\begin{aligned}
\frac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}} &= \left(\frac{1}{p}+\frac{p}{q}\right)\div\left(\frac{q}{p}-\frac{1}{q}\right)\\
&= \left(\frac{1\times q}{p\times q}+\frac{p\times p}{q\times p}\right)\div\left(\frac{q\times q}{p\times q}-\frac{1\times p}{q\times p}\right) \qquad [\text{LCD}(p,q)=pq]\\
&= \left(\frac{q}{pq}+\frac{p^2}{pq}\right)\div\left(\frac{q^2}{pq}-\frac{p}{pq}\right) = \left(\frac{q+p^2}{pq}\right)\div\left(\frac{q^2-p}{pq}\right)\\
&= \frac{q+p^2}{\cancel{pq}}\times\frac{\cancel{pq}}{q^2-p}
\end{aligned}$$

$$\frac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}} = \boldsymbol{\frac{p^2+q}{q^2-p}}$$

Problem 8: $\dfrac{\dfrac{1}{r}+\dfrac{1}{t}}{\dfrac{1}{r^2}-\dfrac{1}{t^2}}$

Solution:

$$\begin{aligned}
\frac{\dfrac{1}{r}+\dfrac{1}{t}}{\dfrac{1}{r^2}-\dfrac{1}{t^2}} &= \left(\frac{1}{r}+\frac{1}{t}\right)\div\left(\frac{1}{r^2}-\frac{1}{t^2}\right)\\
&= \left(\frac{1\times t}{rt}+\frac{1\times r}{rt}\right)\div\left(\frac{1\times t^2}{r^2\times t^2}-\frac{1\times r^2}{t^2\times r^2}\right) \qquad [\mathrm{LCD}(r,t)=rt]\\
&= \left(\frac{t}{rt}+\frac{r}{rt}\right)\div\left(\frac{t^2}{r^2t^2}-\frac{r^2}{r^2t^2}\right) = \left(\frac{t+r}{rt}\right)\div\left(\frac{t^2-r^2}{(rt)^2}\right)\\
&= \left(\frac{t+r}{rt}\right)\times\left(\frac{(rt)^2}{t^2-r^2}\right) = \frac{\cancel{t+r}}{\cancel{rt}}\times\frac{(rt)^{\cancel{2}}}{\cancel{(t+r)}(t-r)}
\end{aligned}$$

$$\boldsymbol{\frac{\dfrac{1}{r}+\dfrac{1}{t}}{\dfrac{1}{r^2}-\dfrac{1}{t^2}} = \frac{rt}{t-r}}$$

Problem 9: $\dfrac{\dfrac{m}{n}+\dfrac{1}{n}}{\dfrac{1}{n}-\dfrac{n}{m}}$

Solution:

$$\begin{aligned}
\frac{\dfrac{m}{n}+\dfrac{1}{n}}{\dfrac{1}{n}-\dfrac{n}{m}} &= \left(\frac{m}{n}+\frac{1}{n}\right)\div\left(\frac{1}{n}-\frac{n}{m}\right)\\
&= \left(\frac{m+1}{n}\right)\div\left(\frac{1\times m}{n\times m}-\frac{n\times n}{m\times n}\right) \qquad [\mathrm{LCD}(m,n)=mn]\\
&= \left(\frac{m+1}{n}\right)\div\left(\frac{m}{nm}-\frac{n^2}{mn}\right) = \left(\frac{m+1}{n}\right)\div\left(\frac{m-n^2}{mn}\right)\\
&= \frac{m+1}{\cancel{n}}\times\frac{m\cancel{n}}{m-n^2}
\end{aligned}$$

$$\boldsymbol{\frac{\dfrac{m}{n}+\dfrac{1}{n}}{\dfrac{1}{n}-\dfrac{n}{m}} = \frac{m(m+1)}{m-n^2}}$$

Chapter 6

SYSTEMS OF MEASUREMENT

6.1 U.S. System

U.S. System Units	
Length	**Volume**
1 foot (ft) = 12 inches (in)	3 teaspoons (t) = 1 tablespoon
1 yard (yd) = 3 feet (ft)	16 Tablespoons (T) = 1 cup (C)
1 mile (mi) = 5280 feet (ft)	1 cup (C) = 8 fluid ounces (fl oz)
	1 pint (pt) = 2 cups (C)
	1 quart (qt) = 2 pints (pt)
	1 gallon (gal) = 4 quarts (qt)
Weight	**Time**
pound (lb) = 16 ounces (oz)	1 minute (min) = 60 seconds (s)
1 ton = 2000 pounds (lb)	1 hour (h) = 60 minutes (min)
	1 day = 24 hours (h)
	1 week (wk) = 7 days
	1 year (yr) = 365 days

Exercise - 6.1

In the following exercises, convert the units

Problem 1: A park bench is 6 feet long. Convert the length to inches

Solution:

$$\begin{aligned} 1 \text{ foot} &= 12 \text{ inches} && \text{[Formula]} \\ 6 \text{ feet} &= 6 \times 12 \text{ inches} \\ 6 \text{ feet} &= 72 \text{ inches} \end{aligned}$$

Hence the park bench is **72 inches** long

Problem 2: A ribbon is 18 inches long. Convert the length to feet

Solution:

$$\begin{aligned} 1 \text{ foot} &= 12 \text{ inches} && \text{[Formula]} \\ \text{Then, } 1 \text{ inch} &= \frac{1}{12} \text{ feet} \\ 18 \text{ inches} &= \cancel{18}^{3} \times \frac{1}{\cancel{12}^{2}} \text{ feet} \\ &= \frac{3}{2} \text{ feet} \end{aligned}$$

Hence the ribbon is **1.5 feet** long

Problem 3: Salim is 6 feet 4 inches tall. Convert his height to inches

Solution:

$$\begin{aligned} 1 \text{ foot} &= 12 \text{ inches} && \text{[Formula]} \\ 6 \text{ feet} &= 6 \times 12 \text{ inches} \\ 6 \text{ feet} &= 72 \text{ inches} \\ 6 \text{ feet } 4 \text{ inches} &= (72 + 4) \text{ inches} \end{aligned}$$

Hence, Salim is **76 inches** tall

Problem 4: A football field is 160 feet wide. Convert the width to yards.

Solution:

$$\begin{aligned} 1 \text{ yard} &= 3 \text{ feet} && \text{[Formula]} \\ \text{Then, } 1 \text{ feet} &= \frac{1}{3} \text{ yard} \\ 160 \text{ feet} &= 160 \times \frac{1}{3} \text{ yards} \end{aligned}$$

Hence the football field is $\mathbf{53 \cdot \bar{3}}$ yards wide

Problem 5: On a baseball diamond, the distance from home plate to first base is 30 yards. Convert the distance to feet.

Solution:

$$\begin{aligned} 1 \text{ yard} &= 3 \text{ feet} && \text{[Formula]} \\ 30 \text{ yards} &= 30 \times 3 \text{ feet} \\ &= 90 \text{ feet} \end{aligned}$$

Hence the required distance is **90 feet**

Problem 6: Humaid lives 1.5 miles from school. Convert the distance to feet.

Solution:

$$\begin{aligned} 1 \text{ mile} &= 5280 \text{ feet} && \text{[Formula]} \\ 1.5 \text{ miles} &= 1.5 \times 5280 \text{ feet} \end{aligned}$$

Hence the required distance is **7920 feet**

Problem 7: Omer's dog Beans weighs 8 pounds. Convert its weight to ounces.

Solution:

$$\begin{aligned} 1 \text{ pound} &= 16 \text{ ounces} && \text{[Formula]} \\ 8 \text{ pounds} &= 8 \times 16 \text{ ounces} \end{aligned}$$

Hence the required weight is **128 ounces**

Problem 8: Baby Omer weighed 7 pounds 3 ounces at birth. Convert his weight to ounces.

Solution:

$$\begin{aligned} 1 \text{ pound} &= 16 \text{ ounces} && \text{[Formula]} \\ 7 \text{ pounds} &= 7 \times 16 \text{ ounces} \\ &= 112 \text{ ounces} \\ 7 \text{ pounds } 3 \text{ ounces} &= 112 + 3 \text{ ounces} \end{aligned}$$

Hence the required weight is **115 ounces**

Problem 9: Blue whales can weigh as much as 150 tons. Convert the weight to pounds.

Solution:

$$\begin{aligned} 1 \text{ ton} &= 2000 \text{ pounds} && \text{[Formula]} \\ 150 \text{ tons} &= 150 \times 2000 \text{ pounds} \end{aligned}$$

Hence the required weight is **300,000 ounces**

In the following exercises, solve and write your answer in mixed units

Problem 10: Azzan caught three fish. The weights of the fish were 2 pounds 4 ounces, 1 pound 11 ounces, and 4 pounds 14 ounces. What was the total weight of the three fishes?

Solution:

First fish	2 pounds 04 ounces	
Second fish	1 pounds 11 ounces	
Third fish	4 pounds 14 ounces	[+]
Total	7 pounds 29 ounces	
	1 pound = 16 ounces $\left[\frac{29}{16} = 1\frac{13}{16}\right]$ 29 ounces = 1 pound and 13 ounces 7 pounds 29 ounces = 7 pounds + 1 pound and 13 ounces	[Formula]
Total	8 pounds 13 ounces	

Hence the total weight of the fishes is **8 pounds 13 ounces**

Problem 11: Maryam bought 1 pound 6 ounces of almonds, 2 pounds 3 ounces of walnuts, and 8 ounces of cashews. What was the total weight of the nuts?

Solution:

Almond	1 pounds 6 ounces	
Walnut	2 pounds 3 ounces	
Cashew	0 pounds 8 ounces	[+]
Total	3 pounds 17 ounces	
	1 pound = 16 ounces	[Formula]
	$\left[\frac{17}{16} = 1\frac{1}{16}\right]$	
	17 ounces = 1 pound and 1 ounce	
	3 pounds 17 ounces = 3 pounds + 1 pound and 1 ounce	
Total	4 pounds 1 ounce	

Hence the total weight of the nuts is **4 pounds 1 ounce**

Problem 12: Lamees attached a 6-feet-6-inch extension cord to her computers 3-feet-8-inch power cord. What was the total length of the cords?

Solution:

Extension cord length	6 feet 6 inches	
Computer power cord length	3 feet 8 inches	
Total	9 feet 14 inches	
	1 foot = 12 inches	[Formula]
	$\left[\frac{14}{12} = 1\frac{2}{12}\right]$	
	14 inches = 1 foot and 2 inches	
9 feet 14 inches	= 9 feet + 1 foot and 2 inches	
Total	10 feet 2 inches	

Hence the total length of the cord is **10 feet 2 inches**

6.2 Metric System

Metric Measurements		
Length	**Mass**	**Volume/ Capacity**
1 kilometer (km) = 1000 m	1 kilogram (kg) = 1000 g	1 kilolitre (kL) = 1000 L
1 hectometer (hm) = 100 m	1 hectogram (hg) = 100 g	1 hectolitre (hL) = 100 L
1 decameter (dam) = 10 m	1 decagram (dag) = 10 g	1 decalitre (daL) = 10 L
1 decimeter (dm) = 0.1 m	1 decigram (dg) = 0.1 g	1 decilitre (dL) = 0.1 L
1 centimeter (cm) = 0.01 m	1 centigram (cg) = 0.01 g	1 centilitre (cL) = 0.01 L
1 millimeter (mm) = 0.001 m	1 milligram (mg) = 0.001 g	1 millilitre (mL) = 0.001 L
1 meter = 100 centimeters	1 gram = 100 centigrams	1 litre = 100 centilitres
1 meter = 1000 millimeters	1 gram = 1000 milligrams	1 litre = 1000 millilitres

Exercise - 6.2

In the following exercises, convert the units.

Problem 1: Nabhan ran 5 kilometers. Convert the length to meters.

Solution:

$$\begin{aligned} \text{1 kilometer} &= \text{1000 meters} && \text{[Formula]} \\ \text{5 kilometers} &= 5 \times \text{1000 meters} \\ \text{5 kilometers} &= \text{5000 meters} \end{aligned}$$

Hence the required length is **5000 meters**

Problem 2: Al Jabal Al Akhdar is 2980 meters tall. Convert the height to kilometers

Solution:

$$\begin{aligned} \text{1 kilometer} &= \text{1000 meters} && \text{[Formula]} \\ \text{Then, 1 meter} &= \frac{1}{1000} \text{ kilometers} \\ \text{2980 meters} &= 2980 \times \frac{1}{1000} \text{ kilometers} \\ \text{2980 meters} &= \text{2.980 kilometers} \end{aligned}$$

Hence, Al Jabal Al Akhdar is **2.98 kilometers** tall

Problem 3: Bushra is 1.55 meters tall. Convert her height to centimeters.

Solution:

$$\begin{aligned} 1 \text{ meter} &= 100 \text{ centimeters} && \text{[Formula]} \\ 1.55 \text{ meters} &= 1.55 \times 100 \text{ centimeters} \\ 1.55 \text{ meters} &= 155 \text{ centimeters} \end{aligned}$$

Hence, Bushra is **155 centimeters** tall

Problem 4: A multivitamin contains 1500 milligrams of calcium. Convert this to grams

Solution:

$$\begin{aligned} 1 \text{ milligram} &= 0.001 \text{ grams} && \text{[Formula]} \\ 1500 \text{ milligrams} &= 1500 \times 0.001 \text{ grams} \\ 1500 \text{ milligrams} &= 1.5 \text{ grams} \end{aligned}$$

Hence, the multivitamin contains **1.5 grams** of calcium

Problem 5: A bottle of medicine contained 300 millilitres. Convert this to litres.

Solution:

$$\begin{aligned} 1 \text{ millilitre} &= 0.001 \text{ litres} && \text{[Formula]} \\ 300 \text{ millilitres} &= 300 \times 0.001 \text{ litres} \\ 300 \text{ milligrams} &= 0.3 \text{ litres} \end{aligned}$$

Hence, the bottle of medicine contained **0.3 litres**

Problem 6: One stick of butter contains 91.6 grams of fat. Convert this to milligrams

Solution:

$$\begin{aligned} 1 \text{ gram} &= 1000 \text{ milligrams} && \text{[Formula]} \\ 91.6 \text{ grams} &= 91.6 \times 1000 \text{ milligrams} \\ 91.6 \text{ grams} &= 91600 \text{ milligrams} \end{aligned}$$

Hence, one stick of butter contains **91600 milligrams** of fat

In the following exercises, solve and write your answer in mixed units.

Problem 7: Ahmed mailed 5 packages that weighed 420 grams each. What was the total weight of the packages in kilograms?

Solution:

$$\begin{aligned} \text{One pack weight} &= 420 \text{ grams} \\ \text{Then, 5 pack weight} &= 5 \times 420 \text{ grams} \\ &= 2100 \text{ grams} \\ \text{1 gram} &= 0.001 \text{ kilograms} \qquad \text{[Formula]} \\ \text{2100 grams} &= 2100 \times 0.001 \text{kilograms} \\ &= 2.1 \text{kilograms} \end{aligned}$$

Hence, the total weight of the packages is **2.1 kilograms**

Problem 8: Khalil is 1.6 meters tall. His sister is 95 centimeters tall. How much taller, in centimeters, is Khalil than his sister?

Solution:

$$\begin{aligned} \text{Khalil height} &= 1.6 \text{ meters} \\ \text{1 meter} &= 100 \text{ centimeters} \qquad \text{[Formula]} \\ \text{Then, 1.6 meters} &= 1.6 \times 100 \text{ centimeters} \\ \text{Hence, Khalil height} &= 160 \text{ centimeters} \\ \text{His sister's height} &= 95 \text{ centimeters} \\ \text{Difference} &= \text{Khalil height - His sister's height} \\ &= 160 - 95 \text{ centimeters} \\ &= 65 \text{ centimeters} \end{aligned}$$

Hence, Khalil is **65cm** taller than his sister.

Problem 9: Murshid drinks 200 millilitres of water 8 times a day. How many litres of water does Murshid drink in a day?

Solution:

$$\begin{aligned} \text{One time consumption} &= 200 \text{ millilitres} \\ \text{Then, 8 times} &= 8 \times 200 \text{ millilitres} \\ &= 1600 \text{ millilitres} \\ \text{1 millilitre} &= 0.001 \text{ litres} \qquad \text{[Formula]} \\ \text{1600 millilitres} &= 1600 \times 0.001 \text{ litres} \\ &= 1.6 \text{litres} \end{aligned}$$

Hence, Murshid drinks **1.6 litres** of water in a day

6.3 Conversion Between U.S. and Metric Systems

Conversion Factors Between U.S. and Metric Systems		
Length	**Weight**	**Volume**
1 in = 2.54 cm	1 lb = 0.45 kg	1 qt = 0.95 L
1 ft = 0.305 m	1 oz = 28 g	1 fl oz = 30 mL
1 yd = 0.914 m	1 kg = 2.2 lb	1 gallon = 3.785 L
1 mi = 1.61 km		1 L = 1.06 qt
1 m = 3.28 ft		

Exercise - 6.3

In the following exercises, convert between U.S. and metric units. Round to the nearest tenth.

Problem 1: Majid is 61 inches tall. Convert his height to centimeters.

Solution:

$$\begin{aligned} 1 \text{ inch} &= 2.54 \text{ centimeters} && \text{[Formula]} \\ 61 \text{ inches} &= 61 \times 2.54 \text{ centimeters} \\ 61 \text{ inches} &= 154.94 \text{ centimeters} \end{aligned}$$

Hence, Majid is **154.94 centimeters** tall

Problem 2: A college basketball court is 84 feet long. Convert this length to meters.

Solution:

$$\begin{aligned} 1 \text{ foot} &= 0.305 \text{ meters} && \text{[Formula]} \\ 84 \text{ feet} &= 84 \times 0.305 \text{ meters} \\ 84 \text{ feet} &= 25.62 \text{ meters} \end{aligned}$$

Hence, college basketball court is **25.64 meters** long

Problem 3: Rahma walked 2.5 miles. Convert this distance to kilometers.

Solution:

$$\begin{aligned} 1 \text{ mile} &= 1.61 \text{ kilometers} && \text{[Formula]} \\ 2.5 \text{ miles} &= 2.5 \times 1.61 \text{ kilometers} \\ 2.5 \text{ miles} &= 4.025 \text{ kilometers} \end{aligned}$$

Hence, Rahma walked **4.03 kilometers**

Problem 4: Khalfan weighs 78 kilograms. Convert his weight to pounds.

Solution:

$$\begin{aligned} 1 \text{ kilogram} &= 2.2 \text{ pounds} && \text{[Formula]} \\ 78 \text{ kilograms} &= 78 \times 2.2 \text{ pounds} \\ 78 \text{ kilograms} &= 171.6 \text{ pounds} \end{aligned}$$

Hence, Khalfan weighs **171.60 pounds**

Problem 5: Steves car holds 30 gallons of gas. Convert this to litres.

Solution:

$$\begin{aligned} 1 \text{ galoon} &= 3.785 \text{ litres} && \text{[Formula]} \\ 30 \text{ galoons} &= 30 \times 3.785 \text{ litres} \\ 30 \text{ galoons} &= 113.55 \text{ litres} \end{aligned}$$

Hence, Steves car holds **113.55 litres** of gas

Problem 6: A box of books weighs 25 pounds. Convert this weight to kilograms.

Solution:

$$\begin{aligned} 1 \text{ pound} &= 0.45 \text{ kilograms} && \text{[Formula]} \\ 25 \text{ pounds} &= 25 \times 0.45 \text{ kilograms} \\ 25 \text{ pounds} &= 11.25 \text{ kilograsm} \end{aligned}$$

Hence, the weight of the box of books is **11.25 kilograms**

6.4 Measurements of Temperature

Temperature Conversion			
	Celsius	**Fahrenheit**	**Kelvin**
Celsius		$C = \frac{5}{9}(F-32)$	$C = K - 273$
Fahrenheit	$F = \frac{9}{5}C + 32$		$F = \frac{9}{5}(K-273) + 32$
Kelvin	$K = C + 273$	$K = \frac{5}{9}(F-32) + 273$	

Exercise - 6.4

In the following exercises, convert the Fahrenheit temperature to degrees Celsius. Round to the nearest tenth

Problem 1: 86° F

Solution:

$$\begin{aligned} \mathrm{C} &= \frac{5}{9}(\mathrm{F}-32) && \text{[Formula]} \\ \mathrm{C} &= \frac{5}{9}(86-32) \\ &= \frac{5}{\cancel{9}} \times \cancel{54}^{6} \\ &= 5 \times 6 \\ \mathrm{C} &= 30 \implies \boxed{86^\circ \text{ F} = 30^\circ \text{ C}} \end{aligned}$$

Problem 2: 77° F

Solution:

$$\begin{aligned} \mathrm{C} &= \frac{5}{9}(\mathrm{F}-32) && \text{[Formula]} \\ \mathrm{C} &= \frac{5}{9}(77-32) \\ &= \frac{5}{\cancel{9}} \times \cancel{45}^{5} \\ &= 5 \times 5 \\ \mathrm{C} &= 25 \implies \boxed{77^\circ \text{ F} = 25^\circ \text{ C}} \end{aligned}$$

Problem 3: 104° F

Solution:

$$\begin{aligned} C &= \frac{5}{9}(F-32) && \text{[Formula]} \\ C &= \frac{5}{9}(104-32) \\ &= \frac{5}{\cancel{9}} \times \cancel{72}^{8} \\ &= 5 \times 8 \\ C &= 40 \implies \boxed{104^\circ\ \text{F} = 40^\circ\ \text{C}} \end{aligned}$$

Problem 4: 14° F

Solution:

$$\begin{aligned} C &= \frac{5}{9}(F-32) && \text{[Formula]} \\ C &= \frac{5}{9}(14-32) \\ &= \frac{5}{\cancel{9}} \times (-\cancel{18}^{2}) \\ &= 5 \times (-2) \\ C &= -10 \implies \boxed{14^\circ\ \text{F} = -10^\circ\ \text{C}} \end{aligned}$$

Problem 5: 72° F

Solution:

$$\begin{aligned} C &= \frac{5}{9}(F-32) && \text{[Formula]} \\ C &= \frac{5}{9}(72-32) \\ &= \frac{5}{9} \times 40 \\ &= \frac{200}{9} \\ C &= 22.\bar{2} \implies \boxed{72^\circ\ \text{F} = 22.\bar{2}^\circ\ \text{C}} \end{aligned}$$

Problem 6: 4° F

Solution:

$$
\begin{aligned}
C &= \frac{5}{9}(F-32) && \text{[Formula]}\\
C &= \frac{5}{9}(4-32)\\
&= \frac{5}{9}\times(-28)\\
&= -\frac{140}{9}\\
C &= -15.\bar{5} \implies \boxed{14^\circ\ \text{F}= -15.\bar{5}^\circ\ \text{C}}
\end{aligned}
$$

Problem 7: 0° F

Solution:

$$
\begin{aligned}
C &= \frac{5}{9}(F-32) && \text{[Formula]}\\
C &= \frac{5}{9}(0-32)\\
&= \frac{5}{9}\times(-32)\\
&= -\frac{160}{9}\\
C &= -17.\bar{7} \implies \boxed{0^\circ\ \text{F}= -17.\bar{7}^\circ\ \text{C}}
\end{aligned}
$$

Problem 8: 120° F

Solution:

$$
\begin{aligned}
C &= \frac{5}{9}(F-32) && \text{[Formula]}\\
C &= \frac{5}{9}(120-32)\\
&= \frac{5}{9}\times(88)\\
&= \frac{440}{9}\\
C &= 48.\bar{8} \implies \boxed{120^\circ\ \text{F}= 48.\bar{8}^\circ\ \text{C}}
\end{aligned}
$$

In the following exercises, convert the Celsius temperatures to degrees Fahrenheit. Round to the nearest tenth.

Problem 9: 5° C

Solution:

$$\begin{aligned} F &= \frac{9}{5}C + 32 && \text{[Formula]} \\ F &= \frac{9}{\cancel{5}} \times (\cancel{5}) + 32 \\ &= 9 + 32 \\ &= 41 \\ F &= 48.9 \implies \boxed{5^\circ \text{ C} = 41^\circ \text{ F}} \end{aligned}$$

Problem 10: 25° C

Solution:

$$\begin{aligned} F &= \frac{9}{5}C + 32 && \text{[Formula]} \\ F &= \frac{9}{\cancel{5}} \times (\cancel{25}^5) + 32 \\ &= (9 \times 5) + 32 \\ &= 45 + 32 \\ F &= 77 \implies \boxed{25^\circ \text{ C} = 77^\circ \text{ F}} \end{aligned}$$

Problem 11: -10° C

Solution:

$$\begin{aligned} F &= \frac{9}{5}C + 32 && \text{[Formula]} \\ F &= \frac{9}{\cancel{5}} \times (-\cancel{10}^2) + 32 \\ &= (9 \times (-2)) + 32 \\ &= -18 + 32 \\ F &= 14 \implies \boxed{-10^\circ \text{ C} = 14^\circ \text{ F}} \end{aligned}$$

Problem 12: -15° C

Solution:

$$\begin{aligned} F &= \frac{9}{5}C + 32 && \text{[Formula]} \\ F &= \frac{9}{\cancel{5}} \times (-\cancel{15}^{3}) + 32 \\ &= (9 \times (-3)) + 32 \\ &= -27 + 32 \\ F &= 5 \implies \boxed{\text{-15° C= 5° F}} \end{aligned}$$

Problem 13: 22° C

Solution:

$$\begin{aligned} F &= \frac{9}{5}C + 32 && \text{[Formula]} \\ F &= \frac{9}{5} \times (22) + 32 \\ &= \frac{198}{5} + 32 \\ &= 39.6 + 32 \\ F &= 71.6 \implies \boxed{\text{22° C= 71.6° F}} \end{aligned}$$

Problem 14: 8° C

Solution:

$$\begin{aligned} F &= \frac{9}{5}C + 32 && \text{[Formula]} \\ F &= \frac{9}{5} \times (8) + 32 \\ &= \frac{72}{5} + 32 \\ &= 14.4 + 32 \\ F &= 46.4 \implies \boxed{\text{8° C= 46.4° F}} \end{aligned}$$

Problem 15: 43° C

Solution:

$$\begin{aligned} F &= \frac{9}{5}C + 32 && \text{[Formula]} \\ F &= \frac{9}{5} \times (43) + 32 \\ &= \frac{387}{5} + 32 \\ &= 77.4 + 32 \\ F &= 109.4 \implies \boxed{43^\circ \text{ C} = 109.4^\circ \text{ F}} \end{aligned}$$

Problem 16: 16° C

Solution:

$$\begin{aligned} F &= \frac{9}{5}C + 32 && \text{[Formula]} \\ F &= \frac{9}{5} \times (16) + 32 \\ &= \frac{144}{5} + 32 \\ &= 28.8 + 32 \\ F &= 60.8 \implies \boxed{16^\circ \text{ C} = 60.8^\circ \text{ F}} \end{aligned}$$

Problem 17: To conserve energy, room temperatures are kept at 68° F in the winter and 77° F in the summer. What are these temperatures on the Celsius scale?

Solution:

Formula: $C = \frac{5}{9}(F - 32)$

$$\begin{aligned} F &= 68^\circ \text{ F} \\ C &= \frac{5}{9}(68 - 32) \\ &= \frac{5}{\cancel{9}} \times (\cancel{36}^{4}) \\ &= 5 \times 4 \\ C &= 20 \implies \boxed{68^\circ \text{ F} = 20^\circ \text{ C}} \end{aligned}$$

$$\begin{aligned} F &= 77^\circ \text{ F} \\ C &= \frac{5}{9}(77 - 32) \\ &= \frac{5}{\cancel{9}} \times (\cancel{45}^{5}) \\ &= 5 \times 5 \\ C &= 25 \implies \boxed{77^\circ \text{ F} = 25^\circ \text{ C}} \end{aligned}$$

Problem 18: What is the Fahrenheit temperature of a person with a 39° C fever?

Solution:

$$
\begin{aligned}
F &= \frac{9}{5}C + 32 && \text{[Formula]} \\
F &= \frac{9}{5} \times (39) + 32 \\
&= \frac{351}{5} + 32 = 70.2 + 32 \\
F &= 102.2 \implies \boxed{39^\circ \text{ C} = 102.2^\circ \text{ F}}
\end{aligned}
$$

Problem 19: Frost damage to most plants occurs at temperatures of 28° F or lower. What is this temperature on the Kelvin scale?

Solution:

$$
\begin{aligned}
K &= \frac{5}{9}(F - 32) + 273 && \text{[Formula]} \\
K &= \frac{5}{9}(28 - 32) + 273 \\
&= \frac{5}{9} \times (-4) + 273 = -\frac{20}{9} + 273 = -2.2 + 273 \\
K &= 270.8 \implies \boxed{28^\circ \text{ F} = 270.8 \text{ K}}
\end{aligned}
$$

Problem 20: One of the hottest temperatures ever recorded on the surface of Earth was 134° F in Death Valley, CA. What is this temperature in Celsius degrees? What is this temperature in Kelvin?

Solution:

$$
\begin{aligned}
C &= \frac{5}{9}(F - 32) && \text{[Formula]} \\
C &= \frac{5}{9}(134 - 32) = \frac{5}{9} \times (102) = \frac{510}{9} \\
C &= 56.7 \implies \boxed{134^\circ \text{ F} = 56.7^\circ \text{ C}} \\
K &= C + 273 = 56.7 + 273 \implies \boxed{134^\circ \text{ F} = 329.7 \text{ K}}
\end{aligned}
$$

Problem 21: At what temperature do the Fahrenheit and Celsius scales have the same numerical value?

Solution:

$$\begin{aligned}
C &= \frac{5}{9}(F-32) && \text{[Formula]}\\
C = F \implies C &= \frac{5}{9}(C-32)\\
9C &= 5(C-32)\\
9C-5C &= -160\\
4C &= -160\\
C &= -\frac{\cancel{160}^{40}}{\cancel{4}}\\
C &= -40
\end{aligned}$$

Fahrenheit and Celsius scales have the same numerical value at -40

Problem 22: At what temperature do the Fahrenheit and Kelvin scales have the same numerical value?

Solution:

$$\begin{aligned}
K &= \frac{5}{9}(F-32)+273 && \text{[Formula]}\\
K = F \implies K &= \frac{5}{9}(K-32)+273\\
K-273 &= \frac{5}{9}(K-32)\\
9(K-273) &= 5(K-32)\\
9K-2457 &= 5K-160\\
9K-5K &= 2457-160\\
4K &= 2297\\
K &= \frac{2297}{4}\\
K &= 574.25
\end{aligned}$$

Fahrenheit and Kelvin scales have the same numerical value at 574.25

Chapter 7

LINEAR EQUATIONS & INEQUALITIES

7.1 Linear Equations

A linear equation is of the form $ax + b = 0$, where a and b are any two real numbers and x is an unknown variable.

Solution of an equation

A solution of an equation is a value of a variable that makes a true statement when it is substituted into the equation

Exercise - 7.1

In the following exercises, solve each linear equation.

Problem 1: $3(10 - 2x) + 54 = 0$

Solution:

$$\begin{aligned}
& 3(10 - 2x) + 54 = 0 \\
\Rightarrow\ & 30 - 6x + 54 = 0 && \text{[Simplification of terms inside the bracket]} \\
\Rightarrow\ & 30 + 54 - 6x = 0 \\
\Rightarrow\ & 84 - 6x = 0 \\
\Rightarrow\ & -6x = -84 \\
\Rightarrow\ & x = \frac{\cancel{-84}^{14}}{\cancel{-6}} = 14 \implies \boxed{x = 14}
\end{aligned}$$

Problem 2: $\boldsymbol{-2(11-7x)+54=4}$

Solution:

$$
\begin{aligned}
& -2(11-7x)+54=4 \\
\Rightarrow\ & -22+14x+54=4 && \text{[Simplification of terms inside the bracket]} \\
\Rightarrow\ & 54-22+14x=4 \\
\Rightarrow\ & 32+14x=4 \\
\Rightarrow\ & 14x=4-32 \\
\Rightarrow\ & 14x=-28 \\
\Rightarrow\ & x=\frac{-\cancel{28}^{2}}{\cancel{14}}=-2\boxed{x=-2}
\end{aligned}
$$

Problem 3: $\boldsymbol{\frac{2}{3}(9c-3)=22}$

Solution:

$$
\begin{aligned}
& \frac{2}{3}(9c-3)=22 \\
\Rightarrow\ & \left(\frac{2}{\cancel{3}}\times\cancel{9}^{3}c\right)-\left(\frac{2}{\cancel{3}}\times\cancel{3}\right)=22 && \text{[Simplification of terms inside the bracket]} \\
\Rightarrow\ & 2\times 3c-2=22 \\
\Rightarrow\ & 6c-2=22 \\
\Rightarrow\ & 6c=22+2 \\
\Rightarrow\ & c=\frac{\cancel{24}^{4}}{\cancel{6}}=4 \implies \boxed{c=4}
\end{aligned}
$$

Problem 4: $\boldsymbol{-15+4(2-5y)=-7(y-4)+4}$

Solution:

$$
\begin{aligned}
& -15+4(2-5y)=-7(y-4)+4 \\
\Rightarrow\ & -15+8-20y=-7y+28+4 && \text{[Simplification of terms inside the bracket]} \\
\Rightarrow\ & -7-20y=-7y+32 \\
\Rightarrow\ & 7y-20y=7+32 && \text{[Separate the variables and the numbers]} \\
\Rightarrow\ & -13y=39 \\
\Rightarrow\ & y=\frac{\cancel{39}^{3}}{-\cancel{13}}=-3 \implies \boxed{y=-3}
\end{aligned}
$$

Problem 5: $4(p-4)-(p+7)=5(p-3)$

Solution:

$$
\begin{aligned}
& 4(p-4)-(p+7)=5(p-3) \\
\Rightarrow\ & 4p-16-p-7=5p-15 && \text{[Simplification of terms inside the bracket]} \\
\Rightarrow\ & 4p-p-16-7=5p-15 \\
\Rightarrow\ & 3p-23=5p-15 \\
\Rightarrow\ & 3p-5p=23-15 && \text{[Separate the variables and the numbers]} \\
\Rightarrow\ & -2p=8 \\
\Rightarrow\ & p=\frac{\cancel{8}^{4}}{-\cancel{2}}=-4 \\
\Rightarrow\ & \boxed{p=-4}
\end{aligned}
$$

Problem 6: $4[5-8(4c-3)]=12(1-13c)-8$

Solution:

$$
\begin{aligned}
& 4[5-8(4c-3)]=12(1-13c)-8 \\
\Rightarrow\ & 4[5-32c+24]=12(1-13c)-8 && \text{[Simplification of terms inside the inner bracket]} \\
\Rightarrow\ & 4[29-32c]=12(1-13c)-8 \\
\Rightarrow\ & 116-128c=12-156c-8 && \text{[Simplification of terms inside the outer bracket]} \\
\Rightarrow\ & 116-128c=4-156c \\
\Rightarrow\ & 156c-128c=4-116 && \text{[Separate the variables and the numbers]} \\
\Rightarrow\ & 28c=-112 \\
\Rightarrow\ & c=\frac{-\cancel{112}^{4}}{\cancel{28}}=-4 \\
\Rightarrow\ & \boxed{c=-4}
\end{aligned}
$$

In the following exercises, solve each equation with fraction coefficients.

Problem 7: $\frac{1}{4}x - \frac{1}{2} = -\frac{3}{4}$

Solution:

$$\frac{1}{4}x - \frac{1}{2} = -\frac{3}{4}$$

$$\Rightarrow \quad 4\left(\frac{1}{4}x - \frac{1}{2}\right) = 4\left(-\frac{3}{4}\right) \qquad \text{[Multiply by 4 on both sides]}$$

$$\Rightarrow \quad \cancel{4}\left(\frac{1}{\cancel{4}}x\right) - \cancel{4}^2\left(\frac{1}{\cancel{2}}\right) = \cancel{4}\left(-\frac{3}{\cancel{4}}\right)$$

$$\Rightarrow \quad x - 2 = -3$$

$$\Rightarrow \quad x = -3 + 2$$

$$\Longrightarrow \quad \boxed{x = -1}$$

Problem 8: $\frac{3}{4}x - \frac{1}{2} = \frac{1}{4}$

Solution:

$$\frac{3}{4}x - \frac{1}{2} = \frac{1}{4}$$

$$\Rightarrow \quad 4\left(\frac{3}{4}x - \frac{1}{2}\right) = 4\left(\frac{1}{4}\right) \qquad \text{[Multiply by 4 on both sides]}$$

$$\Rightarrow \quad \cancel{4}\left(\frac{3}{\cancel{4}}x\right) - \cancel{4}^2\left(\frac{1}{\cancel{2}}\right) = \cancel{4}\left(\frac{1}{\cancel{4}}\right)$$

$$\Rightarrow \quad 3x - 2 = 1$$

$$\Rightarrow \quad 3x = 1 + 2$$

$$\Rightarrow \quad 3x = 3$$

$$\Rightarrow \quad x = \frac{\cancel{3}}{\cancel{3}} = 1$$

$$\boxed{x = 1}$$

Problem 9: $\mathbf{\dfrac{3x+4}{2}+1=\dfrac{5x+10}{8}}$

Solution:

$$
\begin{aligned}
& \frac{3x+4}{2}+1=\frac{5x+10}{8} \\
\Rightarrow\ & 8\left(\frac{3x+4}{2}+1\right)=8\left(\frac{5x+10}{8}\right) && \text{[Multiply by 8 on both sides]} \\
\Rightarrow\ & \cancel{8}^{4}\left(\frac{3x+4}{\cancel{2}}\right)+8=\cancel{8}\left(\frac{5x+10}{\cancel{8}}\right) \\
\Rightarrow\ & 4(3x+4)+8=5x+10 && \text{[Simplification of terms inside the bracket]} \\
\Rightarrow\ & 12x+16+8=5x+10 \\
\Rightarrow\ & 12x-5x=10-24 && \text{[Separate the variables and the numbers]} \\
\Rightarrow\ & 7x=-14 \\
\Rightarrow\ & x=\frac{-\cancel{14}^{2}}{\cancel{7}}=-2 \implies \boxed{x=-2}
\end{aligned}
$$

In the following exercises, solve the formula for the specific variable

Problem 10: $\mathbf{8x+y=15}$ **for** $\mathbf{x}$

Solution:

$$
\begin{aligned}
& 8x+y=15 \\
\Rightarrow\ & 8x=15-y \\
\Rightarrow\ & \boxed{x=\frac{15-y}{8}}
\end{aligned}
$$

Problem 11: $\mathbf{-4x+y=-6}$ **for** $\mathbf{y}$

Solution:

$$
\begin{aligned}
& -4x+y=-6 \\
\Rightarrow\ & \boxed{y=4x-6}
\end{aligned}
$$

Problem 12: $\mathbf{A=\frac{1}{2}bh}$ **for** $\mathbf{b}$

Solution:

$$A=\frac{1}{2}bh \implies 2A=bh \implies \boxed{b=\frac{2A}{h}}$$

7.2 Applications of Linear Equations

Introduction

Linear equations have a wide range of applications and may be used in a variety of real-life scenarios. To use algebra to deal with real-life issues, we convert the circumstance into mathematical statements. Determine the situation's unknowns and assign variables to these unknown numbers. Re-frame the answer to the problem statement and examine if it perfectly fits the problem using systematic equation solving procedures.

Exercise - 7.2

In the following exercises, solve using the problem solving strategy for word problems. Remember to write a complete sentence to answer each question

Problem 1: There are 16 girls in a school club. The number of girls is four more than twice the number of boys. Find the number of boys.

Solution:
Let the number of girls be G and boys be B.
Given that, $G = 16$ and $G = 4 + 2B$

$$\begin{aligned} \text{Hence, } 4 + 2B &= 16 \\ 2B &= 16 - 4 \\ 2B &= 12 \\ B &= \frac{\cancel{12}^{6}}{\cancel{2}} \\ B &= 6 \end{aligned}$$

Hence the number of boys is 6

Problem 2: The difference of a number and 12 is three. Find the number.

Solution:
Let the number be x. Given that, $x - 12 = 3$ or $12 - x = 3$

$$\begin{aligned} \text{Hence, } x - 12 &= 3 \\ x &= 3 + 12 \\ x &= 15 \end{aligned}$$

$$\begin{aligned} \text{Hence, } 12 - x &= 3 \\ 12 - 3 &= x \\ 9 &= x \end{aligned}$$

Hence the required number is 9 or 15

Problem 3: The sum of three times a number and eight is 23. Find the number.

Solution:

Let the number be x. Given that, $3x + 8 = 23$

$$\begin{aligned}
\text{Hence, } 3x + 8 &= 23 \\
3x &= 23 - 8 \\
3x &= 15 \\
x &= \frac{\cancel{15}^{5}}{\cancel{3}} \\
x &= 5
\end{aligned}$$

Hence the required number is 5

Problem 4: One number is five more than the other. Their sum is 33. Find the numbers.

Solution:

Let one number be x and the other could be $(x - 5)$. Given that, $x + (x - 5) = 33$

$$\begin{aligned}
\text{Hence, } x + (x - 5) &= 33 \\
x + x - 5 &= 33 \\
2x - 5 &= 33 \\
2x &= 33 + 5 \\
x &= \frac{\cancel{38}^{19}}{\cancel{2}} \\
x &= 19 \\
\implies \quad x - 5 &= 19 - 5 = 14
\end{aligned}$$

Hence the required numbers are 19 and 14

Problem 5: The sum of two numbers is 20. One number is four less than the other. Find the numbers.

Solution:

Let one number be y. Given that, the other number is $y - 4$ and $y + (y - 4) = 20$

$$\begin{aligned}
\text{Hence, } y + (y - 4) &= 20 \\
y + y - 4 &= 20 \\
2y - 4 &= 20 \\
2y &= 20 + 4 \\
y &= \frac{\cancel{24}^{12}}{\cancel{2}} \\
y &= 12 \\
\implies \quad y - 4 &= 12 - 4 = 8
\end{aligned}$$

Hence the required numbers are 12 and 8

Problem 6: Find three consecutive integers whose sum is -3.

Solution:
Let the first number be x. Then the next two numbers are $x+1$ and $x+2$. Given that, $x+(x+1)+(x+2)=-3$

$$
\begin{aligned}
\text{Hence, } x+(x+1)+(x+2) &= -3 \\
3x+3 &= -3 \\
3x &= -3-3 \\
3x &= -6 \\
x &= \frac{-\cancel{6}^{2}}{\cancel{3}} \\
x &= -2 \\
\implies \quad x+1 &= -2+1=-1 \\
\implies \quad x+2 &= -2+2=0
\end{aligned}
$$

Hence the required numbers are -2, -1 and 0

Problem 7: Find three consecutive integers whose sum is -33.

Solution:
Let the first number be x. Then the next two numbers are $x+1$ and $x+2$. Given that, $x+(x+1)+(x+2)=-33$

$$
\begin{aligned}
\text{Hence, } x+(x+1)+(x+2) &= -33 \\
3x+3 &= -33 \\
3x &= -33-3 \\
3x &= -36 \\
x &= \frac{-\cancel{36}^{12}}{\cancel{3}} \\
x &= -12 \\
\implies \quad x+1 &= -12+1=-11 \\
\implies \quad x+2 &= -12+2=-10
\end{aligned}
$$

Hence the required numbers are -12, -11 and -10

Problem 8: Find three consecutive even integers whose sum is 60.

Solution:
Let the first even number be x. Then the next two even numbers are $x+2$ and $x+4$. Given that, $x+(x+2)+(x+4)=60$

$$\begin{aligned}
\text{Hence, } x+(x+2)+(x+4) &= 60\\
3x+6 &= 60\\
3x &= 60-6\\
3x &= 54\\
x &= \frac{\cancel{54}^{18}}{\cancel{3}}\\
x &= 18\\
\implies \quad x+2 &= 18+2=20\\
\implies \quad x+4 &= 18+4=22
\end{aligned}$$

Hence the required even numbers are 18, 20 and 22

Problem 9: Find four consecutive even integers whose sum is 92.

Solution:

Let the first even number be x. Then the next three even numbers are $x+2, x+4$ and $x+6$. Given that, $x+(x+2)+(x+4)+(x+6)=92$

$$\begin{aligned}
\text{Hence, } x+(x+2)+(x+4)+(x+6) &= 92\\
4x+12 &= 92\\
4x &= 92-12\\
4x &= 80\\
x &= \frac{\cancel{80}^{20}}{\cancel{4}}\\
x &= 20\\
\implies \quad x+2 &= 20+2=22\\
\implies \quad x+4 &= 20+4=24\\
\implies \quad x+6 &= 20+6=26
\end{aligned}$$

Hence the required even numbers are 20, 22, 24 and 26

Problem 10: Find three consecutive odd integers whose sum is -99.

Solution:

Let the first odd number be x. Then the next two odd numbers are $x+2$ and $x+4$. Given that, $x+(x+2)+(x+4)=-99$

$$\begin{aligned}
\text{Hence, } x+(x+2)+(x+4) &= -99\\
3x+6 &= -99\\
3x &= -99-6\\
3x &= -105 \implies x = \frac{-\cancel{105}^{35}}{\cancel{3}} \implies x=-35\\
\implies \quad x+2 &= -35+2=-33\\
\implies \quad x+4 &= -35+4=-31
\end{aligned}$$

Hence the required numbers are -31, -33 and -35

Problem 11: Find four consecutive odd integers whose sum is 48.

Solution:
Let the first odd number be x. Then the next three odd numbers are $x+2, x+4$ and $x+6$. Given that, $x+(x+2)+(x+4)+(x+6)=48$

$$\begin{aligned}
\text{Hence, } x+(x+2)+(x+4)+(x+6) &= 48\\
4x+12 &= 48\\
4x &= 48-12\\
4x &= 36\\
x &= \frac{\cancel{36}^{9}}{\cancel{4}}\\
x &= 9\\
\implies \quad x+2 &= 9+2=11\\
\implies \quad x+4 &= 9+4=13\\
\implies \quad x+6 &= 9+6=15
\end{aligned}$$

Hence the required even numbers are 9, 11, 13 and 15

In the following exercises, solve using a geometry formula.

Problem 12: The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.

Solution:
Let the smaller angle be $x°$. Given that, two smaller angles of a right triangle have equal measures, hence the other two angels are $x°$ and $90°$. We know that the sum of all the angels of a triangle is $180°$. $x°+x°+90°=180°$

$$\begin{aligned}
\text{Hence, } x+x+90 &= 180\\
2x+90 &= 180\\
2x &= 180-90\\
2x &= 90\\
x &= \frac{\cancel{90}^{45}}{\cancel{2}}\\
x &= 45
\end{aligned}$$

Hence the required angles of the right triangle are $45°, 45°$ and $90°$

Problem 13: The angles in a triangle are such that one angle is twice the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

Solution:
Let the first smallest angle be $x°$. Given that, the second angle is twice the smallest one, hence it is $2x°$ and the third angle is three times the smallest one, hence it is $3x°$. We know that the sum of all the angels of a triangle is $180°$. Hence,
$x° + 2x° + 3x° = 180°$

$$\begin{aligned}
\text{Hence, } x + 2x + 3x &= 180 \\
6x &= 180 \\
x &= \frac{\cancel{180}^{30}}{\cancel{6}} \\
x &= 30 \\
\implies \quad 2x &= 2 \times 30 = 60 \\
\implies \quad 3x &= 3 \times 30 = 90
\end{aligned}$$

Hence the required angles of the triangle are $30°, 60°$ and $90°$

Problem 14: One side of a triangle is seven inches more than the first side. The third side is four inches less than three times the first. The perimeter is 28 inches. Find the length of the three sides of the triangle.

Solution:
Let the first side of the triangle be x inches. Given that, the second side is seven inches more than the first side, hence the second side is $x + 7$ inches and the third side is four inches less than three times the first side, hence the second side is $3x - 4$ inches. Also given that, the perimeter is 28 inches.

$$\begin{aligned}
\text{Hence, } x + (x + 7) + (3x - 4) &= 28 \\
5x + 3 &= 28 \\
5x &= 28 - 3 \\
5x &= 25 \\
x &= \frac{\cancel{25}^{5}}{\cancel{5}} \\
x &= 5 \\
\implies \quad x + 7 &= 5 + 7 = 12 \\
\implies \quad 3x - 4 &= (3 \times 5) - 4 = 15 - 4 = 11
\end{aligned}$$

Hence the length of the three sides are 5 inches, 11 inches and 12 inches

Problem 15: One side of a triangle is three feet less than the first side. The third side is five feet less than twice the first. The perimeter is 20 feet. Find the length of the three sides of the triangle.

Solution:
Let the first side of the triangle be x feet. Given that, the second side of a triangle is three feet less than the first side, hence the second side is $x - 3$ feet and the third side is five feet less than twice the first, hence the second side is $2x - 5$ feet. Also given that, the perimeter is 20 feet.

$$\begin{aligned} \text{Hence, } x + (x-3) + (2x-5) &= 20 \\ 4x - 8 &= 20 \\ 4x &= 20 + 8 \\ 4x &= 28 \\ x &= \frac{\cancel{28}^{7}}{\cancel{4}} \\ x &= 7 \\ \implies \quad x - 3 &= 7 - 3 = 4 \\ \implies \quad 2x - 5 &= (2 \times 7) - 5 = 14 - 5 = 9 \end{aligned}$$

Hence the length of the three sides are 4 feet, 7 feet and 9 feet.

7.3 Linear Inequalities

A linear inequality is much like a linear equation-but the equal sign is replaced with an inequality sign.

Linear Inequality

A linear inequality in one variable is an inequality that can be written in one of the following forms where a, b, and c are real numbers and $a \neq 0$:

$$ax + b < c, \qquad ax + b \leq c, \qquad ax + b > c, \qquad \text{or } ax + b \geq c$$

Addition and Subtraction Property of Inequalities

For any numbers a, b, and c, if $a < b$, then

$$\text{a + c < b + c} \qquad \text{a - c < b - c}$$

Multiplication and Division Property of Inequalities

For any numbers a, b, and c,

multiply or divide by a positive

if $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

multiply or divide by a negative

if $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

Exercise - 7.3

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line and write in interval form.

Problem 1:

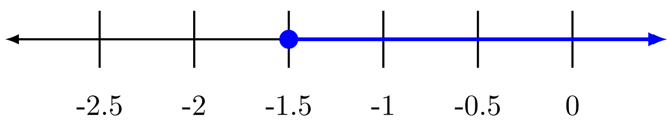
a. $\boldsymbol{x \leq 5}$

$x \leq 5$

$(-\infty, 5]$

b. $\boldsymbol{x \geq -1.5}$

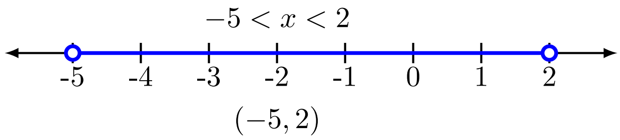
$x \geq -1.5$



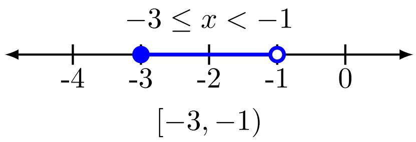
$[-1.5, \infty)$

Problem 2:

a. $\boldsymbol{-5 < x < 2}$

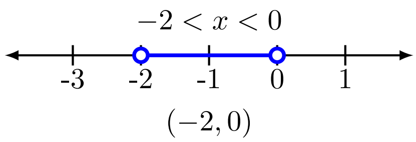


b. $\boldsymbol{-3 \leq x < -1}$

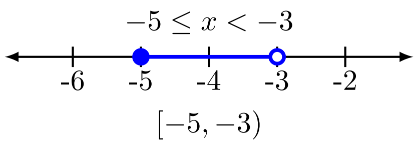


Problem 3:

a. $\boldsymbol{-2 < x < 0}$



b. $\boldsymbol{-5 \leq x < -3}$



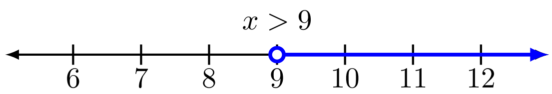
In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval form.

Problem 4: $\boldsymbol{8x > 72}$

Solution:

$$8x > 72 \implies \frac{\cancel{8}x}{8} > \frac{\cancel{72}^{9}}{\cancel{8}} \quad \text{[Divided by 8 on both sides]}$$
$$\implies \boxed{\text{x} > 9}$$

Graph:



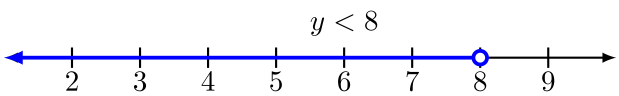
$(9, \infty)$

Problem 5: $\boldsymbol{6y < 48}$

Solution:

$$6y < 48 \implies \frac{\cancel{6}y}{6} < \frac{\cancel{48}^{8}}{\cancel{6}} \quad \text{[Divided by 6 on both sides]}$$
$$\implies \boxed{\text{y} < 8}$$

Graph:



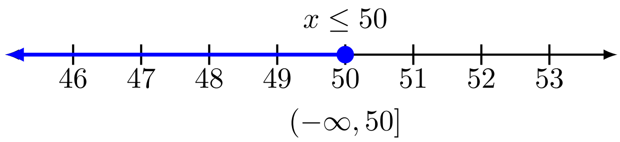
$(-\infty, 8)$

Problem 6: $\boldsymbol{20 \geq \frac{2}{5}x}$

Solution:

$$20 \geq \frac{2}{5}x$$
$$\implies 20 \times 5 \geq \cancel{5} \times \frac{2}{\cancel{5}}x \quad \text{[Multiply by 5 on both sides]}$$
$$\implies 100 \geq 2x$$
$$\implies \frac{\cancel{100}^{50}}{\cancel{2}} \geq \frac{\cancel{2}x}{\cancel{2}} \quad \text{[Divided by 2 on both sides]}$$
$$\implies \boxed{x \leq 50}$$

Graph:



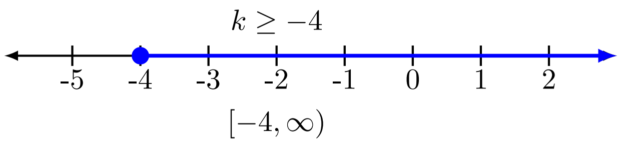
Problem 7: $-5k \leq 20$

Solution:

$$-5k \leq 20 \implies \frac{\cancel{-5}k}{\cancel{-5}} \geq \frac{\cancel{20}^{4}}{-\cancel{5}} \quad \text{[Divided by } -5 \text{ on both sides]}$$

$$\implies \boxed{k \geq -4}$$

Graph:



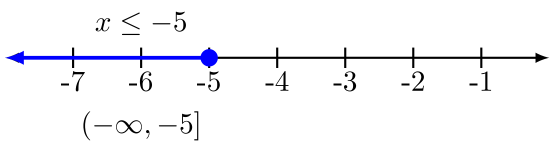
Problem 8: $3x \leq -15$

Solution:

$$3x \leq -15 \implies \frac{\cancel{3}x}{\cancel{3}} \leq \frac{-\cancel{15}^{5}}{\cancel{3}} \quad \text{[Divided by 3 on both sides]}$$

$$\implies \boxed{x \leq -5}$$

Graph:

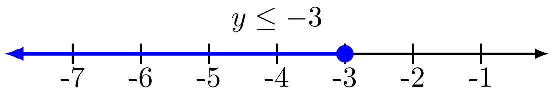


Problem 9: $\mathbf{12 \leq -4y}$

Solution:

$$\begin{aligned} 12 \leq -4y \quad &\implies \frac{\cancel{12}^{3}}{-\cancel{4}} \geq \frac{\cancel{-4}y}{\cancel{-4}} \quad \text{[Divided by } -4 \text{ on both sides]} \\ &\implies \boxed{y \leq -3} \end{aligned}$$

Graph:



$(-\infty, -3]$

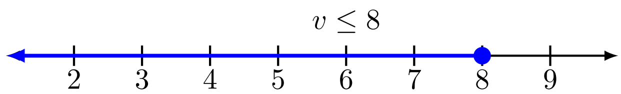
In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation

Problem 10: $\mathbf{4v \geq 9v - 40}$

Solution:

$$\begin{aligned} 4v \geq 9v - 40 \quad &\implies 40 \geq 9v - 4v \\ &\implies 40 \geq 5v \\ &\implies \frac{\cancel{40}^{8}}{\cancel{5}} \geq \frac{\cancel{5}v}{\cancel{5}} \qquad \text{[Divided by 5 on both sides]} \\ &\implies \boxed{v \leq 8} \end{aligned}$$

Graph:



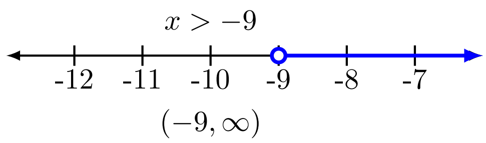
$(-\infty, 8]$

Problem 11: $\mathbf{12x + 3(x + 7) > 10x - 24}$

Solution:

$$\begin{aligned} 12x + 3(x+7) > 10x - 24 \quad &\implies 12x + 3x + 21 > 10x - 24 \\ &\implies 15x + 21 > 10x - 24 \\ &\implies 15x - 10x > -24 - 21 \\ &\implies 5x > -45 \\ &\text{[Divided by 5 on both sides]} \\ &\implies \frac{\cancel{5}x}{\cancel{5}} > \frac{-\cancel{45}^{9}}{\cancel{5}} \\ &\implies \boxed{x > -9} \end{aligned}$$

Graph:

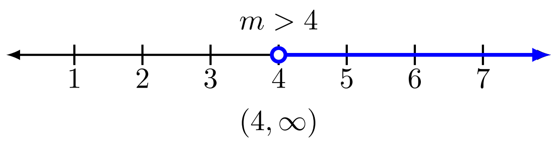


Problem 12: $\mathbf{8m - 2(3 - m) \geq 2(m + 7) + 3m}$

Solution:

$$\begin{aligned}
8m - 2(3-m) \geq 2(m+7) + 3m \quad &\Longrightarrow 8m - 6 + 2m \geq 2m + 14 + 3m \\
&\Longrightarrow 10m - 6 > 5m + 14 \\
&\Longrightarrow 10m - 5m > 14 + 6 \\
&\Longrightarrow 5m > 20 \\
&\text{[Divided by 5 on both sides]} \\
&\Longrightarrow \frac{\cancel{5}m}{\cancel{5}} > \frac{\cancel{20}^4}{\cancel{5}} \\
&\Longrightarrow \boxed{m > 4}
\end{aligned}$$

Graph:



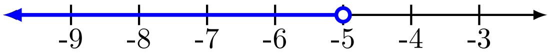
Problem 13: $\mathbf{9y + 5(y + 3) < 4y - 35}$

Solution:

$$\begin{aligned}
9y + 5(y+3) < 4y - 35 \quad &\Longrightarrow 9y + 5y + 15 < 4y - 35 \\
&\Longrightarrow 14y + 15 < 4y - 35 \\
&\Longrightarrow 14y - 4y < -35 - 15 \\
&\Longrightarrow 10y < -50 \\
&\text{[Divided by 10 on both sides]} \\
&\Longrightarrow \frac{\cancel{10}y}{\cancel{10}} < \frac{-\cancel{50}^5}{\cancel{10}} \\
&\Longrightarrow \boxed{y < -5}
\end{aligned}$$

Graph:

$y < -5$



$(-\infty, -5)$

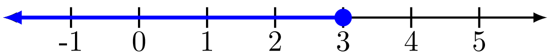
Problem 14: $\boldsymbol{6h - 4(h-1) \geq 7h - 11}$

Solution:

$$\begin{aligned}
6h - 4(h-1) \geq 7h - 11 \quad & \Longrightarrow \; 6h - 4h + 4 \geq 7h - 11 \\
& \Longrightarrow \; 2h + 4 \geq 7h - 11 \\
& \Longrightarrow \; 11 + 4 \geq 7h - 2h \\
& \Longrightarrow \; 5h \leq 15 \\
& \text{[Divided by 5 on both sides]} \\
& \Longrightarrow \; \frac{\cancel{5}h}{\cancel{5}} < \frac{\cancel{15}^{3}}{\cancel{5}} \\
& \Longrightarrow \; \boxed{h \leq 3}
\end{aligned}$$

Graph:

$h \leq 3$



$(-\infty, 3]$

Problem 15: $\boldsymbol{4k - (k-2) \geq 7k - 26}$

Solution:

$$\begin{aligned}
4k - (k-2) \geq 7k - 26 \quad & \Longrightarrow \; 4k - k + 2 \geq 7k - 26 \\
& \Longrightarrow \; 3k + 2 \geq 7k - 26 \\
& \Longrightarrow \; 26 + 2 \geq 7k - 3k \; \Longrightarrow \; 4k \leq 28 \\
& \text{[Divided by 4 on both sides]} \\
& \Longrightarrow \; \frac{\cancel{4}k}{\cancel{4}} \leq \frac{\cancel{28}^{7}}{\cancel{4}} \\
& \Longrightarrow \; \boxed{k \leq 7}
\end{aligned}$$

Graph:

$k \leq 7$

$(-\infty, 7]$

Problem 16: $\frac{2}{3}b - \frac{3}{4}b < \frac{5}{12}b - \frac{1}{2}$

Solution:

$$\frac{2}{3}b - \frac{3}{4}b < \frac{5}{12}b - \frac{1}{2} \quad \text{[Multiply by 12 on both sides]}$$

$$\implies \left(\cancel{12}^4 \times \frac{2}{\cancel{3}}b\right) - \left(\cancel{12}^3 \times \frac{3}{\cancel{4}}b\right) < \left(\cancel{12} \times \frac{5}{\cancel{12}}b\right) - \left(\cancel{12}^6 \times \frac{1}{\cancel{2}}\right)$$

$$\implies 8b - 9b < 5b - 6$$

$$\implies -b < 5b - 6$$

$$\implies 5b + b > 6$$

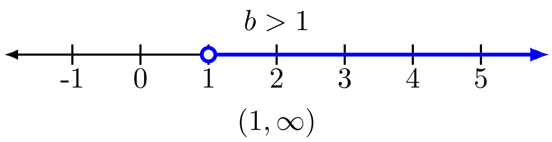
$$\implies 6b > 6$$

[Divided by 6 on both sides]

$$\implies \frac{\cancel{6}b}{\cancel{6}} > \frac{\cancel{6}}{\cancel{6}}$$

$$\implies \boxed{b > 1}$$

Graph:



Problem 17: $6n - 12(3 - n) \leq 9(n - 4) + 9n$

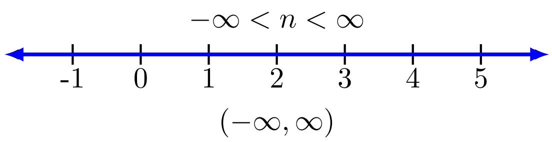
Solution:

$$6n - 12(3 - n) \leq 9(n - 4) + 9n \implies 6n - 36 + 12n \leq 9n - 36 + 9n$$

$$\implies 18n - 36 \leq 18n - 36$$

$$\implies 0 \leq 0, \text{True}$$

All the real values of n will satisfy this inequality $\implies \boxed{-\infty < n < \infty}$



Problem 18: $\mathbf{18q - 4(10 - 3q) < 5(6q - 8)}$

Solution:

$$\begin{aligned} 18q - 4(10 - 3q) < 5(6q - 8) &\implies 18q - 40 + 12q < 30q - 40 \\ &\implies 30q - 40 < 30q - 40 \\ &\implies 0 < 0, \text{Not True} \end{aligned}$$

This inequality will not be satisfied by any real numbers

Problem 19: $\mathbf{-\frac{9}{4}x \geq -\frac{5}{12}}$

Solution:

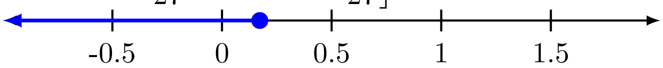
$$-\frac{9}{4}x \geq -\frac{5}{12} \quad \text{[Multiply by } -12 \text{ on both sides]}$$

$$\implies \left(-\cancel{12}^{3} \times \frac{(-9)}{\cancel{4}}x\right) \leq \left(-\cancel{12} \times \frac{(-5)}{\cancel{12}}\right)$$

$$\implies 27x \leq 5 \implies \boxed{x \leq \frac{5}{27}}$$

Graph:

$$x \leq \frac{5}{27} \implies \left(-\infty, \frac{5}{27}\right]$$



In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

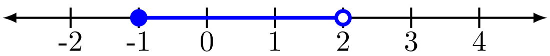
Problem 20: $\mathbf{-5 \leq 4x - 1 < 7}$

Solution:

$$\begin{aligned} -5 \leq 4x - 1 < 7 \quad &\text{[Add 1 on each side of the inequality]} \\ &\implies -5 + 1 \leq 4x - 1 + 1 < 7 + 1 \\ &\implies -4 \leq 4x < 8 [\div \text{ by 4 on each side of the inequality]} \\ &\implies -\frac{\cancel{4}}{\cancel{4}} \leq \frac{\cancel{4}x}{\cancel{4}} < \frac{\cancel{8}^{2}}{\cancel{4}} \\ &\implies \boxed{-1 \leq x < 2} \end{aligned}$$

Graph:

$$-1 \leq x < 2$$



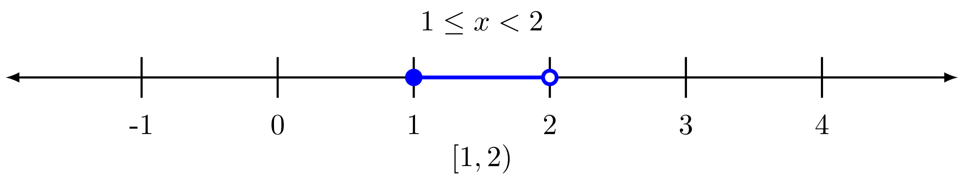
$[-1, 2)$

Problem 21: $\mathbf{5 < 4x + 1 < 9}$

Solution:

$$
\begin{aligned}
5 < 4x + 1 < 9 \quad & \text{[Add } -1 \text{ on each side of the inequality]} \\
& \implies 5 - 1 \leq 4x + 1 - 1 < 9 - 1 \\
& \implies 4 \leq 4x < 8 \\
& \text{[Divided by 4 on each side of the inequality]} \\
& \implies \frac{\cancel{4}}{\cancel{4}} \leq \frac{\cancel{4}x}{\cancel{4}} < \frac{\cancel{8}^2}{\cancel{4}} \\
& \implies \boxed{1 \leq x < 2}
\end{aligned}
$$

Graph:

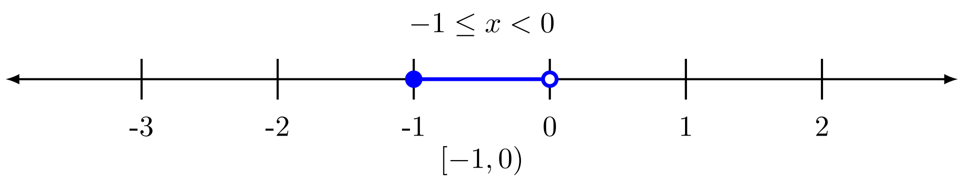


Problem 22: $\mathbf{-6 \leq 4x - 2 < -2}$

Solution:

$$
\begin{aligned}
-6 \leq 4x - 2 < -2 \quad & \text{[Add 2 on each side of the inequality]} \\
& \implies -6 + 2 \leq 4x - 2 + 2 < -2 + 2 \\
& \implies -4 \leq 4x < 0 \\
& \text{[Divided by 4 on each side of the inequality]} \\
& \implies -\frac{\cancel{4}}{\cancel{4}} \leq \frac{\cancel{4}x}{\cancel{4}} < \frac{\cancel{0}}{\cancel{4}} \\
& \implies \boxed{-1 \leq x < 0}
\end{aligned}
$$

Graph:



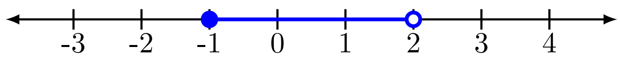
Problem 23: $-1 < 3x + 2 < 8$

Solution:

$$
\begin{aligned}
-1 < 3x+2 < 8 \quad & [\text{Add } -2 \text{ on each side of the inequality}] \\
& \implies -1-2 \le 3x+2-2 < 8-2 \\
& \implies -3 \le 3x < 6 \\
& [\text{Divided by 3 on each side of the inequality}] \\
& \implies -\frac{\cancel{3}}{\cancel{3}} \le \frac{\cancel{3}x}{\cancel{3}} < \frac{\cancel{6}^2}{\cancel{3}} \\
& \implies \boxed{-1 \le x < 2}
\end{aligned}
$$

Graph:

$-1 \le x < 2$



$[-1, 2)$

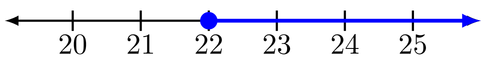
In the following exercises, translate and solve. Then graph the solution on the number line and write the solution in interval notation.

Problem 24: **Three more than** h **is no less than 25**

Solution:

Three more than h is no less than 25 $\implies 3 + h \ge 25 \implies h \ge 25 - 3 \implies h \ge 22$

$h \ge 22$



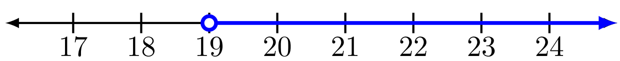
$[22, \infty)$

Problem 25: **Six more than** k **exceeds 25**

Solution:

Six more than k exceeds 25 $\implies 6 + k > 25 \implies k > 25 - 6 \implies k > 19$

$k > 19$



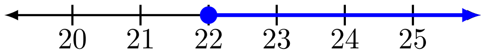
$(19, \infty)$

Problem 26: Fifteen less than a is at least 7

Solution:

Fifteen less than a is at least 7 $\implies a - 15 \geq 7 \implies a \geq 7 + 15 \implies a \geq 22$

$a \geq 22$



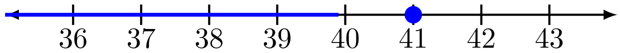
$[22, \infty)$

Problem 27: Nineteen less than b is at most 22

Solution:

Nineteen less than b is at most 22 $\implies b - 19 \leq 22 \implies b \leq 22 + 19 \implies b \leq 41$

$b \leq 41$



$(-\infty, 41]$

Solve the following exercises

Problem 28: Ryan charges his neighbors \$20 to wash their car. How many cars must he wash next summer if his goal is to earn at least \$1500?

Solution:

For a single car wash Ryan earns \$20, his goal is to earn at least \$1500.
Let x be number of cars which must be washed to achieve his goal.

$$\begin{aligned} \text{Hence,} \quad & 20x \geq 1500 \\ \implies \quad & x \geq \frac{\cancel{1500}^{75}}{\cancel{20}} \\ \implies \quad & x \geq 75, \end{aligned}$$

He must wash at least **75** cars to achieve his goal.

Problem 29: Maryam got a OMR 20 gift card for the coffee shop. Her favorite iced drink costs OMR 3. What is the maximum number of drinks she can buy with the gift card?

Solution:
Maryam's favorite iced drink costs is OMR 3, and she got OMR 20 gift card.
Let x be the number of iced drink she can buy, hence the maximum number of her favorite drink can be calculated by solving and maximizing the inequality $3x \leq 20$.
Here the maximum possible answer is 6 (Since, $3 \times 6 \leq 20$).
She can buy maximum 6 iced drinks by using the gift card.

Problem 30: Ahmed is a personal chef. He charges OMR 4 per meal. His monthly expenses are OMR 400. How many meals must he sell in order to make a profit of at least OMR 600?

Solution: Let x be the required number of meals to be sold.
Given that 400 is the monthly expense and cost of the single meal is 4.
Hence the total profit is $4x - 400$, But we want profit atleast 600.

$$\begin{aligned} \text{Then,} \quad & 4x - 400 \geq 600 \\ \implies \quad & 4x \geq 1000 \\ \implies \quad & x \geq \frac{1000}{4} \\ \implies \quad & x \geq 250 \end{aligned}$$

Hence he has to sell **250 meals** to get a minimum profit of OMR 600.

Chapter 8

QUADRATIC EQUATIONS

8.1 Solving Quadratic Equation using Formula

1. The solutions to a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$ are given by,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. The nature of the solution for the given quadratic equation $ax^2+bx+c = 0, a \neq 0$ is given by the Discriminant $b^2 - 4ac$

   (a) If $b^2 - 4ac > 0$, the equation has two real distinct solutions

   (b) If $b^2 - 4ac = 0$, the equation has two equal real solutions

   (c) If $b^2 - 4ac < 0$, the equation has no real solution

Exercise - 8.1

Solving Quadratic Equations Using the Formula

Solve the following Quadratic equations using the Quadratic Formula.

Problem 1: $4m^2 + m - 3 = 0$

Solution:

Formula: $m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 4,\ b = 1,\ c = -3$

$$\begin{aligned}
m &= \frac{-1 \pm \sqrt{(1)^2 - 4(4)(-3)}}{2(4)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\
&= \frac{-1 \pm \sqrt{1+48}}{8} \\
&= \frac{-1 \pm \sqrt{49}}{8} \\
m &= \frac{-1 \pm 7}{8}
\end{aligned}$$

$$\begin{aligned}
\text{Hence, } m &= \frac{-1+7}{8}, & m &= \frac{-1-7}{8} \\
m &= \frac{\cancel{6}^{3}}{\cancel{8}^{4}} & m &= \frac{-8}{8} \\
m &= \frac{3}{4} & x &= -1
\end{aligned}$$

The solutions for the given quadratic equation is $\boxed{m = \frac{3}{4} \ \& \ m = -1}$

Problem 2: $\mathbf{4n^2 - 9n + 5 = 0}$

Solution:

Formula: $n = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 4, \ b = -9, \ c = 5$

$$\begin{aligned}
n &= \frac{-(-9) \pm \sqrt{(-9)^2 - 4(4)(5)}}{2(4)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\
&= \frac{9 \pm \sqrt{81-80}}{8} \\
&= \frac{9 \pm \sqrt{1}}{8} \\
n &= \frac{9 \pm 1}{8}
\end{aligned}$$

$$\begin{aligned}
\text{Hence, } n &= \frac{9+1}{8}, & n &= \frac{9-1}{8} \\
n &= \frac{\cancel{10}^{5}}{\cancel{8}^{4}} & x &= \frac{8}{8} \\
n &= \frac{5}{4} & n &= 1
\end{aligned}$$

The solutions for the given quadratic equation is $\boxed{n = \frac{5}{4} \ \& \ n = 1}$

Problem 3: $\mathbf{2p^2 - 7p + 3 = 0}$

Solution:

Formula: $p = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 2,\ b = -7,\ c = 3$

$$
\begin{aligned}
p &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(3)}}{2(2)} \quad \text{[Substitute } a, b, c \text{ values in the formula]}\\
&= \frac{7 \pm \sqrt{49 - 24}}{4}\\
&= \frac{7 \pm \sqrt{25}}{4}\\
p &= \frac{7 \pm 5}{4}
\end{aligned}
$$

$$
\begin{aligned}
\text{Hence}, p = \frac{7+5}{4}, &\qquad p = \frac{7-5}{4}\\
p = \frac{\cancel{12}^{3}}{\cancel{4}} &\qquad p = \frac{\cancel{2}}{\cancel{4}^{2}}\\
p = 3 &\qquad p = \frac{1}{2}
\end{aligned}
$$

The solutions for the given quadratic equation is $\boxed{p = \frac{1}{2} \ \&\ p = 3}$

Problem 4: $\mathbf{3q^2 + 8q - 3 = 0}$

Solution:

Formula: $q = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 3,\ b = 8,\ c = -3$

$$
\begin{aligned}
q &= \frac{-8 \pm \sqrt{(8)^2 - 4(3)(-3)}}{2(3)} \quad \text{[Substitute } a, b, c \text{ values in the formula]}\\
&= \frac{-8 \pm \sqrt{64 + 36}}{6}\\
&= \frac{-8 \pm \sqrt{100}}{6}\\
q &= \frac{-8 \pm 10}{6}
\end{aligned}
$$

$$\text{Hence, } q = \frac{-8+10}{6}, \qquad q = \frac{-8-10}{6}$$
$$q = \frac{\cancel{2}}{\cancel{6}^{3}} \qquad q = \frac{-18}{6}$$
$$q = \frac{1}{3} \qquad q = -3$$

The solutions for the given quadratic equation is $\boxed{q = \frac{1}{3} \ \& \ q = -3}$

Problem 5: $p^2 + 7p + 12 = 0$

Solution:

Formula: $p = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 1, \ b = 7, \ c = 12$

$$\begin{aligned} p &= \frac{-(7) \pm \sqrt{(7)^2 - 4(1)(12)}}{2(1)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{-7 \pm \sqrt{49 - 48}}{2} \\ &= \frac{-7 \pm \sqrt{1}}{2} \\ p &= \frac{-7 \pm 1}{2} \end{aligned}$$

$$\text{Hence, } p = \frac{-7+1}{2}, \qquad p = \frac{-7-1}{2}$$
$$p = \frac{-\cancel{6}^{3}}{\cancel{2}} \qquad p = \frac{-\cancel{8}^{4}}{\cancel{2}}$$
$$p = -3 \qquad p = -4$$

The solutions for the given quadratic equation is $\boxed{p = -3 \ \& \ p = -4}$

Problem 6: $q^2 + 3q - 18 = 0$

Solution:

Formula: $q = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 1, \ b = 3, \ c = -18$

$$\begin{aligned} q &= \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-18)}}{2(1)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{-3 \pm \sqrt{9 + 72}}{2} \\ &= \frac{-3 \pm \sqrt{81}}{2} \\ q &= \frac{-3 \pm 9}{2} \end{aligned}$$

$$\begin{aligned} \text{Hence, } q = \frac{-3+9}{2}, &\qquad q = \frac{-3-9}{2} \\ q = \frac{\cancel{6}^3}{\cancel{2}} &\qquad q = \frac{-\cancel{12}^6}{\cancel{2}} \\ q = 3 &\qquad q = -6 \end{aligned}$$

The solutions for the given quadratic equation is $\boxed{q = 3 \;\&\; q = -6}$

Problem 7: $\boldsymbol{r^2 - 8r - 33 = 0}$

Solution:

Formula: $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 1,\ b = -8,\ c = -33$

$$\begin{aligned} r &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-33)}}{2(1)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{8 \pm \sqrt{64 + 132}}{2} \\ &= \frac{8 \pm \sqrt{196}}{2} \\ r &= \frac{8 \pm 14}{2} \end{aligned}$$

$$\begin{aligned} \text{Hence, } r = \frac{8+14}{2}, &\qquad r = \frac{8-14}{2} \\ r = \frac{\cancel{22}^{1}1}{\cancel{2}} &\qquad r = \frac{-\cancel{6}^3}{\cancel{2}} \\ r = 11 &\qquad r = -3 \end{aligned}$$

The solutions for the given quadratic equation is $\boxed{r = 11 \;\&\; r = -3}$

Problem 8: $t^2 + 13t + 40 = 0$

Solution:

Formula: $t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 1,\ b = 13,\ c = 40$

$$\begin{aligned} t &= \frac{-(13) \pm \sqrt{(13)^2 - 4(1)(40)}}{2(1)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{-13 \pm \sqrt{169 - 160}}{2} \\ &= \frac{-13 \pm \sqrt{9}}{2} \\ t &= \frac{-13 \pm 3}{2} \end{aligned}$$

$$\text{Hence, } t = \frac{-13+3}{2}, \qquad t = \frac{-13-3}{2}$$

$$t = \frac{-\cancel{10}^{5}}{\cancel{2}} \qquad t = \frac{-\cancel{16}^{8}}{\cancel{2}}$$

$$t = -5 \qquad t = -8$$

The solutions for the given quadratic equation is $\boxed{t = -5 \ \&\ t = -8}$

Problem 9: $3u^2 + 7u - 2 = 0$

Solution:

Formula: $u = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 3,\ b = 7,\ c = -2$

$$\begin{aligned} u &= \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-2)}}{2(3)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{-7 \pm \sqrt{49 + 24}}{6} \\ u &= \frac{-7 \pm \sqrt{73}}{6} \end{aligned}$$

The solutions for the given quadratic equation is $\boxed{u = \dfrac{-7 + \sqrt{73}}{6} \ \&\ u = \dfrac{-7 - \sqrt{73}}{6}}$

Problem 10: $\mathbf{6z^2 - 9z + 1 = 0}$

Solution:

Formula: $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 6,\ b = -9,\ c = 1$

$$\begin{aligned} z &= \frac{-(-9) \pm \sqrt{(-9)^2 - 4(6)(1)}}{2(6)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{9 \pm \sqrt{81 - 24}}{12} \\ &= \frac{9}{12} \pm \frac{\sqrt{57}}{12} \\ z &= \frac{3}{4} \pm \frac{\sqrt{57}}{12} \end{aligned}$$

The solutions for the given quadratic equation is $\boxed{z = \frac{3}{4} + \frac{\sqrt{57}}{12} \ \& \ z = \frac{3}{4} - \frac{\sqrt{57}}{12}}$

Problem 11: $\mathbf{2a^2 - 6a + 3 = 0}$

Solution:

Formula: $a = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$, here, $A = 2,\ B = -6,\ C = 3$

$$\begin{aligned} a &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(3)}}{2(2)} \quad \text{[Substitute } A, B, C \text{ values in the formula]} \\ &= \frac{6 \pm \sqrt{36 - 24}}{4} \\ &= \frac{6 \pm \sqrt{12}}{4} \\ a &= \frac{6 \pm 2\sqrt{3}}{4} \end{aligned}$$

$$\begin{aligned} \text{Hence, } a &= \frac{6 + 2\sqrt{3}}{4}, \qquad & a &= \frac{6 - 2\sqrt{3}}{4} \\ a &= \frac{\cancel{2}(3 + \sqrt{3})}{\cancel{4}^{2}}, \qquad & a &= \frac{\cancel{2}(3 - \sqrt{3})}{\cancel{4}^{2}} \\ a &= \frac{3 + \sqrt{3}}{2} \qquad & a &= \frac{3 - \sqrt{3}}{2} \end{aligned}$$

The solutions for the given quadratic equation is $\boxed{a = \frac{3 + \sqrt{3}}{2} \ \& \ a = \frac{3 - \sqrt{3}}{2}}$

Problem 12: $\mathbf{2x^2 + 3x + 9 = 0}$

Solution:

Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 2,\ b = 3,\ c = 9$

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 4(2)(9)}}{2(2)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{-3 \pm \sqrt{9 - 72}}{4} \\ x &= \frac{-3 \pm \sqrt{-63}}{4} \end{aligned}$$

We cannot take the square root of a negative number.

$\implies$ $\boxed{\text{There is no real solution}}$

Problem 13: $\mathbf{3t(t - 2) = 2}$

Solution:

$$\begin{aligned} 3t(t-2) = 2 \quad &\implies \quad 3t(t) - 3t(2) = 2 \\ &\implies \quad 3t^2 - 6t = 2 \\ &\implies \quad 3t^2 - 6t - 2 = 0 \end{aligned}$$

Formula: $t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 3,\ b = -6,\ c = -2$

$$\begin{aligned} t &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-2)}}{2(3)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{6 \pm \sqrt{36 + 24}}{6} \\ &= \frac{6 \pm \sqrt{60}}{6} \\ &= \frac{6 \pm 2\sqrt{15}}{6} \\ &= \frac{\cancel{6}}{\cancel{6}} \pm \frac{\cancel{2}\sqrt{15}}{\cancel{6}3} \\ t &= 1 \pm \frac{\sqrt{15}}{3} \end{aligned}$$

Hence the solutions are given by, $\boxed{t = 1 + \dfrac{\sqrt{15}}{3} \ \&\ t = 1 - \dfrac{\sqrt{15}}{3}}$

Problem 14: $\boldsymbol{v(v+5)-10=0}$

Solution:

$$\begin{aligned} v(v+5)-10=0 &\implies v(v)+v(5)-10=0 \\ &\implies v^2+5v-10=0 \end{aligned}$$

Formula: $v=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, here, $a=1,\ b=5,\ c=-10$

$$\begin{aligned} t &= \frac{-(5)\pm\sqrt{(5)^2-4(1)(-10)}}{2(1)} \quad \text{[Substitute } a,b,c \text{ values in the formula]} \\ &= \frac{-5\pm\sqrt{25+40}}{2} \\ v &= \frac{-5\pm\sqrt{65}}{2} \end{aligned}$$

Hence the solutions are given by, $\boxed{v=\dfrac{-5+\sqrt{65}}{2}\ \&\ v=\dfrac{-5-\sqrt{65}}{2}}$

Problem 15: $\boldsymbol{3w(w-2)-8=0}$

Solution:

$$\begin{aligned} 3w(w-2)-8=0 &\implies 3w(w)-3w(2)-8=0 \\ &\implies 3w^2-6w-8=0 \end{aligned}$$

Formula: $w=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, here, $a=3,\ b=-6,\ c=-8$

$$\begin{aligned} w &= \frac{-(-6)\pm\sqrt{(-6)^2-4(3)(-8)}}{2(3)} \quad \text{[Substitute } a,b,c \text{ values in the formula]} \\ &= \frac{6\pm\sqrt{36+96}}{6} \\ &= \frac{6\pm2\sqrt{33}}{6} \\ &= \frac{\cancel{6}}{\cancel{6}}\pm\frac{\cancel{2}\sqrt{33}}{\cancel{6}3} \\ w &= 1\pm\frac{\sqrt{33}}{3} \end{aligned}$$

Hence the solutions are given by, $\boxed{w=1+\dfrac{\sqrt{33}}{3}\ \&\ w=1-\dfrac{\sqrt{33}}{3}}$

Problem 16: $\mathbf{6y^2 - 5y + 2 = 0}$

Solution:

Formula: $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 6,\ b = -5,\ c = 2$

$$\begin{aligned} y &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(6)(2)}}{2(6)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{5 \pm \sqrt{25 - 48}}{12} \\ y &= \frac{5 \pm \sqrt{-23}}{12} \end{aligned}$$

We cannot take the square root of a negative number.
$\Longrightarrow$ $\boxed{\text{There is no real solution}}$

Problem 17: $\mathbf{2a^2 + 12a + 5 = 0}$

Solution:

Formula: $a = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$, here, $A = 2,\ B = 12,\ C = 5$

$$\begin{aligned} a &= \frac{-(12) \pm \sqrt{(12)^2 - 4(2)(5)}}{2(2)} \quad \text{[Substitute } A, B, C \text{ values in the formula]} \\ &= \frac{-12 \pm \sqrt{144 - 40}}{4} \\ &= \frac{-12 \pm \sqrt{104}}{4} \\ &= \frac{-12 \pm 2\sqrt{26}}{4} \\ &= -\frac{\cancel{12}^{3}}{\cancel{4}} \pm \frac{\cancel{2}\sqrt{26}}{\cancel{4}^{2}} \\ a &= -3 \pm \frac{\sqrt{26}}{2} \end{aligned}$$

The solutions for the given quadratic equation is $\boxed{a = -3 + \dfrac{\sqrt{26}}{2} \ \&\ a = -3 - \dfrac{\sqrt{26}}{2}}$

Problem 18: $\mathbf{16y^2 + 8y + 1 = 0}$

Solution:

Formula: $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 16,\ b = 8,\ c = 1$

$$\begin{aligned}
y &= \frac{-(8) \pm \sqrt{(8)^2 - 4(16)(1)}}{2(16)} \quad \text{[Substitute } a, b, c \text{ values in the formula]}\\
&= \frac{-8 \pm \sqrt{64 - 64}}{32}\\
&= \frac{-8 \pm \sqrt{0}}{32}\\
&= \frac{-\cancel{8}}{\cancel{32}^{4}}\\
y &= \frac{-1}{4}
\end{aligned}$$

The solutions are given by, $\boxed{y = \dfrac{-1}{4}\ \&\ y = \dfrac{-1}{4}}$

In the following exercises, determine the number of solutions to each quadratic equation.

Problem 19: $\mathbf{4x^2 - 5x + 16 = 0}$

Solution:

Formula: Discriminant$= B^2 - 4AC$ here, $A = 4,\ B = -5,\ C = 16$

$$\begin{aligned}
\text{Discriminant} &= (-5)^2 - 4(4)(16)\\
&= 25 - 256\\
&= -231 < 0
\end{aligned}$$

Hence, the equation has no real solution

Problem 20: $\mathbf{9v^2 - 15v + 25 = 0}$

Solution:

Formula: Discriminant$= B^2 - 4AC$ here, $A = 9,\ B = -15,\ C = 25$

$$\begin{aligned}
\text{Discriminant} &= (-15)^2 - 4(9)(25)\\
&= 225 - 900\\
&= -675 < 0
\end{aligned}$$

Hence, the equation has no real solution

Problem 21: $\boldsymbol{r^2 + 12r + 36 = 0}$

Solution:
Formula: Discriminant= $B^2 - 4AC$ here, $A = 1,\ B = 12,\ C = 36$

$$\begin{aligned}\text{Discriminant} &= (12)^2 - 4(1)(36)\\ &= 144 - 144\\ &= 0\end{aligned}$$

Hence, the equation has two real and equal solutions

Problem 22: $\boldsymbol{6m^2 + 3m - 5 = 0}$

Solution:
Formula: Discriminant= $B^2 - 4AC$ here, $A = 6,\ B = 3,\ C = -5$

$$\begin{aligned}\text{Discriminant} &= (3)^2 - 4(6)(-5)\\ &= 9 + 120\\ &= 129 > 0\end{aligned}$$

Hence, the equation has two real distinct solutions

Problem 23: $\boldsymbol{4u^2 - 12u + 9 = 0}$

Solution:
Formula: Discriminant= $B^2 - 4AC$ here, $A = 4,\ B = -12,\ C = 9$

$$\begin{aligned}\text{Discriminant} &= (-12)^2 - 4(4)(9)\\ &= 144 - 144\\ &= 0\end{aligned}$$

Hence, the equation has two real and equal solutions

Problem 24: $\boldsymbol{3v^2 - 5v - 1 = 0}$

Solution:
Formula: Discriminant= $B^2 - 4AC$ here, $A = 3,\ B = -5,\ C = -1$

$$\begin{aligned}\text{Discriminant} &= (-5)^2 - 4(3)(-1)\\ &= 25 + 12\\ &= 39 > 0\end{aligned}$$

Hence, the equation has two real distinct solutions

Problem 25: $5c^2 + 7c - 10 = 0$

Solution:
Formula: Discriminant$= B^2 - 4AC$ here, $A = 5,\ B = 7,\ C = -10$

$$\begin{aligned} \text{Discriminant} &= (7)^2 - 4(5)(-10) \\ &= 49 + 200 \\ &= 249 > 0 \end{aligned}$$

Hence, the equation has two real distinct solutions

Problem 26: $8t^2 - 11t + 5 = 0$

Solution:
Formula: Discriminant$= B^2 - 4AC$ here, $A = 8,\ B = -11,\ C = 5$

$$\begin{aligned} \text{Discriminant} &= (-11)^2 - 4(8)(5) \\ &= 121 - 160 \\ &= -39 < 0 \end{aligned}$$

Hence, the equation has no real solution

8.2 Applications of Quadratic Equations

In this section we solve real life problems using quadratic equation.

Exercise - 8.2

In the following exercises, solve by using the Quadratic Formula.

Problem 1: The product of two consecutive integers is 72. Find the numbers

Solution:

Let the numbers be x and $x + 1$, hence $x(x + 1) = 72$

$$\begin{aligned} x(x+1) = 72 \quad &\Longrightarrow \quad x(x) + x(1) = 72 \\ &\Longrightarrow \quad x^2 + x = 72 \\ &\Longrightarrow \quad x^2 + x - 72 = 0 \end{aligned}$$

Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 1,\ b = 1,\ c = -72$

$$\begin{aligned} x &= \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(-72)}}{2(1)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{-1 \pm \sqrt{1 + 288}}{2} \\ &= \frac{-1 \pm \sqrt{289}}{2} \\ x &= \frac{-1 \pm 17}{2} \end{aligned}$$

$$\begin{aligned} \text{Hence, } x = \frac{-1 + 17}{2}, &\qquad x = \frac{-1 - 17}{2} \\ x = \frac{\cancel{16}^{8}}{\cancel{2}} &\qquad x = \frac{-\cancel{18}^{9}}{\cancel{2}} \\ x = 8 &\qquad x = -9 \end{aligned}$$

If $x = 8 \implies x + 1 = 8 + 1 = 9$
If $x = -9 \implies x + 1 = -9 + 1 = -8$
Hence the required numbers are {8 & 9} or {-8 & -9}

Problem 2: The product of two consecutive odd numbers is 63. Find the numbers

Solution:

Let the first odd number be x, hence the next one is $x + 2$, then $x(x + 2) = 63$

$$\begin{aligned} x(x + 2) = 63 &\implies x(x) + x(2) = 63 \\ &\implies x^2 + 2x = 63 \\ &\implies x^2 + 2x - 63 = 0 \end{aligned}$$

Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 1,\ b = 2,\ c = -63$

$$\begin{aligned} x &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-63)}}{2(1)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{-2 \pm \sqrt{4 + 252}}{2} \\ &= \frac{-2 \pm \sqrt{256}}{2} \\ x &= \frac{-2 \pm 16}{2} \end{aligned}$$

$$\text{Hence, } x = \frac{-2+16}{2}, \qquad x = \frac{-2-16}{2}$$
$$x = \frac{\cancel{14}^{7}}{\cancel{2}} \qquad x = \frac{-\cancel{18}^{9}}{\cancel{2}}$$
$$x = 7 \qquad x = -9$$

If $x = 7 \implies x + 2 = 7 + 2 = 9$
If $x = -9 \implies x + 2 = -9 + 2 = -7$
Hence the required consecutive odd numbers are {7 & 9} or {-7 & -9}

Problem 3: The product of two consecutive even numbers is 80. Find the numbers

Solution:

Let the first even number be x, hence the next one is $x + 2$, then $x(x + 2) = 80$

$$\begin{aligned} x(x+2) = 80 \quad &\implies \quad x(x) + x(2) = 80 \\ &\implies \quad x^2 + 2x = 80 \\ &\implies \quad x^2 + 2x - 80 = 0 \end{aligned}$$

Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 1,\ b = 2,\ c = -80$

$$\begin{aligned} x &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-80)}}{2(1)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{-2 \pm \sqrt{4 + 320}}{2} \\ &= \frac{-2 \pm \sqrt{324}}{2} \\ x &= \frac{-2 \pm 18}{2} \end{aligned}$$

$$\text{Hence, } x = \frac{-2+18}{2}, \qquad x = \frac{-2-18}{2}$$
$$x = \frac{\cancel{16}^{8}}{\cancel{2}} \qquad x = \frac{-\cancel{20}^{10}}{\cancel{2}}$$
$$x = 8 \qquad x = -10$$

If $x = 8 \implies x + 2 = 8 + 2 = 10$
If $x = -10 \implies x + 2 = -10 + 2 = -8$
Hence the required consecutive even numbers are {8 & 10} or {-8 & -10}

Problem 4: The product of two consecutive even numbers is 24. Find the numbers

Solution:

Let the first even number be x, hence the next one is $x+2$, then $x(x+2)=24$

$$\begin{aligned} x(x+2)=24 \quad &\implies \quad x(x)+x(2)=24 \\ &\implies \quad x^2+2x=24 \\ &\implies \quad x^2+2x-24=0 \end{aligned}$$

Formula: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, here, $a=1,\ b=2,\ c=-24$

$$\begin{aligned} x &= \frac{-(2)\pm\sqrt{(2)^2-4(1)(-24)}}{2(1)} \quad \text{[Substitute } a,b,c \text{ values in the formula]} \\ &= \frac{-2\pm\sqrt{4+96}}{2} \\ &= \frac{-2\pm\sqrt{100}}{2} \\ x &= \frac{-2\pm 10}{2} \end{aligned}$$

$$\text{Hence, } x=\frac{-2+10}{2}, \qquad x=\frac{-2-10}{2}$$

$$x=\frac{\cancel{8}^4}{\cancel{2}} \qquad x=\frac{-\cancel{12}^6}{\cancel{2}}$$

$$x=4 \qquad x=-6$$

If $x=4 \implies x+2=4+2=6$
If $x=-6 \implies x+2=-6+2=-4$
Hence the required consecutive even numbers are {4 & 6} or {-4 & -6}

Problem 5: The product of two consecutive odd numbers is 35. Find the numbers

Solution:

Let the first odd number be x, hence the next one is $x+2$, then $x(x+2)=35$

$$\begin{aligned} x(x+2)=35 \quad &\implies \quad x(x)+x(2)=35 \\ &\implies \quad x^2+2x=35 \\ &\implies \quad x^2+2x-35=0 \end{aligned}$$

Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 1,\ b = 2,\ c = -35$

$$\begin{aligned}
x &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-35)}}{2(1)} \quad \text{[Substitute } a, b, c \text{ values in the formula]}\\
&= \frac{-2 \pm \sqrt{4 + 140}}{2}\\
&= \frac{-2 \pm \sqrt{144}}{2}\\
x &= \frac{-2 \pm 12}{2}
\end{aligned}$$

$$\text{Hence, } x = \frac{-2 + 12}{2}, \qquad x = \frac{-2 - 12}{2}$$

$$x = \frac{\cancel{10}^5}{\cancel{2}} \qquad x = \frac{-\cancel{14}^7}{\cancel{2}}$$

$$x = 5 \qquad x = -7$$

If $x = 5 \implies x + 2 = 5 + 2 = 7$
If $x = -7 \implies x + 2 = -7 + 2 = -5$
Hence the required consecutive odd numbers are {5 & 7} or {-5 & -7}

Problem 6: The product of two consecutive odd numbers is 99. Find the numbers

Solution:

Let the first odd number be x, hence the next one is $x + 2$, then $x(x + 2) = 99$

$$\begin{aligned}
x(x + 2) = 99 &\implies x(x) + x(2) = 99\\
&\implies x^2 + 2x = 99\\
&\implies x^2 + 2x - 99 = 0
\end{aligned}$$

Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 1,\ b = 2,\ c = -99$

$$\begin{aligned}
x &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-99)}}{2(1)} \quad \text{[Substitute } a, b, c \text{ values in the formula]}\\
&= \frac{-2 \pm \sqrt{4 + 396}}{2}\\
&= \frac{-2 \pm \sqrt{400}}{2}\\
x &= \frac{-2 \pm 20}{2}
\end{aligned}$$

$$\text{Hence, } x = \frac{-2+20}{2}, \qquad x = \frac{-2-20}{2}$$
$$x = \frac{\cancel{18}^{9}}{\cancel{2}} \qquad x = \frac{-\cancel{22}^{1}1}{\cancel{2}}$$
$$x = 9 \qquad x = -11$$

If $x = 9 \implies x + 2 = 9 + 2 = 11$
If $x = -11 \implies x + 2 = -11 + 2 = -9$
Hence the required consecutive odd numbers are {9 & 11} or {-9 & -11}

Problem 7: The product of two consecutive even numbers is 48. Find the numbers

Solution:

Let the first even number be x, hence the next one is $x + 2$, then $x(x + 2) = 48$

$$\begin{aligned} x(x+2) = 48 &\implies x(x) + x(2) = 48 \\ &\implies x^2 + 2x = 48 \\ &\implies x^2 + 2x - 48 = 0 \end{aligned}$$

Formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here, $a = 1,\ b = 2,\ c = -48$

$$\begin{aligned} x &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-48)}}{2(1)} \quad \text{[Substitute } a, b, c \text{ values in the formula]} \\ &= \frac{-2 \pm \sqrt{4 + 192}}{2} \\ &= \frac{-2 \pm \sqrt{196}}{2} \\ x &= \frac{-2 \pm 14}{2} \end{aligned}$$

$$\text{Hence, } x = \frac{-2+14}{2}, \qquad x = \frac{-2-14}{2}$$
$$x = \frac{\cancel{12}^{6}}{\cancel{2}} \qquad x = \frac{-\cancel{16}^{8}}{\cancel{2}}$$
$$x = 6 \qquad x = -8$$

If $x = 6 \implies x + 2 = 6 + 2 = 8$
If $x = -8 \implies x + 2 = -8 + 2 = -6$
Hence the required consecutive even numbers are {6 & 8} or {-6 & -8}

Problem 8: A triangle with area 45 square inches has a height that is two less than four times the width. Find the height and width of the triangle. [Hint: For a triangle with base b and height h , the area, A , is given by the formula $A = \frac{1}{2}bh$]

Solution:

Given that, Area, $A = \frac{1}{2}bh = 45$ and $h = 4b - 2$

Hence, $45 = \dfrac{1}{2}bh = \dfrac{1}{2}b(4b-2) = \frac{1}{\cancel{2}}b(\cancel{2})(2b-1) = 2b^2 - b \implies 2b^2 - b - 45 = 0$

Formula: $b = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$, here, $A = 2,\ B = -1,\ C = -45$

$$\begin{aligned} b &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-45)}}{2(2)} && \text{[Substitute } A, B, C \text{ values in the formula]} \\ &= \frac{1 \pm \sqrt{1 + 360}}{4} \\ b &= \frac{1 \pm \sqrt{361}}{4} \implies b = \frac{1 + \sqrt{361}}{4},\ b = \frac{1 - \sqrt{361}}{4} \end{aligned}$$

Since b is a base, so it can not be negative, hence $b = \dfrac{1 + \sqrt{361}}{4}$,

hence $h = 4b - 2 \implies h = \cancel{4}(\dfrac{1 + \sqrt{361}}{\cancel{4}}) - 2 = 1 + \sqrt{361} - 2 = \sqrt{361} - 1$

Hence the height and width of the triangle is $\sqrt{361} - 1$ inches & $\dfrac{1 + \sqrt{361}}{4}$ inches

Problem 9: The width of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the height and width of the triangle

Solution:

Given that, Area, $A = \frac{1}{2}bh = 88$ and $b = 2h + 6$

Hence, $88 = \frac{1}{2}bh = \frac{1}{2}(2h+6)h = \frac{1}{\cancel{2}}h(\cancel{2})(h+3) = h^2 + 3h \implies h^2 + 3h - 88 = 0$

Formula: $h = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$, here, $A = 1,\ B = 3,\ C = -88$

$$
\begin{aligned}
h &= \frac{-3 \pm \sqrt{3^2 - 4(1)(-88)}}{2(1)} && \text{[Substitute } A, B, C \text{ values in the formula]}\\
&= \frac{-3 \pm \sqrt{9+352}}{2}\\
&= \frac{-3 \pm \sqrt{361}}{2}\\
h &= \frac{-3 \pm 19}{2}\\
\implies & \quad h = \frac{-3+19}{2}, \; h = \frac{-3-19}{2}\\
\implies & \quad h = \frac{\cancel{16}^{8}}{\cancel{2}}, \; h = \frac{-\cancel{22}^{11}}{\cancel{2}} \implies h = 8, h = -11
\end{aligned}
$$

Since h refers height, so it can not be negative, hence $h = 8$,
hence $b = 2h + 6 \implies b = 2(8) + 6 = 16 + 6 = 22$
Hence the height and width of the triangle is 8 yards & 22 yards

Problem 10: A rectangular sign board has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign board

Solution:

Given that, Area, $A = lw = 30$ and $l = w + 1$
Hence, $30 = lw = (w+1)w = w^2 + w \implies w^2 + w - 30 = 0$

Formula: $w = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$, here, $A = 1, \; B = 1, \; C = -30$

$$
\begin{aligned}
w &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-30)}}{2(1)} && \text{[Substitute } A, B, C \text{ values in the formula]}\\
&= \frac{-1 \pm \sqrt{121}}{2}\\
w &= \frac{-1 \pm 11}{2}\\
\implies & \quad w = \frac{-1+11}{2}, \; w = \frac{-1-11}{2}\\
\implies & \quad w = \frac{\cancel{10}^{5}}{\cancel{2}}, \; w = \frac{-\cancel{12}^{6}}{\cancel{2}}\\
\implies & \quad w = 5, w = -6
\end{aligned}
$$

Since w refers width, so it can not be negative, hence $w = 5$,
hence $l = w + 1 \implies l = 5 + 1 = 6$
Hence the length and width of the triangle is 6 feet & 5 feet

Problem 11: A rectangular lawn has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the lawn

Solution:

Given that, Area, $A = lw = 180$ and $w = l - 3$
Hence, $180 = lw = l(l-3) = l^2 - 3l \implies l^2 - 3l - 180 = 0$

Formula: $l = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$, here, $A = 1,\ B = -3,\ C = -180$

$$\begin{aligned}
l &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-180)}}{2(1)} \quad \text{[Substitute } A, B, C \text{ values in the formula]}\\
&= \frac{3 \pm \sqrt{9 + 720}}{2}\\
&= \frac{3 \pm \sqrt{729}}{2}\\
l &= \frac{3 \pm 27}{2}\\
\implies & \quad l = \frac{3+27}{2},\ l = \frac{3-27}{2}\\
\implies & \quad l = \frac{\cancel{30}^{15}}{\cancel{2}},\ l = \frac{-\cancel{24}^{12}}{\cancel{2}}\\
\implies & \quad l = 15, l = -12
\end{aligned}$$

Since w refers width, so it can not be negative, hence $l = 15$,
hence $w = l - 3 \implies w = 15 - 3 = 12$
Hence the length and width of the triangle is 15 feet & 12 feet

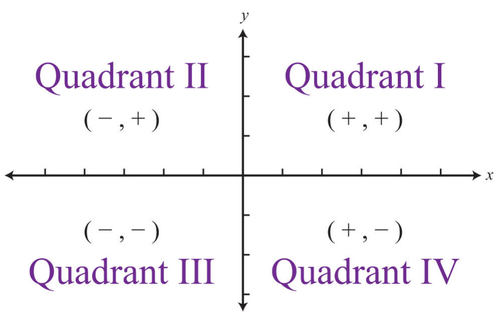
Chapter 9

COORDINATE GEOMETRY

9.1 Rectangular Coordinate System

An ordered pair (x, y) represents the position of a point relative to the origin. Using this system, every position (point) in the plane is uniquely identified. This system is often called the Cartesian coordinate system.

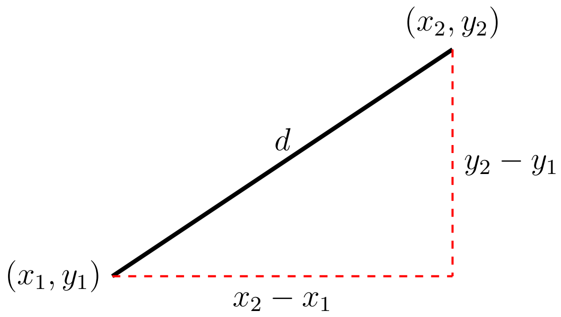
The x- and y-axes break the plane into four regions called quadrants, named using roman numerals I, II, III, and IV, as pictured.



The distance between two points

Given two points, (x_1, y_1) and (x_2, y_2), then the distance, d, between them is given by the distance formula:

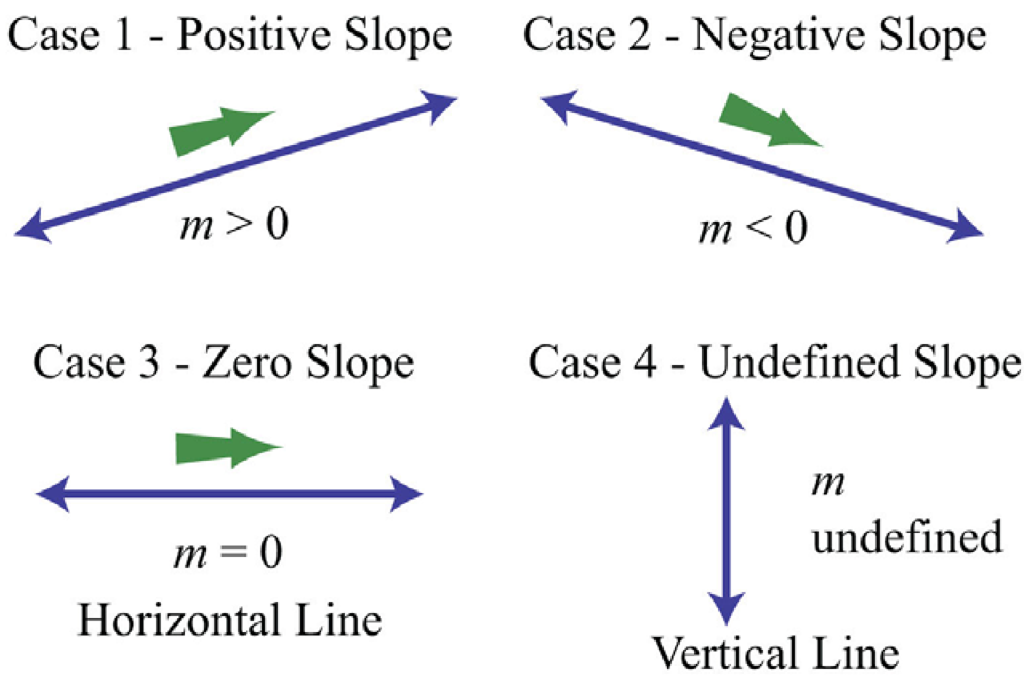
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Slope of a Straight line

The slope of the line can be measured as the ratio of the vertical change with respect to the horizontal change. Given any two points (x_1, y_1) and (x_2, y_2), the slope is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



Exercise - 9.1

Plot the given set of ordered pairs.

Problem 1:

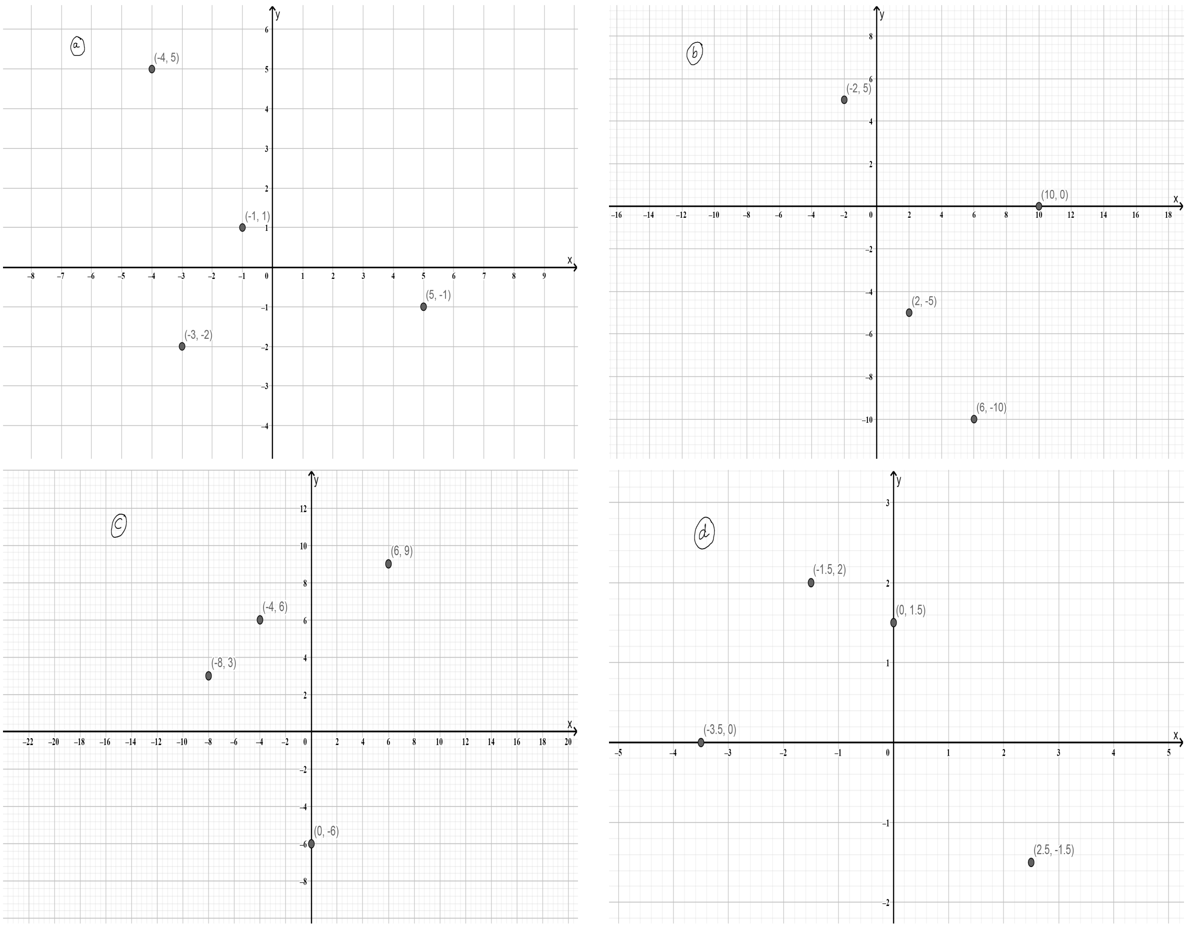
a $\{(-4, 5), (-1, 1), (-3, -2), (5, -1)\}$

b $\{(-2, 5), (10, 0), (2, -5), (6, -10)\}$

c $\{(-8, 3), (-4, 6), (0, -6), (6, 9)\}$

d $\{(-3.5, 0), (-1.5, 2), (0, 1.5), (2.5, -1.5)\}$

Solution:



State the quadrant in which the given point lies

Problem 2: $(-3, 2)$

Solution:

$(-3, 2)$: $x-$ coordinate is negative and $y-$ coordinate is positive. Hence this point belongs to the **second quadrant**

Problem 3: $(5, 7)$

Solution:

$(5, 7)$: $x-$ coordinate is positive and $y-$ coordinate is positive. Hence this point belongs to the **first quadrant**

Problem 4: $(-12, -15)$

Solution:

$(-12, -15)$: $x-$ coordinate is negative and $y-$ coordinate is negative. Hence this point belongs to the **third quadrant**

Problem 5: $(7, -8)$

Solution:

$(7, -8)$: $x-$ coordinate is positive and $y-$ coordinate is negative. Hence this point belongs to the **fourth quadrant**

Problem 6: $(-3.8, 4.6)$

Solution:

$(-3.8, 4.6)$: $x-$ coordinate is negative and $y-$ coordinate is positive. Hence this point belongs to the **second quadrant**

Problem 7: $(-18, -58)$

Solution:

$(-18, -58)$: $x-$ coordinate is negative and $y-$ coordinate is negative. Hence this point belongs to the **third quadrant**

Calculate the distance between the given two points

Problem 8: $(-5, 3)$ and $(-9, 6)$

Solution:
Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, here $x_1 = -5,\ y_1 = 3,\ x_2 = -9,\ y_2 = 6$

$$\begin{aligned} d &= \sqrt{((-9) - (-5))^2 + (6 - 3)^2} \\ &= \sqrt{(-9 + 5)^2 + (3)^2} \\ &= \sqrt{(-4)^2 + 9} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ d &= 5 \end{aligned}$$

$\boxed{\textbf{d((-5, 3), (-9, 6)) = 5}}$

Problem 9: $((6, -2))$ and $(-2, 4)$

Solution:
Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, here $x_1 = 6,\ y_1 = -2,\ x_2 = -2,\ y_2 = 4$

$$\begin{aligned} d &= \sqrt{(-2 - 6)^2 + (4 - (-2))^2} \\ &= \sqrt{(-8)^2 + (6)^2} \\ &= \sqrt{64 + 36} \\ &= \sqrt{100} \\ d &= 10 \end{aligned}$$

$\boxed{\textbf{d((6, -2), (-2, 4)) = 10}}$

Problem 10: $((0, 0))$ and $(5, 12)$

Solution:
Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, here $x_1 = 0,\ y_1 = 0,\ x_2 = 5,\ y_2 = 12$

$$\begin{aligned} d &= \sqrt{(5 - 0)^2 + (12 - 0)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ d &= 13 \end{aligned}$$

$\boxed{\textbf{d((0, 0), (5, 12)) = 13}}$

Problem 11: $(-6, -8)$ and $(0, 0)$

Solution:

Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, here $x_1 = -6,\ y_1 = -8,\ x_2 = 0,\ y_2 = 0$

$$\begin{aligned} d &= \sqrt{(0-(-6))^2 + (0-(-8))^2} \\ &= \sqrt{6^2 + 8^2} \\ &= \sqrt{36 + 64} \\ &= \sqrt{100} \\ d &= 10 \end{aligned}$$

$\boxed{\textbf{d((-6,-8), (0, 0)) = 10}}$

Problem 12: $(-7, 8)$ and $(5, -1)$

Solution:

Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, here $x_1 = -7,\ y_1 = 8,\ x_2 = 5,\ y_2 = -1$

$$\begin{aligned} d &= \sqrt{(5-(-7))^2 + (-1-(8))^2} \\ &= \sqrt{(5+7)^2 + (-1-8)^2} \\ &= \sqrt{12^2 + (-9)^2} \\ &= \sqrt{144 + 81} \\ &= \sqrt{225} \\ d &= 15 \end{aligned}$$

$\boxed{\textbf{d((-7,8), (5, -1)) = 15}}$

Problem 13: $(1, 2)$ and $(4, 3)$

Solution:

Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, here $x_1 = 1,\ y_1 = 2,\ x_2 = 4,\ y_2 = 3$

$$\begin{aligned} d &= \sqrt{(4-1)^2 + (3-2)^2} \\ &= \sqrt{(3)^2 + (1)^2} \\ &= \sqrt{9 + 1} \\ d &= \sqrt{10} \end{aligned}$$

$\boxed{d((1,2),\ (4,3)) \ = \ \sqrt{10}}$

Problem 14: $(2, -4)$ and $(-3, -2)$

Solution:
Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, here $x_1 = 2,\ y_1 = -4,\ x_2 = -3,\ y_2 = -2$

$$\begin{aligned} d &= \sqrt{(-3-2)^2 + ((-2)-(-4))^2} \\ &= \sqrt{(-5)^2 + (-2+4)^2} \\ &= \sqrt{25 + (2)^2} \\ &= \sqrt{25+4} \\ d &= \sqrt{29} \end{aligned}$$

$$\boxed{d((2,-4),\ (-3,-2)) \ = \ \sqrt{29}}$$

Problem 15: $(-7, -3)$ and $(2, 6)$

Solution:
Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, here $x_1 = -7,\ y_1 = -3,\ x_2 = 2,\ y_2 = 6$

$$\begin{aligned} d &= \sqrt{(2-(-7))^2 + (6-(-3))^2} \\ &= \sqrt{(2+7)^2 + (6+3)^2} \\ &= \sqrt{9^2 + 9^2} \\ &= \sqrt{81+81} \\ d &= \sqrt{2 \times 81} \end{aligned}$$

$$\boxed{d((-7,-3),\ (2,6)) \ = \ 9\sqrt{2}}$$

Problem 16: $(0, 1)$ and $(1, 0)$

Solution:
Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, here $x_1 = 0,\ y_1 = 1,\ x_2 = 1,\ y_2 = 0$

$$\begin{aligned} d &= \sqrt{(1-0)^2 + (0-1)^2} \\ &= \sqrt{1+1} \\ d &= \sqrt{2} \end{aligned}$$

$$\boxed{d((0,1),\ (1,0)) \ = \ \sqrt{2}}$$

Find the slope of the line joining the given two points.

Problem 17: $(3, 2)$ and $(5, 1)$

Solution:

Formula: Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here $x_1 = 3,\ y_1 = 2,\ x_2 = 5,\ y_2 = 1$

$$m = \frac{1-2}{5-3} \implies \boxed{m = \frac{-1}{2}}$$

Problem 18: $(7, 8)$ and $(-3, 5)$

Solution:

Formula: Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here $x_1 = 7,\ y_1 = 8,\ x_2 = -3,\ y_2 = 5$

$$m = \frac{5-8}{-3-7} = \frac{\cancel{-}3}{\cancel{-}10} \implies \boxed{m = \frac{3}{10}}$$

Problem 19: $(2, -3)$ and $(-3, 2)$

Solution:

Formula: Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here $x_1 = 2,\ y_1 = -3,\ x_2 = -3,\ y_2 = 2$

$$m = \frac{2-(-3)}{-3-2} = \frac{2+3}{-5} = \frac{\cancel{5}}{-\cancel{5}} \implies \boxed{m = -1}$$

Problem 20: $(-3, 2)$ and $(7, -5)$

Solution:

Formula: Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here $x_1 = -3,\ y_1 = 2,\ x_2 = 7,\ y_2 = -5$

$$m = \frac{-5-2}{7-(-3)} = \frac{-7}{7+3} \implies \boxed{m = \frac{-7}{10}}$$

Problem 21: $(-1,-6)$ and $(3,2)$

Solution:

Formula: Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here $x_1 = -1,\ y_1 = -6,\ x_2 = 3,\ y_2 = 2$

$$m = \frac{2-(-6)}{3-(-1)} = \frac{2+6}{3+1} = \frac{\cancel{8}^2}{\cancel{4}} \implies \boxed{m = 2}$$

Problem 22: $(5,3)$ and $(4,12)$

Solution:

Formula: Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here $x_1 = 5,\ y_1 = 3,\ x_2 = 4,\ y_2 = 12$

$$m = \frac{12-3}{4-5} = \frac{9}{-1} \implies \boxed{m = -9}$$

Problem 23: $(-3,1)$ and $(-14,1)$

Solution:

Formula: Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here $x_1 = -3,\ y_1 = 1,\ x_2 = -14,\ y_2 = 1$

$$m = \frac{1-1}{-14-(-3)} = \frac{0}{-14+3} \implies \boxed{m = 0}$$

Problem 24: $(-4,-4)$ and $(5,5)$

Solution:

Formula: Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here $x_1 = -4,\ y_1 = -4,\ x_2 = 5,\ y_2 = 5$

$$m = \frac{5-(-4)}{5-(-4)} = \frac{5+4}{5+4} = \frac{\cancel{9}}{\cancel{9}} \implies \boxed{m = 1}$$

Problem 25: $(-2,3)$ and $(-2,-4)$

Solution:

Formula: Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here $x_1 = -2,\ y_1 = 3,\ x_2 = -2,\ y_2 = -4$

$$m = \frac{-4-3}{-2-(-2)} = \frac{-7}{0} \implies \boxed{m = \text{undefined}}$$

Problem 26: Find b if the slope of the line passing through $(-2,3)$ and $(4,b)$ is 12

Solution:

Formula: Slope, $m = \dfrac{y_2-y_1}{x_2-x_1}$, here $x_1=-2,\ y_1=3,\ x_2=4,\ y_2=b$ & $m=12$

$$\begin{aligned} 12 &= \frac{b-3}{4-(-2)} = \frac{b-3}{4+2} = \frac{b-3}{6} \\ \implies b-3 &= 12\times 6 \\ \implies b-3 &= 72 \\ \implies b &= 72+3 \implies \boxed{b=75} \end{aligned}$$

Problem 27: Find b if the slope of the line passing through $(5,b)$ and $(-4,2)$ is 0

Solution:

Formula: Slope, $m = \dfrac{y_2-y_1}{x_2-x_1}$, here $x_1=5,\ y_1=b,\ x_2=-4,\ y_2=2$ & $m=0$

$$0 = \frac{2-b}{-4-5} \implies 2-b=0 \implies \boxed{b=2}$$

Problem 28: Find a if the slope of the line passing through $(-3,2)$ and $(a,5)$ is undefined

Solution:

Formula: Slope, $m = \dfrac{y_2-y_1}{x_2-x_1}$, here
$x_1=-3,\ y_1=2,\ x_2=a,\ y_2=5$ & $m=$undefined

$$\text{undefined} = \frac{5-2}{a-(-3)} = \frac{3}{a+3} \implies a+3=0 \implies \boxed{a=-3}$$

9.2 Straight Lines

Slope - Intercept Form of a Line

The equation of the form $y = mx + b$ is known as slope - intercept form of a non-vertical line. Here, m represents the slope and $(0, b)$ represent the $y-$intercept point.

Point - Slope form of a line

The equation of any non-vertical line with slope m and passing through the point (x_1, y_1) can be written in the point - slope form $y - y_1 = m(x - x_1)$.

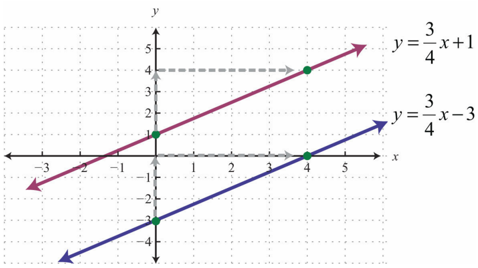
Standard equation of a straight line

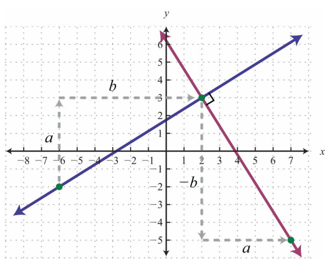
The standard equation of a straight line is $ax + by + c = 0$, where a, b and c are real numbers and a and b are not both 0. Here, $ax + by + c = 0 \implies by = -ax - c \implies$ $y = \dfrac{-a}{b}x - \dfrac{c}{b}$,

- Slope, $m = -\dfrac{a}{b}$
- x- intercept $= \left(-\dfrac{c}{a}, 0\right)$
- y- intercept $= \left(0, -\dfrac{c}{b}\right)$

Parallel lines and Perpendicular Lines

Parallel lines are lines in the same plane that never intersect. ***Perpendicular lines*** are lines in the same plane that intersect at right angles (90 degrees).





- **If m is the slope of a straight line, then**
  - The slope of a parallel line is, $m_{\parallel} = m$.

- The slope of a perpendicular line is, $m_\perp = -\dfrac{1}{m}$.

- **If m_1 and m_2 are the slopes of two straight lines L_1 and L_2, then**
  - L_1 and L_2 are parallel if $m_1 = m_2$.
  - L_1 and L_2 are perpendicular $m_1 \cdot m_2 = -1$.

Exercise - 9.2

Find the $x-$ intercepts and $y-$ intercepts.

Problem 1: $\boldsymbol{5x - 4y = 20}$

Solution:
Formula: For the straight line $\boldsymbol{ax + by + c = 0}$,
$\boldsymbol{x -}$ intercept $= \left(\boldsymbol{-\dfrac{c}{a}, 0}\right)$; $\boldsymbol{y -}$ intercept $= \left(\boldsymbol{0, -\dfrac{c}{b}}\right)$
Given that, $5x - 4y = 20 \implies 5x - 4y - 20 = 0$
Here, $a = 5,\ b = -4,\ c = -20$

$$\textbf{x- intercept} = \left(-\frac{c}{a}, 0\right) = \left(-\frac{(-\cancel{20}^{4})}{\cancel{5}}, 0\right) = \boldsymbol{(4, 0)}$$

$$\textbf{y- intercept} = \left(0, -\frac{c}{b}\right) = \left(0, -\frac{(\cancel{-20}^{5})}{\cancel{-4}}\right) = \boldsymbol{(0, -5)}$$

Problem 2: $\boldsymbol{-2x + 7y = -28}$

Solution:
Formula: For the straight line $\boldsymbol{ax + by + c = 0}$,
$\boldsymbol{x -}$ intercept $= \left(\boldsymbol{-\dfrac{c}{a}, 0}\right)$; $\boldsymbol{y -}$ intercept $= \left(\boldsymbol{0, -\dfrac{c}{b}}\right)$
Given that, $-2x + 7y = -28 \implies -2x + 7y + 28 = 0$
Here, $a = -2,\ b = 7,\ c = 28$

$$\textbf{x- intercept} = \left(-\frac{c}{a}, 0\right) = \left(-\frac{\cancel{28}^{14}}{-\cancel{2}}, 0\right) = \boldsymbol{(14, 0)}$$

$$\textbf{y- intercept} = \left(0, -\frac{c}{b}\right) = \left(0, -\frac{\cancel{28}^{4}}{\cancel{7}}\right) = \boldsymbol{(0, -4)}$$

Problem 3: $\boldsymbol{x - y = 3}$

Solution:

Formula: For the straight line $\boldsymbol{ax + by + c = 0}$,

$\boldsymbol{x-}$ intercept $= \boldsymbol{\left(-\frac{c}{a}, 0\right)}$; $\boldsymbol{y-}$ intercept $= \boldsymbol{\left(0, -\frac{c}{b}\right)}$

Given that, $x - y = 3 \implies x - y - 3 = 0$

Here, $a = 1,\ b = -1,\ c = -3$

x- intercept $= \left(-\frac{c}{a}, 0\right) = \left(-\frac{(-3)}{1}, 0\right) = \mathbf{(3, 0)}$

y- intercept $= \left(0, -\frac{c}{b}\right) = \left(0, -\frac{(-3)}{(-1)}\right) = \mathbf{(0, -3)}$

Problem 4: $\boldsymbol{y = 6}$

Solution:

Formula: For the straight line $\boldsymbol{ax + by + c = 0}$,

$\boldsymbol{x-}$ intercept $= \boldsymbol{\left(-\frac{c}{a}, 0\right)}$; $\boldsymbol{y-}$ intercept $= \boldsymbol{\left(0, -\frac{c}{b}\right)}$

Given that, $y = 6 \implies 0x + y - 6 = 0$

Here, $a = 0,\ b = 1,\ c = -6$

Since, $a = 0$, the curve does not intersect $x-$ axis.

y- intercept $= \left(0, -\frac{c}{b}\right) = \left(0, -\frac{(-6)}{1}\right) = \mathbf{(0, 6)}$

Problem 5: $\boldsymbol{x = -1}$

Solution:

Formula: For the straight line $\boldsymbol{ax + by + c = 0}$,

$\boldsymbol{x-}$ intercept $= \boldsymbol{\left(-\frac{c}{a}, 0\right)}$; $\boldsymbol{y-}$ intercept $= \boldsymbol{\left(0, -\frac{c}{b}\right)}$

Given that, $x = -1 \implies x + 0y + 1 = 0$

Here, $a = 1,\ b = 0,\ c = 1$

x- intercept $= \left(-\frac{c}{a}, 0\right) = \left(-\frac{1}{1}, 0\right) = \mathbf{(-1, 0)}$

Since, $b = 0$, the curve does not intersect $y-$ axis.

Problem 6: $\boldsymbol{y = mx + b}$

Solution:

Formula: For the straight line $\boldsymbol{ax + by + c = 0}$,

$\boldsymbol{x-}$ intercept $= \boldsymbol{\left(-\frac{c}{a}, 0\right)}$; $\boldsymbol{y-}$ intercept $= \boldsymbol{\left(0, -\frac{c}{b}\right)}$

Given that, $y = mx + b \implies mx - y + b = 0$
Here, $a = m,\ b = -1,\ c = b$

x- intercept $= \left(-\frac{c}{a}, 0\right) = \left(\boldsymbol{-\frac{b}{m}, 0}\right)$

y- intercept $= \left(0, -\frac{c}{b}\right) = \left(0, -\frac{b}{(-1)}\right) = \boldsymbol{(0, b)}$

Express the given linear equation in slope - intercept form and identify the slope and $y-$ intercept.

Problem 7: $\boldsymbol{6x - 5y = 30}$

Solution:

$$\begin{aligned} 6x - 5y = 30 &\implies 6x - 30 = 5y \\ &\implies y = \frac{6x - 30}{5} \\ &\implies y = \frac{6}{5}x - \frac{\cancel{30}^{6}}{\cancel{5}} \\ &\implies y = \frac{6}{5}x - 6 \end{aligned}$$

This is of the form $y = mx + b$, hence, Slope, $m = \frac{6}{5}$
and $y-$ intercept$= (0, b) = (0, -6)$

Problem 8: $\boldsymbol{-2x + 7y = 28}$

Solution:

$$\begin{aligned} -2x + 7y = 28 &\implies 7y = 2x + 28 \\ &\implies y = \frac{2x + 28}{7} \\ &\implies y = \frac{2}{7}x + \frac{\cancel{28}^{4}}{\cancel{7}} \\ &\implies y = \frac{2}{7}x + 4 \end{aligned}$$

This is of the form $y = mx + b$, hence, Slope, $m = \frac{2}{7}$
and $y-$ intercept$= (0, b) = (0, 4)$

Problem 9: $x - 3y = 18$

Solution:

$$\begin{aligned} x - 3y = 18 &\implies x - 18 = 3y \\ &\implies y = \frac{x-18}{3} \\ &\implies y = \frac{1}{3}x - \frac{\cancel{18}^{6}}{\cancel{3}} \\ &\implies y = \frac{1}{3}x - 6 \end{aligned}$$

This is of the form $y = mx + b$, hence, Slope, $m = \frac{1}{3}$ and $y-$ intercept$= (0, b) = (0, -6)$

Problem 10: $2x - 3y = 0$

Solution:

$$\begin{aligned} 2x - 3y = 0 &\implies 2x = 3y \\ &\implies y = \frac{2}{3}x \end{aligned}$$

This is of the form $y = mx + b$, hence, Slope, $m = \frac{2}{3}$ and $y-$ intercept$= (0, b) = (0, 0)$

Problem 11: $-6x + 3y = 0$

Solution:

$$\begin{aligned} -6x + 3y = 0 &\implies 3y = 6x \\ &\implies y = \frac{\cancel{6}^{2}}{\cancel{3}}x \\ &\implies y = 2x \end{aligned}$$

This is of the form $y = mx + b$, hence, Slope, $m = 2$ and $y-$ intercept$= (0, b) = (0, 0)$

Problem 12: $9x - y = 17$

Solution:

$$\begin{aligned} 9x - y = 17 &\implies 9x - 17 = y \\ &\implies y = 9x - 17 \end{aligned}$$

This is of the form $y = mx + b$, hence, Slope, $m = 9$ and $y-$ intercept$= (0, b) = (0, -17)$

Given the slope and $y-$ intercept, determine the equation of the line.

Problem 13: $\boldsymbol{m = 4; (0, -1)}$

Solution:
Formula: The equation of the straight line in slope and $y-$ intercept form is $y = mx + b$, here, m is slope and $y-$ intercept is $(0, b)$
Given that, $m = 4$; and y- intercept is$(0, -1)$
Hence the required equation is, $y = 4x + (-1) \implies \boxed{\text{y=4x-1}}$

Problem 14: $\boldsymbol{m = -3; (0, 9)}$

Solution:
Formula: The equation of the straight line in slope and $y-$ intercept form is $y = mx + b$, here, m is slope and $y-$ intercept is $(0, b)$
Given that, $m = -3$; and y- intercept is$(0, 9)$
Hence the required equation is, $\boxed{\text{y=-3x+9}}$

Problem 15: $\boldsymbol{m = 0; (0, -1)}$

Solution:
Formula: The equation of the straight line in slope and $y-$ intercept form is $y = mx + b$, here, m is slope and $y-$ intercept is $(0, b)$
Given that, $m = 0$; and y- intercept is$(0, -1)$
Hence the required equation is, $y = 0x + (-1) \implies \boxed{\text{y=-1}}$

Problem 16: $\boldsymbol{m = \frac{1}{2}; (0, 5)}$

Solution:
Formula: The equation of the straight line in slope and $y-$ intercept form is $y = mx + b$, here, m is slope and $y-$ intercept is $(0, b)$
Given that, $m = \frac{1}{2}$; and y- intercept is$(0, 5)$
Hence the required equation is, $y = \frac{1}{2}x + 5 \implies \boxed{\text{2y=x+10}}$

Problem 17: $\boldsymbol{m = -\frac{2}{3}; (0, -4)}$

Solution:
Formula: The equation of the straight line in slope and $y-$ intercept form is $y = mx + b$, here, m is slope and $y-$ intercept is $(0, b)$
Given that, $m = -\frac{2}{3}$; and y- intercept is$(0, -4)$
Hence the required equation is,
$y = -\frac{2}{3}x + (-4) \implies 3y = -2x - 12 \implies \boxed{\text{2x+3y+12=0}}$

Problem 18: $\boldsymbol{m = 5; (0, 0)}$

Solution:
Formula: The equation of the straight line in slope and $y-$ intercept form is $y = mx + b$, here, m is slope and $y-$ intercept is $(0, b)$
Given that, $m = 5$; and y- intercept is$(0, 0)$
Hence the required equation is, $y = 5x + 0 \implies \boxed{\text{y=5x}}$

Use the point - slope formula to find the equation of the line passing through the two points.

Problem 19: $\boldsymbol{(-4, 0), (0, 5)}$

Solution:
Given, $x_1 = -4,\ y_1 = 0,\ x_2 = 0,\ y_2 = 5$
Formula: Equation of the line, point - slope form is $y - y_1 = m(x - x_1)$
Here, Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$,
$m = \dfrac{5 - 0}{0 - (-4)} = \dfrac{5}{4}$

Equation of the Straight line is given by,

$$\begin{aligned}
& y - y_1 = m(x - x_1) \\
\implies\ & y - 0 = \left(\frac{5}{4}\right)(x - (-4)) \\
\implies\ & y = \left(\frac{5}{4}\right)(x + 4) \\
\implies\ & 4y = 5(x + 4) \\
\implies\ & 4y = 5x + 20 \\
\implies\ & \boxed{\text{5x-4y+20=0}}
\end{aligned}$$

Problem 20: $(-1, 2), (0, 3)$

Solution:
Given, $x_1 = -1,\ y_1 = 2,\ x_2 = 0,\ y_2 = 3$
Formula: Equation of the line, point - slope form is $y - y_1 = m(x - x_1)$
Here, Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$,
$m = \dfrac{3-2}{0-(-1)} = \dfrac{1}{1} = 1$

Equation of the Straight line is given by,

$$\begin{aligned}
& y - y_1 = m(x - x_1) \\
\implies & y - 2 = (1)\,(x - (-1)) \\
\implies & y - 2 = x + 1 \\
\implies & y = x + 1 + 2 \\
\implies & \boxed{\text{y=x+3}}
\end{aligned}$$

Problem 21: $(-3, -2), (3, 2)$

Solution:
Given, $x_1 = -3,\ y_1 = -2,\ x_2 = 3,\ y_2 = 2$
Formula: Equation of the line, point - slope form is $y - y_1 = m(x - x_1)$
Here, Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$,
$m = \dfrac{2-(-2)}{3-(-3)} = \dfrac{2+2}{3+3} = \dfrac{\cancel{4}^{2}}{\cancel{6}^{3}} = \dfrac{2}{3}$

Equation of the Straight line is given by,

$$\begin{aligned}
& y - y_1 = m(x - x_1) \\
\implies & y - (-2) = \left(\frac{2}{3}\right)(x - (-3)) \\
\implies & y + 2 = \left(\frac{2}{3}\right)(x + 3) \\
\implies & 3(y + 2) = 2(x + 3) \\
\implies & 3y + 6 = 2x + 6 \\
\implies & 2x - 3y - 6 + 6 = 0 \\
\implies & \boxed{\text{2x-3y=0}}
\end{aligned}$$

Problem 22: $(-3, -1), (3, 3)$

Solution:
Given, $x_1 = -3,\ y_1 = -1,\ x_2 = 3,\ y_2 = 3$
Formula: Equation of the line, point - slope form is $y - y_1 = m(x - x_1)$
Here, Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$,
$m = \dfrac{3-(-1)}{3-(-3)} = \dfrac{3+1}{3+3} = \dfrac{\cancel{4}^2}{\cancel{6}^3} = \dfrac{2}{3}$

Equation of the Straight line is given by,

$$\begin{aligned}
& y - y_1 = m(x - x_1) \\
\implies\quad & y - (-1) = \left(\frac{2}{3}\right)(x - (-3)) \\
\implies\quad & y + 1 = \left(\frac{2}{3}\right)(x + 3) \\
\implies\quad & 3(y+1) = 2(x+3) \\
\implies\quad & 3y + 3 = 2x + 6 \\
\implies\quad & 2x - 3y + 6 - 3 = 0 \\
\implies\quad & \boxed{\text{2x-3y+3=0}}
\end{aligned}$$

Problem 23: $(1, 5), (0, 5)$

Solution:
Given, $x_1 = 1,\ y_1 = 5,\ x_2 = 0,\ y_2 = 5$
Formula: Equation of the line, point - slope form is $y - y_1 = m(x - x_1)$
Here, Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$,
$m = \dfrac{5-5}{0-1} = 0$

Equation of the Straight line is given by,

$$\begin{aligned}
& y - y_1 = m(x - x_1) \\
\implies\quad & y - (5) = (0)(x - 1) \\
\implies\quad & y - 5 = 0 \\
\implies\quad & \boxed{\text{y=5}}
\end{aligned}$$

Problem 24: $(-8, 0), (6, 0)$

Solution:
Given, $x_1 = -8,\ y_1 = 0,\ x_2 = 6,\ y_2 = 0$
Formula: Equation of the line, point - slope form is $y - y_1 = m(x - x_1)$
Here, Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$,
$m = \dfrac{0 - 0}{6 - (-8)} = 0$

Equation of the Straight line is given by,

$$\begin{aligned} & y - y_1 = m(x - x_1) \\ \implies & y - 0 = (0)\,(x - (-8)) \\ \implies & \boxed{\text{y=0}} \end{aligned}$$

Determine the slopes of the lines that are parallel and perpendicular to the given lines:

Problem 25: $y = -34x + 8$

Solution:
The equation of the given straight line is, $y = -34x + 8$ and it is of the form $y = mx + c$, here $m = -34$
Slope of the parallel line = -34
Slope of the perpendicular line is $= \dfrac{-1}{m} = \dfrac{-1}{-34} = \dfrac{1}{34}$

Problem 26: $-2x + 7y = 28$

Solution:

$$\begin{aligned} & -2x + 7y = 28 \\ \implies & 7y = 2x + 28 \\ \implies & y = \frac{2}{7}x + \frac{28}{7} \end{aligned}$$

Hence, the equation of the given straight line is, $y = \dfrac{2}{7}x + \dfrac{28}{7}$ and it is of the form $y = mx + c$, here $m = \dfrac{2}{7}$

Slope of the parallel line $= \dfrac{2}{7}$

Slope of the perpendicular line is $= \dfrac{-1}{m} = \dfrac{-1}{2/7} = \dfrac{-7}{2}$

Problem 27: $\boldsymbol{y = 4x + 4}$

Solution:

The equation of the given straight line is, $y = 4x + 4$ and it is of the form $y = mx + c$, here $m = 4$

Slope of the parallel line $= 4$

Slope of the perpendicular line is $= \dfrac{-1}{m} = \dfrac{-1}{4} = \dfrac{-1}{4}$

Problem 28: $\boldsymbol{4x + 3y = 0}$

Solution:

$$\begin{aligned} & 4x + 3y = 0 \\ \implies & 3y = -4x \\ \implies & y = -\frac{4}{3}x \end{aligned}$$

Hence, the equation of the given straight line is, $y = -\frac{4}{3}x$ and it is of the form $y = mx + c$, here $m = -\frac{4}{3}$

Slope of the parallel line $= \dfrac{-4}{3}$

Slope of the perpendicular line is $= \dfrac{-1}{m} = \dfrac{-1}{-4/3} = \dfrac{3}{4}$

Problem 29: $\boldsymbol{-2x + 7y = 14}$

Solution:

$$\begin{aligned} & -2x + 7y = 14 \\ \implies & 7y = 2x + 14 \\ \implies & y = \frac{2}{7}x + \frac{14}{7} \end{aligned}$$

Hence, the equation of the given straight line is, $y = \frac{2}{7}x + \frac{14}{7}$ and it is of the form $y = mx + c$, here $m = \frac{2}{7}$

Slope of the parallel line $= \dfrac{2}{7}$

Slope of the perpendicular line is $= \dfrac{-1}{m} = \dfrac{-1}{2/7} = \dfrac{-7}{2}$

Problem 30: $-x - y = 15$

Solution:
$-x - y = 15 \implies y = -x - 15$
Hence, the equation of the given straight line is, $y = -x - 15$ and it is of the form $y = mx + c$, here $m = -1$
Slope of the parallel line $= -1$

Slope of the perpendicular line is $= \dfrac{-1}{m} = \dfrac{-1}{-1} = 1$

Problem 31: $y = 5$

Solution:
$y = 5 \implies y = 0x + 5$
Hence, the equation of the given straight line is, $y = 0x + 5$ and it is of the form $y = mx + c$, here $m = 0$
Slope of the parallel line $= 0$

Since $m = 0$**, hence slope of the perpendicular line is** $\dfrac{-1}{m} = \dfrac{-1}{0}$ **is undefined**

Problem 32: $x = -12$

Solution:
The equation of the given straight line is, $x = -12$ hence it is parallel to $y-$ axis and slope is not defined for this line
Slope of the parallel lines are also not defined
Given line is parallel to $y-$ **axis hence its perpendicular line is** $x-$ **axis, hence the slope of the perpendicular line is** 0

Problem 33: $x - y = 0$

Solution:
$x - y = 0 \implies y = x$
Hence, the equation of the given straight line is, $y = x$ and it is of the form $y = mx + c$, here $m = 1$
Slope of the parallel line $= 1$

Slope of the perpendicular line is $= \dfrac{-1}{m} = \dfrac{-1}{1} = -1$

Determine whether the lines are parallel, perpendicular, or neither.

Problem 34:

$$\begin{cases} y = \frac{2}{3}x + 3 \\ y = \frac{2}{3}x - 3 \end{cases}$$

Solution:

$$\begin{cases} y = \frac{2}{3}x + 3 & \implies m_1 = \frac{2}{3} \\ y = \frac{2}{3}x - 3 & \implies m_2 = \frac{2}{3} \end{cases}$$

Since the slopes are same, hence the given **lines are parallel**

Problem 35:

$$\begin{cases} y = \frac{3}{4}x - 1 \\ y = \frac{3}{4}x + 3 \end{cases}$$

Solution:

$$\begin{cases} y = \frac{3}{4}x - 1 & \implies m_1 = \frac{3}{4} \\ y = \frac{3}{4}x + 3 & \implies m_2 = \frac{3}{4} \end{cases}$$

Since the slopes are same, hence the given **lines are parallel**

Problem 36:

$$\begin{cases} y = -2x + 1 \\ y = \frac{1}{2}x + 8 \end{cases}$$

Solution:

$$\begin{cases} y = -2x + 1 & \implies m_1 = -2 \\ y = \frac{1}{2}x + 8 & \implies m_2 = \frac{1}{2} \end{cases}$$

Since $m_1 \cdot m_2 = (-2) \cdot \frac{1}{2} = -1$, hence the given **lines are perpendicular to each other**.

Problem 37:

$$\begin{cases} \boldsymbol{x - y = 7} \\ \boldsymbol{3x + 3y = 2} \end{cases}$$

Solution:

$$\begin{cases} x - y = 7 & \Longrightarrow y = x - 7 & & \Longrightarrow m_1 = 1 \\ 3x + 3y = 2 & \Longrightarrow 3y = -3x + 2 & \Longrightarrow y = -x + \dfrac{2}{3} & \Longrightarrow m_2 = -1 \end{cases}$$

Since $m_1 \cdot m_2 = 1 \times (-1) = -1$, hence the given **lines are perpendicular to each other**.

Problem 38:

$$\begin{cases} \boldsymbol{2x - 6y = 4} \\ \boldsymbol{-x + 3y = -2} \end{cases}$$

Solution:

$$\begin{cases} 2x - 6y = 4 & \Longrightarrow 6y = 2x - 4 & \Longrightarrow y = \dfrac{\cancel{2}}{\cancel{6}^3}x - \dfrac{4}{6} & \Longrightarrow m_1 = \dfrac{1}{3} \\ -x + 3y = -2 & \Longrightarrow 3y = x - 2 & \Longrightarrow y = \dfrac{1}{3}x - \dfrac{2}{3} & \Longrightarrow m_2 = \dfrac{1}{3} \end{cases}$$

Since the slopes are same, hence the given **lines are parallel**

Problem 39:

$$\begin{cases} \boldsymbol{3x - 5y = 15} \\ \boldsymbol{5x + 3y = 9} \end{cases}$$

Solution:

$$\begin{cases} 3x - 5y = 15 & \Longrightarrow 5y = 3x - 15 & \Longrightarrow y = \dfrac{3}{5}x - \dfrac{15}{5} & \Longrightarrow m_1 = \dfrac{3}{5} \\ 5x + 3y = 9 & \Longrightarrow 3y = -5x + 9 & \Longrightarrow y = -\dfrac{5}{3}x + \dfrac{9}{3} & \Longrightarrow m_2 = -\dfrac{5}{3} \end{cases}$$

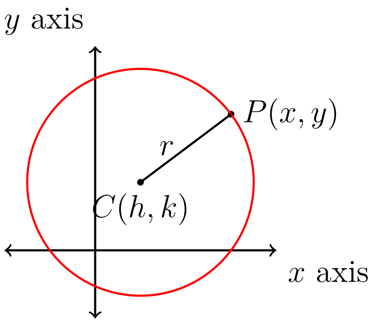
Since $m_1 \cdot m_2 = \dfrac{3}{5} \times \left(\dfrac{-5}{3}\right) = -1$, hence the given **lines are perpendicular to each other**.

9.3 Circle

A circle is all points in a plane at a fixed distance from a given point in the plane. The given point is called the center, (h, k), and the fixed distance is called the radius, r, of the circle.

Standard Form of the Equation a Circle

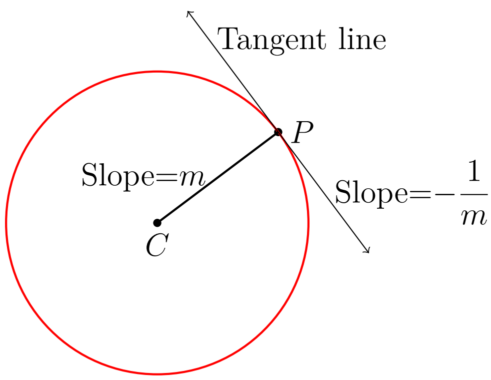
The standard form of the equation of a circle with center (h, k), and radius r, is $(x - h)^2 + (y - k)^2 = r^2$



Tangent lines

Tangent: A tangent to a circle is a line which intersects the circle in exactly one point. A tangent to a circle can also be defined as a straight line which touches the circle at only one point. This point is called the point of tangency or the point of contact.

The tangent to a circle is always perpendicular to the radius at the point of contact.



Finding the equation of the tangent line at a point on the circle.

- Find the center of the circle, $C = (h, k)$.
- Take the point as $P = (x_1, y_1)$.
- Find the slope of the radius PC, $m = \dfrac{y_1 - k}{x_1 - h}$.
- The tangent line at P is perpendicular to the radius PC. Therefore, slope of the tangent is $m' = -\dfrac{1}{m}$.
- The tangent at P is the line passing through (x_1, y_1) and having slope $m' = -\dfrac{1}{m}$.
- Equation of the tangent is $y - y_1 = m'(x - x_1)$.

Exercise - 9.3

In the following exercises, write the standard form of the equation of the circle with the given radius and center $(0, 0)$

Problem 1: Radius: 7

Solution:
Formula: Equation of the circle having center at $(0, 0)$ and radius r is given by $x^2 + y^2 = r^2$
Given $r = 7$, $x^2 + y^2 = r^2 \implies x^2 + y^2 = 7^2 \implies \boxed{x^2 + y^2 = 49}$

Problem 2: Radius: 9

Solution:
Formula: Equation of the circle having center at $(0, 0)$ and radius r is given by $x^2 + y^2 = r^2$
Given $r = 9$, $x^2 + y^2 = r^2 \implies x^2 + y^2 = 9^2 \implies \boxed{x^2 + y^2 = 81}$

Problem 3: Radius: $\sqrt{2}$

Solution:

Formula: Equation of the circle having center at $(0,0)$ and radius r is given by $x^2 + y^2 = r^2$

Given $r = \sqrt{2}$, $x^2 + y^2 = r^2 \implies x^2 + y^2 = (\sqrt{2})^2 \implies \boxed{x^2 + y^2 = 2}$

Problem 4: Radius: $\sqrt{5}$

Solution:

Formula: Equation of the circle having center at $(0,0)$ and radius r is given by $x^2 + y^2 = r^2$

Given $r = \sqrt{5}$, $x^2 + y^2 = r^2 \implies x^2 + y^2 = (\sqrt{5})^2 \implies \boxed{x^2 + y^2 = 5}$

In the following exercises, write the standard form of the equation of the circle with the given radius and center.

Problem 5: Radius: 1, center: $(3, 5)$

Solution:

Formula: Equation of the circle having center at (h,k) and radius r is given by $(x-h)^2 + (x-k)^2 = r^2$

Given that, $(h,k) = (3,5)$ and $r = 1$,

$(x-3)^2 + (y-5)^2 = 1^2 \implies \boxed{(x-3)^2 + (y-5)^2 = 1}$

Problem 6: Radius: 10, center: $(-2, 6)$

Solution:

Formula: Equation of the circle having center at (h,k) and radius r is given by $(x-h)^2 + (x-k)^2 = r^2$

Given that, $(h,k) = (-2,6)$ and $r = 10$,

$(x-(-2))^2 + (y-6)^2 = 10^2 \implies \boxed{(x+2)^2 + (y-6)^2 = 100}$

Problem 7: Radius: $\sqrt{15}$, center: $(0, 0)$

Solution:
Formula: Equation of the circle having center at (h, k) and radius r is given by $(x-h)^2 + (x-k)^2 = r^2$
Given that, $(h, k) = (0, 0)$ and $r = \sqrt{15}$,
$(x-0)^2 + (y-0)^2 = (\sqrt{15})^2 \implies \boxed{x^2 + y^2 = 15}$

Problem 8: Radius: 5, center: $(-1, 0)$

Solution:
Formula: Equation of the circle having center at (h, k) and radius r is given by $(x-h)^2 + (x-k)^2 = r^2$
Given that, $(h, k) = (-1, 0)$ and $r = 5$,
$(x-(-1))^2 + (y-0)^2 = 5^2 \implies \boxed{(x+1)^2 + y^2 = 25}$

Problem 9: Radius: 2.5, center: $(1.5, -3.5)$

Solution:
Formula: Equation of the circle having center at (h, k) and radius r is given by $(x-h)^2 + (x-k)^2 = r^2$
Given that, $(h, k) = (1.5, -3.5)$ and $r = 2.5$, $(x-h)^2 + (x-k)^2 = r^2$
$\implies (x-(1.5))^2 + (y-(-3.5))^2 = (2.5)^2$
$\implies \boxed{(x-1.5)^2 + (y+3.5)^2 = 6.25}$

Problem 10: Radius: $3\sqrt{2}$, center: $(-5.5, -6.5)$

Solution:
Formula: Equation of the circle having center at (h, k) and radius r is given by $(x-h)^2 + (x-k)^2 = r^2$
Given that, $(h, k) = (-5.5, -6.5)$ and $r = 3\sqrt{2}$,
$(x-h)^2 + (x-k)^2 = r^2 \implies (x-(-5.5))^2 + (y-(-6.5))^2 = (3\sqrt{2})^2$
$\implies \boxed{(x+5.5)^2 + (y+6.5)^2 = 18}$

For the following exercises, write the standard form of the equation of the circle with the given center with point on the circle.

Problem 11: Center $(3, -2)$ with point $(3, 6)$

Solution:
The radius is the distance from the center to the point on the circle.
So radius, $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,
here $x_1 = 3, y_1 = -2, x_2 = 3, y_2 = 6$

$$\begin{aligned} r &= \sqrt{(3-3)^2 + (6-(-2))^2} \\ &= \sqrt{0 + (6+2)^2} \\ &= \sqrt{8^2} \\ r &= 8 \end{aligned}$$

Formula: Equation of the circle having center at (h, k) and radius r is given by $(x-h)^2 + (x-k)^2 = r^2$
Given that, $(h, k) = (3, -2)$ and $r = 8$,
$(x-3)^2 + (y-(-2))^2 = 8^2 \implies \boxed{(x-3)^2 + (y+2)^2 = 64}$

Problem 12: Center $(6, -6)$ with point $(2, -3)$

Solution:
The radius is the distance from the center to the point on the circle.
So radius, $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,
here $x_1 = 6, y_1 = -6, x_2 = 2, y_2 = -3$

$$\begin{aligned} r &= \sqrt{(2-6)^2 + (-3-(-6))^2} \\ &= \sqrt{(-4)^2 + (-3+6)^2} \\ &= \sqrt{16 + 3^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ r &= 5 \end{aligned}$$

Formula: Equation of the circle having center at (h, k) and radius r is given by $(x - h)^2 + (x - k)^2 = r^2$
Given that, $(h, k) = (6, -6)$ and $r = 5$,
$(x - 6)^2 + (y - (-6))^2 = 5^2 \implies \boxed{(x - 6)^2 + (y + 6)^2 = 25}$

Problem 13: Center $(0, 0)$ with point $(-6, 8)$

Solution:
The radius is the distance from the center to the point on the circle.
So radius, $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,
here $x_1 = 0, y_1 = 0, x_2 = -6, y_2 = 8$

$$\begin{aligned} r &= \sqrt{(-6 - 0)^2 + (8 - 0)^2} \\ &= \sqrt{6^2 + 8^2} \\ &= \sqrt{36 + 64} \\ &= \sqrt{100} \\ r &= 10 \end{aligned}$$

Formula: Equation of the circle having center at (h, k) and radius r is given by $(x - h)^2 + (x - k)^2 = r^2$
Given that, $(h, k) = (0, 0)$ and $r = 10$,
$(x - 0)^2 + (y - 0)^2 = 10^2 \implies \boxed{x^2 + y^2 = 100}$

Problem 14: Center $(-3, 4)$ with point $(0, 0)$

Solution:
The radius is the distance from the center to the point on the circle.
So radius, $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,
here $x_1 = -3, y_1 = 4, x_2 = 0, y_2 = 0$

$$\begin{aligned} r &= \sqrt{(0 - (-3))^2 + (0 - 4)^2} \\ &= \sqrt{3^2 + (-4)^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ r &= 5 \end{aligned}$$

Formula: Equation of the circle having center at (h, k) and radius r is given by $(x-h)^2 + (x-k)^2 = r^2$
Given that, $(h, k) = (-3, 4)$ and $r = 5$,
$(x-(-3))^2 + (y-4)^2 = 5^2 \implies \boxed{(x+3)^2 + (y-4)^2 = 25}$

Problem 15: Center $(4, 4)$ with point $(2, 2)$

Solution:
The radius is the distance from the center to the point on the circle.
So radius, $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,
here $x_1 = 4, y_1 = 4, x_2 = 2, y_2 = 2$

$$\begin{aligned} r &= \sqrt{(2-4)^2 + (2-4)^2} \\ &= \sqrt{(-2)^2 + (-2)^2} \\ &= \sqrt{4+4} \\ r &= \sqrt{8} \end{aligned}$$

Formula: Equation of the circle having center at (h, k) and radius r is given by $(x-h)^2 + (x-k)^2 = r^2$
Given that, $(h, k) = (4, 4)$ and $r = \sqrt{8}$,
$(x-4)^2 + (y-4)^2 = (\sqrt{8})^2 \implies \boxed{(x-4)^2 + (y-4)^2 = 8}$

Problem 16: Center $(-5, 6)$ with point $(-2, 3)$

Solution:
The radius is the distance from the center to the point on the circle.
So radius, $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,
here $x_1 = -5, y_1 = 6, x_2 = -2, y_2 = 3$

$$\begin{aligned} r &= \sqrt{(-2-(-5))^2 + (3-6)^2} \\ &= \sqrt{(-2+5)^2 + (-3)^2} \\ &= \sqrt{3^2 + 9} \\ &= \sqrt{9+9} \\ r &= \sqrt{18} \end{aligned}$$

Formula: Equation of the circle having center at (h, k) and radius r is given by $(x-h)^2+(x-k)^2=r^2$
Given that, $(h, k)=(-5, 6)$ and $r=\sqrt{18}$,
$(x-(-5))^2+(y-6)^2=(\sqrt{18})^2 \implies \boxed{(x+5)^2+(y-6)^2=18}$

In the following exercises, find the center and radius of each circle.

Problem 17: $(x+5)^2+(y+3)^2=1$

Solution:
$(x+5)^2+(y+3)^2=1 \implies (x-(-5))^2+(y-(-3))^2=1^2$
It represents the standard form of the circle, $(x-h)^2+(x-k)^2=r^2$ having center at (h, k) and radius r
Hence, $\boxed{\text{center, } (h, k)=(-5,-3) \text{ and radius, } r=1}$

Problem 18: $(x-2)^2+(y-3)^2=9$

Solution:
$(x-2)^2+(y-3)^2=9 \implies (x-2)^2+(y-3)^2=3^2$
It represents the standard form of the circle, $(x-h)^2+(x-k)^2=r^2$ having center at (h, k) and radius r
Hence, $\boxed{\text{center, } (h, k)=(2, 3) \text{ and radius, } r=3}$

Problem 19: $x^2+(y+2)^2=75$

Solution:
$x^2+(y+2)^2=75 \implies (x-0)^2+(y-(-2))^2=(5\sqrt{3})^2$
It represents the standard form of the circle, $(x-h)^2+(x-k)^2=r^2$ having center at (h, k) and radius r
Hence, $\boxed{\text{center, } (h, k)=(0,-2) \text{ and radius, } r=5\sqrt{3}}$

Problem 20: $(x+2)^2+(y+5)^2=4$

Solution:
$(x+2)^2+(y+5)^2=4 \implies (x-(-2))^2+(y-(-5))^2=2^2$
It represents the standard form of the circle, $(x-h)^2+(x-k)^2=r^2$
having center at (h,k) and radius r
Hence, $\boxed{\text{center, } (h,k)=(-2,-5) \text{ and radius, } r=2}$

Problem 21: $(x-4)^2+(y+2)^2=32$

Solution:
$(x-4)^2+(y+2)^2=32 \implies (x-4)^2+(y-(-2))^2=(4\sqrt{2})^2$
It represents the standard form of the circle, $(x-h)^2+(x-k)^2=r^2$
having center at (h,k) and radius r
Hence, $\boxed{\text{center, } (h,k)=(4,-2) \text{ and radius, } r=4\sqrt{2}}$

Problem 22: $(x-1)^2+y^2=36$

Solution:
$(x-1)^2+y^2=36 \implies (x-1)^2+(y-0)^2=6^2$
It represents the standard form of the circle, $(x-h)^2+(x-k)^2=r^2$
having center at (h,k) and radius r
Hence, $\boxed{\text{center, } (h,k)=(1,0) \text{ and radius, } r=6}$

Tangent line Problems

Problem 23: **Find the equation of the tangent to the circle $(x+9)^2+(y+2)^2=125$ at $(1,3)$.**

Solution:
STEP: 1
$(x+9)^2+(y+2)^2=125 \implies (x-(-9))^2+(y-(-2))^2=125$

It represents the standard form of the circle, $(x-h)^2+(x-k)^2=r^2$ having center at (h,k) and radius r
Hence, **center** is, $(h,k)=(-9,-2)$

STEP: 2

☛ To find the equation of the tangent to the circle, we need to find the slope of the tangent equation.
☛ To find the slope of the equation, we need to find the slope of the radius line(which is perpendicular to the tangent line), it is a line joining the center and a point on the circle.
Here, center is $(-9,-2)$ and the point on the line is $(1,3)$
Slope, $m=\dfrac{y_2-y_1}{x_2-x_1}$, here, $x_1=-9, y_1=-2, x_2=1, y_2=3$

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{3-(-2)}{1-(-9)}=\frac{3+2}{1+9}=\frac{\cancel{5}}{\cancel{10}}=\frac{1}{2}$$

Hence the **slope of the tangent line,** $m'=\dfrac{-1}{m}=\dfrac{-1}{1/2}=-2$

STEP: 3

Equation of the Straight line is given by,

$$\begin{aligned}
& y-y_1=m'(x-x_1), \text{ here, } x_1=1, y_1=3 \\
\implies & y-3=(-2)(x-1) \\
\implies & y-3=-2x+2 \\
\implies & 2x+y=2+3 \\
\implies & \boxed{\text{2x+y=5}}
\end{aligned}$$

This is the equation of the required tangent line

Problem 24: Find the equation of the tangent to the circle $x^2+y^2=29$ at $(-2,5)$.

Solution:

STEP: 1

$x^2+y^2=29 \implies (x-0)^2+(y-0)^2=29$
It represents the standard form of the circle, $(x-h)^2+(x-k)^2=r^2$ having center at (h,k) and radius r

Hence, **center** is, $(h,k)=(0,0)$

STEP: 2

☛ To find the equation of the tangent to the circle, we need to find the slope of the tangent equation.

☛ To find the slope of the equation, we need to find the slope of the radius line(which is perpendicular to the tangent line), it is a line joining the center and a point on the circle.

Here, center is $(0,0)$ and the point on the line is $(-2,5)$

Slope, $m=\dfrac{y_2-y_1}{x_2-x_1}$, here, $x_1=0, y_1=0, x_2=-2, y_2=5$

$m=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{5-0}{-2-0}=\dfrac{5}{-2}=\dfrac{-5}{2}$

Hence the **slope of the tangent line,** $m'=\dfrac{-1}{m}=\dfrac{-1}{(-5)/2}=\dfrac{2}{5}$

STEP: 3

Equation of the Straight line is given by,

$$\begin{aligned}
& y-y_1=m'(x-x_1),\ \text{here, } x_1=-2, y_1=5\\
\implies\quad & y-5=\left(\frac{2}{5}\right)(x-(-2))\\
\implies\quad & 5(y-5)=2(x+2)\\
\implies\quad & 5y-25=2x+4\\
\implies\quad & 2x-5y+25+4=0\\
\implies\quad & \boxed{\text{2x-5y+29=0}}
\end{aligned}$$

This is the equation of the required tangent line

Problem 25: Find the equation of the tangent to the circle $(x-3)^2+y^2-8=0$ at $(1,-2)$.

Solution:

STEP: 1

$(x-3)^2+y^2-8=0 \implies (x-3)^2+(y-0)^2=8$

It represents the standard form of the circle, $(x-h)^2+(x-k)^2=r^2$ having center at (h,k) and radius r

Hence, **center** is, $(h, k) = (3, 0)$

STEP: 2

☛ To find the equation of the tangent to the circle, we need to find the slope of the tangent equation.

☛ To find the slope of the equation, we need to find the slope of the radius line(which is perpendicular to the tangent line), it is a line joining the center and a point on the circle.

Here, center is $(3, 0)$ and the point on the line is $(1, -2)$

Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here, $x_1 = 3, y_1 = 0, x_2 = 1, y_2 = -2$

$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-2-0}{1-3} = \dfrac{\cancel{-2}}{\cancel{-2}} = 1$

Hence the **slope of the tangent line,** $m' = \dfrac{-1}{m} = \dfrac{-1}{1} = -1$

STEP: 3

Equation of the Straight line is given by,

$$\begin{aligned}
& y - y_1 = m'(x - x_1), \text{ here, } x_1 = 1, y_1 = -2 \\
\implies & y - (-2) = (-1)(x - 1) \\
\implies & y + 2 = -x + 1 \\
\implies & x + y + 2 - 1 = 0 \\
\implies & \boxed{\text{x+y+1=0}}
\end{aligned}$$

This is the equation of the required tangent line

Problem 26: Find the equation of the tangent to the circle $(x+5)^2 + (y-7)^2 = 20$ at $(-7, 3)$.

Solution:

STEP: 1

$(x+5)^2 + (y-7)^2 = 20 \implies (x - (-5))^2 + (y-7)^2 = 20$

It represents the standard form of the circle, $(x-h)^2 + (x-k)^2 = r^2$ having center at (h, k) and radius r

Hence, **center** is, $(h, k) = (-5, 7)$

STEP: 2

☛ To find the equation of the tangent to the circle, we need to find the slope of the tangent equation.
☛ To find the slope of the equation, we need to find the slope of the radius line(which is perpendicular to the tangent line), it is a line joining the center and a point on the circle.
Here, center is $(-5, 7)$ and the point on the line is $(-7, 3)$
Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here, $x_1 = -5, y_1 = 7, x_2 = -7, y_2 = 3$
$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-7}{-7-(-5)} = \dfrac{-4}{-7+5} = \dfrac{\cancel{-4}^{2}}{\cancel{-2}} = 2$
Hence the **slope of the tangent line,** $m' = \dfrac{-1}{m} = \dfrac{-1}{2}$
STEP: 3
Equation of the Straight line is given by,

$$
\begin{aligned}
& y - y_1 = m'(x - x_1), \text{ here, } x_1 = -7, y_1 = 3 \\
\implies & y - 3 = \left(\frac{-1}{2}\right)(x - (-7)) \\
\implies & 2(y-3) = -1(x+7) \\
\implies & 2y - 6 = -x - 7 \\
\implies & x + 2y - 6 + 7 = 0 \\
\implies & \boxed{\text{x+2y+1=0}}
\end{aligned}
$$

This is the equation of the required tangent line

Problem 27: Find the slope of the tangent to the circle $(x-1)^2 + (y+1)^2 = 5$ at $(2, -3)$.

Solution:
STEP: 1
$(x-1)^2 + (y+1)^2 = 5 \implies (x-1)^2 + (y-(-1))^2 = 5$
It represents the standard form of the circle, $(x-h)^2 + (x-k)^2 = r^2$ having center at (h, k) and radius r
Hence, **center** is, $(h, k) = (1, -1)$
STEP: 2

☛ To find the equation of the tangent to the circle, we need to find the slope of the tangent equation.
☛ To find the slope of the equation, we need to find the slope of the radius line(which is perpendicular to the tangent line), it is a line joining the center and a point on the circle.
Here, center is $(1, -1)$ and the point on the line is $(2, -3)$
Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here, $x_1 = 1, y_1 = -1, x_2 = 2, y_2 = -3$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - (-1)}{2 - 1} = \frac{-3 + 1}{1} = -2$$

Hence the **slope of the tangent line** $= \dfrac{-1}{m} = \dfrac{\cancel{-}1}{\cancel{-}2} = \dfrac{1}{2}$

$$\Longrightarrow \boxed{\mathbf{\frac{1}{2}} \textbf{ is the slope of the required tangent line}}$$

Problem 28: Find the slope of the tangent to the circle $(x - 2)^2 + (y - 3)^2 = 18$ at $(-1, 6)$.

Solution:

STEP: 1

$(x - 2)^2 + (y - 3)^2 = 18$, it represents the standard form of the circle, $(x - h)^2 + (x - k)^2 = r^2$ having center at (h, k)
Hence, **center** is, $(h, k) = (2, 3)$

STEP: 2

☛ To find the equation of the tangent to the circle, we need to find the slope of the tangent equation.
☛ To find the slope of the equation, we need to find the slope of the radius line(which is perpendicular to the tangent line), it is a line joining the center and a point on the circle.
Here, center is $(2, 3)$ and the point on the line is $(-1, 6)$
Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, here, $x_1 = 2, y_1 = 3, x_2 = -1, y_2 = 6$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 3}{-1 - 2} = \frac{\cancel{3}}{-\cancel{3}} = -1$$

Hence the **slope of the tangent line** $= \dfrac{-1}{m} = \dfrac{\cancel{-1}}{\cancel{-1}} = 1$

$\Longrightarrow$ $\boxed{\textbf{1 is the slope of the required tangent line}}$

9.4 Testing Equations for Symmetry

x-axis symmetry

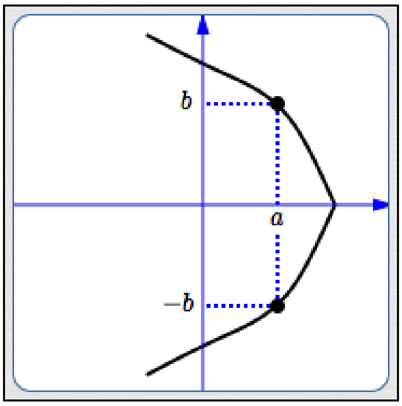
A graph has symmetry about the x-axis if and only if whenever (a, b) is on the graph, so is $(a, -b)$.

Test for x-axis symmetry

Replace every y by $-y$ in the equation.

✔ **symmetric with respect to the x-axis** if the equation is the same.

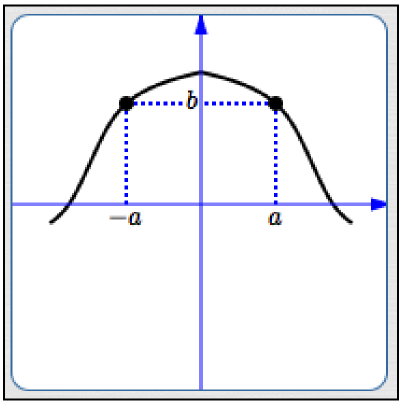
✘ **not symmetric with respect to the x- axis** if the equation is not the same.



y-axis symmetry

A graph has symmetry about the y-axis if and only if whenever (a, b) is on the graph, so is $(-a, b)$.

Test for y-axis symmetry



Replace every x by $-x$ in the equation.

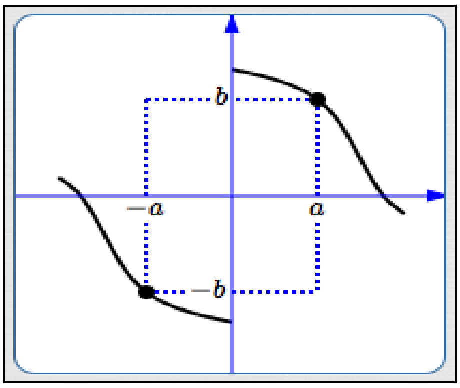
✔ **symmetric with respect to the y-axis** if the equation is the same.

✘ **not symmetric with respect to the y- axis** if the equation is not the same.

Origin symmetry

A graph has symmetry about the origin if and only if whenever (a, b) is on the graph, so is $(-a, -b)$.

Test for Origin symmetry



Replace every x by $-x$ and y by $-y$ in the equation.

✔ **symmetric with respect to the origin** if the equation is the same.

✘ **not symmetric with respect to the origin** if the equation is not the same.

Exercise - 9.4

Problem 1: Suppose that the point $(-2, 4)$ lies on a graph that has $x-$axis symmetry. What other point must lie on the graph?

Solution:

☞ If the graph is symmetry about $x-$axis and the point (a, b) is on the graph, then the point $(a, -b)$ lies on the graph.

☛ Given that, the point $(-2, 4)$ lies on a graph hence, the other point $\mathbf{(-2, -4)}$ also lies on the graph

Problem 2: Suppose that the point $(0, 7)$ lies on a graph that has $x-$axis symmetry. What other point must lie on the graph?

Solution:

☞ If the graph is symmetry about $x-$axis and the point (a, b) is on the graph, then the point $(a, -b)$ lies on the graph.

☛ Given that, the point $(0, 7)$ lies on a graph hence, the other point $\mathbf{(0, -7)}$ also lies on the graph.

Problem 3: Suppose that the point $(-2, 9)$ lies on a graph that has $y-$axis symmetry. What other point must lie on the graph?

Solution:

☞ If the graph is symmetry about $y-$axis and the point (a, b) is on the graph, then the point $(-a, b)$ lies on the graph.
☛ Given that, the point $(-2, 9)$ lies on a graph hence, the other point $(-(-2), 9) = \mathbf{(2, 9)}$ also lies on the graph.

Problem 4: Suppose that the point $(4, -3)$ lies on a graph that has $y-$axis symmetry. What other point must lie on the graph?

Solution:

☞ If the graph is symmetry about $y-$axis and the point (a, b) is on the graph, then the point $(-a, b)$ lies on the graph.
☛ Given that, the point $(4, -3)$ lies on a graph hence, the other point $\mathbf{(-4, -3)}$ also lies on the graph.

Problem 5: Suppose that the point $(1, 10)$ lies on a graph that has origin symmetry. What other point must lie on the graph?

Solution:

☞ If the graph is symmetry about origin and the point (a, b) is on the graph, then the point $(-a, -b)$ lies on the graph.
☛ Given that, the point $(1, 10)$ lies on a graph hence, the other point $\mathbf{(-1, -10)}$ also lies on the graph.

Problem 6: Suppose that the point $(-9,-1)$ lies on a graph that has origin symmetry. What other point must lie on the graph?

Solution:
☞ If the graph is symmetry about origin and the point (a, b) is on the graph, then the point $(-a, -b)$ lies on the graph.
☛ Given that, the point $(-9,-1)$ lies on a graph hence, the other point $(-(-9), -(-1)) = \mathbf{(9, 1)}$ also lies on the graph.

Problem 7: Suppose that the point $(6,-5)$ lies on a graph that has origin symmetry. What other point must lie on the graph?

Solution:
☞ If the graph is symmetry about origin and the point (a, b) is on the graph, then the point $(-a, -b)$ lies on the graph.
☛ Given that, the point $(6,-5)$ lies on a graph hence, the other point $(-6, -(-5)) = \mathbf{(-6, 5)}$ also lies on the graph.

Problem 8: $y = 3x + 5$

Solution:
Given equation is $y = 3x + 5$

Symmetry	Verification
$x-$ axis	❈ **Replace** y **by** $-y \implies -y = 3x + 5$, ✘ Different equation ☛ **Hence the graph is not symmetry about** $x-$ **axis**
$y-$ axis	❈ **Replace** x **by** $-x \implies y = 3(-x) + 5 \implies y = -3x + 5$, ✘ Different equation ☛ **Hence the graph is not symmetry about** $y-$ **axis**
Origin	❈ **Replace** x **with** $-x$ **and** y **with** $-y \implies -y = -3x + 5$, ✘ Different equation ☛ **Hence the graph is not symmetry about origin**

Problem 9: $2y = x^2 - 6$

Solution:
Given equation is $2y = x^2 - 6$

Symmetry	Verification
$x-$ axis	❈ **Replace y by $-y$** $\implies 2(-y) = x^2 - 6 \implies -2y = x^2 - 6,$ ✘ Different equation ☛ **Hence the graph is not symmetry about $x-$ axis**
$y-$ axis	❈ **Replace x by $-x$** $\implies 2y = (-x)^2 - 6 \implies 2y = x^2 - 6,$ ✔ ***Same equation*** ☛ **Hence the graph is symmetry about $y-$ axis**
Origin	❈ **Replace x with $-x$ and y with $-y$** $\implies 2(-y) = (-x)^2 - 6 \implies -2y = x^2 - 6,$ ✘ Different equation ☛ **Hence the graph is not symmetry about origin**

Problem 10: $x + y^2 = 4$

Solution:
Given equation is $x + y^2 = 4$

Symmetry	Verification
$x-$ axis	❈ **Replace y by $-y$** $\implies x + (-y)^2 = 4 \implies x + y^2 = 4,$ ✔ ***Same equation*** ☛ **Hence the graph is symmetry about $x-$ axis**
$y-$ axis	❈ **Replace x by $-x$** $\implies -x + y^2 = 4,$ ✘ Different equation ☛ **Hence the graph is not symmetry about $y-$ axis**
Origin	❈ **Replace x with $-x$ and y with $-y$** $\implies -x + (-y)^2 = 4 \implies -x + y^2 = 4,$ ✘ Different equation ☛ **Hence the graph is not symmetry about origin**

Problem 11: $3x^2 - 5xy + 2y^2 = 25$

Solution:

Given equation is $3x^2 - 5xy + 2y^2 = 25$

Symmetry	Verification
$x-$ axis	❊ **Replace y by $-y$** $\implies 3x^2 - 5x(-y) + 2(-y)^2 = 25 \implies 3x^2 + 5xy + 2y^2 = 25,$ ✘ Different equation ☛ **Hence the graph is not symmetry about $x-$ axis**
$y-$ axis	❊ **Replace x by $-x$** $\implies 3(-x)^2 - 5(-x)y + 2y^2 = 25 \implies 3x^2 + 5xy + 2y^2 = 25,$ ✘ Different equation ☛ **Hence the graph is not symmetry about $y-$ axis**
Origin	❊ **Replace x with $-x$ and y with $-y$** $\implies 3(-x)^2 - 5(-x)(-y) + 2(-y)^2 = 25 \implies 3x^2 - 5xy + 2y^2 = 25,$ ✔ ***Same equation*** ☛ **Hence the graph is symmetry about origin**

Problem 12: $x^3 - 2xy - xy^2 = 0$

Solution:

Given equation is $x^3 - 2xy - xy^2 = 0$

Symmetry	Verification
$x-$ axis	❊ **Replace y by $-y$** $\implies x^3 - 2x(-y) - x(-y)^2 = 0 \implies x^3 + 2xy - xy^2 = 0,$ ✘ Different equation ☛ **Hence the graph is not symmetry about $x-$ axis**
$y-$ axis	❊ **Replace x by $-x$** $\implies (-x)^3 - 2(-x)y - (-x)y^2 = 0 \implies x^3 - 2xy - xy^2 = 0$ ✔ ***Same equation*** ☛ **Hence the graph is symmetry about $y-$ axis**
Origin	❊ **Replace x with $-x$ and y with $-y$** $\implies (-x)^3 - 2(-x)(-y) - (-x)(-y)^2 = 0 \implies -x^3 - 2xy + xy^2 = 0,$ ✘ Different equation ☛ **Hence the graph is not symmetry about origin**

Problem 13: $-4x^4y = 8x^6y^7$

Solution:

Given equation is $-4x^4y = 8x^6y^7$

Symmetry	Verification
$x-$ axis	❇ **Replace** y **by** $-y$ $\implies -4x^4(-y) = 8x^6(-y)^7 \implies 4x^4y = -8x^6y^7,$ ✔ ***Same equation*** ☛ **Hence the graph is symmetry about** $x-$ **axis**
$y-$ axis	❇ **Replace** x **by** $-x$ $\implies -4(-x)^4y = 8(-x)^6y^7 \implies -4x^4y = 8x^6y^7,$ ✔ ***Same equation*** ☛ **Hence the graph is symmetry about** $y-$ **axis**
Origin	❇ **Replace** x **with** $-x$ **and** y **with** $-y$ $\implies -4(-x)^4(-y) = 8(-x)^6(-y)^7 \implies 4x^4y = -8x^6y^7,$ ✔ ***Same equation*** ☛ **Hence the graph is symmetry about origin**

Problem 14: $x^2 + y^2 = 16$

Solution:

Given equation is $x^2 + y^2 = 16$

Symmetry	Verification
$x-$ axis	❇ **Replace** y **by** $-y$ $\implies x^2 + (-y)^2 = 16 \implies x^2 + y^2 = 16,$ ✔ ***Same equation*** ☛ **Hence the graph is symmetry about** $x-$ **axis**
$y-$ axis	❇ **Replace** x **by** $-x$ $\implies (-x)^2 + y^2 = 16 \implies x^2 + y^2 = 16,$ ✔ ***Same equation*** ☛ **Hence the graph is symmetry about** $y-$ **axis**
Origin	❇ **Replace** x **with** $-x$ **and** y **with** $-y$ $\implies (-x)^2 + (-y)^2 = 16 \implies x^2 + y^2 = 16,$ ✔ **Same equation** ☛ **Hence the graph is symmetry about origin**

Problem 15: $y = x^3$

Solution:

Given equation is $y = x^3$

Symmetry	Verification
$x-$ axis	❈ **Replace y by $-y$** $\Longrightarrow \ -y = x^3$, ✘ Different equation ☛ **Hence the graph is not symmetry about $x-$ axis**
$y-$ axis	❈ **Replace x by $-x$** $\Longrightarrow \ y = (-x)^3 \ \Longrightarrow \ y = -x^3$, ✘ Different equation ☛ **Hence the graph is not symmetry about $y-$ axis**
Origin	❈ **Replace x with $-x$ and y with $-y$** $\Longrightarrow \ -y = (-x)^3 \ \Longrightarrow \ y = x^3$, ✔ ***Same equation*** ☛ **Hence the graph is symmetry about origin**

Chapter 10

TRIGONOMETRY

10.1 Angles and Circle

Measure of Angles

1. **Degree:** The measure of an angle is the amount of rotation from the initial side to the terminal. Most familiar unit of angle measurement is the degree. One degree is $\dfrac{1}{360}$ of a circular rotation side. **Notation:** 90 **degrees** $=90^\circ$

2. **Radian:** The radian measure of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle.

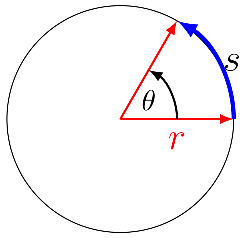
3. **Relationship Between Radians and Degrees**

   - 180 Degrees $=\pi$ Radians
   - 1 Degree $=\dfrac{\pi}{180}$ Radians
   - 1 Radian $=\dfrac{180}{\pi}$ Degrees

Determining the Length of an Arc and Area of a Sector

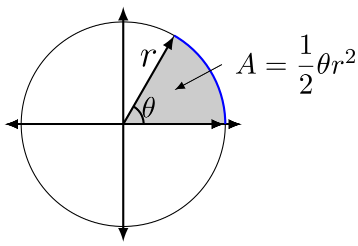
Arc length on a circle

In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians is $\boldsymbol{s = r\theta}$



Area of a sector

The area of a sector of a circle with radius r subtended by an angle θ, measured in radians, is $\boldsymbol{A = \frac{1}{2}\theta r^2}$



Exercise - 10.1

In the following exercises, convert angles in radians to degrees

Problem 1: $\frac{3\pi}{4}$ **radians**

$$
\begin{aligned}
1 \text{ Radian} &= \frac{180}{\pi} \text{ Degree} \qquad \text{[Formula]} \\
\frac{3\pi}{4} \text{ Radian} &= \frac{\cancel{180}^{45}}{\cancel{\pi}} \times \frac{3\cancel{\pi}}{\cancel{4}} \text{ Degree} \\
&= 45 \times 3 \text{ Degree} \\
\frac{3\pi}{4} \text{ Radians} &= 135^\circ
\end{aligned}
$$

Problem 2: $\frac{\pi}{4}$ radians

$$\begin{aligned}
1 \text{ Radian} &= \frac{180}{\pi} \text{ Degree} && \text{[Formula]}\\
\frac{\pi}{4} \text{ Radian} &= \frac{\cancel{180}^{45}}{\cancel{\pi}} \times \frac{\cancel{\pi}}{\cancel{4}} \text{ Degree}\\
\frac{\pi}{4} \text{ Radians} &= 45^\circ
\end{aligned}$$

Problem 3: $\frac{-5\pi}{12}$ radians

$$\begin{aligned}
1 \text{ Radian} &= \frac{180}{\pi} \text{ Degree} && \text{[Formula]}\\
\frac{-5\pi}{12} \text{ Radian} &= \frac{\cancel{180}^{15}}{\cancel{\pi}} \times \frac{-5\cancel{\pi}}{\cancel{12}} \text{ Degree}\\
&= 15 \times (-5) \text{ Degree}\\
\frac{-5\pi}{12} \text{ Radians} &= -75^\circ
\end{aligned}$$

Problem 4: $\frac{\pi}{3}$ radians

$$\begin{aligned}
1 \text{ Radian} &= \frac{180}{\pi} \text{ Degree} && \text{[Formula]}\\
\frac{\pi}{3} \text{ Radian} &= \frac{\cancel{180}^{60}}{\cancel{\pi}} \times \frac{\cancel{\pi}}{\cancel{3}} \text{ Degree}\\
\frac{\pi}{3} \text{ Radians} &= 60^\circ
\end{aligned}$$

For the following exercises, convert angles in degrees to radians.

Problem 5: 90°

$$\begin{aligned}
1 \text{ Degree} &= \frac{\pi}{180} \text{ Radians} && \text{[Formula]}\\
90 \text{ Degree} &= \frac{\pi}{\cancel{180}^{2}} \times \cancel{90} \text{ Radians}\\
90^\circ &= \frac{\pi}{2} \text{ Radians}
\end{aligned}$$

Problem 6: 210°

$$\begin{aligned} 1 \text{ Degree} &= \frac{\pi}{180} \text{ Radians} && \text{[Formula]} \\ 210 \text{ Degree} &= \frac{\pi}{\cancel{180}^{6}} \times \cancel{210}^{7} \text{ Radians} \\ 90^\circ &= \frac{7\pi}{6} \text{ Radians} \end{aligned}$$

Problem 7: $-540°$

$$\begin{aligned} 1 \text{ Degree} &= \frac{\pi}{180} \text{ Radians} && \text{[Formula]} \\ \text{-540 Degree} &= \frac{\pi}{\cancel{180}} \times (-\cancel{540})^{3} \text{ Radians} \\ -540^\circ &= -3\pi \text{ Radians} \end{aligned}$$

Problem 8: $-120°$

$$\begin{aligned} 1 \text{ Degree} &= \frac{\pi}{180} \text{ Radians} && \text{[Formula]} \\ \text{-120 Degree} &= \frac{\pi}{\cancel{180}^{3}} \times (-\cancel{120})^{2} \text{ Radians} \\ -120^\circ &= -\frac{2\pi}{3} \text{ Radians} \end{aligned}$$

In the following exercises, use the given information to find the length of a circular arc.

Problem 9: Find the length of the arc of a circle of radius 9 miles subtended by the central angle of $\frac{\pi}{3}$

Solution:

Formula: Arc length, $s = r\theta$, here, $r = 9,\ \theta = \frac{\pi}{3}$

Arc length, $s = r\theta = (\cancel{9}^{3})\frac{\pi}{\cancel{3}} = 3\pi$

The required **arc length is 3π miles**

Problem 10: Find the length of the arc of a circle of diameter 15 meters subtended by the central angle of $\boldsymbol{\frac{11\pi}{6}}$

Solution:

Formula: Arc length, $s = r\theta$,

here, diameter=15 $\implies$ radius=$\frac{15}{2}$, $\theta = \frac{11\pi}{6}$

Arc length, $s = r\theta = \frac{\cancel{15}^5}{2} \times \frac{11\pi}{\cancel{6}^2} = \frac{55\pi}{4}$

The required **arc length is $\boldsymbol{\frac{55\pi}{4}}$ meters**

Problem 11: Find the length of the arc of a circle of radius 10 centimeters subtended by the central angle of 50°

Solution:

Formula: Arc length, $s = r\theta$, here, $r = 10$, $\theta = 50°$

First, we need to convert the angle measure into radians

$$\text{Radian} = \frac{\pi}{180}\text{ Degree} \implies \text{Radian} = \frac{\pi}{18\cancel{0}} \times 5\cancel{0} = \frac{5\pi}{18}$$

Arc length, $s = r\theta = (\cancel{10}^5)\frac{5\pi}{\cancel{18}^9} = \frac{25\pi}{9}$

The required **arc length is $\boldsymbol{\frac{25\pi}{9}}$ centimeters**

Problem 12: Find the length of the arc of a circle of radius 5 inches subtended by the central angle of 220°

Solution:

Formula: Arc length, $s = r\theta$, here, $r = 5$, $\theta = 220°$

First, we need to convert the angle measure into radians

$$\text{Radian} = \frac{\pi}{180}\text{ Degree} \implies \text{Radian} = \frac{\pi}{\cancel{180}^9} \times \cancel{220}^{11} = \frac{11\pi}{9}$$

Arc length, $s = r\theta = (5)\frac{11\pi}{9} \implies$ **Arc length $= \boldsymbol{\frac{55\pi}{9}}$ inches**

In the following exercises, use the given information to find the area of the sector.

Problem 13: A sector of a circle has a central angle of 45° and a radius 6 cm

Solution:

Formula: Area of the sector, $A = \frac{1}{2}\theta r^2$, here, $r = 6$, $\theta = 45°$

First, we need to convert the angle measure into radians

$$\text{Radian} = \frac{\pi}{180}\text{ Degree} \implies \text{Radian} = \frac{\pi}{\cancel{180}^{4}} \times \cancel{45} = \frac{\pi}{4}$$

Area of the sector, $A = \frac{1}{2}\theta r^2 = \frac{1}{2} \times \frac{\pi}{4} \times 6^2 = \frac{\cancel{36}^{9}\pi}{\cancel{8}^{2}} = \frac{9\pi}{2} = 4.5\pi$

The required **area of the sector is 4.5π cm²**

Problem 14: A sector of a circle has a central angle of 30° and a radius 20 cm

Solution:

Formula: Area of the sector, $A = \frac{1}{2}\theta r^2$, here, $r = 20$, $\theta = 30°$

First, we need to convert the angle measure into radians

$$\text{Radian} = \frac{\pi}{180}\text{ Degree} \implies \text{Radian} = \frac{\pi}{\cancel{180}^{6}} \times \cancel{30} = \frac{\pi}{6}$$

Area of the sector, $A = \frac{1}{2}\theta r^2 = \frac{1}{2} \times \frac{\pi}{6} \times 20^2 = \frac{\cancel{400}^{100}\pi}{\cancel{12}^{3}} = \frac{100\pi}{3}$

The required **area of the sector is** $\boldsymbol{\frac{100\pi}{3}}$ **cm²**

Problem 15: A sector of a circle with diameter 10 feet and an angle of $\frac{\pi}{2}$ radians

Solution:

Formula: Area of the sector, $A = \frac{1}{2}\theta r^2$,

here, $r = \frac{\text{diameter}}{2} = \frac{\cancel{10}^5}{\cancel{2}} = 5,\ \theta = \frac{\pi}{2}$

Area of the sector, $A = \frac{1}{2}\theta r^2 = \frac{1}{2} \times \frac{\pi}{2} \times 5^2 = \frac{25\pi}{4} = 6.25\pi$

The required **area of the sector is 6.25π square feet**

Problem 16: A sector of a circle with diameter 0.7 inches and an angle of π radians

Solution:

Formula: Area of the sector, $A = \frac{1}{2}\theta r^2$,

here, $r = \frac{\text{diameter}}{2} = \frac{0.7}{2} = 0.35,\ \theta = \pi$

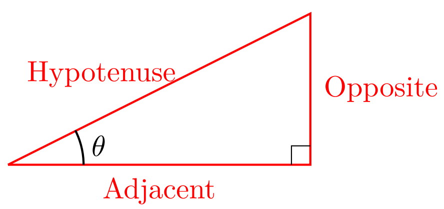
$$A = \frac{1}{2}\theta r^2 = \frac{1}{2} \times \pi \times (0.35)^2 = \frac{0.1225\pi}{2} = 0.06125\pi$$

The required **area of the sector is 0.245π square inches**

10.2 Trigonometric Ratios and Identities

Trigonometric Ratios

Consider the right angled triangle



	Trignometric Ratio		**Reciprocal Functions**	**Name**
1.	$\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}}$	1.	$\sin\theta = \dfrac{1}{\csc\theta}$	[**Sine**]
2.	$\cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}}$	2.	$\cos\theta = \dfrac{1}{\sec\theta}$	[**Cosine**]
3.	$\tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}}$	3.	$\tan\theta = \dfrac{1}{\cot\theta}$	[**Tangent**]
4.	$\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite side}}$	4.	$\csc\theta = \dfrac{1}{\sin\theta}$	[**Cosecant**]
5.	$\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent side}}$	5.	$\sec\theta = \dfrac{1}{\cos\theta}$	[**Secant**]
6.	$\cot\theta = \dfrac{\text{adjacent side}}{\text{opposite side}}$	6.	$\cot\theta = \dfrac{1}{\tan\theta}$	[**Cotangent**]

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$	$1 + \cot^2\theta = \csc^2\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sin^2\theta = 1 - \cos^2\theta$	$\cot^2\theta = \csc^2\theta - 1$	$\tan^2\theta = \sec^2\theta - 1$
$\cos^2\theta = 1 - \sin^2\theta$	$\csc^2\theta - \cot^2\theta = 1$	$\tan^2\theta - \sec^2\theta = 1$

Even & Odd Identities

Even Identity	**Odd Identities**	
$\cos(-\theta) = \cos\theta$	$\sin(-\theta) = -\sin\theta$	$\tan(-\theta) = -\tan\theta$
$\sec(-\theta) = \sec\theta$	$\csc(-\theta) = -\csc\theta$	$\cot(-\theta) = -\cot\theta$

Exercise - 10.2

For the following exercises, use Figures to evaluate each trigonometric ratio of angle θ

Problem 1: Use Figure 1 to evaluate each trigonometric ratio of angle θ

Solution: From Figure 1, hypotenuse=10; adjacent side=6

By Pythagorean Theorem,

$$\begin{aligned}\text{opposite side} &= \sqrt{(\text{hypotenuse})^2 - (\text{adjacent side})^2}\\ &= \sqrt{10^2 - 6^2} = \sqrt{100 - 36} = \sqrt{64} = 8\end{aligned}$$

Trignometric Ratio

1. $\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{\cancel{8}^4}{\cancel{10}^5} = \dfrac{4}{5}$
2. $\cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{\cancel{6}^3}{\cancel{10}^5} = \dfrac{3}{5}$
3. $\tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{\cancel{8}^4}{\cancel{6}^3} = \dfrac{4}{3}$
4. $\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite side}} = \dfrac{\cancel{10}^5}{\cancel{8}^4} = \dfrac{5}{4}$
5. $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent side}} = \dfrac{\cancel{10}^5}{\cancel{6}^3} = \dfrac{5}{3}$
6. $\cot\theta = \dfrac{\text{adjacent side}}{\text{opposite side}} = \dfrac{\cancel{6}^3}{\cancel{8}^4} = \dfrac{3}{4}$

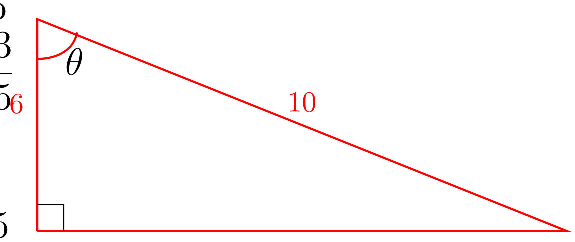


Figure 1

Problem 2: Use Figure 2 to evaluate each trigonometric ratio of angle α

Solution: From the figure 2, opposite side=8; adjacent side=6

By Pythagorean theorem,

$$\begin{aligned} \text{hypotenuse} &= \sqrt{(\text{opposite side})^2 + (\text{adjacent side})^2} \\ &= \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = 10 \end{aligned}$$

Trignometric Ratio

1. $\sin\alpha = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{\cancel{8}^4}{\cancel{10}^5} = \dfrac{4}{5}$
2. $\cos\alpha = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{\cancel{6}^3}{\cancel{10}^5} = \dfrac{3}{5}$
3. $\tan\alpha = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{\cancel{8}^4}{\cancel{6}^3} = \dfrac{4}{3}$
4. $\csc\alpha = \dfrac{\text{hypotenuse}}{\text{opposite side}} = \dfrac{\cancel{10}^5}{\cancel{8}^4} = \dfrac{5}{4}$
5. $\sec\alpha = \dfrac{\text{hypotenuse}}{\text{adjacent side}} = \dfrac{\cancel{10}^5}{\cancel{6}^3} = \dfrac{5}{3}$
6. $\cot\alpha = \dfrac{\text{adjacent side}}{\text{opposite side}} = \dfrac{\cancel{6}^3}{\cancel{8}^4} = \dfrac{3}{4}$

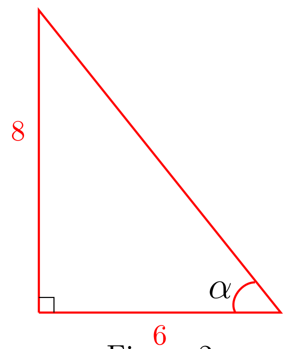


Figure 2

For the following exercises, use the fundamental identities to fully simplify the expression

Problem 3: $\sin x \cos x \sec x$

Solution:

$$\begin{aligned}\sin x \cos x \sec x &= \sin x \cancel{\cos x}\left(\frac{1}{\cancel{\cos x}}\right) && \left[\sec x = \frac{1}{\cos x}\right]\\ &= \sin x \implies \boxed{\sin x \cos x \sec x = \sin x}\end{aligned}$$

Problem 4: $\sin(-x)\cos(-x)\csc(-x)$

Solution:

$$\begin{aligned}&\sin(-x)\cos(-x)\csc(-x)\\ &\quad= (-\sin x)(\cos x)(-\csc x) && \text{[Even \& odd identies]}\\ &\quad= \cancel{\sin x} \times \cos x \times \frac{1}{\cancel{\sin x}} && \left[\csc x = \frac{1}{\sin x}\right]\\ &\quad= \cos x \implies \boxed{\sin(-x)\cos(-x)\csc(-x) = \cos x}\end{aligned}$$

Problem 5: $\tan x \sin x + \sec x \cos^2 x$

Solution:

$$\begin{aligned}&\tan x \sin x + \sec x \cos^2 x\\ &\quad= \frac{\sin x}{\cos x} \times \sin x + \frac{1}{\cos x} \times \cos^2 x && \left[\tan x = \frac{\sin x}{\cos x};\ \sec x = \frac{1}{\cos x}\right]\\ &\quad= \frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} = \frac{\sin^2 x + \cos^2 x}{\cos x}\\ &\quad= \frac{1}{\cos x} && [\sin^2 x + \cos^2 x = 1]\\ &\quad= \sec x && \left[\frac{1}{\cos x} = \sec x\right]\end{aligned}$$

$$\boxed{\tan x \sin x + \sec x \cos^2 x = \sec x}$$

Problem 6: $\csc x + \cos x \cot(-x)$

Solution:

$\csc x + \cos x \cot(-x)$

$$= \csc x + \cos x(-\cot x) \quad [\cot(-x) = -\cot x]$$

$$= \frac{1}{\sin x} - \cos x \times \frac{\cos x}{\sin x} \quad [\csc x = \frac{1}{\sin x};\ \cot x = \frac{\cos x}{\sin x}]$$

$$= \frac{1 - \cos^2 x}{\sin x}$$

$$= \frac{\sin^{\cancel{2}} x}{\cancel{\sin x}} = \sin x \quad [1 - \cos^2 x = \sin^2 x]$$

$$\boxed{\csc x + \cos x \cot(-x) = \sin x}$$

Problem 7: $\dfrac{\cot t + \tan t}{\sec(-t)}$

Solution:

$$\frac{\cot t + \tan t}{\sec(-t)} = \frac{\frac{\cos t}{\sin t} + \frac{\sin t}{\cos t}}{\sec t}$$

$$[\sec(-t) = \sec t,\ \cot t = \frac{\cos t}{\sin t},\ \tan t = \frac{\sin t}{\cos t}]$$

$$= \frac{\cos^2 t + sin^2 t}{\sin t \cos t} \times \frac{1}{\sec t}$$

$$= \frac{1}{\sin t \cos t} \times \frac{1}{\sec t} \quad [\sin^2 t + \cos^2 t = 1]$$

$$= \left(\frac{1}{\sin t}\right)\left(\frac{1}{\cos t}\right) \times \frac{1}{\sec t}$$

$$= (\csc t)\,(\cancel{\sec t}) \times \frac{1}{\cancel{\sec t}} \quad [\csc t = \frac{1}{\sin t},\ \sec t = \frac{1}{\cos t}]$$

$$= \csc t$$

$$\boxed{\frac{\cot t + \tan t}{\sec(-t)} = \csc t}$$

Problem 8: $3\sin^3 t \csc t + \cos^2 t + 2\cos(-t)\cos t$

Solution:

$3\sin^3 t \csc t + \cos^2 t + 2\cos(-t)\cos t$

$$
\begin{aligned}
&= 3\sin^3 t \csc t + \cos^2 t + 2\cos(-t)\cos t \\
&= 3\sin^{\cancel{3}2} t \left(\frac{1}{\cancel{\sin t}}\right) + \cos^2 t + 2(\cos t)\cos t \\
&\qquad\qquad [\csc t = \frac{1}{\sin t},\ \cos(-t) = \cos t] \\
&= 3\sin^2 t + \cos^2 t + 2\cos^2 t \\
&= 3\sin^2 t + 3\cos^2 t \\
&= 3(\sin^2 t + \cos^2 t) \\
&= 3(1) = 3 \qquad [\sin^2 t + \cos^2 t = 1]
\end{aligned}
$$

$$\boxed{3\sin^3 t \csc t + \cos^2 t + 2\cos(-t)\cos t = 3}$$

Problem 9: $-\tan(-x)\cot(-x)$

Solution:

$-\tan(-x)\cot(-x)$

$$
\begin{aligned}
&= -(-\tan x)(-\cot x) \quad [\tan(-x) = -\tan x, \cot(-x) = -\cot x] \\
&= \cancel{\tan x}\left(\frac{-1}{\cancel{\tan x}}\right) \qquad [\cot t = \frac{1}{\tan t}] \\
&= -1
\end{aligned}
$$

$$\boxed{-\tan(-x)\cot(-x) = -1}$$

Problem 10: $\dfrac{-\sin(-x)\cos x \sec x \csc x \tan x}{\cot x}$

Solution:

$$\frac{-\sin(-x)\cos x \sec x \csc x \tan x}{\cot x}$$

$$= \frac{-(-\sin x)\cos x \sec x \csc x \tan x}{\cot x} \qquad [\sin(-x) = -\sin x]$$

$$= \frac{\cancel{\sin x} \times \cancel{\cos x} \times \dfrac{1}{\cancel{\cos x}} \times \dfrac{1}{\cancel{\sin x}} \times \tan x}{\cot x} \qquad [\sec x = \frac{1}{\cos x},\ \csc x = \frac{1}{\sin x}]$$

$$= \tan x \times \tan x \qquad [\tan x = \frac{1}{\cot x}]$$

$$= \tan^2 x$$

$$\boxed{\frac{-\sin(-x)\cos x \sec x \csc x \tan x}{\cot x} = \tan^2 x}$$

Problem 11: $\dfrac{1-\cos^2 x}{\tan^2 x} + 2\sin^2 x$

Solution:

$$\frac{1-\cos^2 x}{\tan^2 x} + 2\sin^2 x$$

$$= \frac{\sin^2 x}{\tan^2 x} + 2\sin^2 x \qquad [1-\cos^2 x = \sin^2 x]$$

$$= \sin^2 x \times \cot^2 x + 2\sin^2 x \qquad [\frac{1}{\tan^2 x} = \cot^2 x]$$

$$= \cancel{\sin^2 x} \times \frac{\cos^2 x}{\cancel{\sin^2 x}} + 2\sin^2 x \qquad [\cot^2 x = \frac{\cos^2 x}{\sin^2 x}]$$

$$= \cos^2 x + \sin^2 x + \sin^2 x$$

$$= 1 + \sin^2 x$$

$$\boxed{\frac{1-\cos^2 x}{\tan^2 x} + 2\sin^2 x = 1 + \sin^2 x}$$

For the following exercises, verify the identity.

Problem 12: $\boldsymbol{\cos x - \cos^3 x = \cos x \sin^2 x}$

Solution:

$$\begin{aligned}
\text{L.H.S} &= \cos x - \cos^3 x \\
&= \cos x\,(1 - \cos^2 x) \\
&= \cos x\,(\sin^2 x) \qquad [1 - \cos^2 x = \sin^2 x] \\
\text{L.H.S} &= \text{R.H.S}
\end{aligned}$$

Hence it is verified

Problem 13: $\boldsymbol{\cos x(\tan x - \sec(-x)) = \sin x - 1}$

Solution:

$$\begin{aligned}
\text{L.H.S} &= \cos x(\tan x - \sec(-x)) \\
&= \cos x(\tan x - \sec x) \qquad [\sec(-x) = \sec x] \\
&= \cos x \times \tan x - \cos x \times \sec x \\
&= \cancel{\cos x} \times \frac{\sin x}{\cancel{\cos x}} - \cancel{\cos x} \times \frac{1}{\cancel{\cos x}} \qquad \left[\tan x = \frac{\sin x}{\cos x};\ \sec x = \frac{1}{\cos x}\right] \\
&= \sin x - 1 \\
\text{L.H.S} &= \text{R.H.S}
\end{aligned}$$

Hence it is verified

Problem 14: $\boldsymbol{\dfrac{1 + \sin^2 x}{\cos^2 x} = 1 + 2\tan^2 x}$

Solution:

$$\begin{aligned}
\text{L.H.S} &= \frac{1 + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} \qquad [\sec(-x) = \sec x] \\
&= \sec^2 x + \tan^2 x \qquad \left[\tan x = \frac{\sin x}{\cos x};\ \sec x = \frac{1}{\cos x}\right] \\
&= 1 + \tan^2 x + \tan^2 x \qquad [\sec^2 x = 1 + \tan^2 x] \\
&= 1 + 2\tan^2 x \\
\text{L.H.S} &= \text{R.H.S} \implies \textbf{Hence it is verified}
\end{aligned}$$

Problem 15: $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

Solution:

$$\begin{aligned}
\text{L.H.S} &= (\sin x + \cos x)^2 \\
&= (\sin x)^2 + 2\sin x \cos x + (\cos x)^2 \quad [(a+b)^2 = a62 + 2ab + b^2] \\
&= \sin^2 x + 2\sin x \cos x + \cos^2 x \\
&= \sin^2 x + \cos^2 x + 2\sin x \cos x \qquad [\tan x = \frac{\sin x}{\cos x};\ \sec x = \frac{1}{\cos x}] \\
&= 1 + 2\sin x \cos x \qquad [\sin^2 x + \cos^2 x = 1] \\
\text{L.H.S} &= \text{R.H.S}
\end{aligned}$$

Hence it is verified

Problem 16: $\cos^2 x - \tan^2 x = 2 - \sin^2 x - \sec^2 x$

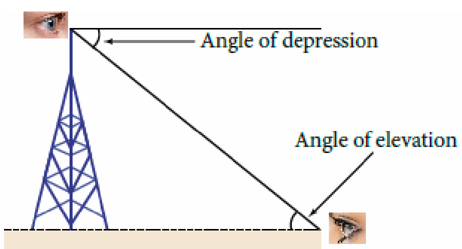
Solution:

$$\begin{aligned}
\text{L.H.S} &= \cos^2 x - \tan^2 x \\
&= 1 - \sin^2 x - (\sec^2 x - 1) \quad [\cos^2 x = 1 - \sin^2 x,\ \tan^2 x = \sec^2 x - 1] \\
&= 1 - \sin^2 x - \sec^2 x + 1 \\
&= 2 - \sin^2 x - \sec^2 x \\
\text{L.H.S} &= \text{R.H.S}
\end{aligned}$$

Hence it is verified

10.3 Applications of Trigonometry

- The **angle of elevation** is the angle formed by the horizontal line of sight and the object when a person is looking up the object
- The **angle of depression** is the angle between the horizontal line of sight and the object when a person is looking down the object.

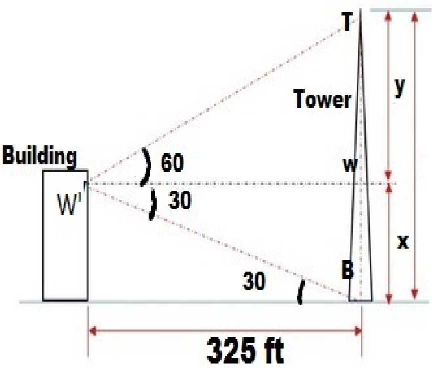


Exercise - 10.3

Problem 1: A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 60 radians, and that the angle of depression to the bottom of the tower is 30 radians. How tall is the tower?

Solution:

$$\begin{aligned}
\text{We know that, } \tan\theta &= \frac{\text{Opposite side}}{\text{Adjacent side}} \\
\text{From the } \triangle\, W'WB, \tan 30^\circ &= \frac{x}{325} \\
\implies x &= \tan 30^\circ \times 325 \\
&= \frac{1}{\sqrt{3}} \times 325 \\
&= \frac{325}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\
x &= \frac{325\sqrt{3}}{3} \\
\text{From the } \triangle\, W'WT, \tan 60^\circ &= \frac{y}{325} \\
\implies y &= \tan 60^\circ \times 325 \\
&= \sqrt{3} \times 325 \\
y &= 325\sqrt{3} \\
\text{Height of the tower} &= x + y \\
&= \frac{325\sqrt{3}}{3} + 325\sqrt{3} \\
&= \frac{325\sqrt{3}}{3} + \frac{975\sqrt{3}}{3} \\
&= \frac{1300\sqrt{3}}{3}
\end{aligned}$$

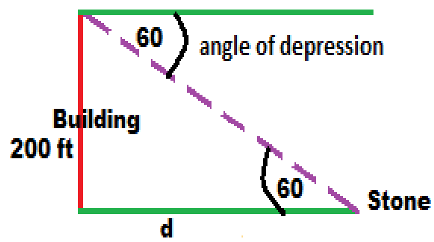


Hence, the height of the tower is $\dfrac{1300\sqrt{3}}{3}$ feet

Problem 2: From the top of a building of height 200 feet, a person determines that the angle of depression of a stone on the ground is 60 degrees. How far is the stone from the base of the building?

Solution:

$$\begin{aligned}
\text{We know that, } \tan\theta &= \frac{\text{Opposite side}}{\text{Adjacent side}} \\
\text{From the figure, } \tan 60^\circ &= \frac{200}{d} \\
\implies d &= \frac{200}{\tan 60^\circ} \\
&= \frac{200}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\
h &= \frac{200\sqrt{3}}{3}
\end{aligned}$$

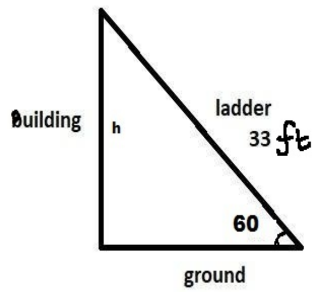


Hence, the distance between the stone and the building is $\dfrac{200\sqrt{3}}{3}$ feet

Problem 3: A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 60°. How high does the ladder reach up the side of the building?

Solution:

$$\begin{aligned}
\text{We know that, } \sin\theta &= \frac{\text{Opposite side}}{\text{Hypotenuse}} \\
\text{From the figure, } \sin 60^\circ &= \frac{h}{33} \\
\implies h &= \sin 60^\circ \times 33 \\
&= \frac{\sqrt{3}}{2} \times 33 \\
h &= \frac{33\sqrt{3}}{2}
\end{aligned}$$

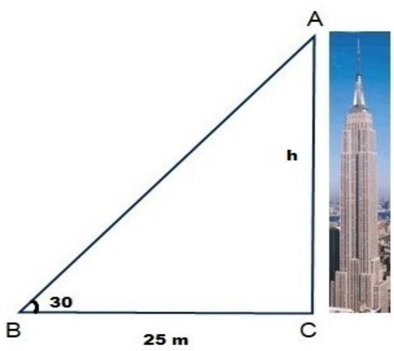


Hence, the required height of the building is $\dfrac{33\sqrt{3}}{2}$ feet

Problem 4: The angle of elevation to the top of a building in New York is found to be 30 degrees from the ground at a distance of 25 meter from the base of the building. Using this information, find the height of the building

Solution:

$$\begin{aligned}
\text{We know that}, \tan\theta &= \frac{\text{Opposite side}}{\text{Adjacent side}} \\
\text{From the figure}, \tan 30^\circ &= \frac{h}{25} \\
\implies h &= \tan 30^\circ \times 25 \\
&= \frac{1}{\sqrt{3}} \times 25 \\
&= \frac{25}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\
h &= \frac{25\sqrt{3}}{3}
\end{aligned}$$



Hence, the height of the building is $\frac{25\sqrt{3}}{3}$ feet

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