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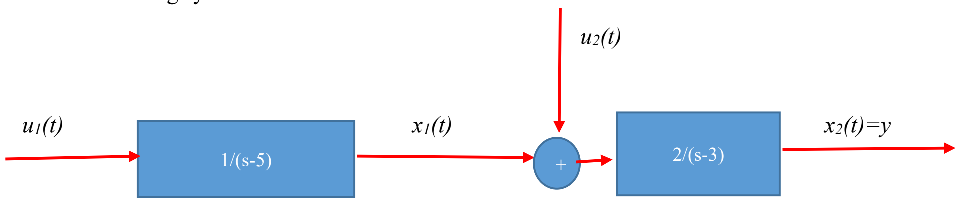
<u>The computer exercise.</u>

Consider the following system



1) Write the state equations of the system
2) Determine the stationary state-feedback control law having structure

$$\underline{\boldsymbol{u}}(\boldsymbol{t}) = -\boldsymbol{L}\underline{\boldsymbol{x}}(\boldsymbol{t})$$

optimizing the infinite horizon cost function

$$\int_{t_0}^{\infty} \left[\underline{\boldsymbol{x}}^{\boldsymbol{T}}(\boldsymbol{t})\boldsymbol{V}\underline{\boldsymbol{x}}(\boldsymbol{t}) + \underline{\boldsymbol{u}}^{\boldsymbol{T}}(\boldsymbol{t})\boldsymbol{P}\underline{\boldsymbol{u}}(\boldsymbol{t})\right] \boldsymbol{dt}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} \mathbf{10} & \mathbf{0} \\ \mathbf{0} & \mathbf{10} \end{bmatrix}$$

and

$$\boldsymbol{P} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$$

3) Check whether this control law ensures that the closed-loop system is asymptotically stable

4) Assume now that the state of the system is not fully accessible (but only the output ***y*** is). Design an asymptotic observer of the state choosing the poles of the observer so that the observer dynamics is considerably "faster" than the dynamics of the original closed-loop system (e.g., ensuring that the time constants of the observer are an order of magnitude smaller than those of that system).

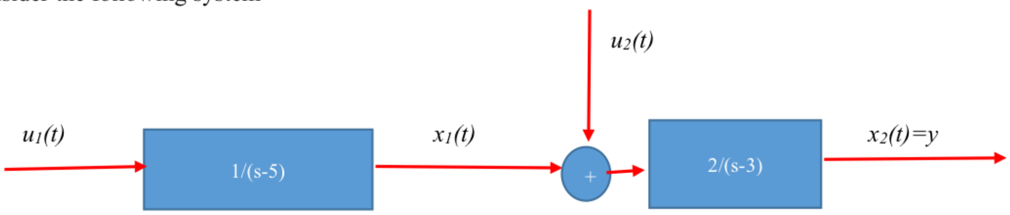
5) Build the overall model (by SIMULINK) of the overall system (the original open-loop system + the feedback controller + the state observer) and analyze the behavior of the system when the input is given by

$$\underline{\boldsymbol{u}}(\boldsymbol{t}) = -\boldsymbol{L}\underline{\boldsymbol{x}}(\boldsymbol{t}) + \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \boldsymbol{r}(\boldsymbol{t})$$

being ***r(t)*** an external sinusoidal input.

6) Determine the closed-loop transfer function $\boldsymbol{T}(\boldsymbol{s}) = \boldsymbol{Y}(\boldsymbol{s})/\boldsymbol{R}(\boldsymbol{s})$ (assuming a very fast convergence of the observed system state to the true system state) and analyze the frequency response (for different frequencies of the sinusoidal input).

Consider the following system



1) Write the state equations of the system

$$\Rightarrow \bullet X_1(s) = \frac{1}{s-5} U_1(s) \quad \Leftrightarrow \quad sX_1(s) - 5X_1(s) = U_1(s)$$

$$\Rightarrow \dot{x}_1(t) = 5x_1(t) + u_1(t)$$

$$\bullet \; X_2(s) = \frac{2}{s-3}\left(X_1(s) + U_2(s)\right)$$

$$sX_2(s) - 3X_2(s) = 2\left(X_1(s) + U_2(s)\right)$$

$$\Rightarrow \dot{x}_2(t) = 2x_1(t) + 3x_2(t) + 2u_2(s)$$

$$\Rightarrow \begin{cases} \dot{\underline{x}}(t) = \begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \underline{u}(t) \\ y(t) = x_2(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x}(t) \end{cases}$$

2) Determine the stationary state-feedback control law having structure

$$\underline{\boldsymbol{u}}(\boldsymbol{t}) = -\boldsymbol{L}\underline{\boldsymbol{x}}(\boldsymbol{t})$$

optimizing the infinite horizon cost function

$$\int_{t_0}^{\infty} \left[\underline{x}^T(t) V \underline{x}(t) + \underline{u}^T(t) P \underline{u}(t)\right] dt$$

where

$$V = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Given the algebraic Riccati Equation:

$$\dot{K}(t) = -K(t)A - A^T K(t) + K(t) B P^{-1} B^T K(t) - V \quad , \quad K[t_f] = [0]$$

We know that, if the system is completely controllable, the limit

$\lim_{t_i \to +\infty} K(0;[0];t_f) = \lim_{t \to -\infty} K(t;[0];t_f)$, that is equal to K_o, <u>exists</u>

and it is positive semidefinite. This matrix K_o (which is not necessarily unique) satisfies: $+K_o(t)A + A^T K_o(t) - K_o(t) B P^{-1} B^T K_o(t) + V = [0]$

To minimize our cost function I simply used the "lqr" Matlab function which gave me three outputs:

1) The optimal gain matrix L
2) The solution of the associated algebraic Riccati Equation
3) The closed loop poles "e"

$$\Rightarrow [L,k,e] = lqr(A,B,V,P,0)$$

$$L = k = \begin{bmatrix} 11{,}0707 & 0{,}7286 \\ 0{,}7286 & 4{,}1427 \end{bmatrix}$$

$$e = \begin{bmatrix} -5{,}6781 + 0{,}4903\,i \\ -5{,}6781 - 0{,}4903\,i \end{bmatrix}$$

3) Check whether this control law ensures that the closed-loop system is asymptotically stable

We know from theory that the closed-loop system controlled by the optimal control law $\underline{u}^\circ(t) = -P^{-1}B^T K_o \underline{x}(t)$ is asymptotically stable if these condition holds:

1) The system is fully controllable (i.e. the matrix [B AB] is full rank
2) Given a matrix H such that $V = H^T H$, the pair (A,H) satisfies the condition $rank(Q) = n$, where $Q = \begin{bmatrix} H \\ HA \\ HA^2 \\ \vdots \\ HA^{n-1} \end{bmatrix}$

From Matlab I obtained $H = \begin{bmatrix} 3.1623 & 0 \\ 0 & 3.1623 \end{bmatrix}$, $rank(Q) = 2$

and rank([B AB]) is maximum too, hence the system is <u>ASYMPTOTICALLY STABLE</u>.

4) Assume now that the state of the system is not fully accessible (but only the output y is). Design an asymptotic observer of the state choosing the poles of the observer so that the observer dynamics is considerably "faster" than the dynamics of the original closed-loop system (e.g., ensuring that the time constants of the observer are an order of magnitude smaller than those of that system).

We want to build an Identity Observer having the structure:

$$\dot{\underline{z}}(t) = (A - GC)\underline{z}(t) + G\underline{y}(t) + B\underline{u}(t)$$

where $F = A - GC$ is given by:

$$F = A - GC = \begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 0 & +g_1 \\ 0 & +g_2 \end{bmatrix} = \begin{bmatrix} 5 & -g_1 \\ 2 & 3-g_2 \end{bmatrix}$$

To assign an arbitrary set of eigenvalues to the matrix $(A-GC)$ (and consequently make it asymptotically stable), we need to check if the pair (A, C) is completely observable, so we need that $\operatorname{rank}(Q_o) = n$

where $Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$.

After checking this condition, I choose the eigenvalues respectively equal to $\lambda_{1/2} = (-10, -20)$, which are positioned quite "far" (in the direction of the negative real part) from the eigenvalues of the original system.

Now we need to compute:

- $\det(\lambda I - F) = \det\left(\begin{bmatrix} \lambda - 5 & g_1 \\ -2 & \lambda - (3-g_2) \end{bmatrix}\right) =$

$$= (\lambda - 5)(\lambda - 3 + g_2) + 2g_1 = \lambda^2 - 3\lambda + \lambda g_2 - 5\lambda + 15 - 5g_2 + 2g_1 = 0$$

① $\lambda_1 = -10 \Rightarrow 100 + 30 - 10g_2 + 50 + 15 - 5g_2 + 2g_1 = 0$

$$\Rightarrow 195 - 15g_2 + 2g_1 = 0$$

② $\lambda_2 = -20 \Rightarrow 400 + 60 - 20g_2 + 100 + 15 - 5g_2 + 2g_1 = 0$

$\Rightarrow 575 - 25g_2 + 2g_1 = 0$

We can now find g_1, g_2: $\begin{cases} ① \\ ② \end{cases} \Rightarrow \begin{cases} g_1 = \frac{375}{2} \\ g_2 = 38 \end{cases}$

The observer structure will become:

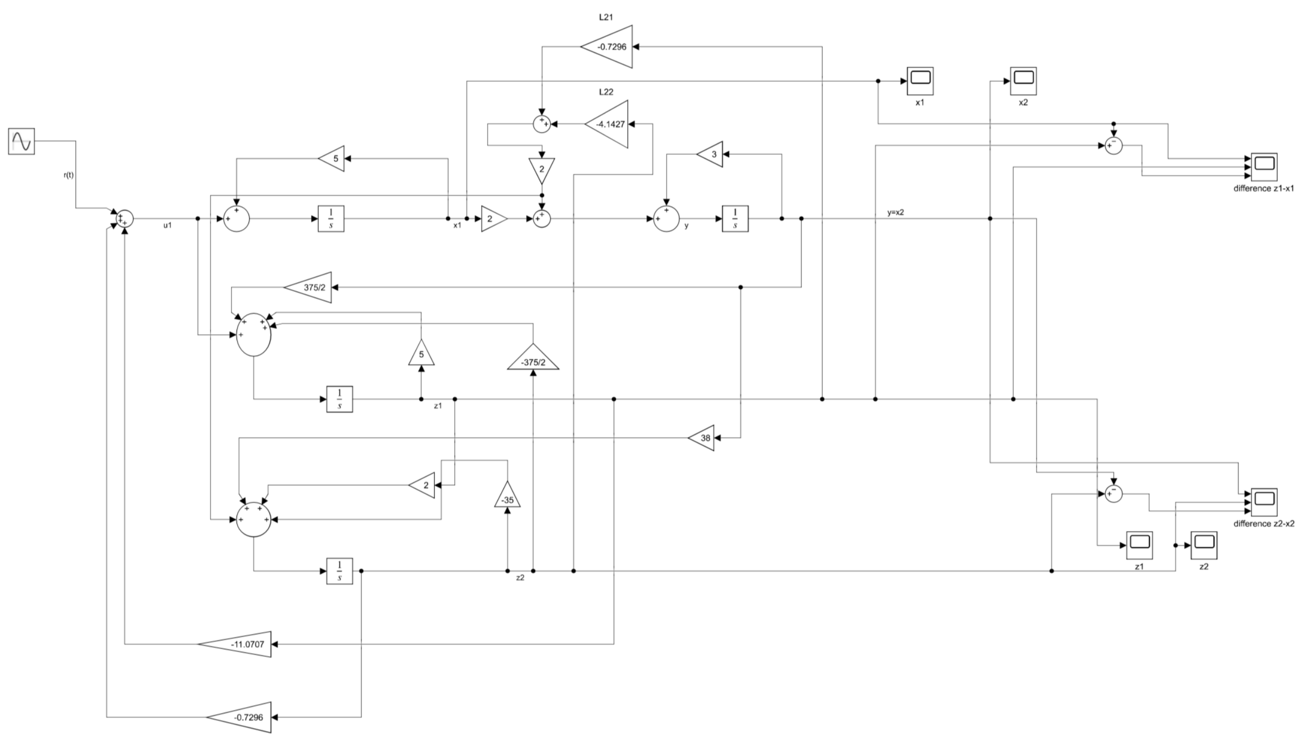
$$\dot{\underline{z}}(t) = \begin{bmatrix} 5 & -\frac{375}{2} \\ 2 & -35 \end{bmatrix} \underline{z}(t) + \begin{bmatrix} 375/2 \\ 38 \end{bmatrix} \underline{y}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \underline{u}(t)$$

5) Build the overall model (by SIMULINK) of the overall system (the original open-loop system + the feedback controller + the state observer) and analyze the behavior of the system when the input is given by

$$\underline{u}(t) = -L\underline{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r(t)$$

being $r(t)$ an external sinusoidal input.

Here's the model of the overall system that I designed via SIMULINK:



Here you can find the plot of z_1, x_1 and their difference (figure 1), z_2, x_2 and their difference (figure 2) when $r(t) = 100\sin(t)$:

Figure 1

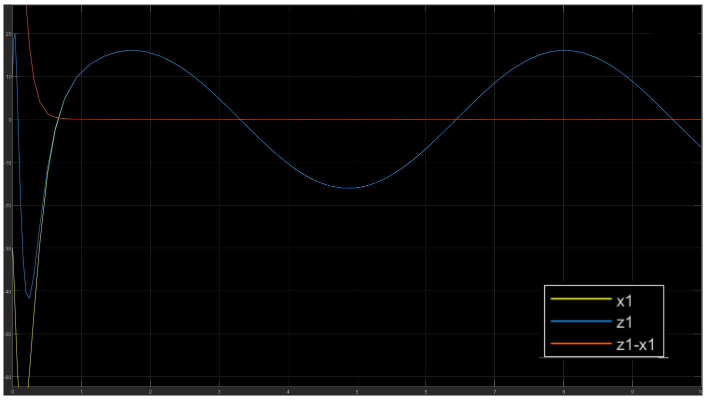
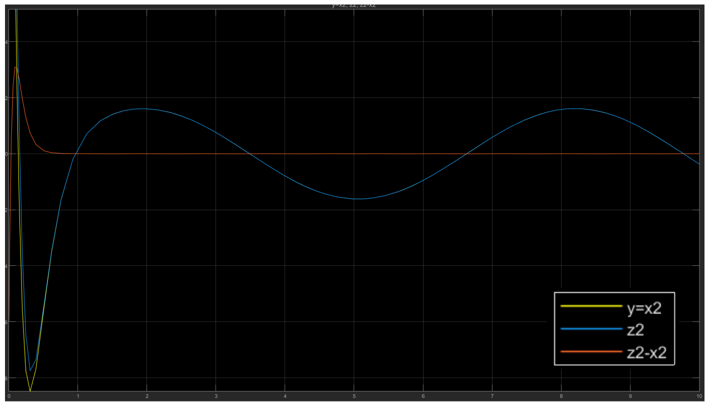


Figure 2



6) Determine the closed-loop transfer function $\boldsymbol{T}(s) = \boldsymbol{Y}(s)/\boldsymbol{R}(s)$ (assuming a very fast convergence of the observed system state to the true system state) and analyze the frequency response (for different frequencies of the sinusoidal input).

We now have:

$$\begin{cases} \dot{\underline{x}}(t) = A\underline{x}(t) + B\left(-L\underline{x}(t) + \begin{bmatrix}1\\0\end{bmatrix} r(t)\right) \\ y(t) = C\underline{x}(t) \end{cases}$$

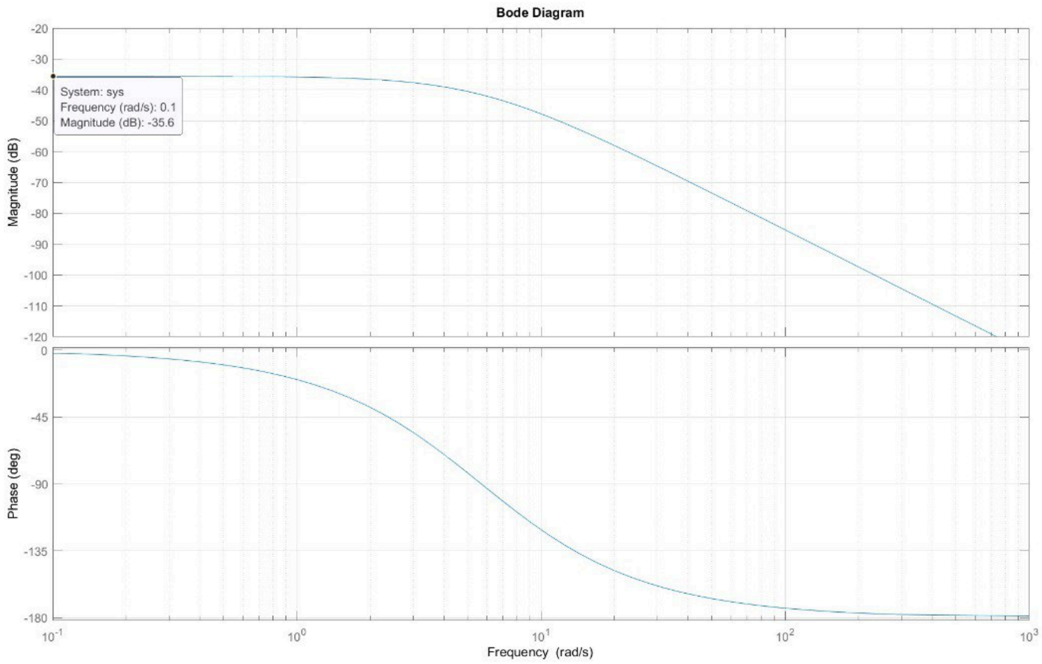
Which, in number, is:

$$\dot{\underline{x}}(t) = \begin{bmatrix} -6.0707 & -0.7286 \\ 0.5408 & -5.2854 \end{bmatrix} \underline{x}(t) + \begin{bmatrix}1\\0\end{bmatrix} r(t)$$

Thanks to "ss2tf" function I can now obtain the transfer function:

$$\Rightarrow \quad T(s) = \frac{Y(s)}{R(s)} = \frac{0.5408}{s^2 + 11.3561s + 32.4808}$$

These are the diagrams for the magnitude and the phase:



Given these following external inputs on the left, I obtained the outputs on the right:

- $r(t) = 1 \cdot \sin(10^{-1} t)$ $\rightarrow$ $|T(j\omega)| = 0.0166$
- $r(t) = 1 \cdot \sin(t)$ $\qquad |T(j\omega)| = 0.0162$
- $r(t) = 1 \cdot \sin(10t)$ $\qquad |T(j\omega)| = 0.0041$