Problem: M/J 2. Another ‘‘logical hat" problem from Dick Hess. Two logicians, A and B, are each wearing a hat with a number affixed. The product x × y is written on A’s hat and the sum x + y on B’s, for not necessarily distinct positive integers x and y. Each logician sees the other’s hat but not his own. Each is error-free in reasoning and knows the situation. They speak in turn.

A: There is no way you can know the number on your hat.
B: I don’t know my number.
A: I don’t know my number.
B: I now know my number.

What numbers are on A’s and B’s hats?

My Solution: x = 1, y = 6. So, A has 6 on his head, B has 7.

Work: I reason through the information that each person has after each statement is made.

A: There is no way you can know the number on your hat.

- B doesn’t know the number on his hat. This means x + y cannot be deduced from x * y. x + y can be deduced from x * y when x * y is not composite. So, x * y is composite. Also, A knows this. This is true when x + y is 1 more than a composite.

A knows: x + y
B knows: x * y
Common knowledge:
(a) x * y is composite
(b) x + y - 1 is composite

B: I don’t know my number.

- Since B does not know his number, x * y and x + y - 1 being composite cannot pick out x + y uniquely. Observe that when x * y is composite, if x = 1 (WLOG: letting x <= y), then x + y - 1 = y = x * y is composite. So, for the knowledge available to B to not be enough to know x + y, at least one of the *nontrivial* factorizations of x * y into (j, k) must satisfy j + k - 1 being composite (otherwise only the trivial factorization would work, and B would know his number). This is sufficient for B to not be sure if x = 1 or x = j.

A knows: x + y
B knows: x * y
Common knowledge:
(a) x * y is composite
(b) x + y - 1 is composite

(c) there exists 1 < j <= k such that j * k = x * y and j + k - 1 is composite

Letting F be the set of numbers which have a non-trivial factorization into q and r such that q + r - 1 is composite, and observing that (c) => (a), lets us simplify to:

A knows: x + y
B knows: x * y
Common knowledge:
(a) x * y is in F
(b) x + y - 1 is composite

A: I don’t know my number.

- Since A doesn’t know his number, x + y and x * y is in F doesn’t pick out x * y uniquely. So there must be another pair of numbers (m, n), besides (x, y), which sum to x + y, such that m * n is in F.

A knows: x + y
B knows: x * y
Common knowledge:
(a) x * y is in F
(b) x + y - 1 is composite
(c) There is another pair (m, n) different than (x, y) such that m + n = x + y and m * n is in F

B: I know my number

- B knows x * y (which we know is in F by (a)), that x + y - 1 is composite (b), and that there is another pair (m, n) which sums to x + y such that m * n is in F (c). This is enough for B to know his number. So we want the (hopefully unique) member of F which has only one factoring into x, y such that x + y - 1 is composite and there’s another pair with the same sum whose product is in F.

At this point, I don’t know of any way forward other than writing code to look for numbers with this set of properties. I ultimately find (1, 6), and that no other pair of numbers (x, y) such that x * y <= 10,000 works, so the answer of A having 6 and B having 7 seems to be unique.

We can run through the logic for the pair (1, 6) to verify it works.

1. A sees 7. Since 7 - 1 = 6 is composite, A knows that B sees a composite number, and so B cannot know his number.
2. B sees 6. Possible pairs are (1, 6) or (2, 3). Since both 2 + 3 - 1 = 4 and 1 + 6 - 1 = 6 are composite, both are consistent with A's statement, and B cannot know his number.
3. A cannot rule out having 10, since 10 is also in F and both 2 + 5 - 1 = 6 and 1 + 10 - 1 = 10 are composite. Or, spelled out: if B saw 10, he would still not know his number, since if A saw either 7 or 11 he would say that there is no way B could know his number. So A cannot know his number.
4. B can now rule out the pair (2, 3), i.e. having 5. If B had 5, then A would have not initially know if he was 4 or 6 (pairs were (1, 4) or (2, 3)). But if the pair were (1, 4), then B would have seen 4. Seeing 4, he would consider (1, 4) and (2, 2). But 2 + 2 - 1 = 3 which is prime, and so can be ruled out by A's first statement. i.e., if (2, 2), B would have 4 on his head, which would mean that A couldn't have ruled out having 3 on his head, and so A couldn't have known that B didn't know his number. So, if B had 5, A would know that his number was 6. Since A doesn't know his number, the only remaining possibility is (1, 6), and so B knows he has 7.