Trade-off between prediction and FDR for high-dimensional Gaussian model selection
(Supplementary material)

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Contents

This supplementary material is an appendix of the article "Trade-off between prediction and FDR for high-dimensional Gaussian model selection". This material contains plots for the bounds $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ for each of the four scenarios described in Table 1 of Subsection 7.1. It is a complementary work to Subsection 7.2 of the article.

In this supplementary materials, graphs for scenarios (i) to (iv) described in Table 1 are provided. Relative changes and relative standard deviations for the $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ bounds when $\tilde{K} \in \{1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, \log(n)\}$ are plotted in Figures 1- 6 for scenario (i), in Figures 8- 13 for scenario (ii), in Figures 15- 20 for scenario (iii) and in Figures 22- 27 for scenarios (iv). The empirical estimation functions of $\mathrm{PR}\big(\hat{m}(K)\big)$ and $\mathrm{FDR}\big(\hat{m}(K)\big)$, the estimated risk (4.1) and the $B(K, \hat{\beta}_{\hat{m}(4)}, \hat{\sigma}^2)$ functions for a grid of values of $K > 0$ are plotted on Figure 7 for scenario (i), Figure 14 for scenario (ii), Figure 21 for scenario (iii) and Figure 28 for scenario (iv).
All the R scripts are available at `https://github.com/PerrineLacroix/Trade_off_FDR_PR`.

When we focus on the scenario (i), the higher the D_{m^*} value is, the smaller the empirical FDR is but the larger the empirical PR for large K is. Moreover, the relative change functions decreases when D_m^* increases, as well as the relative standard deviation ones which remain smaller than 0.5. This can be explained since the higher D_{m^*}, the smaller the number of non active variables, so the smaller the number of the selected non active variables and the smaller FDR value. In the opposite trend, the empirical PR increases with D_{m^*} since penalization tends to select too few variables, especially even K moves away from 2.
As expected, concerning the scenario (ii), when coefficients are smaller than the amplitude of the noise (the second configuration), values of the relative change for the $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ bounds explode (until 10^5) and the relative standard deviation values increase until exceed 1. The best results are obtained for the first β^* configuration of the scenario (ii), but results still remain reasonable with the third one. The relative standard deviation functions increase after $\tilde{K} \geq 4$ whereas in all other scenarios, functions always decrease when $\tilde{K}$ increases.
As for the scenario (iii), we observe unsurprisingly that the higher the value of n, the smaller the relative change, the smaller the relative standard deviation, and the tighter the confidence interval of PR (< 0.04 for $n = 30$). However, we note that the computational time to estimate the bounds was significantly higher for $n = 300$.
Lastly, concerning the scenario (iv), as expected, the higher the noise amplitude, the larger the confidence interval for the PR (< 0.45 for $\sigma^2 = 4$), the higher the relative change (which equals 0 when $\sigma^2 = 0.1$ and around 2 when $\sigma^2 = 4$) and the higher the relative standard deviation. However, values remain reasonable even for $\sigma^2 = 4$ excepted for the empirical PR values which are always larger than 5.

1 Scenario (i)

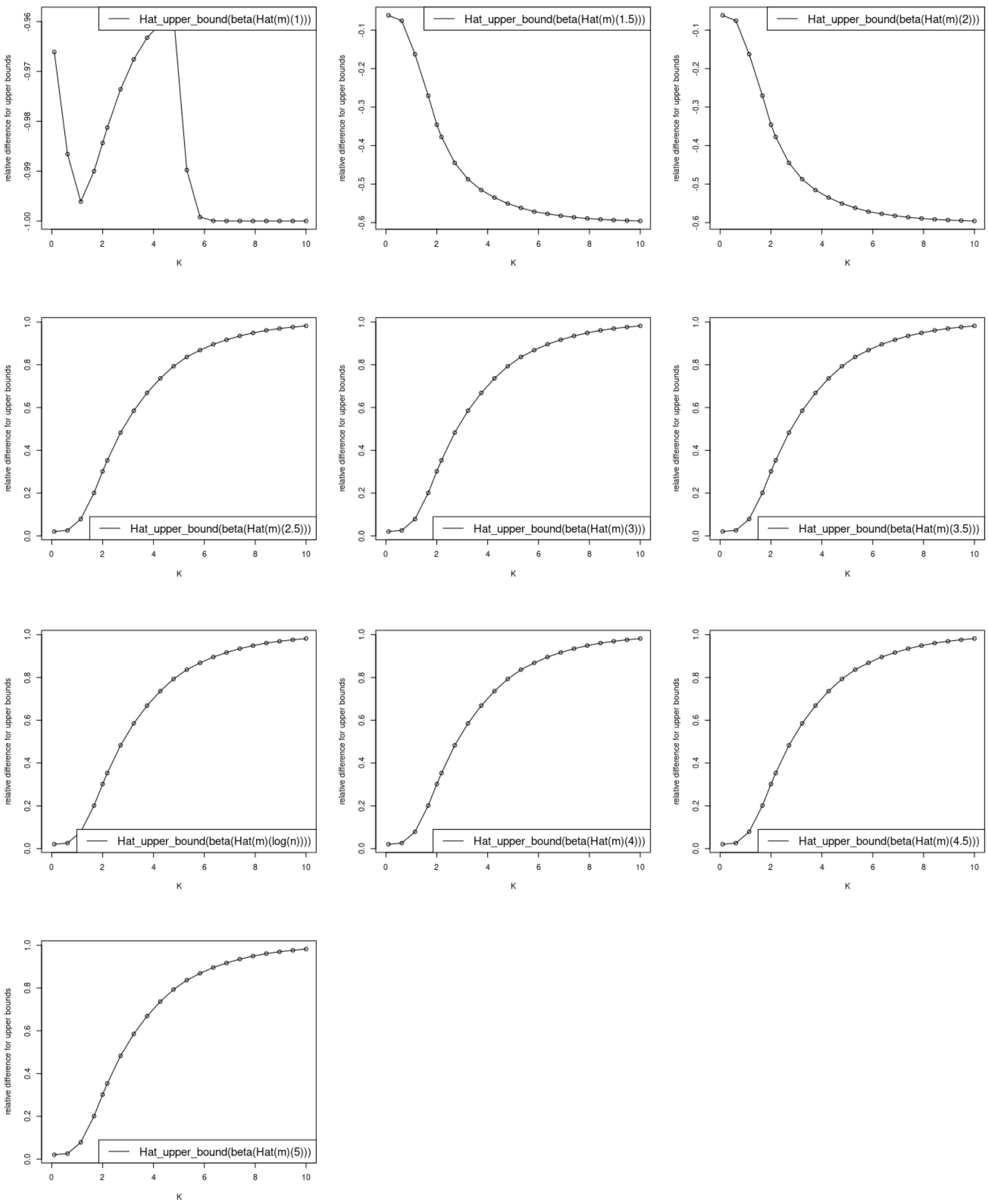


Figure 1: Curves of the relative change values between the functions $B(K, \beta^*, \sigma^2)$ and the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ with respectively $\hat{\beta}_{\hat{m}(1)}$, $\hat{\beta}_{\hat{m}(1.5)}$, $\hat{\beta}_{\hat{m}(2)}$, $\hat{\beta}_{\hat{m}(2.5)}$, $\hat{\beta}_{\hat{m}(3)}$, $\hat{\beta}_{\hat{m}(3.5)}$, $\hat{\beta}_{\hat{m}(4)}$, $\hat{\beta}_{\hat{m}(4.5)}$, $\hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ where estimators are calculating from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $|\beta^*| = 1$.

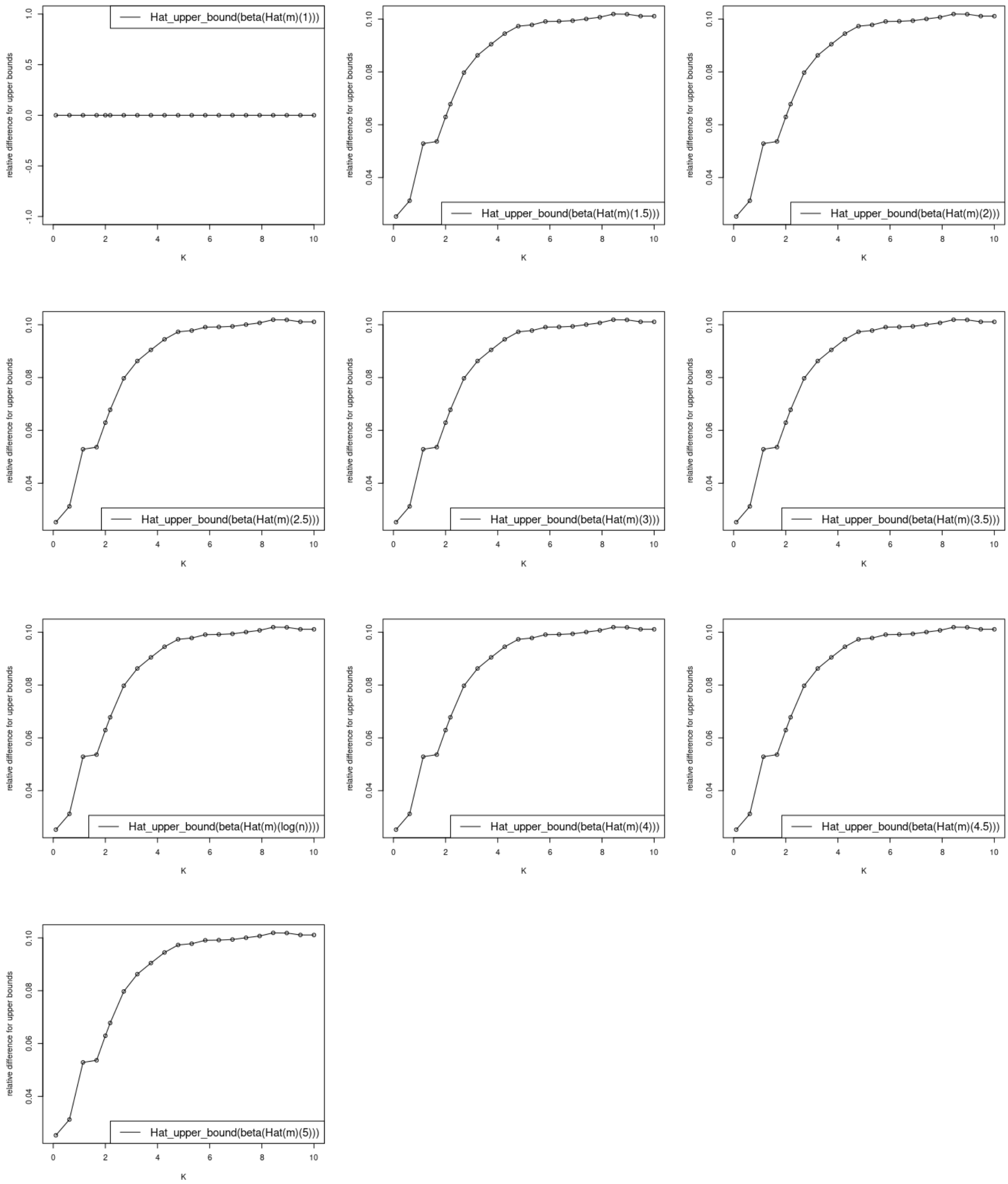


Figure 2: Curves of the relative change values between the functions $B(K, \beta^*, \sigma^2)$ and the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ with respectively $\hat{\beta}_{\hat{m}(1)}$, $\hat{\beta}_{\hat{m}(1.5)}$, $\hat{\beta}_{\hat{m}(2)}$, $\hat{\beta}_{\hat{m}(2.5)}$, $\hat{\beta}_{\hat{m}(3)}$, $\hat{\beta}_{\hat{m}(3.5)}$, $\hat{\beta}_{\hat{m}(4)}$, $\hat{\beta}_{\hat{m}(4.5)}$, $\hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ where estimators are calculating from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $|\beta^*| = 10$.

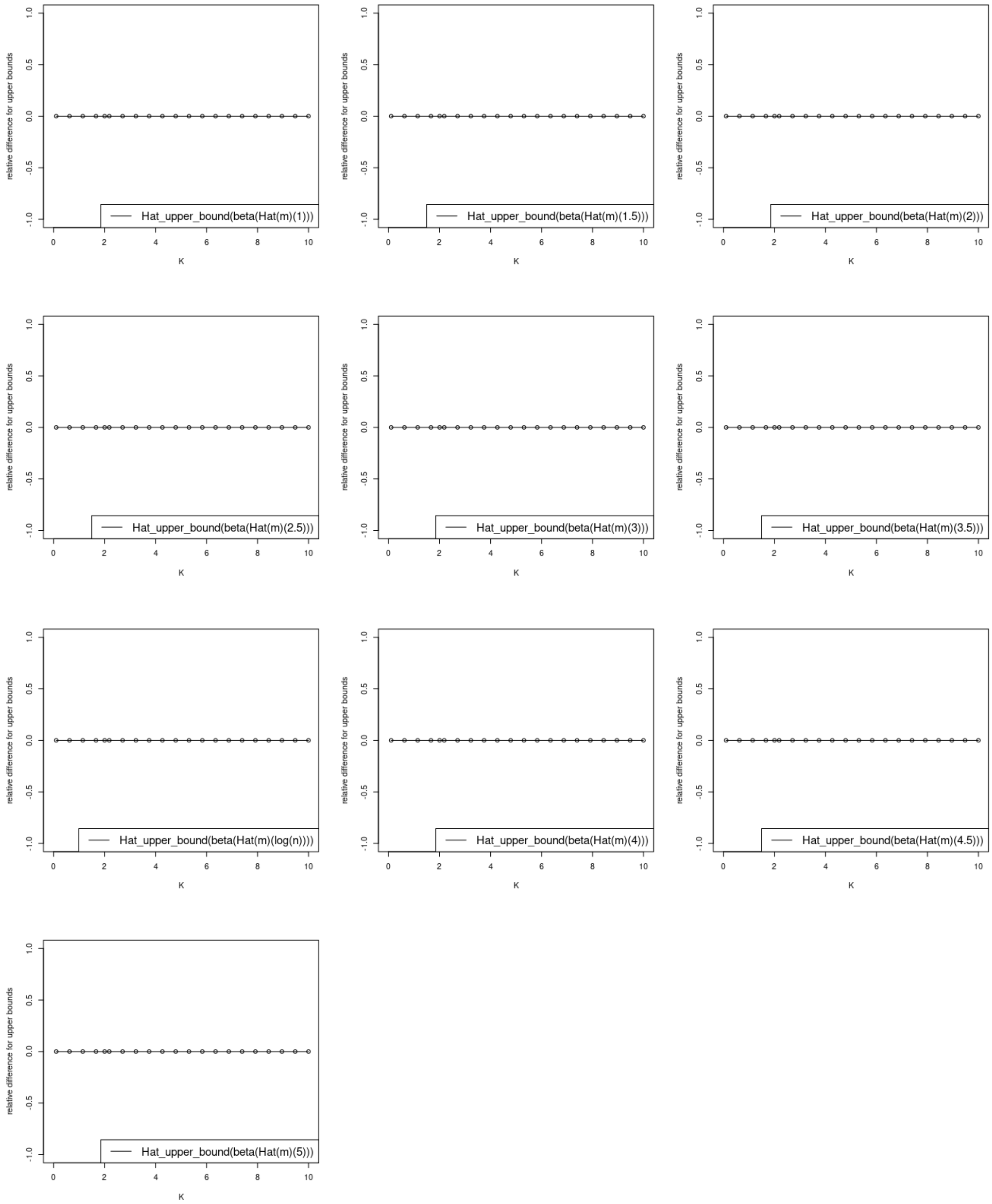


Figure 3: Curves of the relative change values between the functions $B(K, \beta^*, \sigma^2)$ and the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ with respectively $\hat{\beta}_{\hat{m}(1)}$, $\hat{\beta}_{\hat{m}(1.5)}$, $\hat{\beta}_{\hat{m}(2)}$, $\hat{\beta}_{\hat{m}(2.5)}$, $\hat{\beta}_{\hat{m}(3)}$, $\hat{\beta}_{\hat{m}(3.5)}$, $\hat{\beta}_{\hat{m}(4)}$, $\hat{\beta}_{\hat{m}(4.5)}$, $\hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ where estimators are calculating from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $|\beta^*| = 20$.

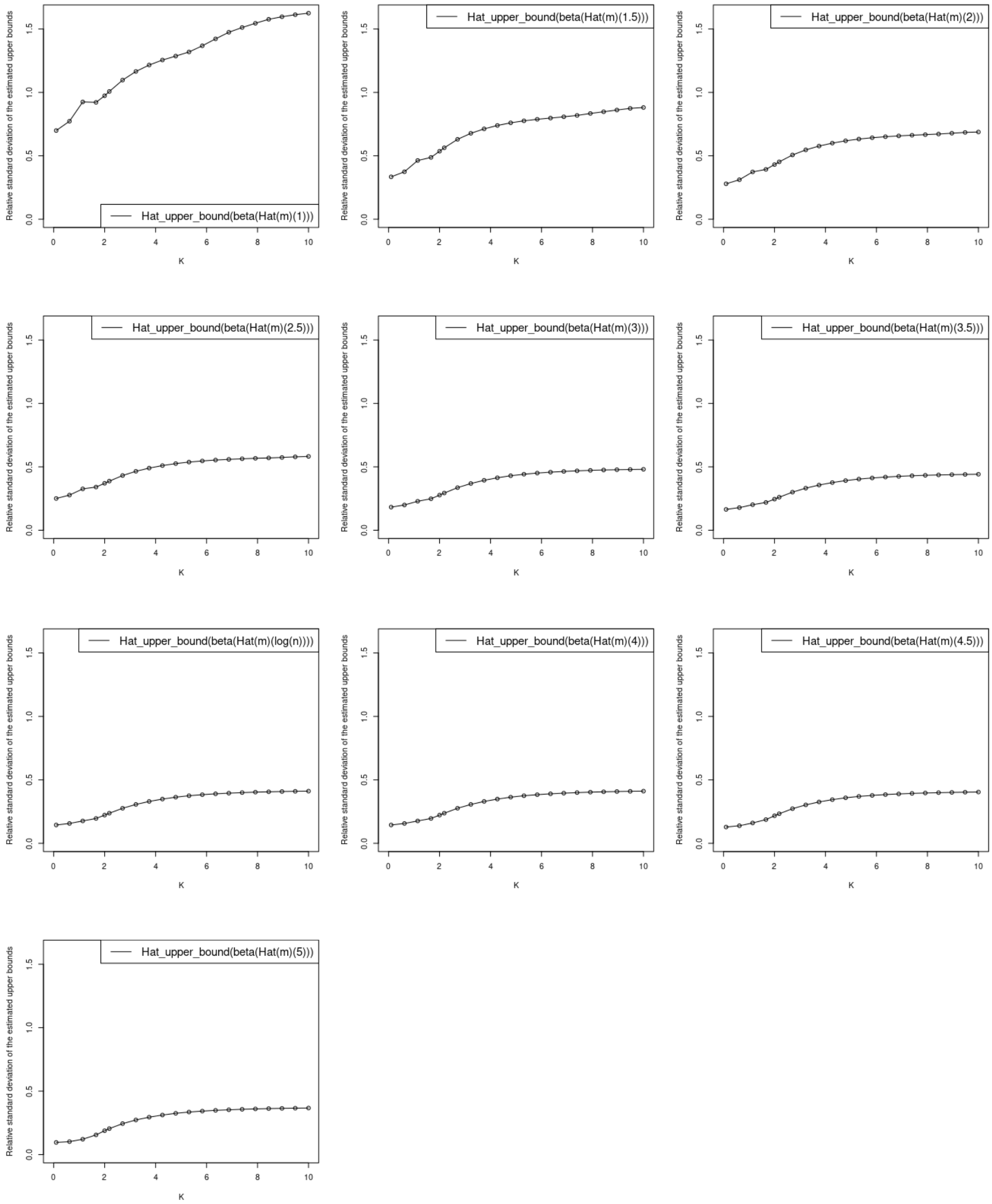


Figure 4: Curves of the relative standard deviation (standard deviation normalized by the mean) of the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ obtained from 100 data sets. With each one, $\hat{\beta}_{\hat{m}(1)}, \hat{\beta}_{\hat{m}(1.5)}, \hat{\beta}_{\hat{m}(2)}, \hat{\beta}_{\hat{m}(2.5)}, \hat{\beta}_{\hat{m}(3)}, \hat{\beta}_{\hat{m}(3.5)}, \hat{\beta}_{\hat{m}(4)}, \hat{\beta}_{\hat{m}(4.5)}, \hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ are calculated given $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$, variance of the 100 $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ functions and then the relative standard deviation with respect to K. These plots are obtained with the *toy data set* described in Subsection 7.1 for $|\beta^*| = 1$.

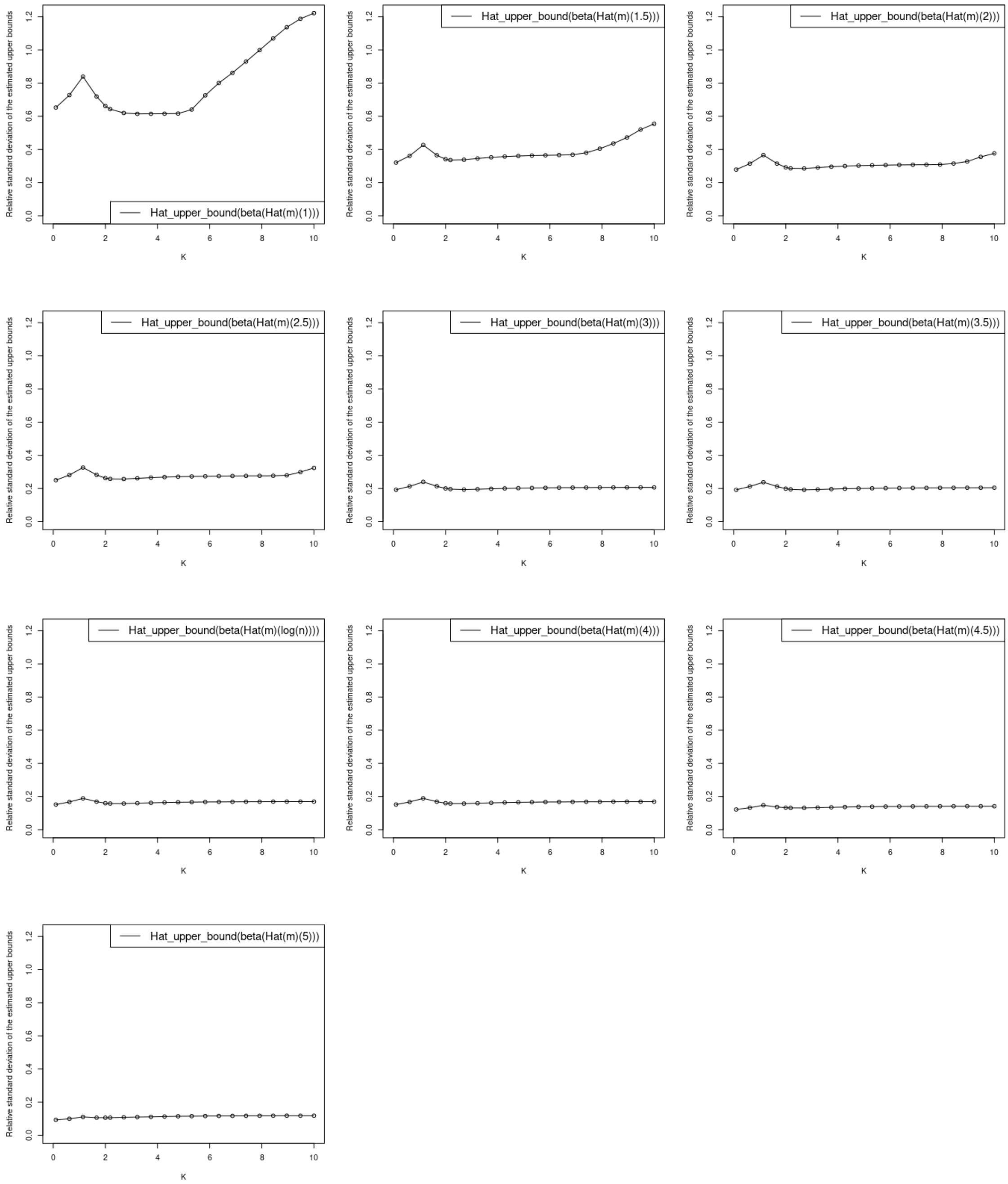


Figure 5: Curves of the relative standard deviation (standard deviation normalized by the mean) of the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ obtained from 100 data sets. With each one, $\hat{\beta}_{\hat{m}(1)}, \hat{\beta}_{\hat{m}(1.5)}, \hat{\beta}_{\hat{m}(2)}, \hat{\beta}_{\hat{m}(2.5)}, \hat{\beta}_{\hat{m}(3)}, \hat{\beta}_{\hat{m}(3.5)}, \hat{\beta}_{\hat{m}(4)}, \hat{\beta}_{\hat{m}(4.5)}, \hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ are calculated given $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$, variance of the 100 $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ functions and then the relative standard deviation with respect to K. These plots are obtained with the *toy data set* described in Subsection 7.1 for $|\beta^*| = 10$.

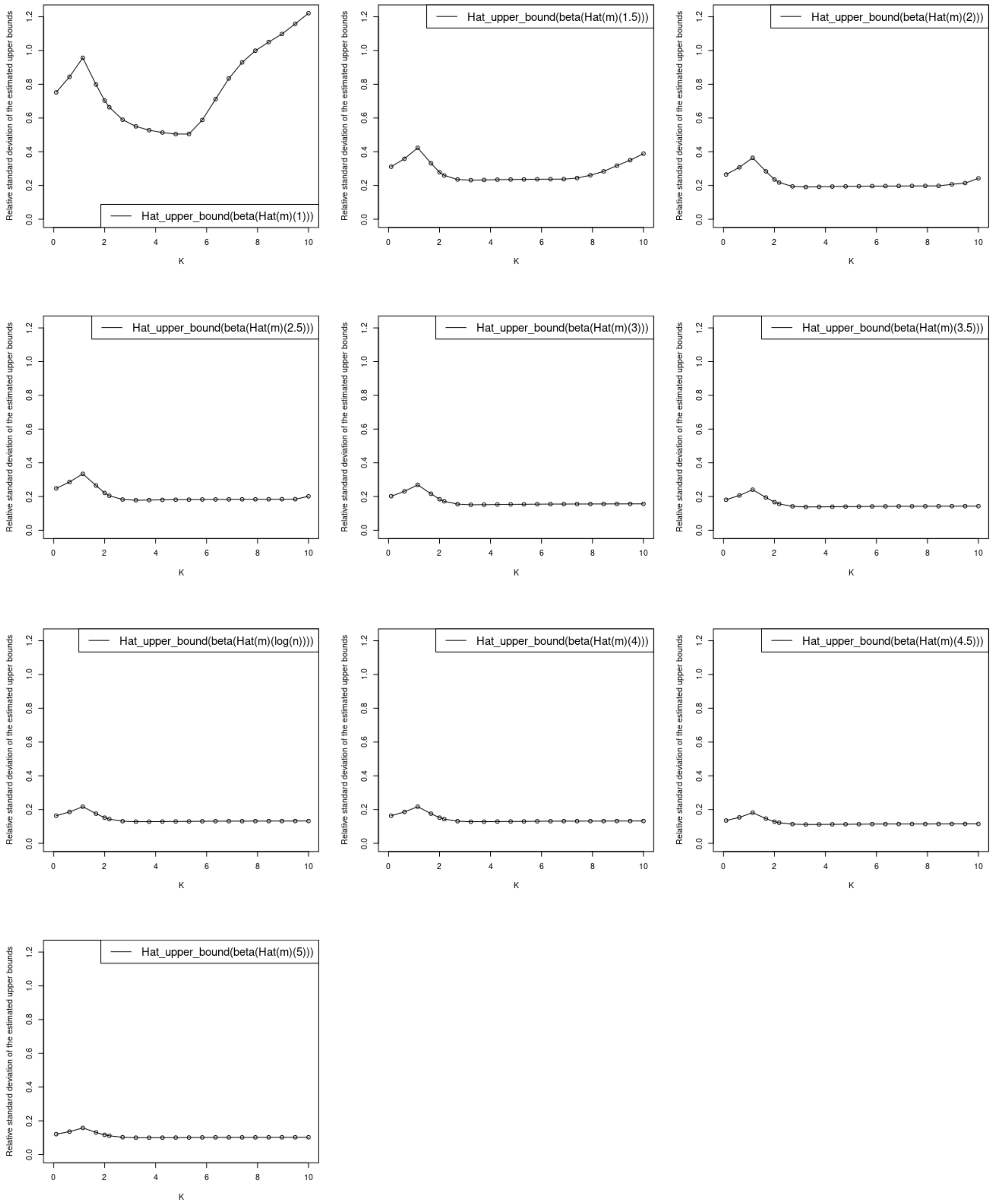
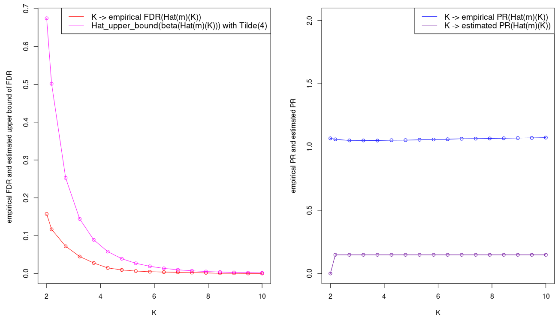
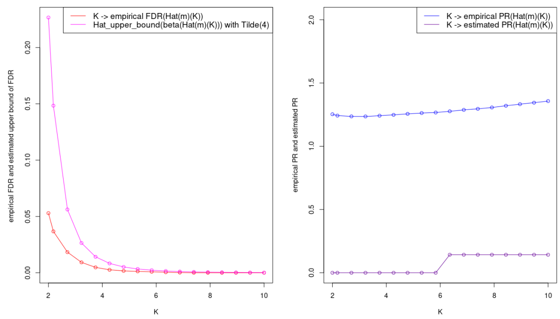


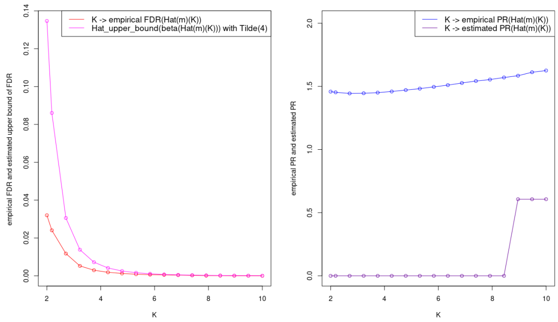
Figure 6: Curves of the relative standard deviation (standard deviation normalized by the mean) of the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ obtained from 100 data sets. With each one, $\hat{\beta}_{\hat{m}(1)}, \hat{\beta}_{\hat{m}(1.5)}, \hat{\beta}_{\hat{m}(2)}, \hat{\beta}_{\hat{m}(2.5)}, \hat{\beta}_{\hat{m}(3)}, \hat{\beta}_{\hat{m}(3.5)}, \hat{\beta}_{\hat{m}(4)}, \hat{\beta}_{\hat{m}(4.5)}, \hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ are calculated given $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$, variance of the 100 $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ functions and then the relative standard deviation with respect to K. These plots are obtained with the *toy data set* described in Subsection 7.1 for $|\beta^*| = 20$.



$|\beta^*| = 1$



$|\beta^*| = 10$



$|\beta^*| = 20$

Figure 7: Curves of the empirical estimation functions of $\mathrm{FDR}\big(\hat{m}(K)\big)$ and $\mathrm{PR}\big(\hat{m}(K)\big)$ for all $K > 0$ by using 1000 datasets and curves of the estimated risk (4.1) and the $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ function obtained in Corollary 3.7 by replacing β^* by $\hat{\beta}_{\hat{m}(4)}$. These two last plots are obtained from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $|\beta^*| = 1, 10, 20$.

2 Scenario (ii)

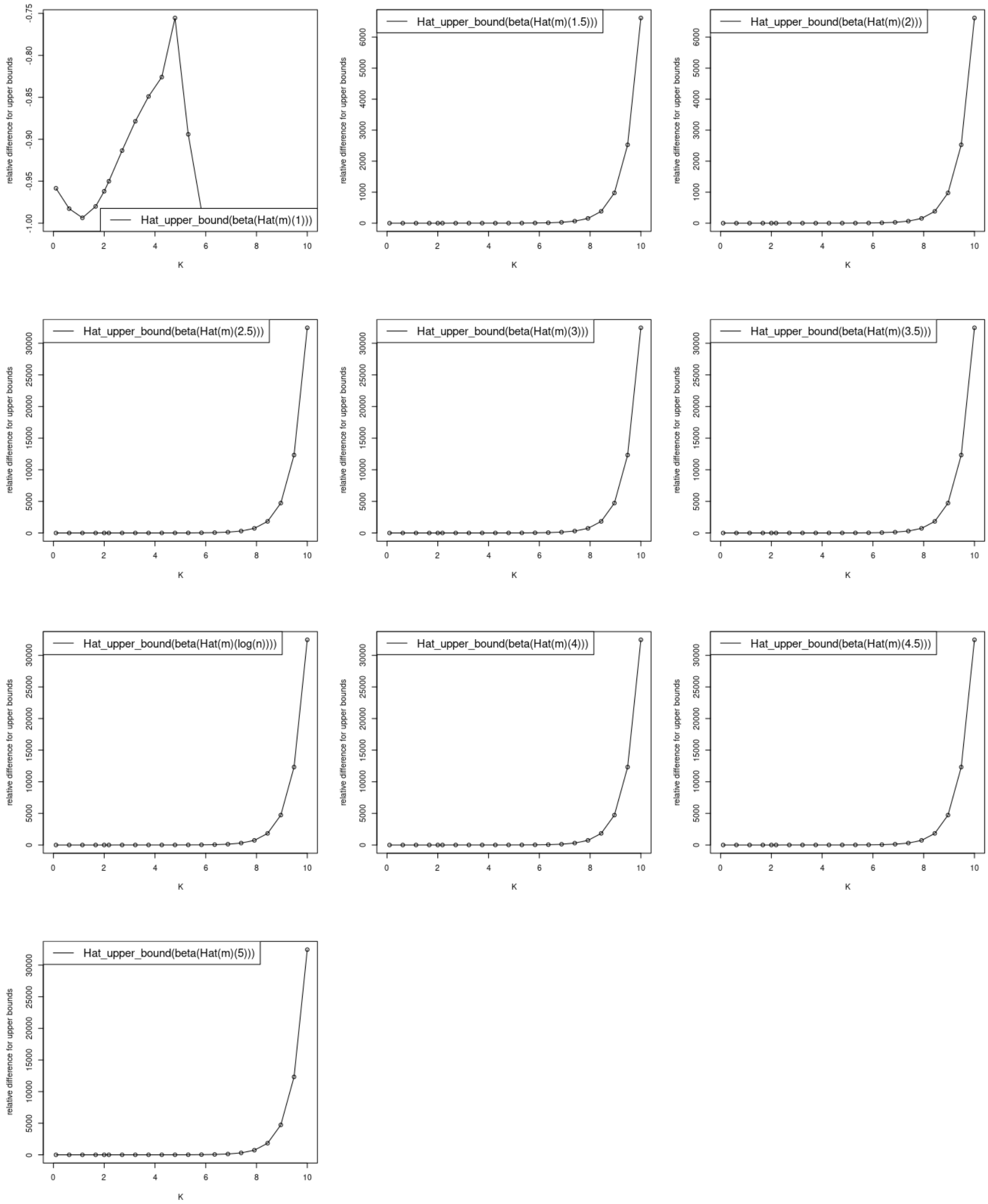


Figure 8: Curves of the relative change values between the functions $B(K,\beta^*,\sigma^2)$ and the functions $B(K,\hat{\beta}_{\hat{m}(\tilde{K})},\hat{\sigma}^2)$ with respectively $\hat{\beta}_{\hat{m}(1)}$, $\hat{\beta}_{\hat{m}(1.5)}$, $\hat{\beta}_{\hat{m}(2)}$, $\hat{\beta}_{\hat{m}(2.5)}$, $\hat{\beta}_{\hat{m}(3)}$, $\hat{\beta}_{\hat{m}(3.5)}$, $\hat{\beta}_{\hat{m}(4)}$, $\hat{\beta}_{\hat{m}(4.5)}$, $\hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ where estimators are calculating from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\beta^*_{10} = \frac{2}{10}$.

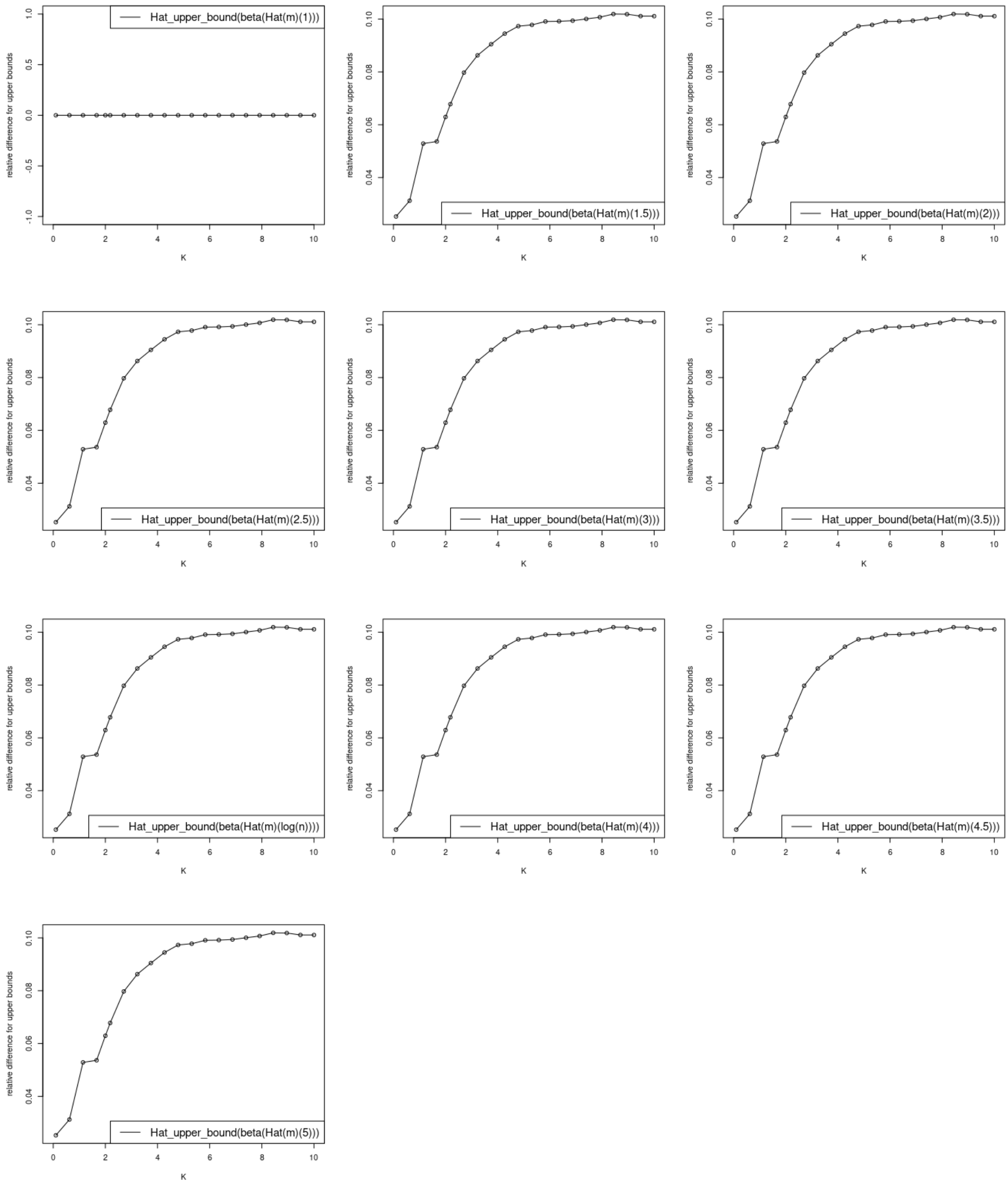


Figure 9: Curves of the relative change values between the functions $B(K, \beta^*, \sigma^2)$ and the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ with respectively $\hat{\beta}_{\hat{m}(1)}$, $\hat{\beta}_{\hat{m}(1.5)}$, $\hat{\beta}_{\hat{m}(2)}$, $\hat{\beta}_{\hat{m}(2.5)}$, $\hat{\beta}_{\hat{m}(3)}$, $\hat{\beta}_{\hat{m}(3.5)}$, $\hat{\beta}_{\hat{m}(4)}$, $\hat{\beta}_{\hat{m}(4.5)}$, $\hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ where estimators are calculating from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\beta^*_{10} = 2$ and distant coefficients.

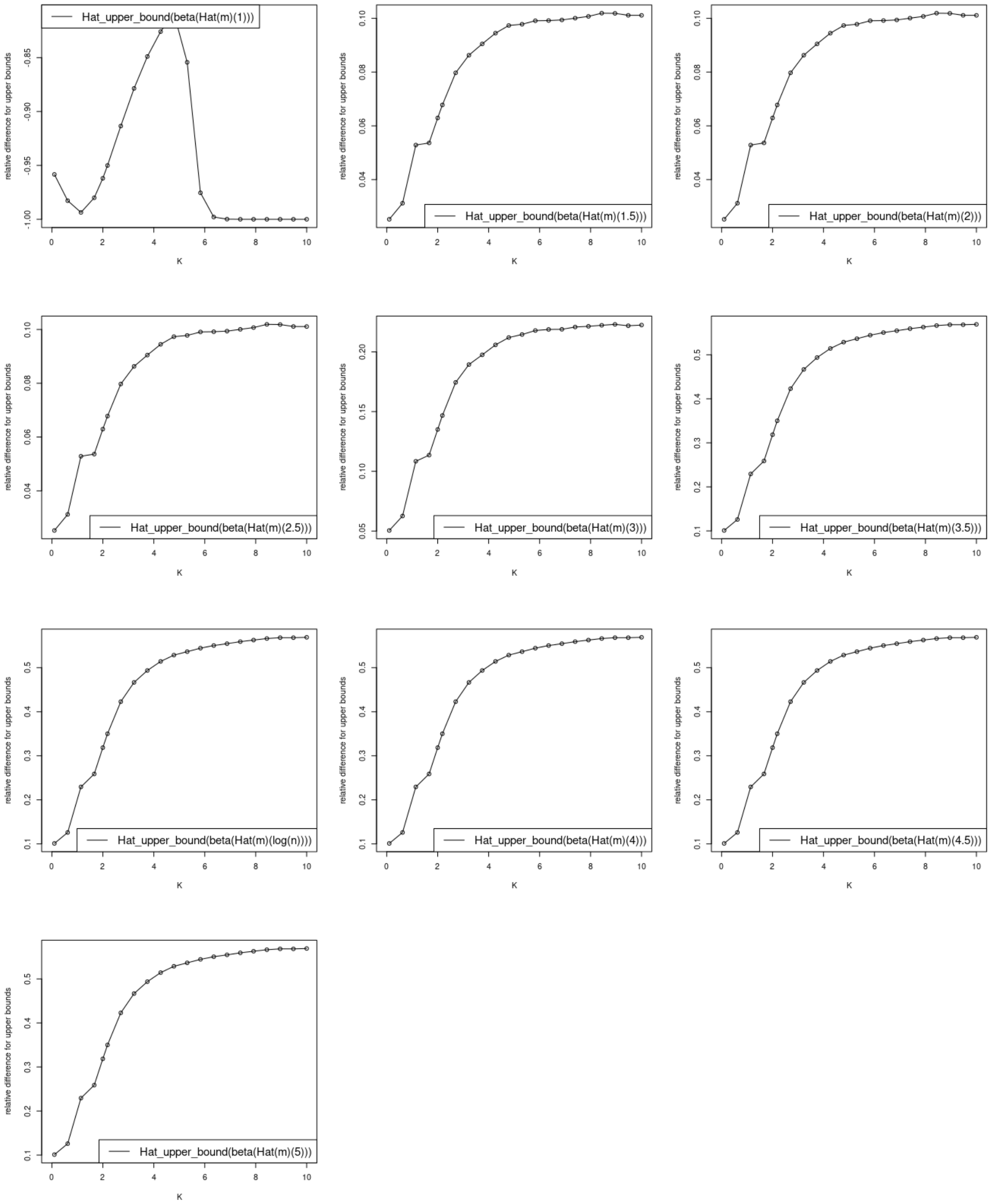


Figure 10: Curves of the relative change values between the functions $B(K, \beta^*, \sigma^2)$ and the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ with respectively $\hat{\beta}_{\hat{m}(1)}$, $\hat{\beta}_{\hat{m}(1.5)}$, $\hat{\beta}_{\hat{m}(2)}$, $\hat{\beta}_{\hat{m}(2.5)}$, $\hat{\beta}_{\hat{m}(3)}$, $\hat{\beta}_{\hat{m}(3.5)}$, $\hat{\beta}_{\hat{m}(4)}$, $\hat{\beta}_{\hat{m}(4.5)}$, $\hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ where estimators are calculating from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\beta^*_{10} = 2$ and close coefficients.

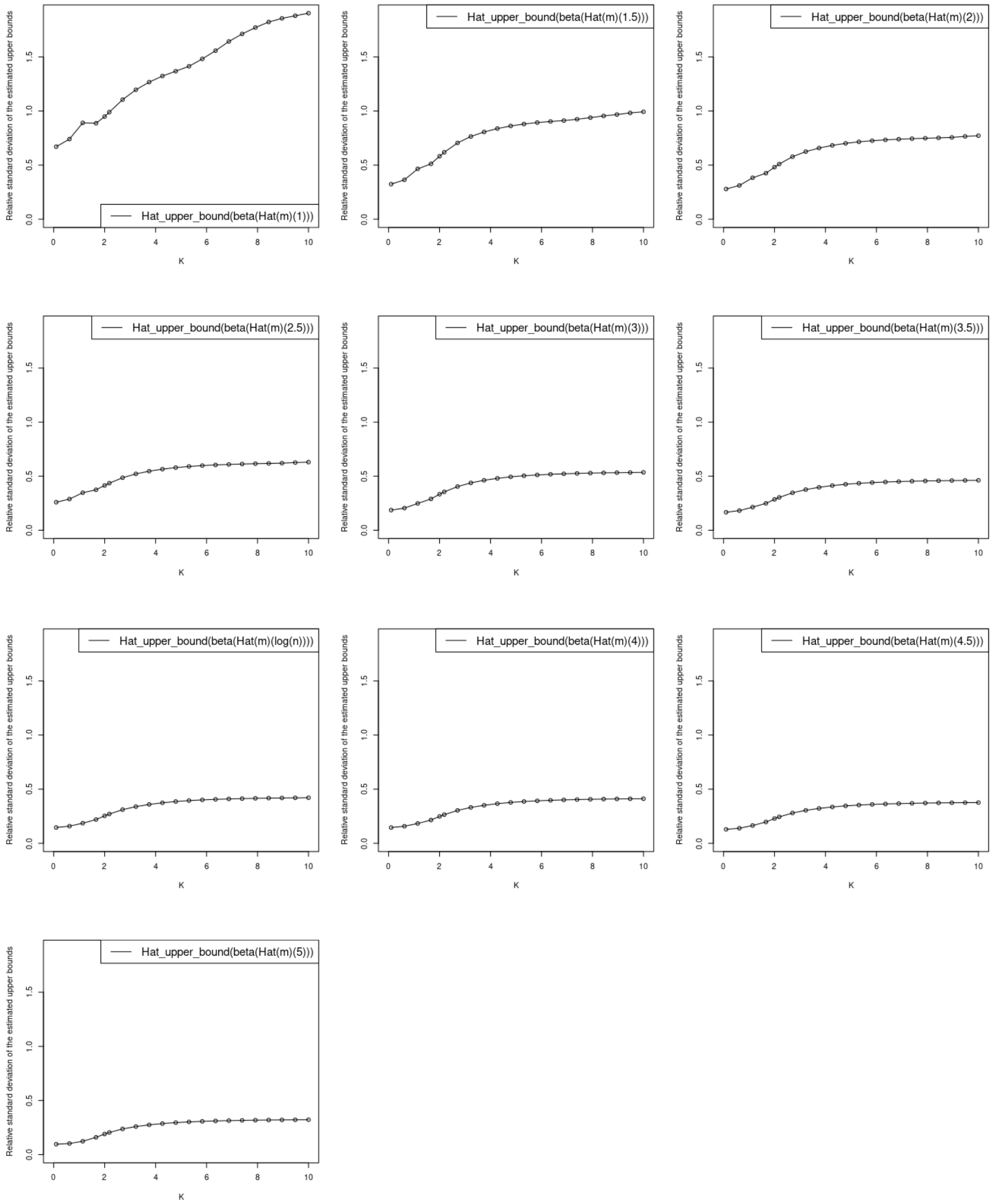


Figure 11: Curves of the relative standard deviation (standard deviation normalized by the mean) of the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ obtained from 100 data sets. With each one, $\hat{\beta}_{\hat{m}(1)}, \hat{\beta}_{\hat{m}(1.5)}, \hat{\beta}_{\hat{m}(2)}, \hat{\beta}_{\hat{m}(2.5)}, \hat{\beta}_{\hat{m}(3)}, \hat{\beta}_{\hat{m}(3.5)}, \hat{\beta}_{\hat{m}(4)}, \hat{\beta}_{\hat{m}(4.5)}, \hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ are calculated given $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$, variance of the 100 $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ functions and then the relative standard deviation with respect to K. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\beta^*_{10} = \frac{2}{10}$.

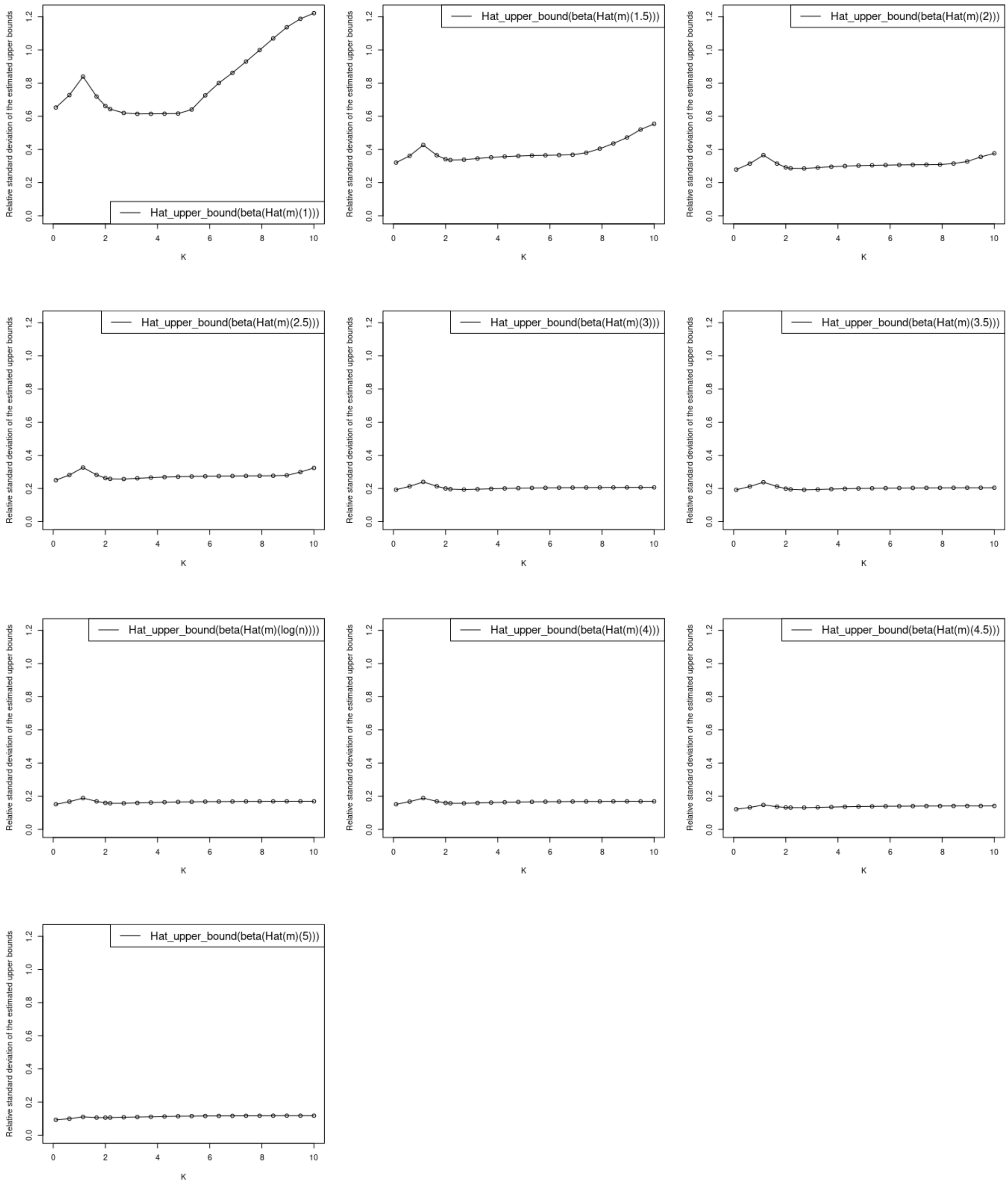


Figure 12: Curves of the relative standard deviation (standard deviation normalized by the mean) of the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ obtained from 100 data sets. With each one, $\hat{\beta}_{\hat{m}(1)}, \hat{\beta}_{\hat{m}(1.5)}, \hat{\beta}_{\hat{m}(2)}, \hat{\beta}_{\hat{m}(2.5)}, \hat{\beta}_{\hat{m}(3)}, \hat{\beta}_{\hat{m}(3.5)}, \hat{\beta}_{\hat{m}(4)}, \hat{\beta}_{\hat{m}(4.5)}, \hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ are calculated given $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$, variance of the 100 $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ functions and then the relative standard deviation with respect to K. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\beta^*_{10} = 2$ and distant coefficients.

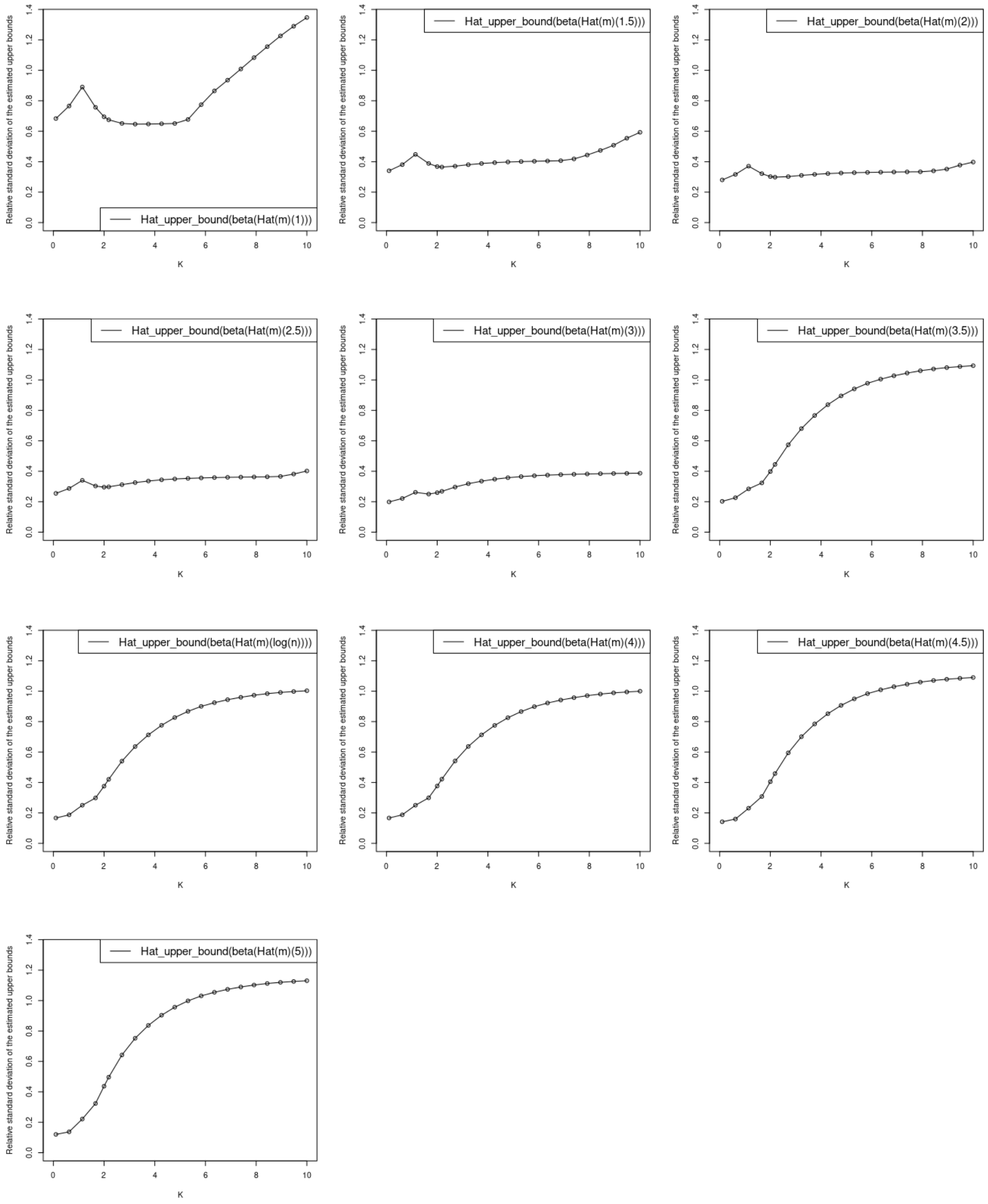
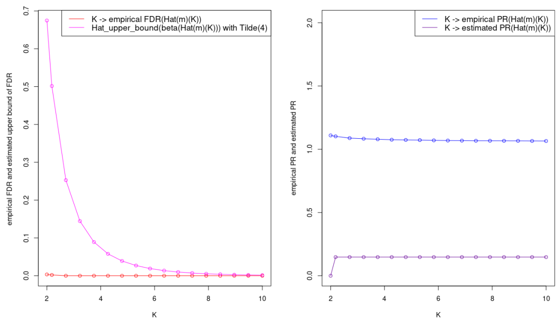
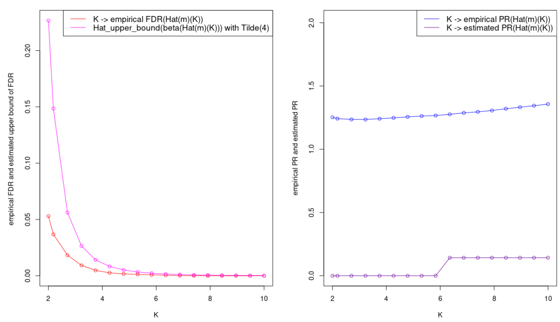


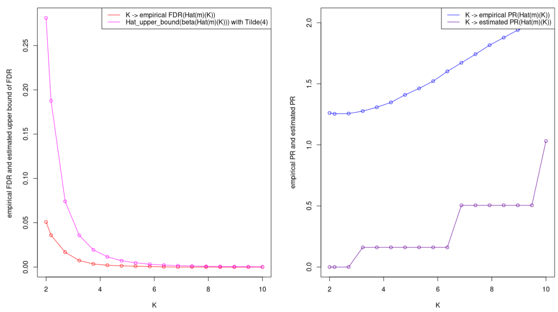
Figure 13: Curves of the relative standard deviation (standard deviation normalized by the mean) of the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ obtained from 100 data sets. With each one, $\hat{\beta}_{\hat{m}(1)}, \hat{\beta}_{\hat{m}(1.5)}, \hat{\beta}_{\hat{m}(2)}, \hat{\beta}_{\hat{m}(2.5)}, \hat{\beta}_{\hat{m}(3)}, \hat{\beta}_{\hat{m}(3.5)}, \hat{\beta}_{\hat{m}(4)}, \hat{\beta}_{\hat{m}(4.5)}, \hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ are calculated given $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$, variance of the 100 $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ functions and then the relative standard deviation with respect to K. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\beta^*_{10} = 2$ and close coefficients.



$\beta_{10}^* = \frac{2}{10}$



$\beta_{10}^* = 2$ and distant coefficients



$\beta_{10}^* = 2$ and close coefficients

Figure 14: Curves of the empirical estimation functions of $\text{FDR}\big(\hat{m}(K)\big)$ and $\text{PR}\big(\hat{m}(K)\big)$ for all $K > 0$ by using 1000 datasets and curves of the estimated risk (4.1) and the $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ function obtained in Corollary 3.7 by replacing β^* by $\hat{\beta}_{\hat{m}(4)}$. These two last plots are obtained from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\beta_{10}^* = \frac{2}{10}$, for $\beta_{10}^* = 2$ and distant coefficients, $\beta_{10}^* = 2$ and close coefficients.

3 Scenario (iii)

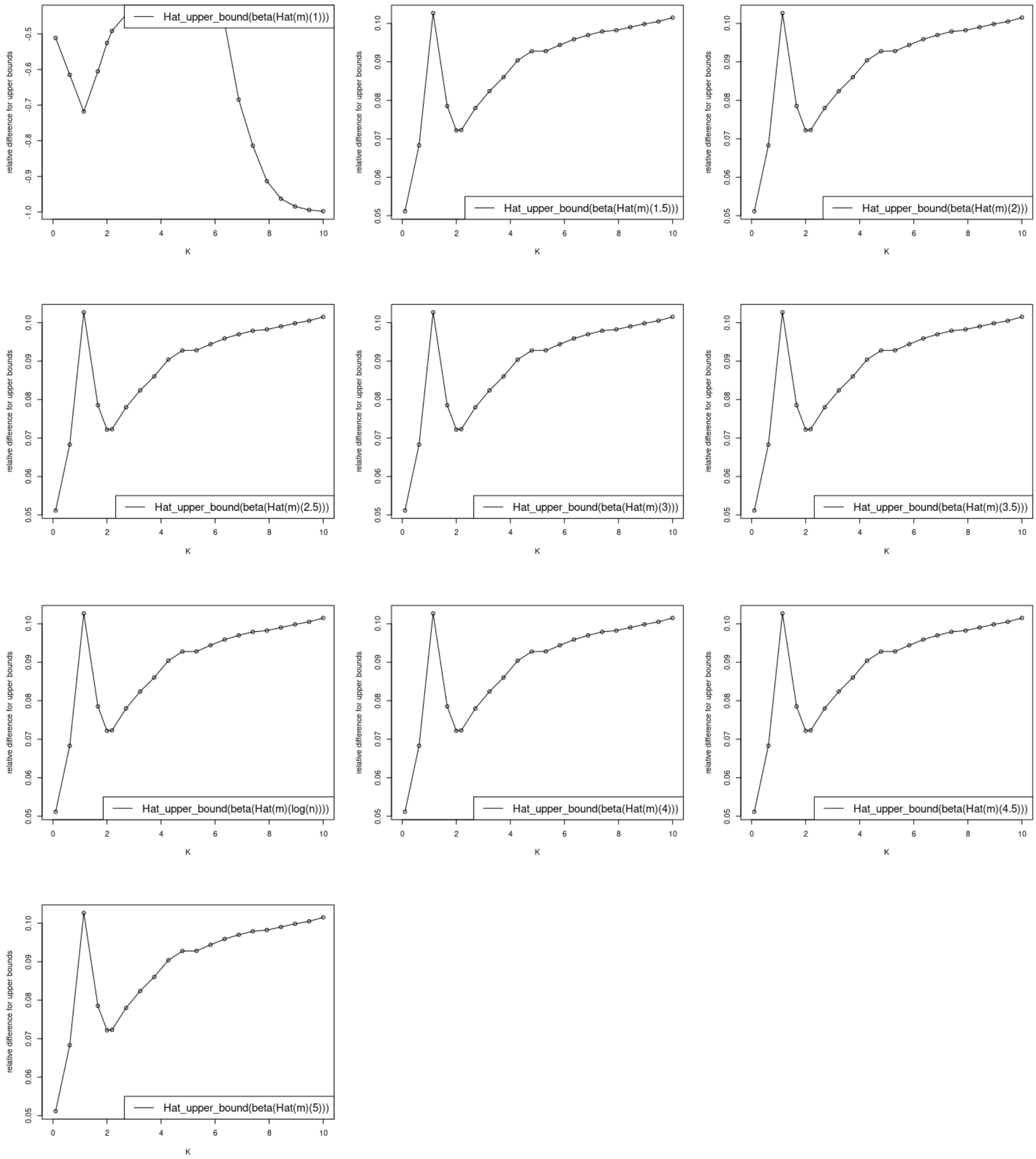


Figure 15: Curves of the relative change values between the functions $B(K, \beta^*, \sigma^2)$ and the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ with respectively $\hat{\beta}_{\hat{m}(1)}$, $\hat{\beta}_{\hat{m}(1.5)}$, $\hat{\beta}_{\hat{m}(2)}$, $\hat{\beta}_{\hat{m}(2.5)}$, $\hat{\beta}_{\hat{m}(3)}$, $\hat{\beta}_{\hat{m}(3.5)}$, $\hat{\beta}_{\hat{m}(4)}$, $\hat{\beta}_{\hat{m}(4.5)}$, $\hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ where estimators are calculating from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $n = 30$.

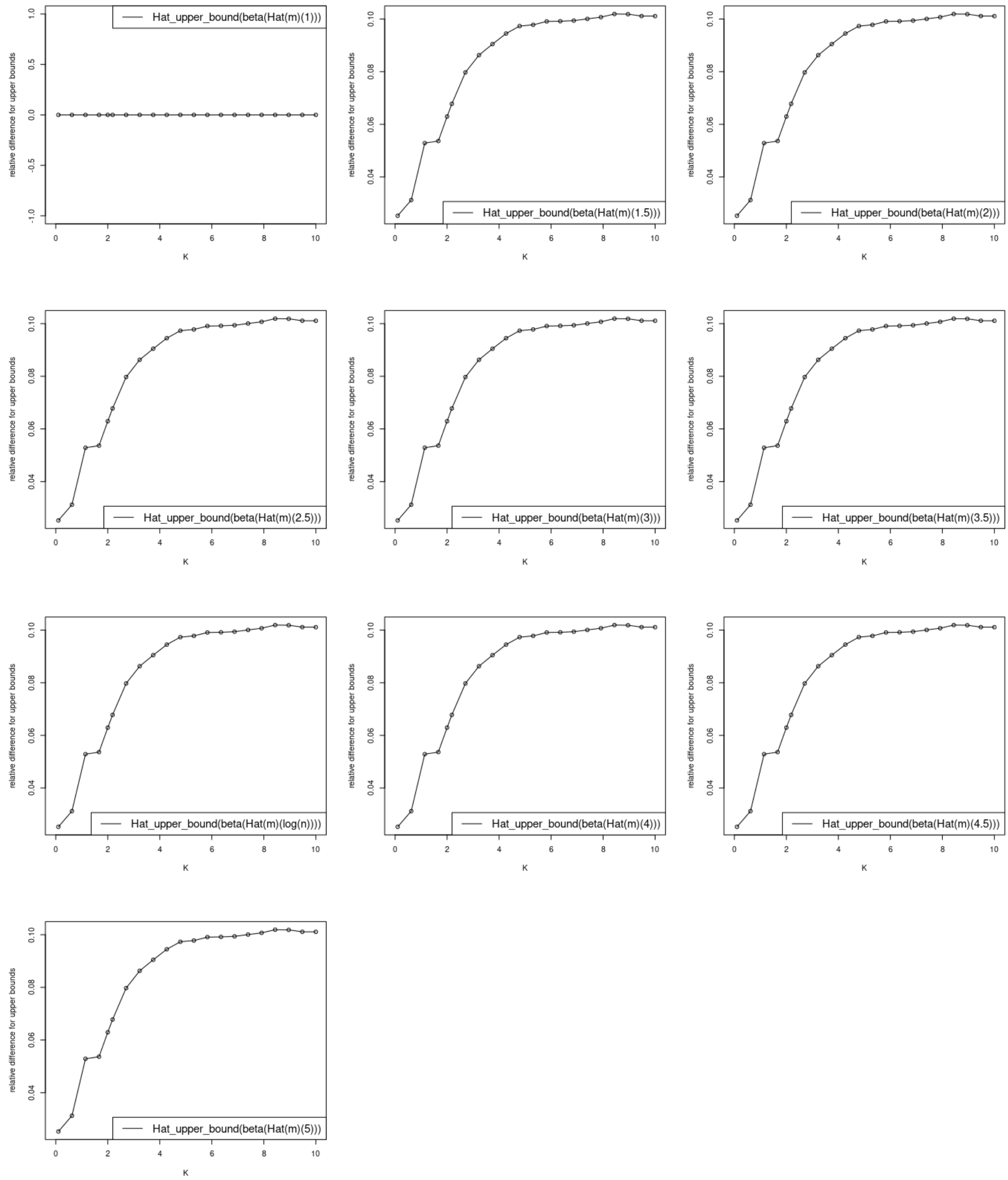


Figure 16: Curves of the relative change values between the functions $B(K, \beta^*, \sigma^2)$ and the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ with respectively $\hat{\beta}_{\hat{m}(1)}$, $\hat{\beta}_{\hat{m}(1.5)}$, $\hat{\beta}_{\hat{m}(2)}$, $\hat{\beta}_{\hat{m}(2.5)}$, $\hat{\beta}_{\hat{m}(3)}$, $\hat{\beta}_{\hat{m}(3.5)}$, $\hat{\beta}_{\hat{m}(4)}$, $\hat{\beta}_{\hat{m}(4.5)}$, $\hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ where estimators are calculating from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $n = 50$.

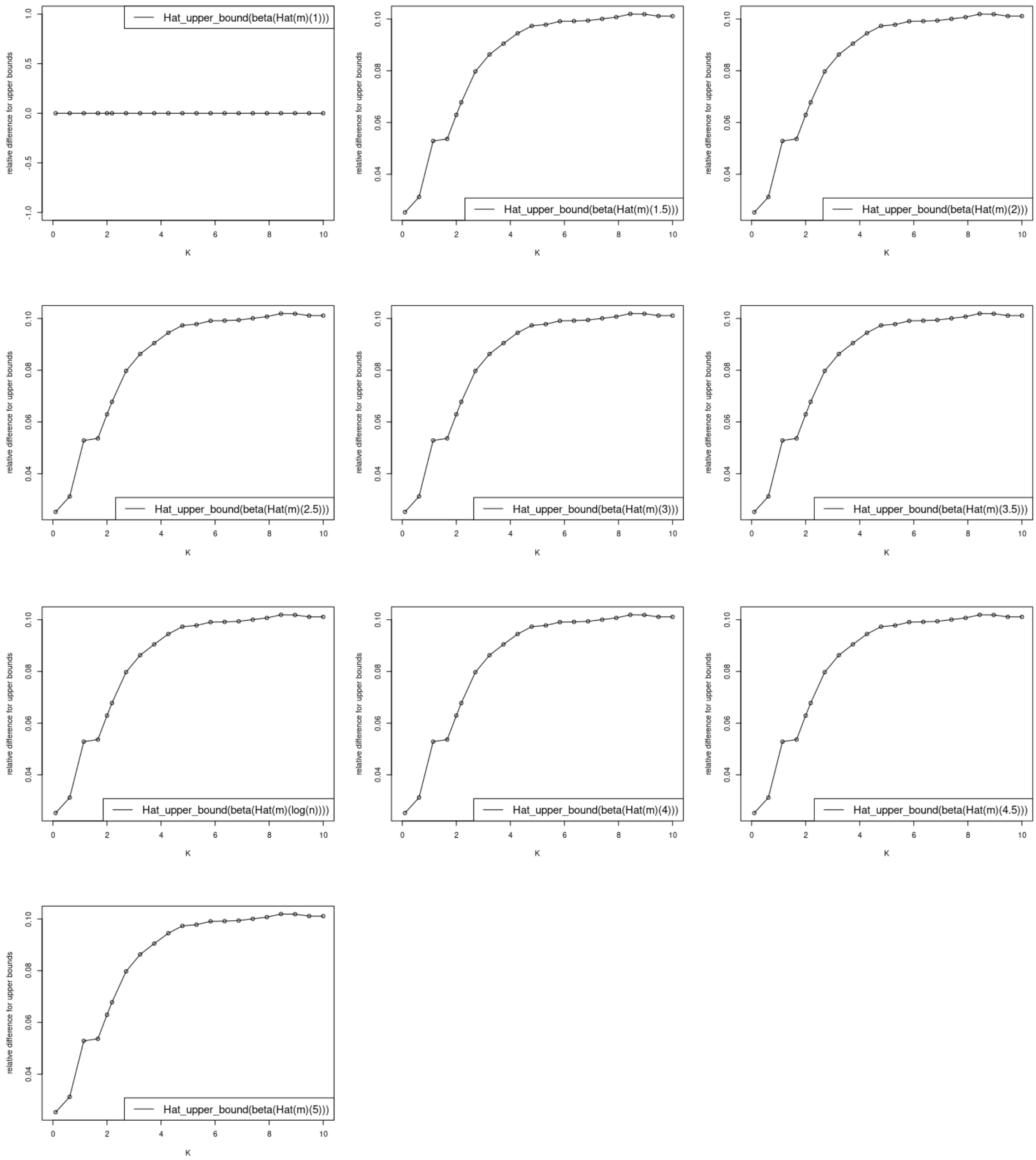


Figure 17: Curves of the relative change values between the functions $B(K, \beta^*, \sigma^2)$ and the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ with respectively $\hat{\beta}_{\hat{m}(1)}$, $\hat{\beta}_{\hat{m}(1.5)}$, $\hat{\beta}_{\hat{m}(2)}$, $\hat{\beta}_{\hat{m}(2.5)}$, $\hat{\beta}_{\hat{m}(3)}$, $\hat{\beta}_{\hat{m}(3.5)}$, $\hat{\beta}_{\hat{m}(4)}$, $\hat{\beta}_{\hat{m}(4.5)}$, $\hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ where estimators are calculating from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $n = 300$.

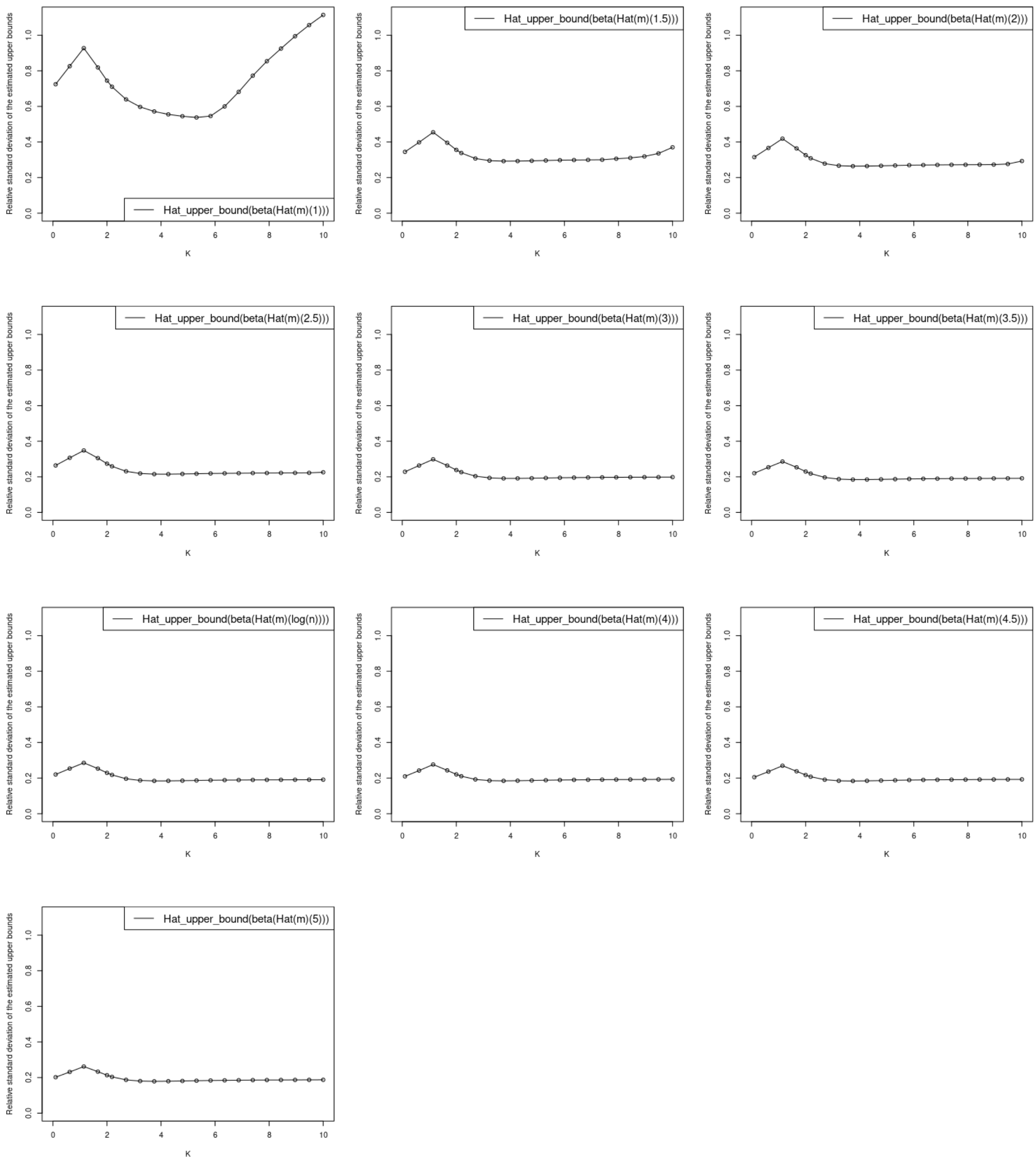


Figure 18: Curves of the relative standard deviation (standard deviation normalized by the mean) of the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ obtained from 100 data sets. With each one, $\hat{\beta}_{\hat{m}(1)}, \hat{\beta}_{\hat{m}(1.5)}, \hat{\beta}_{\hat{m}(2)}, \hat{\beta}_{\hat{m}(2.5)}, \hat{\beta}_{\hat{m}(3)}, \hat{\beta}_{\hat{m}(3.5)}, \hat{\beta}_{\hat{m}(4)}, \hat{\beta}_{\hat{m}(4.5)}, \hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ are calculated given $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$, variance of the 100 $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ functions and then the relative standard deviation with respect to K. These plots are obtained with the *toy data set* described in Subsection 7.1 for $n = 30$.

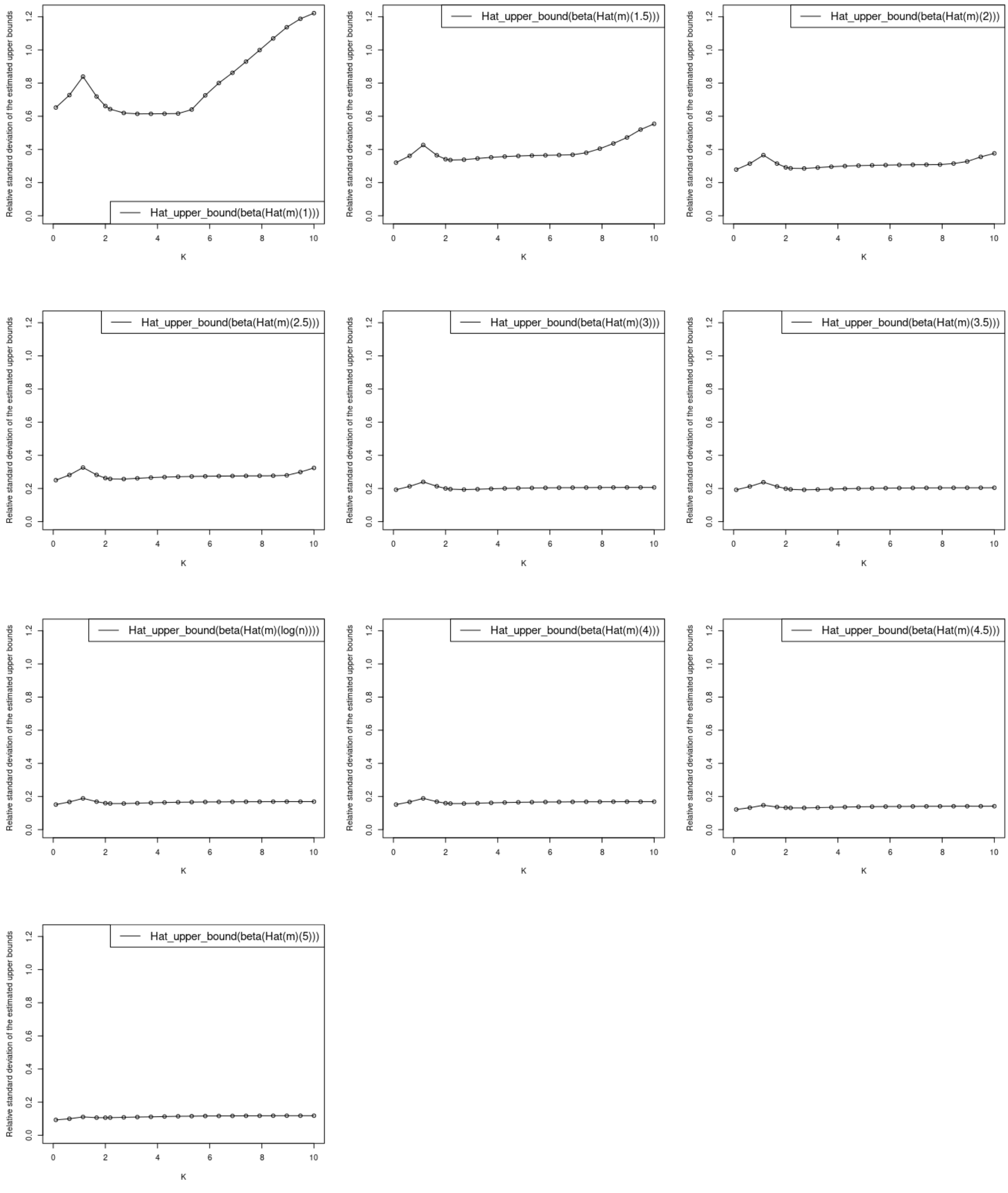


Figure 19: Curves of the relative standard deviation (standard deviation normalized by the mean) of the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ obtained from 100 data sets. With each one, $\hat{\beta}_{\hat{m}(1)}, \hat{\beta}_{\hat{m}(1.5)}, \hat{\beta}_{\hat{m}(2)}, \hat{\beta}_{\hat{m}(2.5)}, \hat{\beta}_{\hat{m}(3)}, \hat{\beta}_{\hat{m}(3.5)}, \hat{\beta}_{\hat{m}(4)}, \hat{\beta}_{\hat{m}(4.5)}, \hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ are calculated given $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$, variance of the 100 $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ functions and then the relative standard deviation with respect to K. These plots are obtained with the *toy data set* described in Subsection 7.1 for $n = 50$.

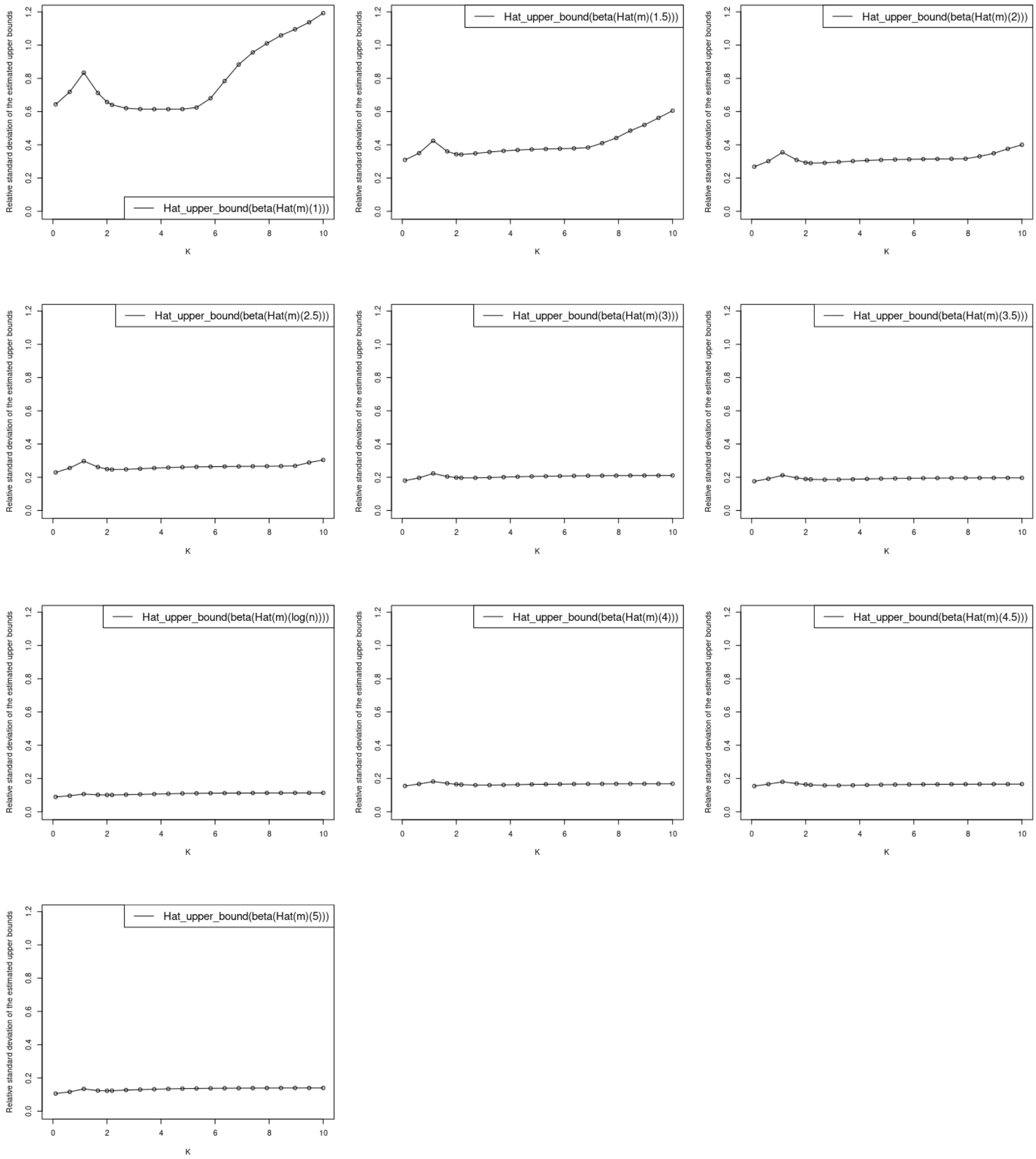
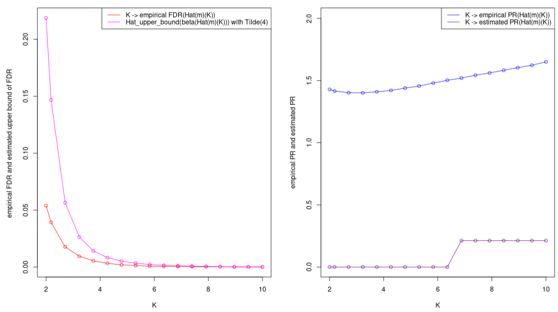
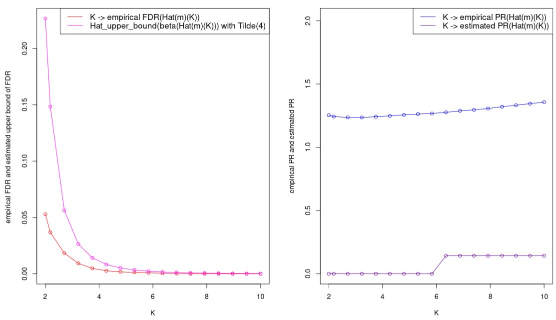


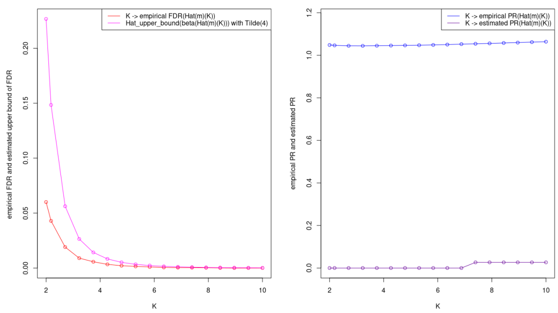
Figure 20: Curves of the relative standard deviation (standard deviation normalized by the mean) of the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ obtained from 100 data sets. With each one, $\hat{\beta}_{\hat{m}(1)}, \hat{\beta}_{\hat{m}(1.5)}, \hat{\beta}_{\hat{m}(2)}, \hat{\beta}_{\hat{m}(2.5)}, \hat{\beta}_{\hat{m}(3)}, \hat{\beta}_{\hat{m}(3.5)}, \hat{\beta}_{\hat{m}(4)}, \hat{\beta}_{\hat{m}(4.5)}, \hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ are calculated given $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$, variance of the 100 $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ functions and then the relative standard deviation with respect to K. These plots are obtained with the *toy data set* described in Subsection 7.1 for $n = 300$.



$n = 30$



$n = 50$



$n = 300$

Figure 21: Curves of the empirical estimation functions of $\mathrm{FDR}\big(\hat{m}(K)\big)$ and $\mathrm{PR}\big(\hat{m}(K)\big)$ for all $K > 0$ by using 1000 datasets and curves of the estimated risk (4.1) and the $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ function obtained in Corollary 3.7 by replacing β^* by $\hat{\beta}_{\hat{m}(4)}$. These two last plots are obtained from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $n = 30, 50, 300$.

4 Scenario (iv)

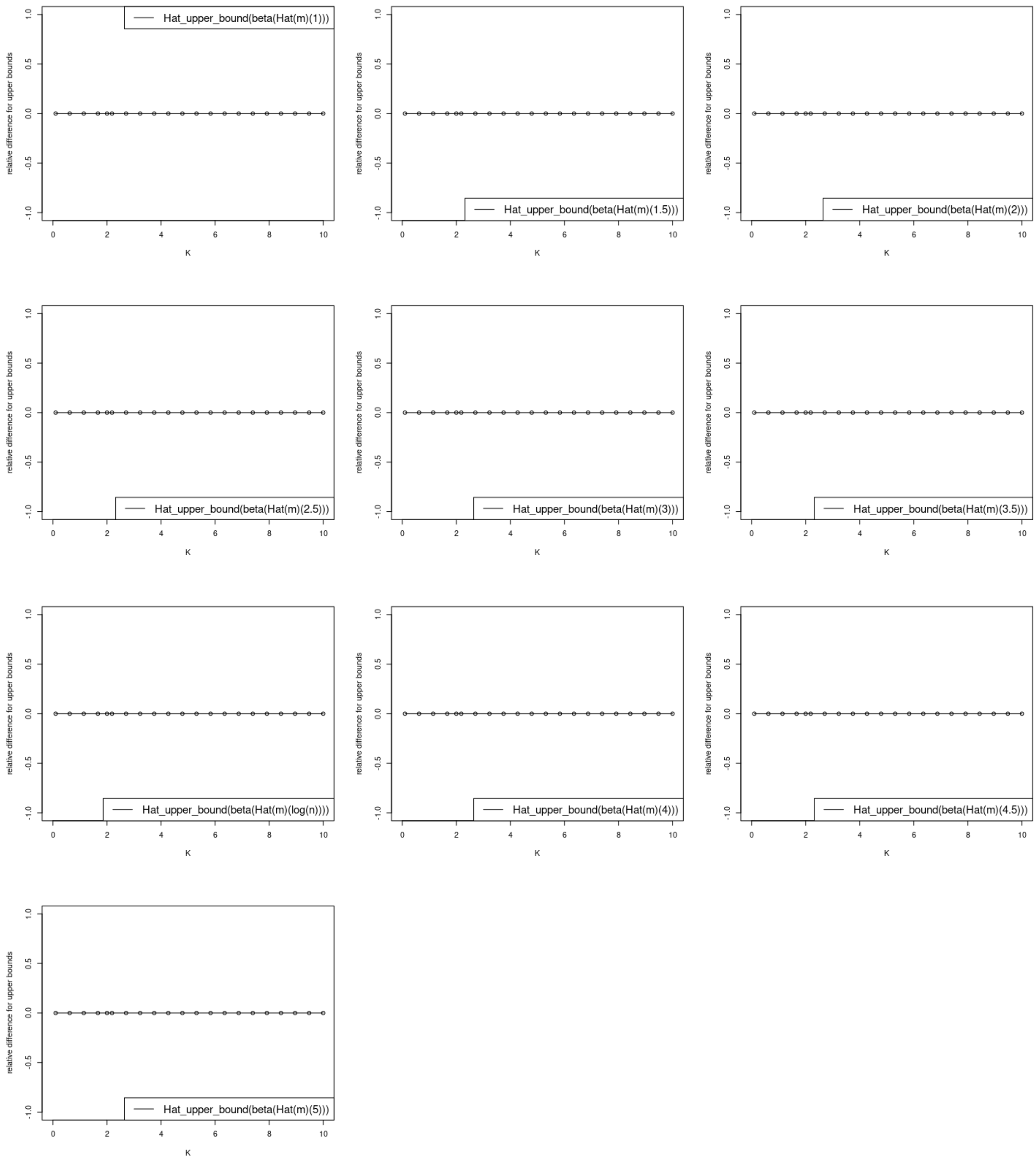


Figure 22: Curves of the relative change values between the functions $B(K, \beta^*, \sigma^2)$ and the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ with respectively $\hat{\beta}_{\hat{m}(1)}$, $\hat{\beta}_{\hat{m}(1.5)}$, $\hat{\beta}_{\hat{m}(2)}$, $\hat{\beta}_{\hat{m}(2.5)}$, $\hat{\beta}_{\hat{m}(3)}$, $\hat{\beta}_{\hat{m}(3.5)}$, $\hat{\beta}_{\hat{m}(4)}$, $\hat{\beta}_{\hat{m}(4.5)}$, $\hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ where estimators are calculating from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\sigma^2 = 0.1$.

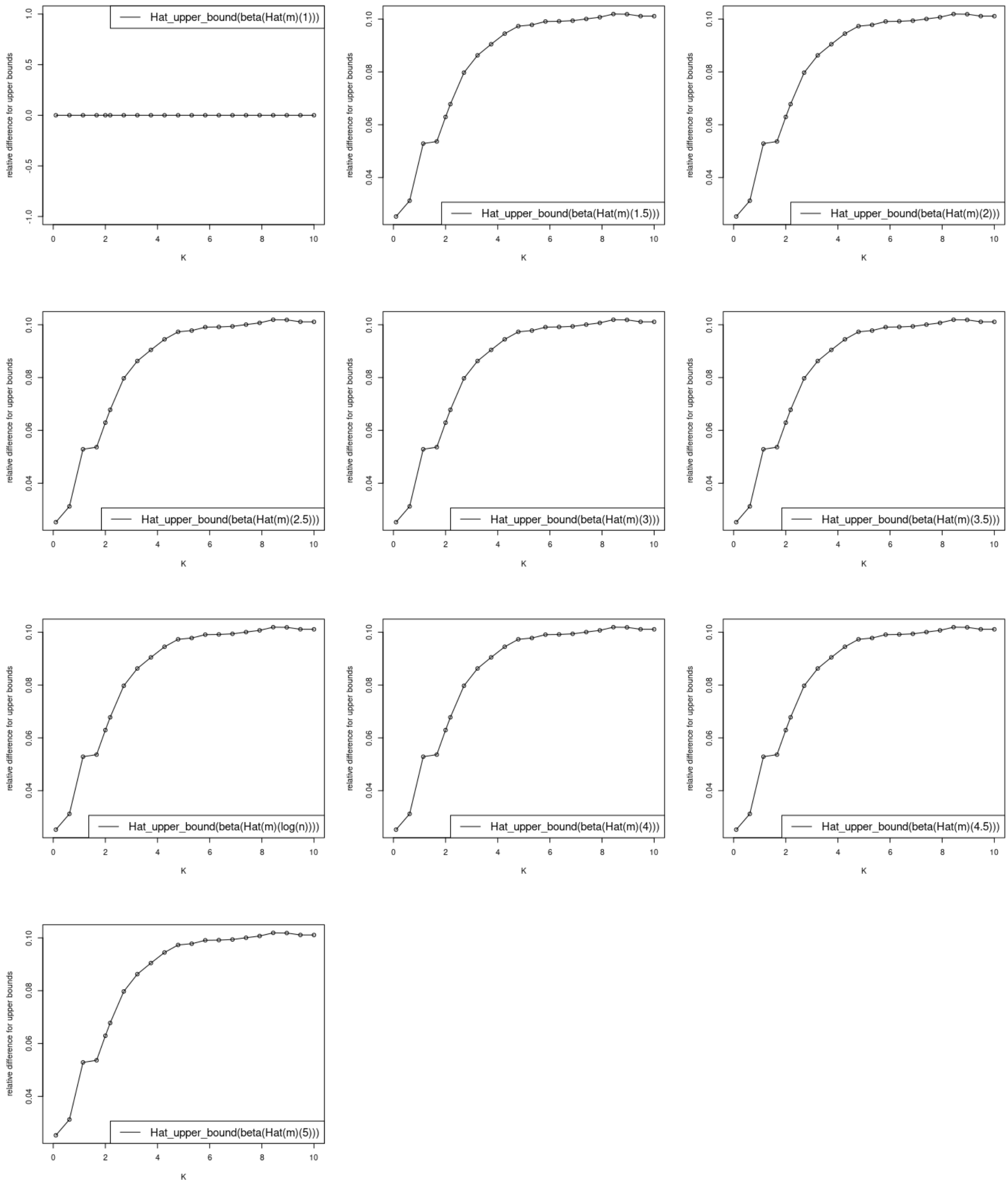


Figure 23: Curves of the relative change values between the functions $B(K, \beta^*, \sigma^2)$ and the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ with respectively $\hat{\beta}_{\hat{m}(1)}$, $\hat{\beta}_{\hat{m}(1.5)}$, $\hat{\beta}_{\hat{m}(2)}$, $\hat{\beta}_{\hat{m}(2.5)}$, $\hat{\beta}_{\hat{m}(3)}$, $\hat{\beta}_{\hat{m}(3.5)}$, $\hat{\beta}_{\hat{m}(4)}$, $\hat{\beta}_{\hat{m}(4.5)}$, $\hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ where estimators are calculating from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\sigma^2 = 1$.

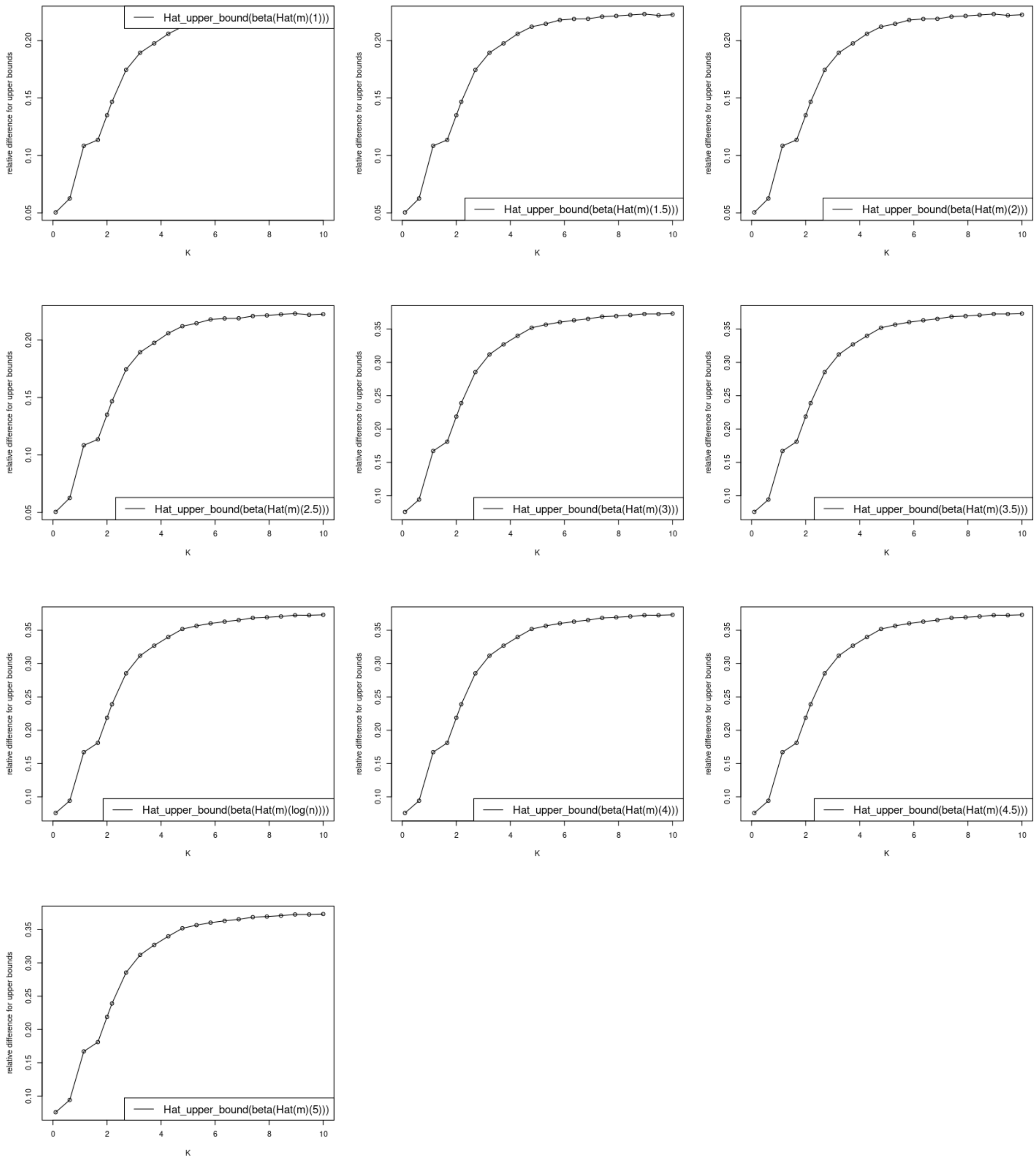


Figure 24: Curves of the relative change values between the functions $B(K, \beta^*, \sigma^2)$ and the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ with respectively $\hat{\beta}_{\hat{m}(1)}$, $\hat{\beta}_{\hat{m}(1.5)}$, $\hat{\beta}_{\hat{m}(2)}$, $\hat{\beta}_{\hat{m}(2.5)}$, $\hat{\beta}_{\hat{m}(3)}$, $\hat{\beta}_{\hat{m}(3.5)}$, $\hat{\beta}_{\hat{m}(4)}$, $\hat{\beta}_{\hat{m}(4.5)}$, $\hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ where estimators are calculating from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\sigma^2 = 4$.

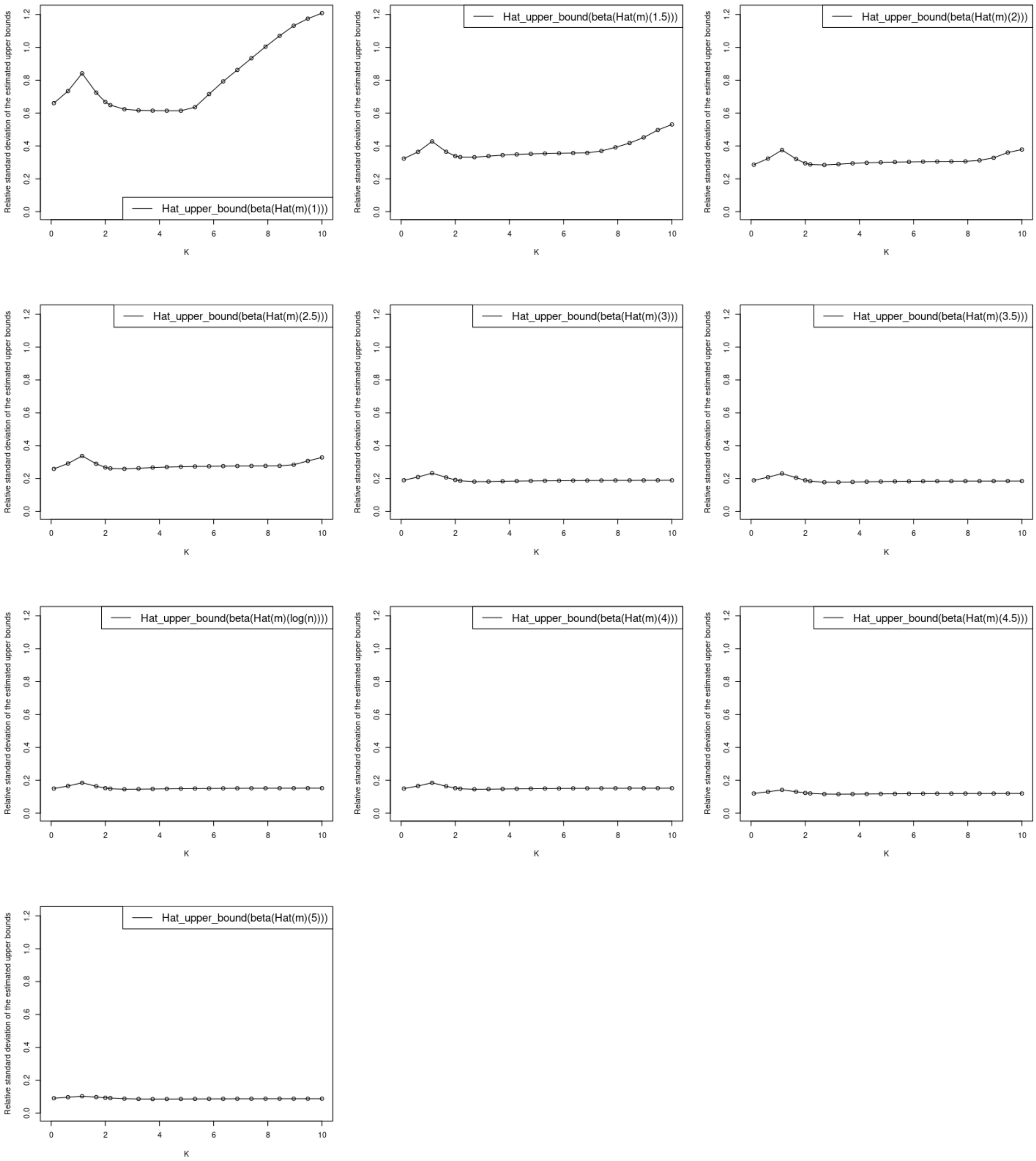


Figure 25: Curves of the relative standard deviation (standard deviation normalized by the mean) of the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ obtained from 100 data sets. With each one, $\hat{\beta}_{\hat{m}(1)}, \hat{\beta}_{\hat{m}(1.5)}, \hat{\beta}_{\hat{m}(2)}, \hat{\beta}_{\hat{m}(2.5)}, \hat{\beta}_{\hat{m}(3)}, \hat{\beta}_{\hat{m}(3.5)}, \hat{\beta}_{\hat{m}(4)}, \hat{\beta}_{\hat{m}(4.5)}, \hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ are calculated given $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$, variance of the 100 $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ functions and then the relative standard deviation with respect to K. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\sigma^2 = 0.1$.

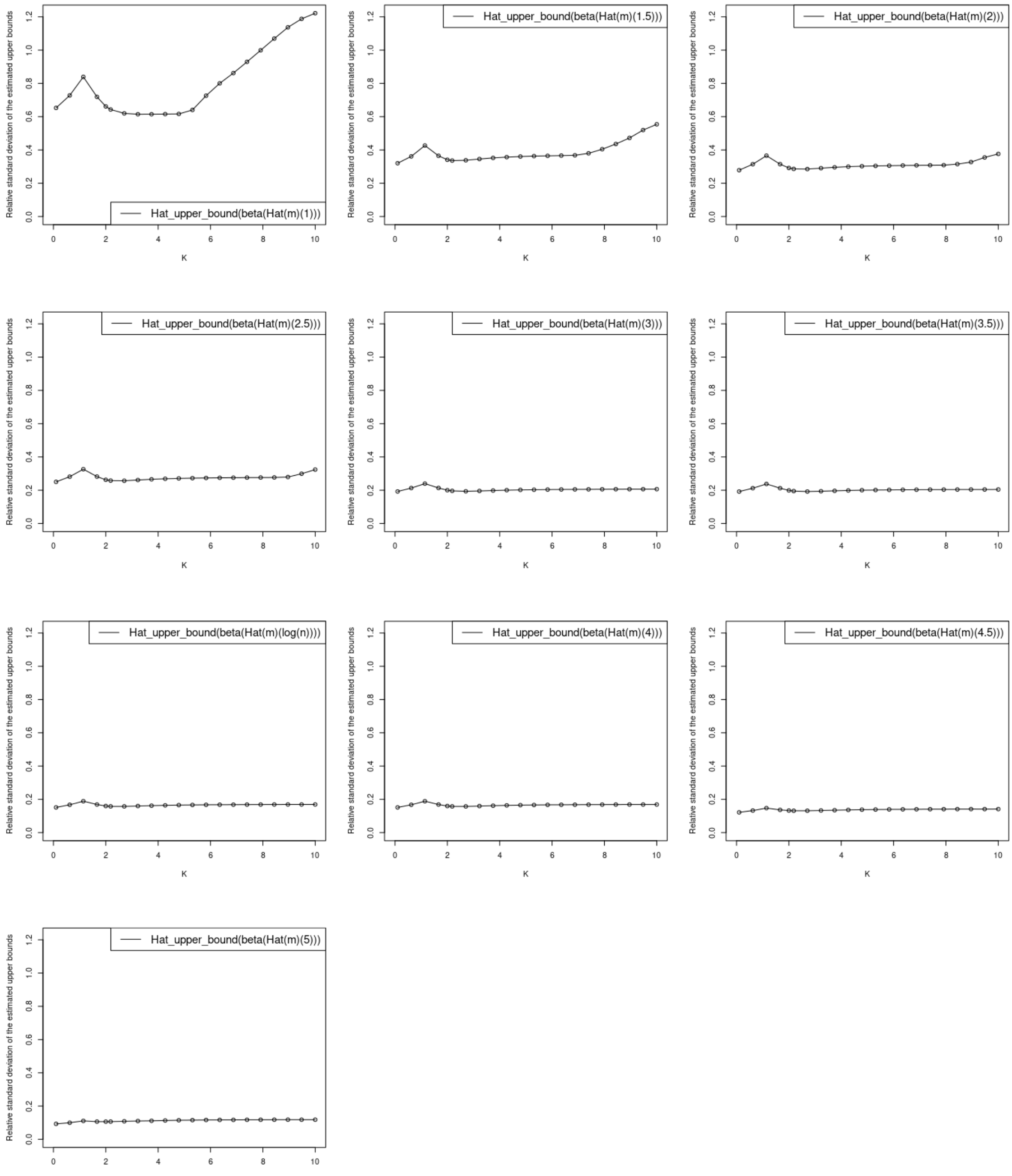


Figure 26: Curves of the relative standard deviation (standard deviation normalized by the mean) of the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ obtained from 100 data sets. With each one, $\hat{\beta}_{\hat{m}(1)}, \hat{\beta}_{\hat{m}(1.5)}, \hat{\beta}_{\hat{m}(2)}, \hat{\beta}_{\hat{m}(2.5)}, \hat{\beta}_{\hat{m}(3)}, \hat{\beta}_{\hat{m}(3.5)}, \hat{\beta}_{\hat{m}(4)}, \hat{\beta}_{\hat{m}(4.5)}, \hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ are calculated given $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$, variance of the 100 $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ functions and then the relative standard deviation with respect to K. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\sigma^2 = 1$.

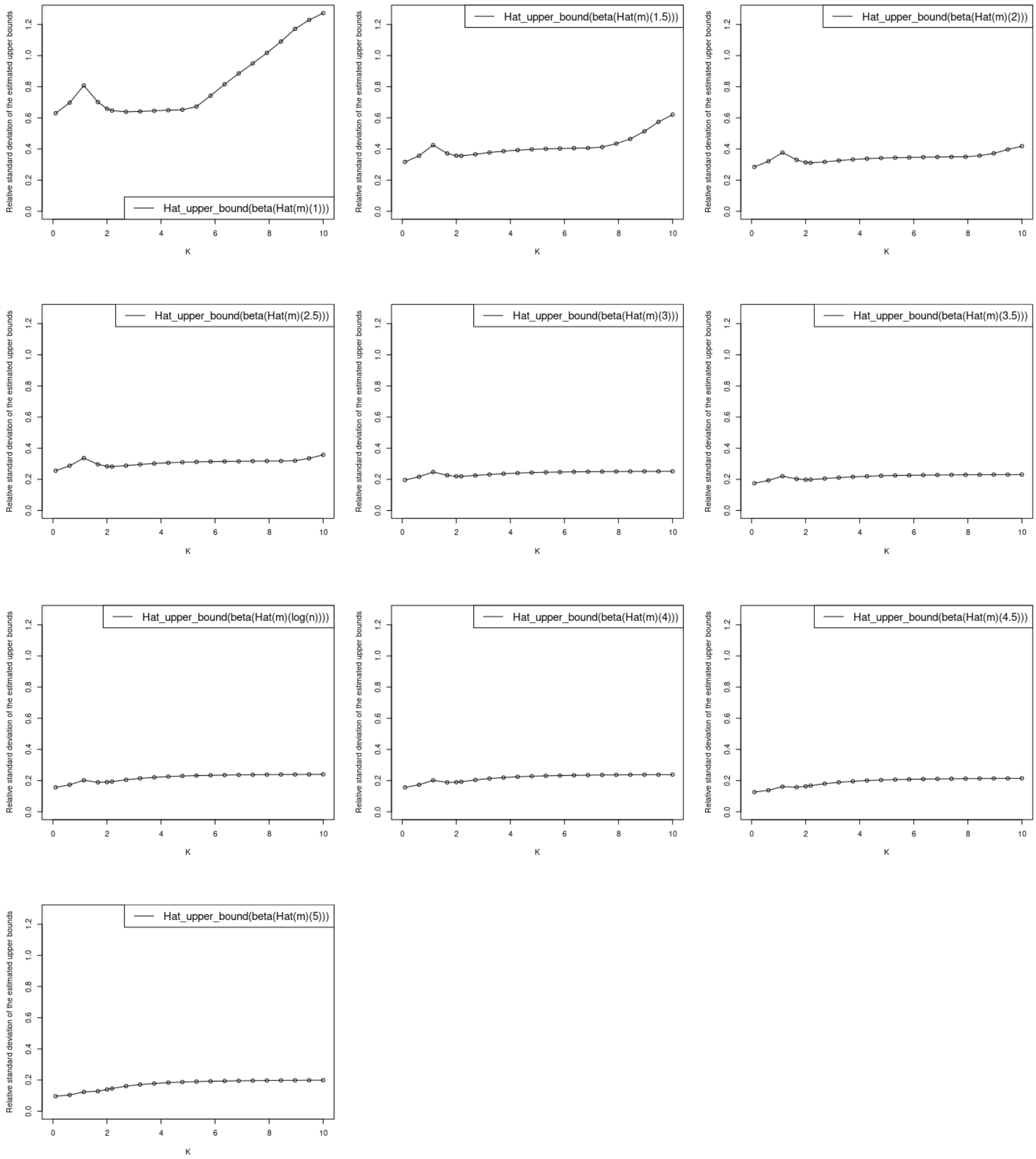
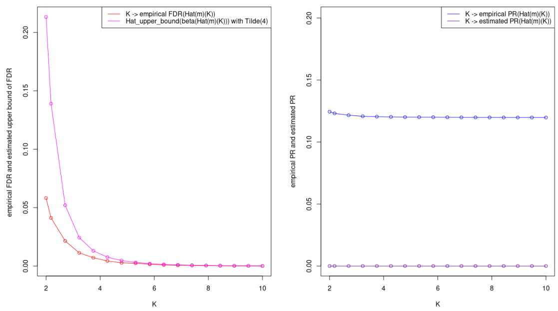
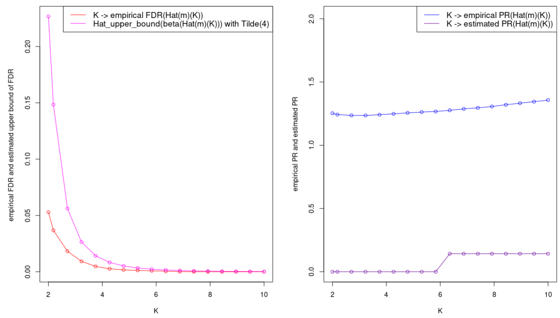


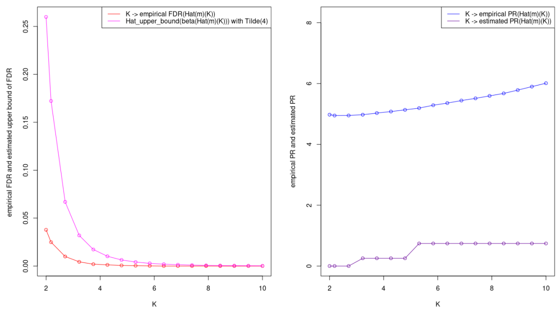
Figure 27: Curves of the relative standard deviation (standard deviation normalized by the mean) of the functions $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ obtained from 100 data sets. With each one, $\hat{\beta}_{\hat{m}(1)}, \hat{\beta}_{\hat{m}(1.5)}, \hat{\beta}_{\hat{m}(2)}, \hat{\beta}_{\hat{m}(2.5)}, \hat{\beta}_{\hat{m}(3)}, \hat{\beta}_{\hat{m}(3.5)}, \hat{\beta}_{\hat{m}(4)}, \hat{\beta}_{\hat{m}(4.5)}, \hat{\beta}_{\hat{m}(5)}$ and $\hat{\beta}_{\hat{m}(\log(n))}$ are calculated given $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$, variance of the 100 $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ functions and then the relative standard deviation with respect to K. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\sigma^2 = 4$.



$\sigma^2 = 0.1$



$\sigma^2 = 1$



$\sigma^2 = 4$

Figure 28: Curves of the empirical estimation functions of $\mathrm{FDR}\big(\hat{m}(K)\big)$ and $\mathrm{PR}\big(\hat{m}(K)\big)$ for all $K > 0$ by using 1000 datasets and curves of the estimated risk (4.1) and the $B(K, \hat{\beta}_{\hat{m}(\tilde{K})}, \hat{\sigma}^2)$ function obtained in Corollary 3.7 by replacing β^* by $\hat{\beta}_{\hat{m}(4)}$. These two last plots are obtained from only one dataset. These plots are obtained with the *toy data set* described in Subsection 7.1 for $\sigma^2 = 0.1, 1, 4$.