Robot Dynamics and Control Assignment

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Introduction

This assignment covers the basics of force/torque equilibrium for a robotic manipulator, a fundamental concept for dynamic control.
It consists of two exercises, in which there are two different kinds of robotic manipulators (i.e. two manipulators that have different types of joint sets) and for each exercise the joint configuration is different.
Our goal is to find the vector of generalized actuator forces $\tau_{\mathbf{eq}}$, which is given by the following relation:

$$\tau_{\mathbf{eq}} = \boldsymbol{J}^T \boldsymbol{W_{ext}} \quad (1)$$

To do that I created two CAD models using **Fusion360** software by Autodesk, one for each exercise. The axes of each part follow the *Denavit-Hartenberg* convention.

After that, I obtained the XML file using a special add-on of **Inventor.** Please note that, in order to use a certain type of material (in our case "steel"), you'll need to re-apply it onto the CAD in Inventor. After doing that, I imported the XML files into **Matlab** (using "*smimport()* function"). The result consists of one data struct for each CAD model, in which are contained useful infos such as:

- **RigidTransform** structure: contains the translation, the angle and the axis of rotation between the i-th frame and the base frame.
- **Solid** structure: contains the mass, the CoM, the MoI and the PoI, all referred to the i-th link.

After that, I computed the transformation matrices with respect to the base frame for each configuration. I used the "*axang2rotm()*" function which returns the corresponding rotation matrix given the axe of rotation and the angle and then I obtained the transformation matrix by considering the distance between each frame with respect to the base.

Now I can project the CoM of all links to the base (by pre-multiplying the i-th CoM vector wrt the i-th frame by the corresponding transformation matrix). Please note that I considered both links as the sum of the rigidly attached motor and the link itself.

Now I can compute all the **Jacobian Matrices** for each CoM with respect to the base, therefore for each CoM $\mathbf{C_i}$ the resulting jacobian has the following form:

$$^{\mathbf{0}}\mathbf{J}_{\mathbf{C_i/0}} = \begin{bmatrix} ^{\mathbf{0}}\mathbf{J}^{\mathbf{A}}_{\mathbf{C_i/0}} \\ ^{\mathbf{0}}\mathbf{J}^{\mathbf{L}}_{\mathbf{C_i/0}} \end{bmatrix} \in \mathbb{R}^{\,6\ x\ n} \ (2)$$

$$where \quad ^{\mathbf{0}}\mathbf{J}^{\mathbf{L(A)}}_{\mathbf{C_i/0}} = \begin{bmatrix} ^{\mathbf{0}}\mathbf{J}^{\mathbf{L(A)}}_{\mathbf{C_i/1}} \ ^{\mathbf{0}}\mathbf{J}^{\mathbf{L(A)}}_{\mathbf{C_i/2}} \ ... ^{\mathbf{0}}\mathbf{J}^{\mathbf{L(A)}}_{\mathbf{C_i/j}} \ \mathbf{0} \ ... \ \mathbf{0} \end{bmatrix} \ (2a(2b))$$
$$i = 1 \ ... \ n \quad | \quad j = 1 \ ... \ i$$

Provided that:

$$\mathbf{J}^{\mathbf{L}}_{\mathbf{C_i/j}} = \begin{bmatrix} (\mathbf{k_j} \times \mathbf{r_{i/j}}) & j \in RJ \\ \mathbf{k_j} & j \in TJ \end{bmatrix} \ (3a) \quad | \quad \mathbf{J}^{\mathbf{A}}_{\mathbf{C_i/j}} = \begin{bmatrix} \mathbf{k_j} & j \in RJ \\ \mathbf{0} & j \in TJ \end{bmatrix} \ (3b)$$

To find $\tau_{\mathbf{eq}}$, we can re-write the equation (1) as:

$$\tau_{\mathbf{eq}} = -\sum_{i=0}^{n} {}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_i/0}} \begin{bmatrix} \mathbf{M^{ext}_i} \\ \mathbf{F^{ext}_i} \end{bmatrix} \quad \in \mathbb{R}^{\,n\ x\ 1} \quad (1a)$$

The last term of the RHS of eq. (1a) corresponds to the i-th **wrench** applied on the i-th CoM ($\in \mathbb{R}^{\,6\ x\ 1}$). When a force or a torque (or both) is applied to a generic point $\mathbf{P_i}$ (different from the CoM of that link), I pre-multiplied the corresponding wrench by the **Rigid Body Jacobian** Matrix $^{\mathbf{0}}\mathbf{S}_{\mathbf{C_i/P_i}}$ which is given by:

$$^{\mathbf{0}}\mathbf{S}_{\mathbf{C_i/P_i}} = \begin{bmatrix} \mathbb{I}_{3x3} & \mathbf{0} \\ [^{\mathbf{0}}\mathbf{r}_{\mathbf{C_i/P_i}} \times] & \mathbb{I}_{3x3} \end{bmatrix} \quad \in \mathbb{R}^{\,6\ x\ 6} \quad (4)$$

Exercise 1

The resulting jacobians for link 1 and link 2 are:

$$ {}^0\mathbf{J}_{\mathbf{C}_1/\mathbf{0}} = \begin{bmatrix} {}^0\mathbf{k_1} & \mathbf{0} \\ {}^0\mathbf{k_1} \times {}^0\mathbf{r}_{\mathbf{C}_1/\mathbf{1}} & \mathbf{0} \end{bmatrix} \quad \in \mathbb{R}^{\,6\;x\;n} \qquad (5a) $$

$$ {}^0\mathbf{J}_{\mathbf{C}_2/\mathbf{0}} = \begin{bmatrix} {}^0\mathbf{k_1} & {}^0\mathbf{k_2} \\ {}^0\mathbf{k_1} \times {}^0\mathbf{r}_{\mathbf{C}_2/\mathbf{1}} & {}^0\mathbf{k_2} \times {}^0\mathbf{r}_{\mathbf{C}_2/\mathbf{2}} \end{bmatrix} \quad \in \mathbb{R}^{\,6\;x\;n} \qquad (5b) $$

1.1

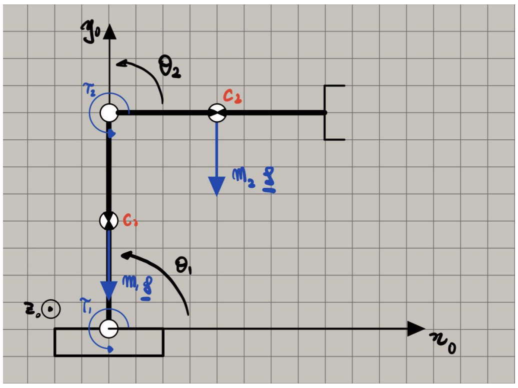

Figure 1

The joint configuration values for the exercise 1.1 are:

$$ \mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \pi/2 \\ -\pi/2 \end{bmatrix} $$

The vector of generalized actuator forces τ_{eq} is equal to:

$$ \tau_{\mathbf{eq}} = -\,({}^0\mathbf{J}^{\mathbf{T}}_{\mathbf{C}_1/\mathbf{0}} \begin{bmatrix} \mathbf{0} \\ m_1 \cdot \mathbf{g} \end{bmatrix} + {}^0\mathbf{J}^{\mathbf{T}}_{\mathbf{C}_2/\mathbf{0}} \begin{bmatrix} \mathbf{0} \\ m_2 \cdot \mathbf{g} \end{bmatrix}) = \begin{bmatrix} 4.5512 \cdot 10^3 \; [N \cdot m] \\ 4.5512 \cdot 10^3 \; [N \cdot m] \end{bmatrix} $$

In this case, only the gravity forces $m_1 \cdot \mathbf{g}$ and $m_2 \cdot \mathbf{g}$ act on the (CoMs of the) body, the torque that needs to be applied results being positive either for the first motor and the second one. This is because the force on $\mathbf{C_1}$ is parallel to the lever arm while the force on the second link has the same x component to the axis of the motors, therefore the applied forces tend to make rotate both joints in a clockwise direction, therefore we need to apply a positive torque to balance those forces and obtain the equilibrium conditions.

1.2

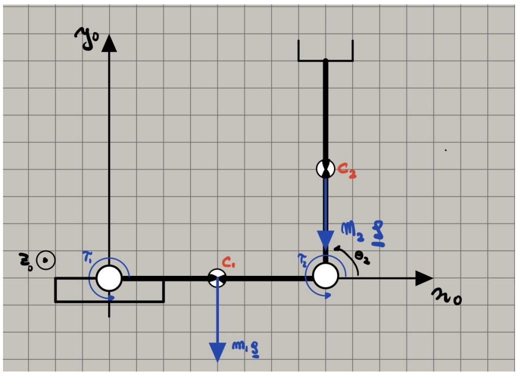

Figure 2

The joint configuration values for the exercise 1.2 are:

$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \pi/2 \end{bmatrix}$$

The vector of generalized actuator forces τ_{eq} is equal to:

$$\tau_{\mathbf{eq}} = -\left({}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_1/0}} \begin{bmatrix} \mathbf{0} \\ m_1 \cdot \mathbf{g} \end{bmatrix} + {}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2/0}} \begin{bmatrix} \mathbf{0} \\ m_2 \cdot \mathbf{g} \end{bmatrix} \right) = \begin{bmatrix} 1.3305 \cdot 10^4 \; [N \cdot m] \\ 0 \; [N \cdot m] \end{bmatrix}$$

As before, only the weights $m_1 \cdot \mathbf{g}$ and $m_2 \cdot \mathbf{g}$ act on the (CoMs of the) body, but in this case, the weight force applied on the second link is parallel to the second link, so the corresponding cross product is null and there's no need to apply a non-null $\tau_\mathbf{2}$ because it is already in equilibrium. The first motor must apply a positive torque to balance $m_1 \cdot \mathbf{g}$ force which would make rotate the chain in a clockwise direction.

1.3

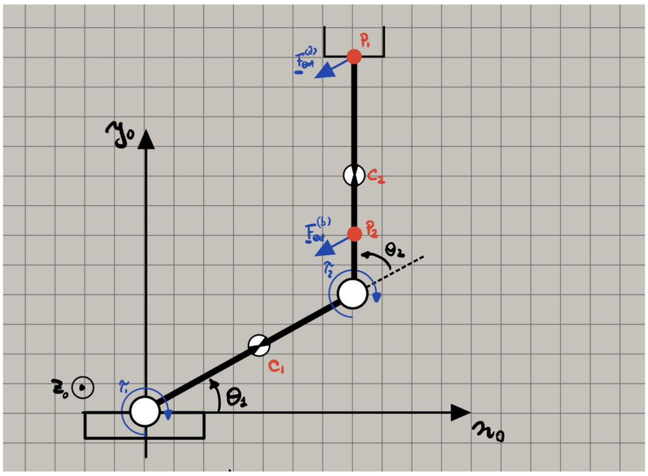

Figure 3

The joint configuration values for the exercise 1.3 are:

$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \pi/6 \\ \pi/3 \end{bmatrix}$$

- Case **(a)**: the vector of generalized actuator forces τ_{eq} is equal to:

$$\tau_{\mathbf{eq_a}} = -\left({}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2/0}} \, {}^{\mathbf{0}}\mathbf{S}^{\mathbf{T}}_{\mathbf{P_1/C_2}} \begin{bmatrix} \mathbf{0} \\ \mathbf{F^a_{ext}} \end{bmatrix} \right) = \begin{bmatrix} -616.9873 \cdot 10^{-3} \; [N \cdot m] \\ -0.700 \; [N \cdot m] \end{bmatrix}$$

- Case **(b)**: the vector of generalized actuator forces τ_{eq} is equal to:

$$\tau_{\mathbf{eq_b}} = -\left({}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2/0}} \, {}^{\mathbf{0}}\mathbf{S}^{\mathbf{T}}_{\mathbf{P_1/C_2}} \begin{bmatrix} \mathbf{0} \\ \mathbf{F^b_{ext}} \end{bmatrix} \right) = \begin{bmatrix} -126.9873 \cdot 10^{-3} \; [N \cdot m] \\ -0.210 \; [N \cdot m] \end{bmatrix}$$

An external force $\mathbf{F_{ext}} = [-0.7; -0.5; 0] \; N$ is applied on $\mathbf{P_1}$ (case 'a') or on $\mathbf{P_2}$ (case 'b'). Either for case (a) and case (b), both links tend to rotate in a counter-clockwise direction and therefore a negative torque must be applied on both of the motors. In case (a), the resulting values are higher because the lever arm that has to be considered is higher, which means that the resulting torque to counter is bigger.

1.4

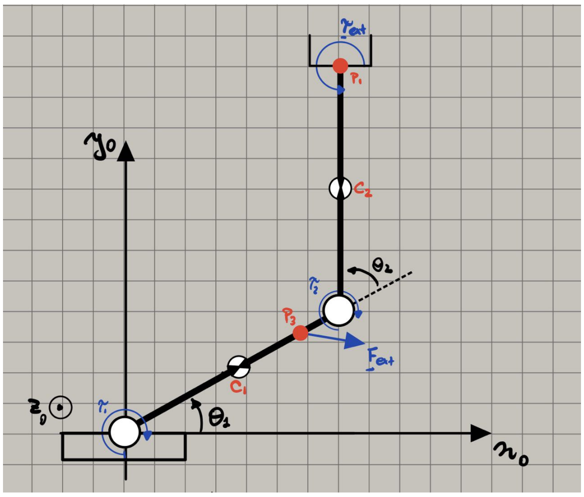

Figure 4

The joint configuration values for the exercise 1.4 are:

$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \pi/6 \\ \pi/3 \end{bmatrix}$$

The vector of generalized actuator forces τ_{eq} is equal to:

$$\tau_{\mathbf{eq}} = -\left({}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_1/0}}\, {}^{\mathbf{0}}\mathbf{S}^{\mathbf{T}}_{\mathbf{P_3/C_1}} \begin{bmatrix} \mathbf{0} \\ \mathbf{F_{ext}} \end{bmatrix} + {}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2/0}} \begin{bmatrix} \tau_{\mathbf{ext}} \\ \mathbf{0} \end{bmatrix}\right) = \begin{bmatrix} -291.1731 \cdot 10^{-3}\ [N \cdot m] \\ -1.2\ [N \cdot m] \end{bmatrix}$$

An external force $\mathbf{F_{ext}} = [1.5; -0.3; 0]\ N$ is applied on $\mathbf{P_3}$ and a torque $\tau_{\mathbf{ext}} = 1.2\ N \cdot m$ is applied about an axis passing through $\mathbf{P_1}$.

For the second link, the external torque is counter-balanced by the torque generated by the second motor, while for the first link the external force on $\mathbf{P_3}$ and the torque on $\mathbf{P_1}$ are counter-balanced by the torque generated by the first motor.

1.5

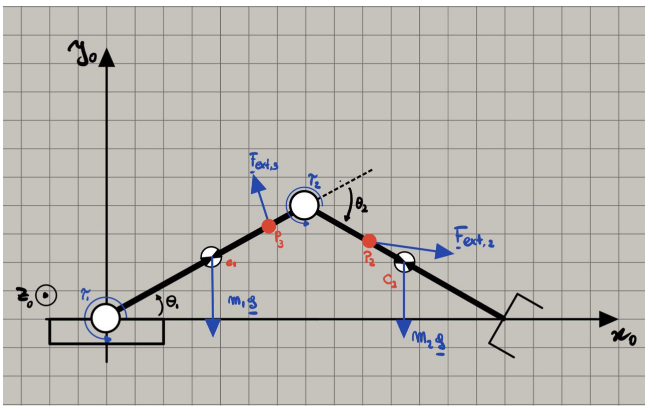

Figure 5

The joint configuration values for the exercise 1.5 are:

$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \pi/6 \\ -\pi/3 \end{bmatrix}$$

The vector of generalized actuator forces τ_{eq} is equal to:

$$\tau_{\mathbf{eq}} = -\left({}^{0}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_1/0}} \begin{bmatrix} \mathbf{0} \\ m_1 \cdot \mathbf{g} \end{bmatrix} + {}^{0}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2/0}} \begin{bmatrix} \mathbf{0} \\ m_2 \cdot \mathbf{g} \end{bmatrix}\right) - \left({}^{0}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_1/0}} {}^{0}\mathbf{S}^{\mathbf{T}}_{\mathbf{P_3/C_1}} \begin{bmatrix} \mathbf{0} \\ \mathbf{F_{ext_3}} \end{bmatrix} + {}^{0}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2/0}} {}^{0}\mathbf{S}^{\mathbf{T}}_{\mathbf{P_2/C_2}} \begin{bmatrix} \mathbf{0} \\ \mathbf{F_{ext_2}} \end{bmatrix}\right) = \begin{bmatrix} 1.5464 \\ 3.9413 \end{bmatrix}$$

In this case the weight force is applied on both CoMs. Moreover, an external force $\mathbf{F_{ext_3}} = [-0.4; 1.2; 0]\ N$ is applied on $\mathbf{P_3}$ and another external force $\mathbf{F_{ext_2}} = [1.2; -0.2; 0]\ N$ is applied on $\mathbf{P_2}$. The resulting torques that need to be applied are clearly positive for both motors, one can observe that the weights are more relevant than the other external forces since their order of magnitude is way bigger than the one related to the other external forces. The weight forces cause both joints to rotate in counter-clockwise direction, hence the torques generated by the motors must be positive.

Exercise 2

The resulting jacobians for link 1 and link 2 are:

$$ {}^{0}\mathbf{J}_{\mathbf{C_1/0}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ {}^{0}\mathbf{k_1} & \mathbf{0} \end{bmatrix} \quad \in \mathbb{R}^{6\ x\ n} \quad (6a)$$

$$ {}^{0}\mathbf{J}_{\mathbf{C_2/0}} = \begin{bmatrix} & {}^{0}\mathbf{k_1} & {}^{0}\mathbf{k_2} \\ {}^{0}\mathbf{k_1} & & {}^{0}\mathbf{k_2} \times {}^{0}\mathbf{r_{C_2/2}} \end{bmatrix} \quad \in \mathbb{R}^{6\ x\ n} \quad (6b)$$

Note: in the following formulas, the CoM of the second link is called $\mathbf{C_2}$, even if in the drawing it is called $\mathbf{C_1}$.

2.1

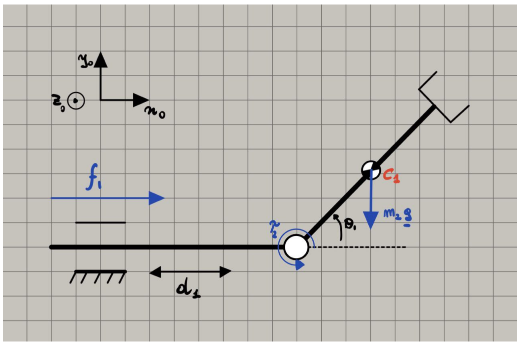

Figure 6

The joint configuration values for the exercise 2.1 are:

$$\mathbf{q} = \begin{bmatrix} d_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \pi/4 \end{bmatrix}$$

The vector of generalized actuator forces τ_{eq} is equal to:

$$\tau_{\mathbf{eq}} = -\,{}^{0}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2/0}} \begin{bmatrix} \mathbf{0} \\ m_2 \cdot \mathbf{g} \end{bmatrix} = \begin{bmatrix} 0\ [N] \\ 3.2182 \cdot 10^3\ [N \cdot m] \end{bmatrix}$$

In this case, only the weight is applied on the CoM of the second link $\mathbf{C_1}$. The resulting equilibrium tau consists of a positive $\tau_{\mathbf{2}}$ to counter-balance the force $m_2 \cdot \mathbf{g}$, while there's no need to apply any force on the prismatic motor because the are no horizontal forces to counter.

2.2

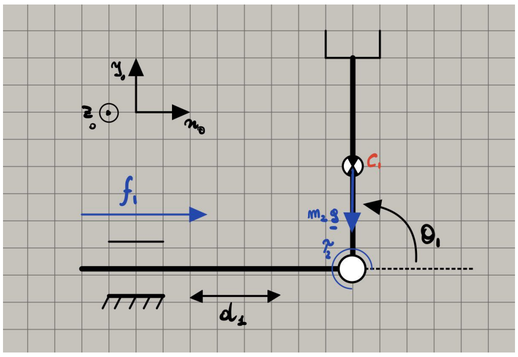

Figure 7

The joint configuration values for the exercise 2.2 are:

$$\mathbf{q} = \begin{bmatrix} d_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \pi/2 \end{bmatrix}$$

The vector of generalized actuator forces τ_{eq} is equal to:

$$\tau_{\mathbf{eq}} = -\,{}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2}/\mathbf{0}} \begin{bmatrix} \mathbf{0} \\ m_2 \cdot \mathbf{g} \end{bmatrix} = \begin{bmatrix} 0\ [N] \\ 0\ [N \cdot m] \end{bmatrix}$$

In this case, only the weight is applied on the CoM of the second link $\mathbf{C_1}$. The force and the torque generated by respectively the first and the second motor are null because the system is already in equilibrium.

2.3

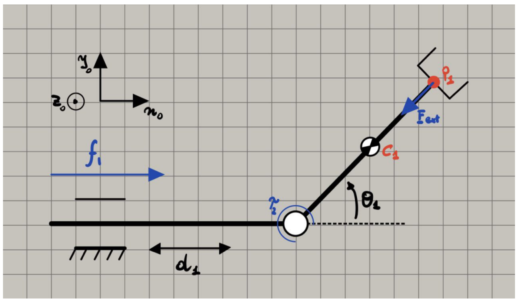

Figure 8

The joint configuration values for the exercise 2.3 are:

$$\mathbf{q} = \begin{bmatrix} d_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \pi/4 \end{bmatrix}$$

The vector of generalized actuator forces $\tau_{\mathbf{eq}}$ is equal to:

$$\tau_{\mathbf{eq}} = \ -\,{}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2}/\mathbf{0}}\,{}^{\mathbf{0}}\mathbf{S}^{\mathbf{T}}_{\mathbf{P_1}/\mathbf{C_2}} \begin{bmatrix} \mathbf{0} \\ \mathbf{F_{ext}} \end{bmatrix} = \begin{bmatrix} 0.8\ [N] \\ 0\ [N \cdot m] \end{bmatrix}$$

An external force $\mathbf{F_{ext}} = [-0.8; -0.8; 0]\ N$ is applied on $\mathbf{P_1}$, the gravity is not considered for this question. The force doesn't generate a torque on the second joint because it is parallel to the second link, while the prismatic motor needs to generate a positive force to counter the first component of $\mathbf{F_{ext}}$.

2.4

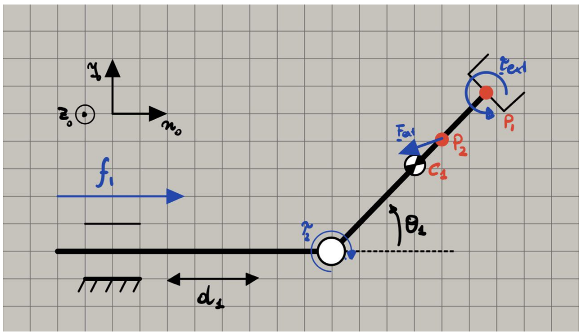

Figure 9

The joint configuration values for the exercise 2.4 are:

$$\mathbf{q} = \begin{bmatrix} d_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \pi/4 \end{bmatrix}$$

The vector of generalized actuator forces τ_{eq} is equal to:

$$\tau_{\mathbf{eq}} = -\left({}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2/0}}\,{}^{\mathbf{0}}\mathbf{S}^{\mathbf{T}}_{\mathbf{P_2/C_2}} \begin{bmatrix} \mathbf{0} \\ \mathbf{F_{ext}} \end{bmatrix} + {}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2/0}} \begin{bmatrix} \tau_{\mathbf{ext}} \\ \mathbf{0} \end{bmatrix}\right) = \begin{bmatrix} 0.8\ [N] \\ -775.7716 \cdot 10^{-3} [N \cdot m] \end{bmatrix}$$

An external force $\mathbf{F_{ext}} = [-0.8; -0.2; 0]\ N$ is applied on $\mathbf{P_2}$ and a torque $\tau_{\mathbf{ext}} = 1.2\ N \cdot m$ is applied about an axis passing through $\mathbf{P_1}$. As in the case before, the first component of the external force is counter-balanced by the prismatic motor while the external torque doesn't affect it. Either the second component and the external torque produce a counter-clockwise rotation, therefore a negative torque must be applied by the revolute motor.

2.5

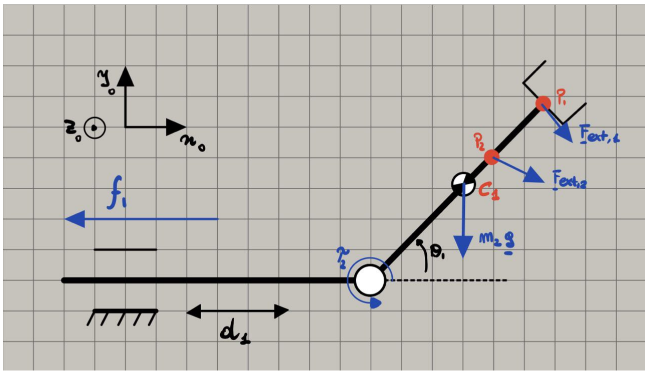

Figure 10

The joint configuration values for the exercise 2.5 are:

$$\mathbf{q} = \begin{bmatrix} d_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \pi/4 \end{bmatrix}$$

The vector of generalized actuator forces τ_{eq} is equal to:

$$\tau_{\mathbf{eq}} = -\,{}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2/0}} \begin{bmatrix} \mathbf{0} \\ m_2 \cdot \mathbf{g} \end{bmatrix} - \left({}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2/0}}\,{}^{\mathbf{0}}\mathbf{S}^{\mathbf{T}}_{\mathbf{P_1/C_2}} \begin{bmatrix} \mathbf{0} \\ \mathbf{F_{ext_1}} \end{bmatrix} + {}^{\mathbf{0}}\mathbf{J}^{\mathbf{T}}_{\mathbf{C_2/0}}\,{}^{\mathbf{0}}\mathbf{S}^{\mathbf{T}}_{\mathbf{P_2/C_2}} \begin{bmatrix} \mathbf{0} \\ \mathbf{F_{ext_2}} \end{bmatrix}\right) = \begin{bmatrix} -1.5\ [N] \\ 3.2196 \cdot 10^{3}\ [N \cdot m] \end{bmatrix}$$

There are two external forces, $\mathbf{F_{ext_1}} = [0.5; -0.6; 0]\ N$ and $\mathbf{F_{ext_2}} = [1.0; -0.4; 0]\ N$, which are applied respectively on $\mathbf{P_1}$ and $\mathbf{P_2}$. There is also the force $m_2 \cdot \mathbf{g}$ acting on the CoM of the second link. Those forces (except for the gravity one) tend to "elongate" the prismatic joint and therefore a negative force must be applied. All the three forces produce a clockwise rotation about the axis passing through the second joint, which needs to be counter-balanced by generating a positive torque τ_2.