SYMMETRY CLASSES OF THE COMPLETE PIEZOELECTRICITY LAW

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Abstract. The piezoelectricity law describes the electrical behavior of a material in response to applied mechanical stress. In its linear formulation, this law involves three vector spaces of constitutive tensors: the space of permittivity tensors $\mathbb{S}$, the space of piezoelectricity tensors $\mathbb{P}\mathrm{iez}$ and the space of elasticity tensors $\mathbb{E}\mathrm{la}$. We already know the symmetry classes for the representation of $\mathrm{O}(3)$ on each of the above three spaces separately. In this paper, we will establish the symmetry classes for the coupled law using Clips operation.

Contents

1. Introduction

Given a Lie group G and a vector space V, we can define a linear map that permits to represent the elements of G using linear applications in $GL(V)$. This representation of G on V permits to divide the elements of V in classes, called symmetry classes, such that two elements of V belong to the same class if their symmetry groups (consisting of the elements of G that fix the element v in question) are conjugate.

Finding the symmetry classes of a Lie group representation has been always an interesting problem. However, it is not always an easy task, all we know is that there exists a finite number of them (see [3]).

For instance, for the $\mathrm{SO}(3)$ representation on the space of elasticity tensors $\mathbb{E}\mathrm{la}$, there exists eight symmetry classes found by Forte and Vianello in 1996 (see [1]). In other words, Forte and Vianello divided the elasticity tensors into eight equivalence classes representing eight different types of symmetry material. Following Forte and Vianello, 16 symmetry classes are obtained for the $\mathrm{O}(3)$ representation on the space of piezoelectricity tensors $\mathbb{P}\mathrm{iez}$ (see [7]). The Piezoelectricity symmetry classes can be also obtained in an easier way using the clips operation (see [4]) as well as the elasticity symmetry classes (see [5]).

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2. Piezoelectricity

The piezoelectricity law describes the electrical behavior of a material subject to mechanical stress.

The mechanical state of a material is characterized by two fields of symmetric second order tensors: the stress tensor σ and the strain tensor ε. The relation between these two fields forms

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the constitutive law that describes the mechanical behaviour of a specific material. In linear elasticity, the relation is linear, given by the known Hook's law, and permits us to introduce the elasticity tensor.

Definition 2.1. The *elasticity* tensor is a fourth order tensor $\mathbf{E}$ that relates the stress and the strain tensors in Hooke's low:

$$\sigma = \mathbf{E} : \varepsilon$$

satisfying the index symmetry:

$$\mathbf{E}_{ijkl} = \mathbf{E}_{jikl} = \mathbf{E}_{ijlk} = \mathbf{E}_{klij}$$

The space of elasticity tensors $\mathbb{E}\mathrm{la}$ is a 21 dimensional vector space.

Similarly to the mechanical state, the electrical state of a material is described by two vector fields: the electric displacement field $\mathbf{d}$ and the electric field $\mathbf{e}$. These two fields are related and the relation between them forms the constitutive law that describes the electrical behavior of a material.

Definition 2.2. The *permittivity* tensor is a second order tensor that relates the displacement and electric field:

$$\mathbf{d} = \mathbf{S}.\mathbf{e}$$

satifying the index symmetry: $\mathbf{S}_{ij} = \mathbf{S}_{ji}$. The space of the permittivity tensors, $\mathbb{S}$, is a 6 dimensional vector space.

One can study the mechanical and the electrical state of a material at the same time using the coupled law given by:

$$\begin{cases} \sigma = \mathbf{E} : \varepsilon - \mathbf{e}.\mathbf{P} \\ \mathbf{d} = \mathbf{P} : \sigma + \mathbf{S}.\mathbf{e} \end{cases}$$

Definition 2.3. The coupled law involves a third order tensor $\mathbf{P}$ called the *piezoelectricity* tensor satisfying the index symmetry:

$$\mathbf{P}_{ijk} = \mathbf{P}_{ikj}$$

The space of piezoelectricity tensors, $\mathbb{P}\mathrm{iez}$, is an 18 dimensional vector space.

Provided that the material is homogeneous, its linear electromechanical behaviour is defined by a triplet $\mathcal{P}$ of constitutive tensors

$$\mathcal{P} := (\mathbf{E}, \mathbf{P}, \mathbf{S}) \in \mathbb{E}\mathrm{la} \oplus \mathbb{P}\mathrm{iez} \oplus \mathbb{S}$$

We will denote by $\mathcal{P}\mathrm{iez}$ the space of piezoelectricity law:

$$\mathcal{P}\mathrm{iez} = \mathbb{E}\mathrm{la} \oplus \mathbb{P}\mathrm{iez} \oplus \mathbb{S}$$

3. Clips operation

In this section we will define the notions of symmetry groups and symmetry classes of a group representation. Next, we will introduce the clips operation that will be the tool to find the symmetry classes of the coupled law.

Consider G a compact Lie group with an action $*$ on a vector space V of finite dimension.

Definition 3.1. A representation of the group G on V is a linear application such that:

$$\begin{aligned} \rho :\ G &\to GL(V) \\ g &\to \rho(g) \quad \forall v \in V\ \rho(g)v = g * v \end{aligned}$$

and that verifies $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ (group morphism).

Given a group representation (V, ρ), we define the symmetry group of an element v of V to be the group of elements of G that fix v:

Definition 3.2. The symmetry group of $v \in V$ is defined by: $G_v = \{g \in G;\ \rho(g)v = v\}$.

Now we regroup the elements of V in symmetry classes:

Definition 3.3. The symmetry class (or isotropy class) of a vector v is the conjugacy class of its symmetry group, where the conjugacy class $[H]$ of a subgroup H is defined as follows:

$$[H] := \{gHg^{-1},\ g \in G\}$$

In other terms, v_1 and v_2 have the same symmetry class if their symmetry groups are conjugate *i.e.*

$$\exists h \in G;\ G_{v_2} = h\ G_{v_1}\ h^{-1}.$$

We denote by $\mathcal{J}(V)$ the set of all symmetry classes of $v \in V$ of the representation V:

$$\mathcal{J}(V) := \{[G_v] : v \in V\}$$

For the O(3)-representation on the space of elasticity tensors $\mathbb{E}$la, Forte and Vianello found in [1] 8 symmetry classes given in this theorem:

Theorem 3.4. *The symmetry classes for an elasticity tensor are*

$$\mathcal{J}(\mathbb{E}\mathrm{la}) = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4], [\mathbb{O}], [\mathrm{O}(2)], [\mathrm{SO}(2)]\}$$

The following theorem gives the symmetry classes of the space $\mathbb{P}$iez. For the clarity of presentation, the notations and definitions of O(3)-subgroups have been moved to Appendix A.

Theorem 3.5. *The space $\mathbb{P}$iez is partitioned into 16 symmetry classes:*

$$\mathcal{J}(\mathbb{P}\mathrm{iez}) = \Big\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{D}_2^v], [\mathbb{D}_3^v], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], [\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4^h], [\mathbb{D}_6^h], [\mathrm{SO}(2)], [\mathrm{O}(2)], [\mathrm{O}(2)^-], [\mathbb{O}^-], [\mathrm{O}(3)]\Big\}$$

Remark 3.6. There are many ways to find the symmetry classes. One can follow the approach elaborated by Forte and Vianello in the elasticity case in [1] to apply it in the piezoelastcity case (see [2] for instance) or one can use the clips operation as done in [4] and [6].

In order to find the symmetry classes of the coupled law, we will define the clips operation between two conjugacy classes.

Definition 3.7. Given two conjugacy classes $[H_1]$ and $[H_2]$ of a group G, we define their clips as the subset of conjugacy classes

$$[H_1] \circledcirc [H_2] := \big\{[H_1 \cap gH_2g^{-1}] : g \in G\big\}$$

This definition immediately extends to two families (finite or infinite) $\mathcal{F}_1$ and $\mathcal{F}_2$ of conjugacy classes:

$$\mathcal{F}_1 \circledcirc \mathcal{F}_2 = \bigcup_{[H_i]\in\mathcal{F}_i} [H_1] \circledcirc [H_2].$$

Example 3.8. We have

- $[\mathbb{1}] \circledcirc [H] = \{[\mathbb{1}]\}$
- $[G] \circledcirc [H] = \{[H]\}$

for every conjugacy class $[H]$ and where $\mathbb{1}$ is the trivial group.

Thanks to the following lemma, we will find the symmetry classes of the O(3)-representation on the space of the piezoelectricity law $\mathcal{P}$iez$= \mathbb{E}\mathrm{la} \oplus \mathbb{P}\mathrm{iez} \oplus \mathbb{S}$.

Lemma 3.9. *Let V_1 and V_2 be two linear representations of G. Then*

$$\mathcal{J}(V_1 \oplus V_2) = \mathcal{J}(V_1) \circledcirc \mathcal{J}(v_2)$$

4. Main result

In this section, we regroup, in graphs, the results that we obtained on the calculation of the clips operation between the type II and III O(3)-subgroups and we will prove those results in the next section.
Let m and n be two integers and $d = \gcd(m, n)$.

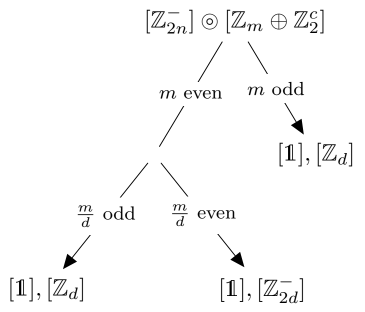

Figure 1. Clips of $\mathbb{Z}_{2n}^-$ with $\mathbb{Z}_m$.

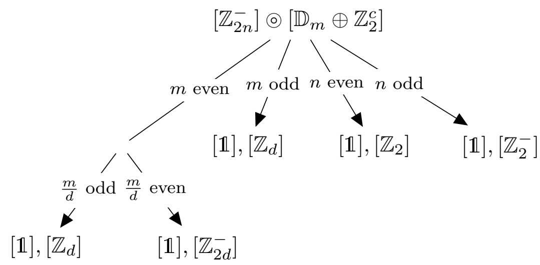

Figure 2. Clips of $\mathbb{Z}_{2n}^-$ with $\mathbb{D}_m$.

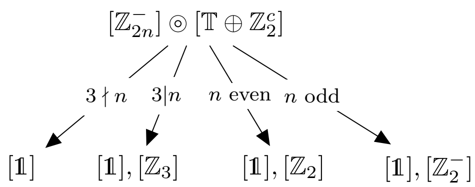

Figure 3. Clips of $\mathbb{Z}_{2n}^-$ with $\mathbb{T}$.

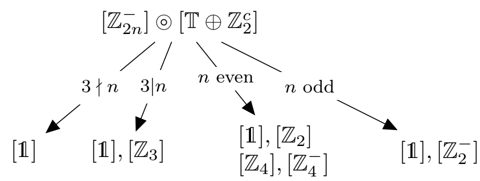

Figure 4. Clips of $\mathbb{Z}_{2n}^-$ with *octa*.

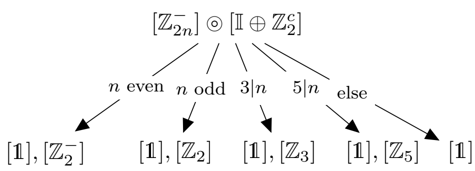

Figure 5. Clips of $\mathbb{Z}_{2n}^-$ with $\mathbb{I}$.

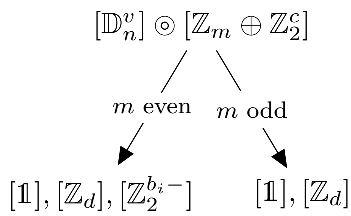

Figure 6. Clips of $\mathbb{D}_n^v$ with $\mathbb{Z}_m$.

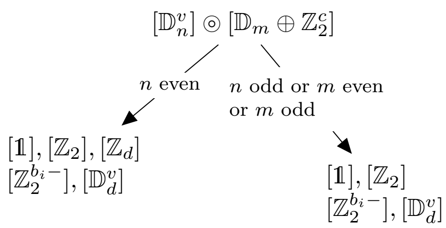

Figure 7. Clips of $\mathbb{D}_n^v$ with $\mathbb{D}_m$.

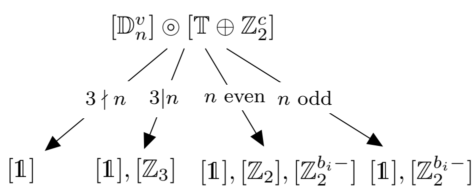

Figure 8. Clips of $\mathbb{D}_n^v$ with $\mathbb{T}$.

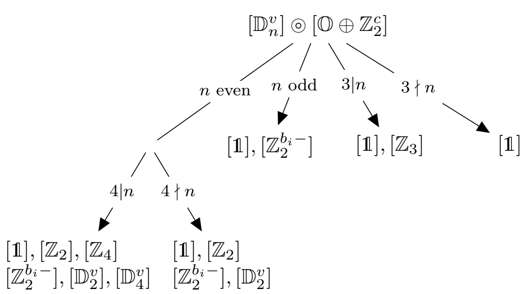

Figure 9. Clips of $\mathbb{D}_n^v$ with $\mathbb{O}$.

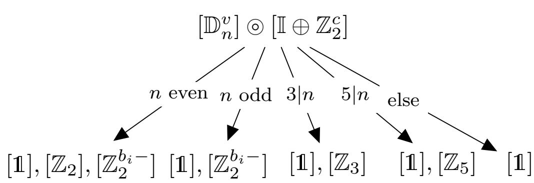

Figure 10. Clips of $\mathbb{D}_n^v$ with $\mathbb{I}$.

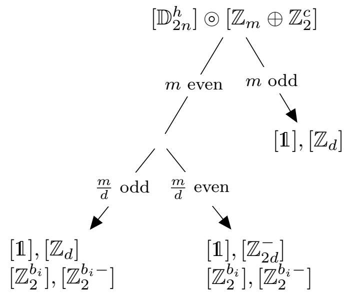

FIGURE 11. Clips of $\mathbb{D}^h_{2n}$ with $\mathbb{Z}_m$.

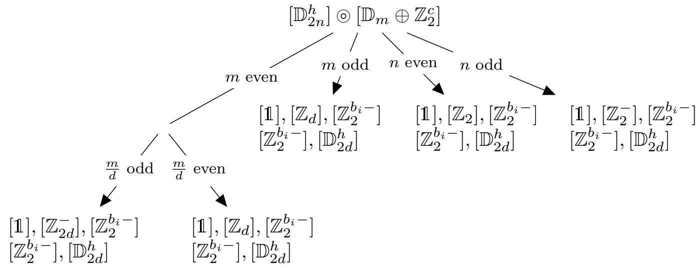

FIGURE 12. Clips of $\mathbb{D}^h_{2n}$ with $\mathbb{D}_m$.

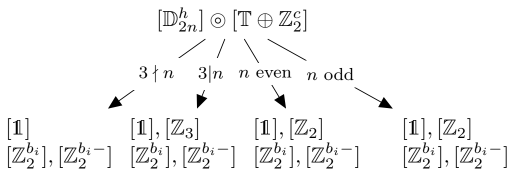

FIGURE 13. Clips of $\mathbb{D}^h_{2n}$ with $\mathbb{T}$.

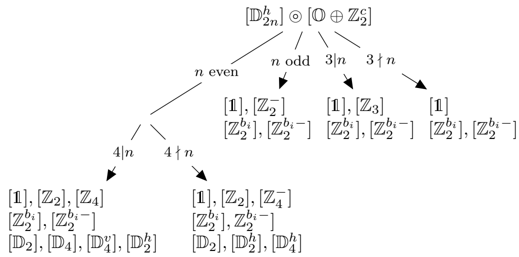

FIGURE 14. Clips of $\mathbb{D}^h_{2n}$ with $\mathbb{O}$.

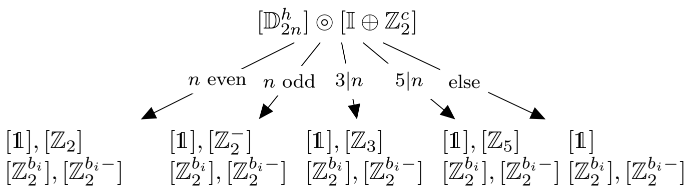

FIGURE 15. Clips of $\mathbb{D}^h_{2n}$ with $\mathbb{I}$.

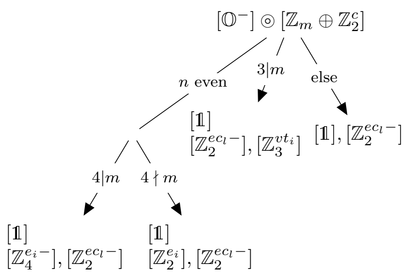

FIGURE 16. Clips of $\mathbb{O}^-$ with $\mathbb{Z}_m$.

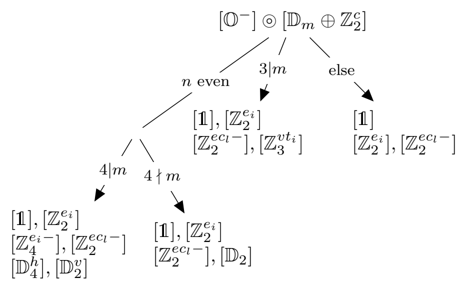

FIGURE 17. Clips of $\mathbb{O}^-$ with $\mathbb{D}_m$.

5. Clips between type II and III O(3)-subgroups

Finding the symmetry classes of the piezoelectricity law involves the calculation of clips operation between O(3)-subgroups of different types. However, the clips operation between subgroups of type I and II has been already calculated in different papers, see for instance [4]. For this reason, in this section, we will be interested in the clips between subgroups of type II and III only.

First let us introduce the O(3)-subgroups, there exists three types. Given Γ a subgroup of O(3), Γ belongs to one of the three types of the following table.

O(3)-subgroups		
Type I	**Type II**	**Type III**
Γ is a subgroup of SO(3)	$-\mathrm{I} \in \Gamma$	Γ is not a subgroup of SO(3) and $-\mathrm{I} \notin \Gamma$

The description of each subgroup is moved to Appendix A.

- For the subgroups of **type I**: Every closed subgroup of SO(3) is conjugate to one of

$$\mathrm{SO}(3), \quad \mathrm{O}(2), \quad \mathrm{SO}(2), \quad \mathbb{D}_n, \quad \mathbb{Z}_n, \quad \mathbb{T}, \quad \mathbb{O}, \quad \mathbb{I}, \text{ or } \quad \mathbb{1}$$

- For the subgroups of **type II**: They are subgroups of type I to which we add the group $\mathbb{Z}_2^c = \{\pm \mathrm{I}\}$:

$$\mathbb{Z}_m \oplus \mathbb{Z}_2^c, \quad \mathbb{D}_m \oplus \mathbb{Z}_2^c, \quad \mathbb{T} \oplus \mathbb{Z}_2^c, \quad \mathbb{O} \oplus \mathbb{Z}_2^c, \quad \mathbb{I} \oplus \mathbb{Z}_2^c$$

- For the subgroups of **type III**: There are four subgroups of O(3) of type III that we construct using a subgroup of type I (see lemma below):

$$\mathbb{Z}_{2n}^-, \quad \mathbb{D}_n^v, \quad \mathbb{D}_{2n}^h, \quad \mathbb{O}^-$$

Lemma 5.1. *Let K a subgroup of type I and $L \subset K$ a subgroup of index 2. Let $g \in K \setminus L$ then $L \cup (-gL)$ is a subgroup of type III.*

Let's illustrate this lemma by an example:

Example 5.2. Let $K = \mathbb{Z}_4 = \left\{\underline{e}, r(e_3, \frac{\pi}{2}), \underline{r(e_3,\pi)}, r(e_3, \frac{3\pi}{2})\right\}$ and $L = \mathbb{Z}_2 = \{e, r(e_3,\pi)\} \subset \mathbb{Z}_4$. Take $g = r(e_3, \frac{\pi}{2}) \in K \setminus L$, then

$$\begin{aligned}\mathbb{Z}_4^- &= \mathbb{Z}_2 \cup (-r(e_3, \frac{\pi}{2})\mathbb{Z}_2) \\ &= \left\{e, r(e_3,\pi), -r(e_3, \frac{\pi}{2}), -r(e_3, \frac{3\pi}{2})\right\}\end{aligned}$$

To find the clips between subgroups of type II and III we will use the following lemma:

Lemma 5.3. *Let Γ be a subgroup of type III and $L = \Gamma \cap \mathrm{SO}(3)$. Then for every subgroup of type II, $K \oplus \mathbb{Z}_2^c$ we have*

$$\Gamma \cap (K \oplus \mathbb{Z}_2^c) = (L \cap K) \cup (-(\gamma L \cap K))$$

where $\gamma \in L$.

Proof. We have $K \oplus \mathbb{Z}_2^c = K \cup (-K)$ and $\Gamma = L \cup (-\gamma L)$ (lemma 5.1). Then

$$\begin{aligned}\Gamma \cap (K \oplus \mathbb{Z}_2^c) &= L \cup (-\gamma L) \cap (K \cup (-K)) \\ &= (L \cap K) \cup (L \cap (-K)) \cup (-\gamma L \cap K) \cup (-\gamma L \cap (-K)) \\ &= (L \cap K) \cup (-(\gamma L \cap K))\end{aligned}$$

Hence the result. ☐

In the next subsections, we will focus on finding $\mathbf{\Gamma} \cap (\mathbf{gKg^{-1}} \oplus \mathbb{Z}_2^c)$ for $g \in \mathrm{SO}(3)$ in order to find $[\Gamma] \circledcirc [K \oplus \mathbb{Z}_2^c]$ since

$$\begin{aligned}[\Gamma] \circledcirc [K \oplus \mathbb{Z}_2^c] &= \left\{[\Gamma \cap g(K \oplus \mathbb{Z}_2^c)g^{-1}], g \in \mathrm{SO}(3)\right\} \\ &= \left\{[\Gamma \cap (gKg^{-1} \oplus \mathbb{Z}_2^c)], g \in \mathrm{SO}(3)\right\}\end{aligned}$$

So by applying the previous lemma, we calculate $\Gamma \cap (gKg^{-1} \oplus \mathbb{Z}_2^c)$ which leads us to $[\Gamma] \circledcirc [K \oplus \mathbb{Z}_2^c]$.

5.1. **Clips with $\mathbb{Z}_{2n}^-$.** In this part we will calculate the clips operation between $\mathbb{Z}_{2n}^-$ and each of the subgroups of type II. First, we construct $\mathbb{Z}_{2n}^-$ from the couple $(\mathbb{Z}_{2n}, \mathbb{Z}_n)$ as we did in example 5.2:

$$\mathbb{Z}_{2n}^- = \left\{ r(e_3, \frac{2k\pi}{n}); k = 0, \dots, n-1, \quad -r(e_3, \frac{(2k+1)\pi}{n}); k = 0, \dots, n-1 \right\}.$$

Lemma 5.4. *Let $m,\ n \geq 2$ be two integers and $d = \gcd(m,n)$. Then*

$$[\mathbb{Z}_{2n}^-] \circledcirc [\mathbb{Z}_m \oplus \mathbb{Z}_2^c] = \left\{ [\mathbb{1}], [\mathbb{Z}_d], [\mathbb{Z}_{2d}^-] \right\}.$$

Proof. Let $H = \mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_m g^{-1} \oplus \mathbb{Z}_2^c)$. By the previous lemma, $L = \mathbb{Z}_{2n}^- \cap \mathrm{SO}(3) = \mathbb{Z}_n$ and

$$H = (\mathbb{Z}_n \cap g\mathbb{Z}_m g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{Z}_m g^{-1}))$$

where $\gamma = r(e_3, \frac{\pi}{n}) \in \mathbb{Z}_{2n} \setminus \mathbb{Z}_n$.
We have $\gamma\mathbb{Z}_n = r(e_3, \frac{\pi}{n}).\left\{r(e_3, \frac{2k\pi}{n}); k = 0, \dots, n-1\right\} = \left\{ r(e_3, \frac{(2k+1)\pi}{n}); k = 0, \dots, n-1 \right\}$.
Hence,

$$\begin{aligned} H = &\left(\left\{ r(e_3, \frac{2k\pi}{n}); k = 0, \dots, n-1 \right\} \cap \left\{ r(ge_3, \frac{2k\pi}{m}); k = 0, \dots, m-1 \right\} \right) \bigcup \\ &- \left(\left\{ r(e_3, \frac{(2k+1)\pi}{n}); k = 0, \dots, n-1 \right\} \cap \left\{ r(ge_3, \frac{2k\pi}{m}); k = 0, \dots, m-1 \right\} \right) \end{aligned}$$

If ge_3 and e_3 are not colinear then $H = \mathbb{1}$.
Otherwise if only $ge_3 = \pm e_3$, for the first intersection we have to solve $\dfrac{2k_1}{n} = \dfrac{2k_2}{m}$:

$$\begin{aligned} \frac{k_1}{n} = \frac{k_2}{m} &\Leftrightarrow \frac{k_1}{n_1 d} = \frac{k_2}{m_1 d} \quad m_1 \wedge n_1 = 1 \\ &\Leftrightarrow k_1 m_1 = k_2 n_1 \\ &\Leftrightarrow n_1 / k_1 \text{ using Gauss lemma} \\ &\Leftrightarrow k_1 = n_1 k' \end{aligned}$$

Replacing in $\frac{k_1}{n} = \frac{k_2}{m}$ we get $k_2 = k' m_1$. Hence on one hand we get $\frac{2k_1\pi}{n} = \frac{2k' n_1 \pi}{dn_1} = \frac{2k'\pi}{d}$ and on the other hand $\frac{2(k_2)\pi}{m} = \frac{2k' m_1 \pi}{m_1 d}$.
We deduce that the intersection is $\mathbb{Z}_d = \left\{ r(e_3, \frac{2k\pi}{d}); k = 0, \dots, n-1 \right\}$.
For the second intersection, we have to solve $\dfrac{2k_1+1}{n} = \dfrac{2k_2}{m}$ which is equivalent to $(2k_1+1)m = 2k_2 n$, we deduce that this has a solution only if m is even:

$$\begin{aligned} (2k_1+1)m = 2k_2 n &\Leftrightarrow (2k_1+1)dm_1 = 2k_2 dn_1 \quad m_1 \wedge n_1 = 1 \\ &\Leftrightarrow m_1 \text{ is necessarily even } m_1 = 2p_1 \\ &\Leftrightarrow (2k_1+1)p_1 = k_2 n_1 \\ &\Leftrightarrow p_1 / k_2 n_1 \quad p_1 \wedge n_1 = 1 \\ &\Leftrightarrow p_1 / k_2 \text{ using Gauss lemma} \\ &\Leftrightarrow k_2 = p_1 k' \end{aligned}$$

Replacing in $(2k_1+1)p_1 = k_2 n_1$ we get $(2k_1+1)p_1 = p_1 k' n_1 \implies k'$ is odd. Hence on one hand we get

$$\frac{2k_2\pi}{m} = \frac{2p_1 k'\pi}{2dp_1} = \frac{k'\pi}{d}$$

and on the other hand

$$\frac{2(k_1+1)\pi}{n} = \frac{k' n_1 \pi}{n_1 d}.$$

We deduce that the intersection is $\left\{r(e_3, \frac{(2k+1)\pi}{d}); k = 0, \dots, n-1\right\}$.
Hence,

$$H = \begin{cases} \mathbb{Z}_{2d-} & \text{if } m \text{ is even and } \frac{m}{d} \text{ is even} \\ \mathbb{Z}_d & \text{else} \end{cases}.$$

$\square$

Lemma 5.5. *Let $m,\ n \geq 2$ be two integers and $d = \gcd(m, n)$. Then*

$$[\mathbb{Z}_{2n}^-] \circledcirc [\mathbb{D}_m \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_d], [\mathbb{Z}_{2d}^-], [\mathbb{Z}_2], [\mathbb{Z}_2^-]\right\}.$$

Proof. We recall that

$$\mathbb{D}_m = \left\{r(e_3, \frac{2k\pi}{m}); k = 0, \dots, m-1, \quad r(b_i, \pi); i = 1, \dots, m\right\}$$

Where b_i are the secondary axis of the dihedral such that $b_1 = e_1$ and $b_k = r(e_3, \frac{\pi}{m})b_{k-1}\ \ \forall k = 2, \dots, m$.
Let $H = \mathbb{Z}_{2n}^- \cap (g\mathbb{D}_m g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3, $L = \mathbb{Z}_{2n}^- \cap \mathrm{SO}(3) = \mathbb{Z}_n$ and we take $\gamma = r(e_3, \frac{\pi}{n}) \in \mathbb{Z}_{2n} \setminus \{\mathbb{Z}_n\}$. Then

$$\begin{aligned} H &= (\mathbb{Z}_n \cap g\mathbb{D}_m g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{D}_m g^{-1})) \\ &= \left(\left\{r(e_3, \frac{2k\pi}{n})\right\} \cap \left\{r(ge_3, \frac{2k\pi}{m}),\ r(gb_i, \pi)\right\}\right) \bigcup - \left(\left\{r(e_3, \frac{(2k+1)\pi}{n})\right\} \cap \left\{r(ge_3, \frac{2k\pi}{m}),\ r(gb_i, \pi)\right\}\right) \end{aligned}$$

If neither ge_3 nor gb_i is colinear to e_3 then $H = \mathbb{1}$.
Otherwise, if only $ge_3 = \pm e_3$ then

$$H = \mathbb{Z}_{2n}^- \cap g(\mathbb{Z}_m \oplus \mathbb{Z}_2^c)g^{-1} = \begin{cases} \mathbb{Z}_d & \text{if } m \text{ is odd} \\ \mathbb{Z}_{2d}^- & \text{if } m \text{ and } \frac{m}{d} \text{ even} \end{cases}.$$

If only $gb_i = \pm e_3$, we have to solve $\dfrac{2k\pi}{n} = \pi$ for the first intersection and $\dfrac{(2k+1)\pi}{n} = \pi$ for the second intersection. Hence we get on one hand $\mathbb{Z}_2$ if n is even and $\mathbb{1}$ if n is odd and on the other hand $\{r(e_3, \pi)\}$ if n is odd and $\emptyset$ if n is even.

We deduce that the union is $\begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even} \\ \mathbb{Z}_2^- & \text{if } n \text{ is odd} \end{cases}$. $\square$

Lemma 5.6. *We have*

$$[\mathbb{Z}_{2n}^-] \circledcirc [\mathbb{T} \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_3], [\mathbb{Z}_2], [\mathbb{Z}_2^-]\right\}.$$

Proof. We recall that

$$\mathbb{T} = \biguplus_{i=1}^{4} \mathbb{Z}_3^{vt_i} \cup \biguplus_{j=1}^{3} \mathbb{Z}_2^{et_j}$$

where vt_i and et_j are the vertices axes and edges axes of the tetrahedron (see [4, Appendix A] for more details on the tetrahedral subgroup).
Let $H = \mathbb{Z}_{2n}^- \cap (g\mathbb{T}g^{-1} \oplus \mathbb{Z}_2^c)$ By lemma 5.3,

$$\begin{aligned} H &= (\mathbb{Z}_n \cap g\mathbb{T}g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{T}g^{-1})) \\ &= \left(\left\{r(e_3, \frac{2k\pi}{n})\right\} \cap \left\{r(gvt_i, \frac{2k\pi}{3}); k = 0, \dots, 2,\ r(get_j, k\pi); k = 0, 1\right\}\right) \bigcup \\ &- \left(\left\{r(e_3, \frac{(2k+1)\pi}{n})\right\} \cap \left\{r(gvt_i, \frac{2k\pi}{3}); k = 0, \dots, 2,\ r(get_j, k\pi); k = 0, 1\right\}\right) \end{aligned}$$

If neither gvt_i nor get_j is colinear to e_3 then $H = \mathbb{1}$.
Otherwise, if only $gvt_i = \pm e_3$ then $H = \mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_m g^{-1} \oplus \mathbb{Z}_2^c)$ where $m = 3$ odd. Hence, $H = \mathbb{Z}_3$ if $3|n$ and $\mathbb{1}$ if not.

Now if only $get_j = \pm e_3$ then we have to solve $\dfrac{2k\pi}{n} = \pi$ which gives $\mathbb{Z}_2$ if n is even and $\mathbb{1}$ if not and we have to solve $\dfrac{(2k+1)\pi}{n} = \pi$ which gives $\{r(e_3,\pi)\}$ if n is odd and $\emptyset$ if not. So we deduce that the union is either $\mathbb{Z}_2$ or $\mathbb{Z}_2^-$. □

Lemma 5.7. *We have*

$$[\mathbb{Z}_{2n}^-] \circledcirc [\mathbb{O} \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_4], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-]\right\}.$$

Proof. We recall that

$$\mathbb{O} = \biguplus_{i=1}^{3} \mathbb{Z}_4^{fc_i} \cup \biguplus_{j=1}^{4} \mathbb{Z}_3^{vc_j} \cup \biguplus_{l=1}^{6} \mathbb{Z}_2^{ec_l}$$

where vc_i, ec_j and fc_l are respectively the vertices, edges, and faces axes of the cube (see [4, Appendix A]).
Let $H = \mathbb{Z}_{2n}^- \cap (g\mathbb{O}g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3,

$$\begin{aligned} H &= (\mathbb{Z}_n \cap g\mathbb{O}g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{O}g^{-1})) \\ &= \left(\left\{r(e_3, \frac{2k\pi}{n})\right\} \cap \left\{r(gfc_i, \frac{2k\pi}{4}); k=0,\dots,3,\ r(gvc_j, \frac{2k\pi}{3}); k=0,1,2,\ r(gec_l,\pi)\right\}\right) \bigcup \\ &\quad - \left(\left\{r(e_3, \frac{(2k+1)\pi}{n})\right\} \cap \left\{r(gfc_i, \frac{2k\pi}{4}); k=0,\dots,3,\ r(gvc_j, \frac{2k\pi}{3}); k=0,1,2,\ r(gec_l,\pi)\right\}\right) \end{aligned}$$

If none of gfc_i, gvc_j and ec_l is colinear to e_3 then $H = \mathbb{1}$.
Otherwise, if only $gfc_i = \pm e_3$ then $H = \mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_m g^{-1} \oplus \mathbb{Z}_2^c)$ where $m = 4$ even.
Hence we have:

- If n is even then we have 2 cases
  - $4|n$ hence $d = \gcd(4,n) = 4$ and $\frac{4}{d} = 1$ is odd so as in the previous lemmas $H = \mathbb{Z}_d = \mathbb{Z}_4$.
  - $4 \nmid n$ hence $d = \gcd(4,n) = 2$ and $\frac{4}{d} = 2$ is even so as in the previous lemmas $H = \mathbb{Z}_{2d}^- = \mathbb{Z}_4^-$.
- If n is odd then $d = \gcd(4,n) = 1$ and $\frac{4}{d} = 4$ is even so $H = \mathbb{Z}_{2d}^- = \mathbb{Z}_2^-$

Now if only $gvc_j = \pm e_3$ then we get $\mathbb{Z}_3$ if $3|n$ and $\mathbb{1}$ if not.
Finally, if only $gec_l = \pm e_3$ then we get $\mathbb{Z}_2$ if n is even and $\mathbb{Z}_2^-$ if n is odd. □

Lemma 5.8. *We have*

$$[\mathbb{Z}_{2n}^-] \circledcirc [\mathbb{I} \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_5], [\mathbb{Z}_2^-]\right\}.$$

Proof. We recall that

$$\mathbb{I} = \biguplus_{i=1}^{6} \mathbb{Z}_5^{fd_i} \cup \biguplus_{j=1}^{10} \mathbb{Z}_3^{vd_j} \cup \biguplus_{l=1}^{15} \mathbb{Z}_2^{ed_l}$$

where vc_i, ec_j and fc_l are respectively the vertices, edges, and faces axes of the dodecahedron (see [4, Appendix A]).
Let $H = \mathbb{Z}_{2n}^- \cap (g\mathbb{I}g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3,

$$\begin{aligned} H &= (\mathbb{Z}_n \cap g\mathbb{I}g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{I}g^{-1})) \\ &= \left(\left\{r(e_3, \frac{2k\pi}{n})\right\} \cap \left\{r(gfd_i, \frac{2k\pi}{5}); k=0,\dots,4,\ r(gvd_j, \frac{2k\pi}{3}); k=0,1,2,\ r(ged_l,\pi)\right\}\right) \bigcup \\ &\quad - \left(\left\{r(e_3, \frac{(2k+1)\pi}{n})\right\} \cap \left\{r(gfd_i, \frac{2k\pi}{5}); k=0,\dots,4,\ r(gvd_j, \frac{2k\pi}{3}); k=0,1,2,\ r(ged_l,\pi)\right\}\right) \end{aligned}$$

If none of gfd_i, gvd_j and ed_l is colinear to e_3 then $H = \mathbb{1}$.
Otherwise, if only $gfd_i = \pm e_3$ then $H = \mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_m g^{-1} \oplus \mathbb{Z}_2^c)$ where $m = 5$. Hence, $H = \mathbb{Z}_5$ if $5|n$ and $\mathbb{1}$ if not.

Now if only $gvd_j = \pm e_3$ then we get $\mathbb{Z}_3$ if $3|n$ and $\mathbb{1}$ if not.
Finally, if only $ged_l = \pm e_3$ then we get $\mathbb{Z}_2$ if n is even and $\mathbb{Z}_2^-$ if n is odd. $\square$

5.2. **Clips with $\mathbb{D}_n^v$.** In this part we will calculate the clips operation between $\mathbb{D}_n^v$ and each of the subgroups of type II. First, we construct $\mathbb{D}_n^v$ from the couple $(\mathbb{D}_n, \mathbb{Z}_n)$, we have

$$\mathbb{D}_n = \mathbb{Z}_n \cup \mathbb{Z}_2^{b_1} \cup \mathbb{Z}_2^{b_2} \cup \ldots \cup \mathbb{Z}_2^{b_n}$$

where b_i are the secondary axis of the dihedral such that $b_1 = e_1$ and $b_k = r(e_3, \frac{\pi}{n})b_{k-1} \quad \forall k = 2, \ldots, n$. Hence,

$$\mathbb{D}_n^v = \left\{ r(e_3, \frac{2k\pi}{n}); k = 0, \ldots, n-1, \; -r(b_1, \pi), \ldots, \; -r(b_n, \pi) \right\}.$$

Lemma 5.9. *Let $m,\ n \geq 2$ be two integers and $d = \gcd(m, n)$. Then*

$$[\mathbb{D}_n^v] \circledcirc [\mathbb{Z}_m \oplus \mathbb{Z}_2^c] = \left\{ [\mathbb{1}], [\mathbb{Z}_d], [\mathbb{Z}_2^{b_i -}] \right\}.$$

Proof. Let $H = \mathbb{D}_n^v \cap (g\mathbb{Z}_m g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3, $L = \mathbb{D}_n^v \cap \mathrm{SO}(3) = \mathbb{Z}_n$ and

$$H = (\mathbb{Z}_n \cap g\mathbb{Z}_m g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{Z}_m g^{-1}))$$

where $\gamma = r(b_1, \pi) \in \mathbb{D}_n \setminus \{\mathbb{Z}_n\}$.
We have $\gamma\mathbb{Z}_n = r(b_1, \pi). \left\{ r(e_3, \frac{2k\pi}{n}); k = 0, \ldots, n-1 \right\} = \{r(b_1, \pi), \ldots, r(b_n, \pi)\}$.
Hence,

$$H = \left(\left\{ r(e_3, \frac{2k\pi}{n}) \right\} \cap \left\{ r(ge_3, \frac{2k\pi}{m}) \right\} \right) \bigcup - \left(\{r(b_1, \pi), \ldots, r(b_n, \pi)\} \cap \left\{ r(ge_3, \frac{2k\pi}{m}) \right\} \right)$$

If ge_3 is neither colinear to e_3 nor to $b_i, \forall i$ then $H = \mathbb{1}$.
Otherwise, if only $ge_3 = \pm e_3$, then $H = \mathbb{Z}_n \cap \mathbb{Z}_m = \{\mathbb{1}, \mathbb{Z}_d\}$ (done in lemma 5.4).
Now if only $ge_3 = \pm b_i$ then $H = \begin{cases} \mathbb{1} & \text{if } m \text{ is odd} \\ \mathbb{1} \cup -r(b_i, \pi) = \mathbb{Z}_2^{b_i -} & \text{if } m \text{ is even} \end{cases}$.

Hence the result. $\square$

Lemma 5.10. *Let $m,\ n \geq 2$ be two integers and $d = \gcd(m, n)$. Then*

$$[\mathbb{D}_n^v] \circledcirc [\mathbb{D}_m \oplus \mathbb{Z}_2^c] = \left\{ [\mathbb{1}], [\mathbb{Z}_d], [\mathbb{Z}_2], [\mathbb{Z}_2^{b_i -}], [\mathbb{D}_d^v] \right\}.$$

Proof. Let $H = \mathbb{D}_n^v \cap (g\mathbb{D}_m g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3, $L = \mathbb{Z}_n$ and $\gamma = r(b_1, \pi)$. Then

$$\begin{aligned} H &= (\mathbb{Z}_n \cap g\mathbb{D}_m g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{D}_m g^{-1})) \\ &= \left(\left\{ r(e_3, \frac{2k\pi}{n}) \right\} \cap \left\{ r(ge_3, \frac{2k\pi}{m}), \; r(gb_i, \pi) \right\} \right) \bigcup - \left(\{r(b_i, \pi)\} \cap \left\{ r(ge_3, \frac{2k\pi}{m}), \; r(gb_i, \pi) \right\} \right) \end{aligned}$$

If ge_3 is neither colinear to e_3 nor to b_i and gb_i is neither colinear to e_3 nor b_i then $H = \mathbb{1}$.
Otherwise,

- if only $ge_3 = \pm e_3$ then $H = \mathbb{Z}_n \cap \mathbb{Z}_m = \{\mathbb{1}, \mathbb{Z}_d\}$ (done in lemma 5.4).
- If only $gb_i = \pm e_3$, then

$$\begin{aligned} H &= \left\{ r(e_3, \frac{2k\pi}{n}); k = 0, \ldots, n-1 \right\} \cap \left\{ r(ge_3, \frac{2k\pi}{m}), r(b_1, \pi), \ldots, \underbrace{r(\pm e_3, \pi)}_{\text{ith position}}, \ldots, r(b_m, \pi) \right\} \\ &= \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even} \\ \mathbb{1} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

- If only $ge_3 = \pm b_i$ then

$$\begin{aligned} H &= \mathbb{1} \bigcup - \left(\{r(b_1, \pi), \ldots, r(b_n, \pi)\} \cap \left\{ r(\pm b_i, \frac{2k\pi}{m}); k = 0, \ldots, m-1, \; r(gb_i, \pi); i = 1, \ldots, m \right\} \right) \\ &= \begin{cases} \mathbb{Z}_2^{b_i -} & \text{if } m \text{ is even} \\ \mathbb{1} & \text{if } m \text{ is odd} \end{cases} \end{aligned}$$

- If only $gb_i = \pm b_i$ then $H = \mathbb{Z}_2^{b_i-}$ $\forall n$
- Finally, if $ge_3 = \pm e_3$ and $gb_i = \pm b_i$, $\forall i$ (such g exists, take for instance $g = \mathrm{I}$) then

$$H = \left(\left\{r(e_3, \frac{2k\pi}{n}); k = 0, \dots, n-1\right\} \cap \left\{r(e_3, \frac{2k\pi}{m}); k = 0, \dots, m-1,\ r(b_i, \pi); i = 1, \dots, m\right\}\right)$$
$$\bigcup - \left(\{r(b_1, \pi), \dots, r(b_n, \pi)\} \cap \left\{r(e_3, \frac{2k\pi}{m}); k = 0, \dots, m-1,\ r(b_i, \pi); i = 1, \dots, m\right\}\right)$$
$$= \left\{r(e_3, \frac{2k\pi}{d}); k = 0, \dots, d-1\right\} \cup -\{r(b_i, \pi), i = 1, \dots, d\}$$
$$= \mathbb{D}_d^v$$

□

Lemma 5.11. *We have*

$$[\mathbb{D}_n^v] \circledcirc [\mathbb{T} \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_2^{b_i-}]\right\}.$$

Proof. Let $H = \mathbb{D}_n^v \cap (g\mathbb{T}g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3,

$$H = (\mathbb{Z}_n \cap g\mathbb{T}g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{T}g^{-1}))$$
$$= \left(\left\{r(e_3, \frac{2k\pi}{n})\right\} \cap \left\{r(gvt_i, \frac{2k\pi}{3}),\ r(get_j, \pi)\right\}\right) \bigcup - \left(\{r(b_i, \pi)\} \cap \left\{r(gvt_i, \frac{2k\pi}{3}),\ r(get_j, \pi)\right\}\right)$$

If $\forall i$ gvt_i is neither colinear to e_3 nor to b_i and $\forall j$ get_j is neither colinear to e_3 nor b_i then $H = \mathbb{1}$.
Otherwise,

- if only $gvt_i = \pm e_3$ then $H = \mathbb{Z}_3$ if $3|n$ and $\mathbb{1}$ if not.
- If only $get_j = \pm e_3$, then $H = \mathbb{Z}_2$ if n is even and $\mathbb{1}$ if not.
- If only $gvt_i = \pm b_i$ then $H = \mathbb{1}$.
- If only $get_j = \pm b_i$ then $H = \mathbb{1} \cup -r(b_i, \pi) = \mathbb{Z}_2^{b_i-}$.

□

Lemma 5.12. *We have*

$$[\mathbb{D}_n^v] \circledcirc [\mathbb{O} \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_4], [\mathbb{Z}_2^{b_i-}], [\mathbb{D}_2^v], [\mathbb{D}_4^v]\right\}.$$

Proof. Let $H = \mathbb{D}_n^v \cap (g\mathbb{O}g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3,

$$H = (\mathbb{Z}_n \cap g\mathbb{O}g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{O}g^{-1}))$$
$$= \left(\left\{r(e_3, \frac{2k\pi}{n})\right\} \cap \left\{r(gfc_i, \frac{2k\pi}{4}), r(gvc_j, \frac{2k\pi}{3}),\ r(gec_l, \pi)\right\}\right) \bigcup$$
$$- \left(\{r(b_i, \pi)\} \cap \left\{r(gfc_i, \frac{2k\pi}{4}), r(gvc_j, \frac{2k\pi}{3}),\ r(gec_l, \pi)\right\}\right)$$

If $\forall i$ gfc_i is neither colinear to e_3 nor to b_i and $\forall j$ gvc_j is neither colinear to e_3 nor b_i and $\forall l$ gec_l is neither colinear to e_3 nor b_i then $H = \mathbb{1}$.
Otherwise,

- If only $gfc_i = \pm e_3$ then $H = \begin{cases} \mathbb{Z}_4 \text{ if } 4|n \\ \mathbb{Z}_2 \text{ if } n \text{ is even but } 4 \nmid n \\ \mathbb{1} \quad \text{if } n \text{ is odd} \end{cases}$
- If only $gvc_j = \pm e_3$ then $H = \mathbb{Z}_3$ if $3|n$ and $\mathbb{1}$ if not.
- If only $gec_l = \pm e_3$ then $H = \mathbb{Z}_2$ if n is even and $\mathbb{1}$ if not.
- If only $gfc_i = \pm b_i$ then $H = \mathbb{Z}_2^{b_i-}$
- If only $gvc_j = \pm b_i$ then $H = \mathbb{1}$.
- If only $gec_l = \pm b_i$ then $H = \mathbb{1} \cup -r(b_i, \pi) = \mathbb{Z}_2^{b_i-}$.
- If $gfc_i = \pm e_3$ and $gec_l = \pm b_i$ (such g exists since if $fc_i = e_3$ then g will be a rotation around e_3 that turns one of the ec_l to one of the b_i) then we have 3 cases:

- If $4|n$ then $H = \mathbb{Z}_4 \cup \{r(b_i, \pi), i = 1, \dots, 4\} = \mathbb{D}_4^v$.
  In fact, if we take g to be the rotation of angle $\frac{\pi}{4}$ that turns the first edge axis to $b_1 = e_1$ (see [4, figure 10]), having that $4|n$ implies $n = 4p$ and so the angle between the axis b_i is at most $\frac{\pi}{4}$ and it is equal to $\frac{\pi}{4p} = \frac{1}{p}\frac{\pi}{4}$. So when we rotate with an angle $\frac{\pi}{4}$ around e_3 we will have at least 2 edge axis and 2 face axis that will superpose with 4 b_i hence the existence of 4 b_i in the second intersection.
- if $4 \nmid n$ and n even then $H = \mathbb{D}_2^v$ (same reasoning as in the first case).
- if n odd then $H = \mathbb{Z}_2^{b_i-}$

• If g is the identity rotation, the discussion as well as the results will be identical to the previous ones.

• If $gfc_i = \pm b_i$ and $gec_l = \pm e_3$ (we can take g to be the rotation around e_1) then
  - If n is even then $H = \mathbb{D}_2^v$ since in the second intersection we will get one $r(b_i, \pi)$ from the fact that $gfc_i = \pm b_i$ and another one from $gec_l = \pm e_3$ since the rotation of ec_l to e_3 will lead the rotation of another ec_l to $e_2 = b_i$ for some i.
  - If n is odd then $H = \mathbb{Z}_2^{b_i-}$.

□

Lemma 5.13. *We have*

$$[\mathbb{D}_n^v] \circledcirc [\mathbb{I} \oplus \mathbb{Z}_2^c] = \left\{ [\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_5], [\mathbb{Z}_2^{b_i}], [\mathbb{Z}_2^{b_i-}] \right\}.$$

Proof. Let $H = \mathbb{D}_n^v \cap (g\mathbb{I}g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3,

$$\begin{aligned} H &= (\mathbb{Z}_n \cap g\mathbb{I}g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{I}g^{-1})) \\ &= \left(\left\{ r(e_3, \frac{2k\pi}{n}) \right\} \cap \left\{ r(gfd_i, \frac{2k\pi}{5}), r(gvd_j, \frac{2k\pi}{3}),\ r(ged_l, \pi) \right\} \right) \\ &\qquad \bigcup - \left(\{r(b_i, \pi)\} \cap \left\{ r(gfd_i, \frac{2k\pi}{5}), r(gvd_j, \frac{2k\pi}{3}),\ r(ged_l, \pi) \right\} \right) \end{aligned}$$

If $\forall i$ gfd_i is neither colinear to e_3 nor to b_i and $\forall j$ gvd_j is neither colinear to e_3 nor b_i and $\forall l$ ged_l is neither colinear to e_3 nor b_i then $H = \mathbb{1}$.
Otherwise,

- If only $gfd_i = \pm e_3$ then $H = \mathbb{Z}_5$ if $5|n$ and $\mathbb{1}$ if not.
- If only $gvd_j = \pm e_3$ then $H = \mathbb{Z}_3$ if $3|n$ and $\mathbb{1}$ if not.
- If only $ged_l = \pm e_3$, then $H = \mathbb{Z}_2$ if n is even and $\mathbb{1}$ if not.
- If only $gfd_i = \pm b_i$ then $H = \mathbb{1}$.
- If only $gvd_j = \pm b_i$ then $H = \mathbb{1}$.
- If only $ged_l = \pm b_i$ then $H = \mathbb{1} \cup -r(b_i, \pi) = \mathbb{Z}_2^{b_i-}$.

□

5.3. **Clips with $\mathbb{D}_{2n}^h$.** In this part we will calculate the clips operation between $\mathbb{D}_{2n}^h$ and each of the subgroups of type II. First, we construct $\mathbb{D}_{2n}^h$ from the couple $(\mathbb{D}_{2n}, \mathbb{D}_n)$, we have

$$\mathbb{D}_{2n} = \mathbb{Z}_{2n} \cup \mathbb{Z}_2^{b_1} \cup \mathbb{Z}_2^{b_2} \cup \ldots \cup \mathbb{Z}_2^{b_{2n}}$$

where the secondary axis of D_{2n} are given by:

$$b_1 = e_1 \text{ and } b_k = r(e_3, \frac{\pi}{2n}) b_{k-1}$$

And $\mathbb{D}_n \subset \mathbb{D}_{2n}$ such that the secondary axis of $\mathbb{D}_n$ are given by:

$$b_1 = e_1 \text{ and } b_k = r(e_3, \frac{\pi}{n}) b_{k-1}$$

So we remark that the secondary axis of $\mathbb{D}_n$ are the b_i of $\mathbb{D}_{2n}$ with odd indices. Hence,

$$\mathbb{D}_{2n}=\left\{\overbrace{\mathrm{I}, r(e_3,\frac{2\pi}{n}), r(e_3,\frac{4\pi}{n}),\dots,r(e_3,\frac{(n-2)\pi}{n}), r(b_1,\pi), r(b_3,\pi),\dots,r(b_{2n-1},\pi)}^{\mathbb{D}_n}\right\}\cup$$
$$\left\{r(e_3,\frac{\pi}{n}), r(e_3,\frac{3\pi}{n}),\dots,r(e_3,\frac{(n-1)\pi}{n}), r(b_2,\pi),\dots,r(b_{2n},\pi)\right\}$$

We deduce,

$$\mathbb{D}_{2n}^h=\left\{r(e_3,\frac{2k\pi}{n}), r(b_{2k+1},\pi); k=0,\dots,n-1\right\}\cup$$
$$\left\{-r(e_3,\frac{(2k+1)\pi}{n}); k=0,\dots,n-1, -r(b_{2k},\pi); k=1,\dots,n\right\}$$

Lemma 5.14. *Let $m,\ n\geq 2$ be two integers and $d=\gcd(m,n)$. Then*

$$[\mathbb{D}_{2n}^h]\circledcirc[\mathbb{Z}_m\oplus\mathbb{Z}_2^c]=\left\{[\mathbb{1}],[\mathbb{Z}_d],[\mathbb{Z}_{2d}^-],[\mathbb{Z}_2^{b_i}],[\mathbb{Z}_2^{b_i-}]\right\}.$$

Proof. Let $H=\mathbb{D}_{2n}^h\cap(g\mathbb{Z}_m g^{-1}\oplus\mathbb{Z}_2^c)$. By lemma 5.3, $L=\mathbb{D}_{2n}^h\cap\mathrm{SO}(3)=\mathbb{D}_n$ and

$$H=(\mathbb{D}_n\cap g\mathbb{Z}_m g^{-1})\cup(-(\gamma\mathbb{D}_n\cap g\mathbb{Z}_m g^{-1}))$$

where $\gamma=r(e_3,\frac{\pi}{n})\in\mathbb{D}_{2n}\setminus\{\mathbb{D}_n\}$. We have

$$\begin{aligned}\gamma\mathbb{D}_n&=r(e_3,\frac{\pi}{n}).\left\{r(e_3,\frac{2k\pi}{n}); k=0,\dots,n-1, r(b_1,\pi), r(b_3,\pi),\dots,r(b_{2n-1},\pi\right\}\\&=\left\{r(e_3,\frac{(2k+1)\pi}{n}), r(b_2,\pi), r(b_4,\pi),\dots,r(b_2n,\pi)\right\}\\&=\left\{r(e_3,\frac{(2k+1)\pi}{n}), r(b_{2(k+1)},\pi); k=0,\dots,n-1\right\}\end{aligned}$$

Indeed, we can remark first that

$$r(b_i,\pi)=r(e_3,\frac{2(i-1)\pi}{2n}).r(b_1,\pi)\quad i=1,\dots,2n$$

Then

$$\begin{aligned}\text{For } i=0,\dots,n-1\quad r(e_3,\frac{\pi}{n}).r(b_{2i+1},\pi)&=r(e_3,\frac{\pi}{n}).r(e_3,\frac{2(2i+1-1)\pi}{2n}).r(b_1,\pi)\\&=r(e_3,\frac{\pi}{n})r(e_3,\frac{2i\pi}{n})r(b_1,\pi)\\&=r(e_3,\frac{(2i+1)\pi}{n}r(b_1,\pi)\\&=r(b_{2(i+1)},\pi)\end{aligned}$$

Hence,

$$H=\left(\left\{r(e_3,\frac{2k\pi}{n}), r(b_{2k+1},\pi); k=0,\dots,n-1\right\}\cap\left\{r(ge_3,\frac{2k\pi}{m}); k=0,\dots,m-1\right\}\right)\bigcup$$
$$-\left(\left\{r(e_3,\frac{(2k+1)\pi}{n}), r(b_{2(k+1)},\pi); k=0,\dots,n-1\right\}\cap\left\{r(ge_3,\frac{2k\pi}{m}); k=0,\dots,m-1\right\}\right)$$

If ge_3 is neither colinear to e_3 nor to $b_i,\forall i$ then $H=\mathbb{1}$.
Otherwise, if only $ge_3=\pm e_3$, then $H=\left\{\mathbb{1},\mathbb{Z}_d,\mathbb{Z}_{2d}^-\right\}$ (done in lemma 5.4).

Now if only $ge_3=\pm b_i$ with i odd then $H=\begin{cases}\mathbb{1} & \text{if } m \text{ is odd}\\ \mathbb{Z}_2^{b_i} & \text{if } m \text{ is even}\end{cases}.$

If only $ge_3=\pm b_i$ with i even then $H=\begin{cases}\mathbb{1} & \text{if } m \text{ is odd}\\ \mathbb{1}\cup -r(b_i,\pi)=\mathbb{Z}_2^{b_i-} & \text{if } m\text{is even}\end{cases}.$ ∎

Lemma 5.15. *Let $m,\ n \geq 2$ be two integers and $d = \gcd(m, n)$. Then*

$$[\mathbb{D}_{2n}^h] \circledcirc [\mathbb{D}_m \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_d], [\mathbb{Z}_{2d}^-], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_2^{b_i}], [\mathbb{Z}_2^{b_i-}], [\mathbb{D}_{2d}^h]\right\}.$$

Proof. Let $H = \mathbb{D}_{2n}^h \cap (g\mathbb{D}_m g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3, $L = \mathbb{D}_n$ and $\gamma = r(e_3, \frac{\pi}{n})$. Then

$$\begin{aligned} H &= (\mathbb{D}_n \cap g\mathbb{D}_m g^{-1}) \cup (-(\gamma\mathbb{D}_n \cap g\mathbb{D}_m g^{-1})) \\ &= \left(\left\{r(e_3, \frac{2k\pi}{n}), r(b_{2k+1}, \pi)\right\} \cap \left\{r(ge_3, \frac{2k\pi}{m}),\ r(gb_i, \pi); i = 1, \dots, m\right\}\right) \bigcup \\ &- \left(\left\{r(e_3, \frac{(2k+1)\pi}{n}), r(b_{2(k+1)}, \pi)\right\} \cap \left\{r(ge_3, \frac{2k\pi}{m}), r(gb_i, \pi); i = 1, \dots, m\right\}\right) \end{aligned}$$

If ge_3 is neither colinear to e_3 nor to b_i $\forall i$ and gb_i is neither colinear to e_3 nor b_i $\forall i$ then $H = \mathbb{1}$. Otherwise,

- if only $ge_3 = \pm e_3$ then $H = \{\mathbb{1}, \mathbb{Z}_d, \mathbb{Z}_{2d}^-\}$.
- If only $gb_i = \pm e_3$, then $H = \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even} \\ \mathbb{Z}_2^- & \text{if } n \text{ is odd} \end{cases}$.
- If only $ge_3 = \pm b_i$ with i is odd then
$$H = \begin{cases} \mathbb{Z}_2^{b_i} & \text{if } m \text{ is even} \\ \mathbb{1} & \text{if } m \text{ is odd} \end{cases}.$$
- If only $ge_3 = \pm b_i$ with i is even then
$$H = \begin{cases} \mathbb{Z}_2^{b_i^-} & \text{if } m \text{ is even} \\ \mathbb{1} & \text{if } m \text{ is odd} \end{cases}.$$
- If only $gb_i = \pm b_i$ with i is odd then $H = \mathbb{Z}_2^{b_i}$ $\forall n, m$.
- If only $gb_i = \pm b_i$ with i is even then $H = \mathbb{Z}_2^{b_i-}$ $\forall n, m$.
- If $ge_3 = \pm e_3$ and $gb_i = \pm b_i$ $\forall i$ (we can take the identity rotation) then

$$\begin{aligned} H = &\left\{r(e_3, \frac{2k\pi}{d}), r(b_{(2k+1)}, \pi); k = 0, \dots, d-1\right\} \cup - \left\{r(e_3, \frac{(2k+1)\pi}{d}), r(b_{2(k+1)}, \pi); k = 0, \dots, d-1\right\} \\ &= \mathbb{D}_{2d}^h. \end{aligned}$$

□

Lemma 5.16. *We have*

$$[\mathbb{D}_{2n}^h] \circledcirc [\mathbb{T} \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_2^-], [\mathbb{Z}_2^{b_i}], [\mathbb{Z}_2^{b_i-}]\right\}.$$

Proof. Let $H = \mathbb{D}_{2n}^h \cap (g\mathbb{T}g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3,

$$\begin{aligned} H &= (\mathbb{D}_n \cap g\mathbb{T}g^{-1}) \cup (-(\gamma\mathbb{D}_n \cap g\mathbb{T}g^{-1})) \\ &= \left(\left\{r(e_3, \frac{2k\pi}{n}), r(b_{2k+1}, \pi)\right\} \cap \left\{r(gvt_i, \frac{2k\pi}{3}),\ r(get_j, \pi)\right\}\right) \bigcup \\ &- \left(\left\{r(e_3, \frac{(2k+1)\pi}{n}), r(b_{2(k+1)}, \pi)\right\} \cap \left\{r(gvt_i, \frac{2k\pi}{3}),\ r(get_j, \pi)\right\}\right) \end{aligned}$$

If $\forall i$ gvt_i is neither colinear to e_3 nor to b_i and $\forall j$ get_j is neither colinear to e_3 nor b_i then $H = \mathbb{1}$.
Otherwise,

- If only $gvt_i = \pm e_3$ then $H = \mathbb{Z}_3$ if $3|n$ and $\mathbb{1}$ if not.
- If only $get_j = \pm e_3$, then $H = \mathbb{Z}_2$ if n is even and $\mathbb{Z}_2^-$ if not.
- If only $gvt_i = \pm b_i$ (with i even or odd) then $H = \mathbb{1}$.
- If only $get_j = \pm b_i$ with i odd then $H = \mathbb{Z}_2^{b_i}$.
- If only $get_j = \pm b_i$ with i even then $H = \mathbb{Z}_2^{b_i-}$.

- If $gvt_i = \pm vt_i, \forall i$ and $get_j = \pm et_j, \forall j$ (g identity rotation) then $H = \mathbb{Z}_2$ if n is even and $\mathbb{Z}_2^-$ if not.

$\square$

Lemma 5.17. *We have*

$$[\mathbb{D}_{2n}^h] \circledcirc [\mathbb{O} \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_4], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], [\mathbb{Z}_2^{b_i}], [\mathbb{Z}_2^{b_i-}], [\mathbb{D}_2], [\mathbb{D}_4], [\mathbb{D}_4^v], [\mathbb{D}_2^h], [\mathbb{D}_4^h]\right\}.$$

Proof. Let $H = \mathbb{D}_{2n}^h \cap (g\mathbb{O}g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3,

$$\begin{aligned} H &= (\mathbb{D}_n \cap g\mathbb{O}g^{-1}) \cup (-(\gamma\mathbb{D}_n \cap g\mathbb{O}g^{-1})) \\ &= \left(\left\{r(e_3, \frac{2k\pi}{n}), r(b_{2k+1}, \pi)\right\} \cap \left\{r(gfc_i, \frac{2k\pi}{4}), r(gvc_j, \frac{2k\pi}{3}), r(gec_l, \pi)\right\}\right) \\ \bigcup - &\left(\left\{r(e_3, \frac{(2k+1)\pi}{n}), r(b_{2(k+1)}, \pi)\right\} \cap \left\{r(gfc_i, \frac{2k\pi}{4}), r(gvc_j, \frac{2k\pi}{3}), r(gec_l, \pi)\right\}\right) \end{aligned}$$

If $\forall i$ gfc_i is neither colinear to e_3 nor to b_i and $\forall j$ gvc_j is neither colinear to e_3 nor b_i and $\forall l$ gec_l is neither colinear to e_3 nor b_i then $H = \mathbb{1}$.
Otherwise,

- If only $gfc_i = \pm e_3$ then $H = \left\{\mathbb{1}, \mathbb{Z}_4, \mathbb{Z}_2^-, \mathbb{Z}_4^-\right\}$ see lemma 5.7.
- If only $gvc_j = \pm e_3$ then $H = \mathbb{Z}_3$ if $3|n$ and $\mathbb{1}$ if not.
- If only $gec_l = \pm e_3$, then $H = \mathbb{Z}_2$ if n is even and $\mathbb{Z}_2^-$ if not.
- If only $gfc_i = \pm b_i$ with i odd then $H = \mathbb{Z}_2^{b_i}$
- If only $gfc_i = \pm b_i$ with i even then $H = \mathbb{Z}_2^{b_i-}$
- If only $gvc_j = \pm b_i$ (with i even or odd) then $H = \mathbb{1}$.
- If only $gec_l = \pm b_i$ with i odd then $H = \mathbb{1} \cup -r(b_i, \pi) = \mathbb{Z}_2^{b_i}$.
- If only $gec_l = \pm b_i$ with i even then $H = \mathbb{1} \cup -r(b_i, \pi) = \mathbb{Z}_2^{b_i-}$.
- If $gfc_i = \pm e_3$ and $gec_l = \pm b_i$ then there exists 3 cases:
  - If $4|n$ then $H = \mathbb{D}_4$ if i is even and $\mathbb{D}_4^v$ if i is odd.
    In fact, when $4|n$, the angle between the b_i in $\mathbb{D}_n$ is equal to $\frac{\pi}{4p}, p \in \mathbb{N}^*$ then two of the edge axes of the cube, ec_{l_1} and ec_{l_2} superpose with two of the b_i's. Hence, when we rotate an edge axis to a b_i with i even, ec_{l_1} and ec_{l_2} will be another two of the b_i's and this rotation will lead e_1 and e_2, which are the face axis of the cube and in the same time two of the b_i's, to bend towards another two of the b_i's with even indices. This will give us four b_i's with even indices in the first intersection, together with $\mathbb{Z}_4$ will give $\mathbb{D}_4$. Same reasoning if the edge axis is rotated to a b_i with odd index, we will end up with four b_i's in the second intersection, together with $\mathbb{Z}_4$ give $\mathbb{D}_4^v$. (see figure below).
  - if $4 \nmid n$ but n even then $H = \mathbb{D}_4^h$. Same reasoning as the previous case except that in this case the two edge axis and e_1, e_2 will rotate to b_i's with indices having alternate parity. So we will have two b_i in the first intersection and another two in the second one, together with $\mathbb{Z}_4^-$ give $\mathbb{D}_4^h$.
  - if n is odd then $H = \mathbb{Z}_2^{b_i}$ if i is odd and $\mathbb{Z}_2^{b_i-}$ if i is even.
- If $gfc_i = \pm b_i$ and $gec_l = \pm e_3$: here i is odd since the only possible rotation satisfying both conditions is the rotation around the axis $e_1 = b_1$ so $i = 1$. This rotation will turn two edge axes to e_3 and e_2 and it will turn e_2 (which is equal to some b_i if n is even) and e_3 to some other two edge axes. Hence we will study two cases:
  - If n is even then $e_2 = b_j$ for some j.
    If $e_2 = b_j$ for j even then $H = \{r(e_3, \pi), r(b_1, \pi)\} \cup -\{r(b_j, \pi)\} = \mathbb{D}_2^h$.
    If $e_2 = b_j$ for j odd then $H = \{r(e_3, \pi), r(b_1, \pi), r(b_j, \pi)\} = \mathbb{D}_2$.
  - If n is odd then $H = \mathbb{1} \cup -\{r(e_3, \pi)\} = \mathbb{Z}_2^-$

$\square$

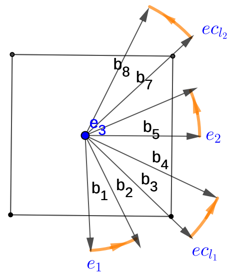

FIGURE 18. Cube viewed from above with $\mathbb{D}_8$

Lemma 5.18. *We have*

$$[\mathbb{D}_{2n}^h] \circledcirc [\mathbb{I} \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_5], [\mathbb{Z}_2^-], [\mathbb{Z}_2^{b_i}], [\mathbb{Z}_2^{b_i-}]\right\}.$$

Proof. Let $H = \mathbb{D}_{2n}^h \cap (\mathbb{I} \oplus \mathbb{Z}_2^c)$. By lemma 5.3,

$$\begin{aligned} H &= (\mathbb{D}_n \cap g\mathbb{I}g^{-1}) \cup (-(\gamma\mathbb{D}_n \cap g\mathbb{I}g^{-1})) \\ &= \left(\left\{r(e_3, \frac{2k\pi}{n}), r(b_{2k+1}, \pi)\right\} \cap \left\{r(gfd_i, \frac{2k\pi}{5}); k = 0, \dots, 4, r(gvd_j, \frac{2k\pi}{3}),\ r(ged_l, \pi)\right\}\right) \\ &\bigcup - \left(\left\{r(e_3, \frac{(2k+1)\pi}{n}), r(b_{2(k+1)}, \pi)\right\} \cap \left\{r(gfd_i, \frac{2k\pi}{5}), r(gvd_j, \frac{2k\pi}{3}),\ r(ged_l, \pi)\right\}\right) \end{aligned}$$

If $\forall i$ gfd_i is neither colinear to e_3 nor to b_i and $\forall j$ gvd_j is neither colinear to e_3 nor b_i and $\forall l$ ged_l is neither colinear to e_3 nor b_i then $H = \mathbb{1}$.
Otherwise,

- If only $gfd_i = \pm e_3$ then $H = \mathbb{Z}_5$ if $5|n$ and $\mathbb{1}$ if not.
- If only $gvd_j = \pm e_3$ then $H = \mathbb{Z}_3$ if $3|n$ and $\mathbb{1}$ if not.
- If only $ged_l = \pm e_3$, then $H = \mathbb{Z}_2$ if n is even and $\mathbb{Z}_2^-$ if not.
- If only $gfd_i = \pm b_i$ $(\forall i)$ then $H = \mathbb{1}$
- If only $gvd_j = \pm b_i$ $(\forall i)$ then $H = \mathbb{1}$.
- If only $ged_l = \pm b_i$ with i odd then $H = \mathbb{Z}_2^{b_i}$.
- If only $ged_l = \pm b_i$ with i even then $H = \mathbb{Z}_2^{b_i-}$.

□

PA: Il faut mettre la figure du cube pour que ça soit plus compréhensible?

5.4. **Clips with $\mathbb{O}^-$.** In this part we will calculate the clips operation between $\mathbb{O}^-$ and each of the subgroups of type II. First, we construct $\mathbb{O}^-$ from the couple $(\mathbb{O}, \mathbb{T})$, we have

$$\mathbb{O} = \left\{r(fc_i, \frac{2k\pi}{4}); i = 1, \dots, 3, r(vc_j, \frac{2k\pi}{3}); j = 1, \dots, 4, r(ec_l, k\pi); l = 1 \dots, 6\right\}$$

where vc_i, ec_j and fc_l are respectively the vertices, edges, and faces axes of the cube.
And

$$\mathbb{T}=\left\{r(vt_i,\frac{2k\pi}{3});i=1,\dots,4,r(et_j,k\pi);j=1,\dots,3\right\}$$

where vt_i and et_j are the vertices axes and edges axes of the tetrahedron. The vertices axes of tetrahedron are the same vertices axes of the cube however the edge axes of the tetrahedron are the faces axes of the cube which are the coordinates axes e_1, e_2 and e_3. Hence,

$$\mathbb{O}^-=\left\{r(vc_j,\frac{2k\pi}{3});j=1,\dots,4,r(fc_i,k\pi),-r(fc_i,\frac{\pi}{2}),-r(fc_i,\frac{3\pi}{2});i=1,\dots,3,-r(ec_l,k\pi);l=1\dots,6\right\}$$

Lemma 5.19. *We have*

$$[\mathbb{O}^-]\circledcirc[\mathbb{Z}_m\oplus\mathbb{Z}_2^c]=\left\{[\mathbb{1}],[\mathbb{Z}_2^{e_i}],[\mathbb{Z}_3^{vt_i}],[\mathbb{Z}_2^{ec_l-}],[\mathbb{Z}_4^{e_i-}]\right\}.$$

Proof. Let $H=\mathbb{O}^-\cap(g\mathbb{Z}_mg^{-1}\oplus\mathbb{Z}_2^c)$. By lemma 5.3, $L=\mathbb{O}^-\cap\mathrm{SO}(3)=\mathbb{T}$ and

$$H=(\mathbb{T}\cap g\mathbb{Z}_mg^{-1})\cup(-(\gamma\mathbb{T}\cap g\mathbb{Z}_mg^{-1}))$$

where $\gamma=r(fc_1,\frac{\pi}{2})=r(e_1,\frac{\pi}{2})\in\mathbb{O}\setminus\{\mathbb{T}\}$. We have

$$\begin{aligned}\gamma\mathbb{T}&=r(e_1,\frac{\pi}{2}).\left\{r(vt_i,\frac{2k\pi}{3});i=1,\dots,4,r(et_j,k\pi);j=1,\dots,3\right\}\\&=\left\{r(ec_l,\pi);l=1,\dots,6,r(\underbrace{ft_j}_{e_j},\frac{\pi}{2}),r(\underbrace{ft_j}_{e_j},\frac{3\pi}{2});j=1,\dots,3\right\}\end{aligned}$$

Hence,

$$\begin{aligned}H=&\left(\left\{r(vt_i,\frac{2k\pi}{3});i=1,\dots,4,r(\underbrace{et_j}_{e_j},k\pi);j=1,\dots,3\right\}\cap\left\{r(ge_3,\frac{2k\pi}{m})\right\}\right)\bigcup\\&-\left(\left\{r(ec_l,\pi);l=1,\dots,6,r(e_j,\frac{\pi}{2}),r(e_j,\frac{3\pi}{2});j=1,\dots,3\right\}\cap\left\{r(ge_3,\frac{2k\pi}{m})\right\}\right)\end{aligned}$$

If ge_3 is not colinear to any of $vt_i,\forall i$, $e_j,\forall j$ and $ec_l,\forall l$ then $H=\mathbb{1}$.
Otherwise, if only $ge_3=\pm vt_i$, then $H=\mathbb{Z}_3^{vt_i}$ if $3|m$ and $\mathbb{1}$ if not.
Now if only $ge_3=\pm e_i$ then we have 3 cases:

- If m is even and $4|m$ then $H=\{e,r(e_i,\pi)\}\cup-\left\{r(e_i,\frac{\pi}{2}),r(e_i,\frac{3\pi}{2})\right\}=\mathbb{Z}_4^{e_i-}$.
- If m is even and $4\nmid m$ then $H=\{e,r(e_i,\pi)\}\cup\emptyset=\mathbb{Z}_2^{e_i}$.
- If m is odd then $H=\mathbb{1}$.

Finally, if only $ge_3=\pm ec_l$ then $H=\mathbb{1}\cup-\{r(ec_l,\pi)\}=\mathbb{Z}_2^{ec_l-}$. □

Lemma 5.20. *We have*

$$[\mathbb{O}^-]\circledcirc[\mathbb{D}_m\oplus\mathbb{Z}_2^c]=\left\{[\mathbb{1}],[\mathbb{Z}_2^{ec_l}],[\mathbb{Z}_2^{ec_l-}],[\mathbb{Z}_2^{e_i}],[\mathbb{Z}_3^{vt_i}],[\mathbb{Z}_4^{e_i-}],[\mathbb{D}_2],[\mathbb{D}_2^v],[\mathbb{D}_4^h]\right\}.$$

Proof. Let $H=\mathbb{O}^-\cap(g\mathbb{D}_mg^{-1}\oplus\mathbb{Z}_2^c)$. By lemma 5.3, $L=\mathbb{T}$ and $\gamma=r(e_1,\frac{\pi}{2})$. Then

$$\begin{aligned}H&=(\mathbb{T}\cap g\mathbb{D}_mg^{-1})\cup(-(\gamma\mathbb{T}\cap g\mathbb{D}_mg^{-1}))\\&=\left(\left\{r(vt_i,\frac{2k\pi}{3}),r(e_j,k\pi)\right\}\cap\left\{r(ge_3,\frac{2k\pi}{m}),\ r(gb_i,\pi)\right\}\right)\bigcup\\&-\left(\left\{r(ec_l,\pi),r(e_j,\frac{\pi}{2}),r(e_j,\frac{3\pi}{2})\right\}\cap\left\{r(ge_3,\frac{2k\pi}{m}),r(gb_i,\pi)\right\}\right)\end{aligned}$$

If ge_3 and gb_i are not colinear to any of $vt_i,\forall i$, $e_j,\forall j$ and $ec_l,\forall l$ then $H=\mathbb{1}$.
Otherwise,

- if only $ge_3=\pm vt_i$ then $H=\mathbb{Z}_3^{vt_i}$ if $3|m$ and $\mathbb{1}$ if not.

- If only $gb_i = \pm vt_i$, then $H = \mathbb{1}$.
- If only $ge_3 = \pm e_i$ then H=$\begin{cases} \mathbb{Z}_4^{e_i-} & \text{if } m \text{ is even and } 4|m \\ \mathbb{Z}_2^{e_i} & \text{if } m \text{ is even and } 4 \nmid m \\ \mathbb{1} & \text{if } m \text{ is odd} \end{cases}$
- If only $gb_i = \pm e_i$ then $H = \mathbb{Z}_2^{e_i}$.
- If only $ge_3 = \pm ec_l$ then $H = \mathbb{Z}_2^{ec_l-}$ if m is even and $\mathbb{1}$ if not.
- If only $gb_i = \pm ec_l$ then $H = \mathbb{Z}_2^{ec_l-}$ $\forall m$.
- If $ge_3 = \pm e_3$ and $gb_i = \pm b_i$ $\forall i$. (we can take the identity rotation) then we have 3 cases:
  - If m is even and $4|m$ then there exits two edge axes ec_{l_1} and ec_{l_2} that coincide with the second and fourth secondary axis in $\mathbb{D}_4$: b_2 and b_4 since the angle between e_1 and an edge axis is $\frac{\pi}{4}$ hence

$$H = \{e, r(e_1,\pi), r(e_2,\pi), r(e_3,\pi)\} \cup -\left\{r(e_3, \frac{\pi}{2}), r(e_3, \frac{3\pi}{2}), r(ec_{l_1},\pi), r(ec_{l_2},\pi)\right\} = \mathbb{D}_4^h.$$

  - If m is even and $4 \nmid m$ then $H = \{e, r(e_1,\pi), r(e_2,\pi), r(e_3,\pi)\} \cup \emptyset = \mathbb{D}_2$.
  - If m is odd then $H = \{e, r(e_1,\pi)\} = \mathbb{Z}_2^{e_1}$.
- If $ge_3 = \pm e_j$ and $gb_i = \pm ec_l$ (we take the rotation around e_3 axis). When b_i turns around e_3 to an edge axis ec_l (which is a secondary axis of $\mathbb{D}_4$), it will leads the other b_i's to turn to some b_j's hence $gb_i = \pm b_j$ and we have the same result as the previous case.
- If $ge_3 = \pm ec_l$ and $gb_i = \pm e_i$ (we take the rotation around e_1). When e_3 turns to an edge axis ec_{l_1} around e_1 it leads e_2 to turn to another edge axis ec_{l_2}. Hence if m is even e_2 will be one of the b_i and we get:

  $$H = \{e, r(e_1,\pi)\} \cup -\{r(ec_{l_1},\pi), r(ec_{l_2},\pi)\} = \mathbb{D}_2^v(\text{but around } e_1 \text{ instead of } e_3)??$$

  If not $H = \mathbb{1}$.

□

Lemma 5.21. *We have*

$$[\mathbb{O}^-] \circledcirc [\mathbb{T} \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_2^{e_i}], [\mathbb{Z}_2^{ec_l-}], [\mathbb{Z}_3^{vt_i}], [\mathbb{T}]\right\}.$$

Proof. Let $H = \mathbb{O}^- \cap (g\mathbb{T}g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3,

$$\begin{aligned} H &= (\mathbb{T} \cap g\mathbb{T}g^{-1}) \cup (-(\gamma\mathbb{T} \cap g\mathbb{T}g^{-1})) \\ &= \left(\left\{r(vt_i, \frac{2k\pi}{3}), r(et_j, k\pi)\right\} \cap \left\{r(gvt_i, \frac{2k\pi}{3}),\ r(get_j,\pi)\right\}\right) \bigcup \\ &- \left(\left\{r(ec_l,\pi), r(e_j, \frac{\pi}{2}), r(e_j, \frac{3\pi}{2})\right\} \cap \left\{r(gvt_i, \frac{2k\pi}{3}),\ r(get_j,\pi)\right\}\right) \end{aligned}$$

Where the edge axis in the tetrahedron et_j are the coordinates axis e_1, e_2 and e_3.
If $\forall i$ gvt_i and $\forall j$ get_j are not colinear to any of $vt_i, \forall i$, $e_j, \forall j$ and $ec_l, \forall l$ then $H = \mathbb{1}$.
Otherwise,

- If only $gvt_i = \pm vt_i$ then $H = \mathbb{Z}_3^{vt_i}$.
- If only $gvt_i = \pm e_j$ then $H = \mathbb{1}$.
- If only $gvt_i = \pm ec_l$ then $H = \mathbb{1}$.
- If only $ge_j = \pm vt_i$ then $H = \mathbb{1}$.
- If only $ge_j = \pm e_j$ then $H = \mathbb{Z}_2^{e_j}$.
- If only $ge_j = \pm ec_l$ then $H = \mathbb{Z}_2^{ec_l-}$.
- If $gvt_i = \pm vt_i, \forall i$ and $ge_j = \pm e_j, \forall j$ (g identity rotation) then $H = \mathbb{T}$.

□

Lemma 5.22. *We have*

$$[\mathbb{O}^-] \circledcirc [\mathbb{O} \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_2^{e_i}], [\mathbb{Z}_2^{ec_l-}], [\mathbb{Z}_3^{vt_i}], [\mathbb{Z}_4^{e_i-}], [\mathbb{D}_4^h], [\mathbb{O}^-]\right\}.$$

Proof. Let $H = \mathbb{O}^- \cap (g\mathbb{O}g^{-1} \oplus \mathbb{Z}_2^c)$. By lemma 5.3,

$$
\begin{aligned}
H &= (\mathbb{T} \cap g\mathbb{O}g^{-1}) \cup (-(\gamma\mathbb{T} \cap g\mathbb{O}g^{-1}))\\
&= \left(\left\{r(vt_i, \frac{2k\pi}{3}), r(e_j, k\pi)\right\} \cap \left\{r(gfc_i, \frac{2k\pi}{4}), r(gvc_j, \frac{2k\pi}{3}), r(gec_l, \pi)\right\}\right)\\
&\bigcup - \left(\left\{r(ec_l, \pi), r(e_j, \frac{\pi}{2}), r(e_j, \frac{3\pi}{2})\right\} \cap \left\{r(gfc_i, \frac{2k\pi}{4}), r(gvc_j, \frac{2k\pi}{3}), r(gec_l, \pi)\right\}\right)
\end{aligned}
$$

where the vertices axis vt_i of the cube and vc_j of the tetrahedron are the same and the edges axes et_j of the tetrahedron are the faces axes of the cube which are the coordinates axes e_1, e_2 and e_3.
If $\forall i$ gfc_i, $\forall j$ gvc_j and $\forall l$, gec_l are not colinear to any of vt_i $\forall i$, e_j $\forall j$ and ec_l $\forall l$ then $H = \mathbb{1}$.
Otherwise,

- If only $gfc_i = \pm vt_i$ then $H = \mathbb{1}$
- If only $gfc_i = \pm e_j$ then $H = \{e, r(e_i, \pi)\} \cup -\left\{r(e_i, \frac{\pi}{2}), r(e_i, \frac{3\pi}{2})\right\} = \mathbb{Z}_4^{e_i-}$.
- If only $gfc_i = \pm ec_l$ then $H = \mathbb{1} \cup -\{r(ec_l, \pi)\} = \mathbb{Z}_2^{ec_l-}$.
- If only $gvc_j = \pm vt_i$ then $H = \mathbb{Z}_3^{vt_i}$.
- If only $gvc_j = \pm e_i$ then $H = \mathbb{1}$.
- If only $gvc_j = \pm ec_l$ then $H = \mathbb{1}$.
- If only $gec_l = \pm vt_i$, then $H = \mathbb{1}$.
- If only $gec_l = \pm e_i$ then $H = \mathbb{Z}_2^{e_i}$.
- If only $gec_l = \pm ec_l$ then $H = \mathbb{Z}_2^{ec_l-}$.
- If $gfc_i = \pm fc_i = \pm e_i$ and $gvc_j = \pm vc_j$ and $gec_l = \pm ec_l$ (g is the identity rotation) then
$$\left\{e, r(e_i, \pi), r(vt_i, \frac{2k\pi}{3})\right\} \cup -\left\{r(e_i, \frac{\pi}{2}), r(e_i, \frac{3\pi}{2}), r(ec_l, \pi)\right\} = \mathbb{O}^-$$
- If $gfc_i = \pm fc_i = \pm e_i$ and $gec_l = e_j$, $j \neq i$ (we can take the rotation around e_i, e_3 for example, that turns a ec_l to e_1 or e_2) then

$$H = \{e, r(e_1, \pi), r(e_2, \pi), r(e_3, \pi)\} \cup -\left\{r(e_i, \frac{\pi}{2}), r(e_i, \frac{3\pi}{2}), r(ec_{l_1}, \pi), r(ec_{l_2}, \pi)\right\} = \mathbb{D}_4^h(\text{ around } e_i).$$

□

Lemma 5.23. *We have*

$$[\mathbb{O}^-] \circledcirc [\mathbb{I} \oplus \mathbb{Z}_2^c] = \left\{[\mathbb{1}], [\mathbb{Z}_2^{e_i}], [\mathbb{Z}_2^{ec_l-}], [\mathbb{Z}_3^{vt_i}], [\mathbb{T}]\right\}.$$

Proof. Let $H = \mathbb{O}^- \cap (\mathbb{I} \oplus \mathbb{Z}_2^c)$. By lemma 5.3,

$$
\begin{aligned}
H &= (\mathbb{T} \cap g\mathbb{I}g^{-1}) \cup (-(\gamma\mathbb{T} \cap g\mathbb{I}g^{-1}))\\
&= \left(\left\{r(vt_i, \frac{2k\pi}{3}), r(e_j, k\pi)\right\} \cap \left\{r(gfd_i, \frac{2k\pi}{5}), r(gvd_j, \frac{2k\pi}{3}),\ r(ged_l, \pi)\right\}\right)\\
&\bigcup - \left(\left\{r(ec_l, \pi), r(e_j, \frac{\pi}{2}), r(e_j, \frac{3\pi}{2})\right\} \cap \left\{r(gfd_i, \frac{2k\pi}{5}), r(gvd_j, \frac{2k\pi}{3}),\ r(ged_l, \pi)\right\}\right)
\end{aligned}
$$

If $\forall i$ gfd_i, $\forall j$ gvd_j and $\forall l$, ged_l are not colinear to any of vt_i $\forall i$, e_j $\forall j$ and ec_l $\forall l$ then $H = \mathbb{1}$.
Otherwise,

- If only $gfd_i = \pm vt_i$ then $H = \mathbb{1}$
- If only $gfd_i = \pm e_j$ then $H = \mathbb{1}$.
- If only $gfd_i = \pm ec_l$ $H = \mathbb{1}$.
- If only $gvd_j = \pm vt_i$ then $H = \mathbb{Z}_3^{vt_i}$.
- If only $gvd_j = \pm e_i$ then $H = \mathbb{1}$.
- If only $gvd_j = \pm ec_l$ then $H = \mathbb{1}$.
- If only $ged_l = \pm vt_i$, then $H = \mathbb{1}$.
- If only $ged_l = \pm e_i$ then $H = \mathbb{Z}_2^{e_i}$.
- If only $ged_l = \pm ec_l$ then $H = \mathbb{Z}_2^{ec_l-}$.

- If $gfd_i = \pm fd_i = \pm e_i$ and $gvd_j = \pm vd_j$ and $ged_l = \pm ed_l$ (g is the identity rotation) then

$$\left\{e, r(vt_i, \frac{2k\pi}{3}), r(e_i, \pi)\right\} = \mathbb{T}.$$

$\square$

6. SYMMETRY CLASSES OF THE PIEZOELECTRICITY LAW

We recall the space of piezoelectricity law $\mathcal{P}\text{iez}$ introduced in the section 2:

$$\mathcal{P}\text{iez} = \mathbb{E}\text{la} \oplus \mathbb{P}\text{iez} \oplus \mathbb{S}$$

In this section, we will find the symmetry classes of $\mathcal{P}\text{iez}$ using lemma 3.9. First we will work on the space

$$\mathcal{P}\text{iela} := \mathbb{E}\text{la} \oplus \mathbb{P}\text{iez}$$

We already know the symmetry classes of both spaces: $\mathbb{E}\text{la}$ and $\mathbb{P}\text{iez}$ (see theorem 3.4 and 3.5).

Appendix A. O(3)-subgroups

There exists three types of O(3)-subgroups:

- For the subgroups of **type I**: Every closed subgroup of SO(3) is conjugate to one of

$$\mathrm{SO}(3), \quad \mathrm{O}(2), \quad \mathrm{SO}(2), \quad \mathbb{D}_n, \quad \mathbb{Z}_n, \quad \mathbb{T}, \quad \mathbb{O}, \quad \mathbb{I}, \text{ or } \quad \mathbb{1}$$

  Where
  - O(2) is the subgroup generated by all the rotations around the z-axis and the order 2 rotation $r : (x, y, z) \to (x, -y, -z)$ around the x-axis.
  - SO(2) is the subgroup of all the rotations around the z-axis.
  - $\mathbb{Z}_n$ is the unique cyclic subgroup of order n of SO(2) ($\mathbb{Z}_1 = \mathrm{I}$).
  - $\mathbb{D}_n$ is the dihedral group. It is generated by $\mathbb{Z}_n$ and $r : (x, y, z) \to (x, -y, -z)$ ($\mathbb{D}_1 = \mathrm{I}$).
  - $\mathbb{T}$ is the tetrahedral group, the (orientation-preserving) symmetry group of the tetrahedron. It has order 12.
  - $\mathbb{O}$ is the octahedral group, the (orientation-preserving) symmetry group of the cube. It has order 24.
  - $\mathbb{I}$ is the icosahedral group, the (orientation-preserving) symmetry group of the dodecahedron. It has order 60.
  - $\mathbb{1}$ is the trivial subgroup, containing only the unit element.
- For the subgroups of **type II**: They are subgroups of type I to which we add the group $\mathbb{Z}_2^c = \{\pm \mathrm{I}\}$:

$$\mathbb{Z}_m \oplus \mathbb{Z}_2^c, \quad \mathbb{D}_m \oplus \mathbb{Z}_2^c, \quad \mathbb{T} \oplus \mathbb{Z}_2^c, \quad \mathbb{O} \oplus \mathbb{Z}_2^c, \quad \mathbb{I} \oplus \mathbb{Z}_2^c$$

- For the subgroups of **type III**: There are four subgroups of O(3) of type III that we construct using a subgroup of type I (see lemma below):

$$\mathbb{Z}_{2n}^-, \quad \mathbb{D}_n^v, \quad \mathbb{D}_{2n}^h, \quad \mathbb{O}^-$$

  Where
  - $\mathbb{Z}_{2n}^-$ ($n \geq 2$) is the group of order $2n$, generated by $\mathbb{Z}_n$ and $-r(e_3, \frac{\pi}{n})$.
  - $\mathbb{D}_n^v$ ($n \geq 2$) is the group of order $2n$ generated by $\mathbb{Z}_n$ and the reflection through the plane normal to e_1 (where $\mathbb{D}_1^v = \mathbb{1}$).
  - $\mathbb{D}_{2n}^h$ ($n \geq 2$) is the group of order $4n$ generated by $\mathbb{D}_n$ and $-r(e_3, \frac{\pi}{n})$.
  - $\mathbb{O}^-$ is generated by SO(2) and the reflection through the plane normal to e_1.

References

[1] S. Forte and M. Vianello. Symmetry classes for elasticity tensors. *Journal of Elasticity*, 43(2):81–108, 1996.
[2] G. Geymonat and T. Weller. Symmetry classes of piezoelectric solids. *Comptes Rendus Mathématique*, 335(10):847–852, 2002.
[3] G. D. Mostow and G. P. Hochschild. *On a conjecture of Montgomery.* Mathematics Division, Air Force Office of Scientific Research, 1957.
[4] M. Olive. Effective computation of so (3) and o (3) linear representation symmetry classes. *Mathematics and Mechanics of Complex Systems*, 7(3):203–237, 2019.
[5] M. Olive and N. Auffray. Symmetry classes for even-order tensors. *Mathematics and Mechanics of Complex Systems*, 1(2):177–210, 2013.
[6] M. Olive and N. Auffray. Symmetry classes for odd-order tensors. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 94(5):421–447, 2014.
[7] T. Weller. *Etude des symétries et modèles de plaques en piézoélectricité linéarisée.* PhD thesis, 2004.

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