1 Introduction

2 Natural Numbers

2.1 Peano Axioms

Axiom A1.2.1. *0 is a natural number.*

Axiom A1.2.2. *If n is a natural number, then $n++$ is also a natural number.*

Definition A1.2.1.3. We define 1 to be the number $0++$, 2 to be the number $(0++)++$, etc.

Proposition A1.2.1.4. *3 is a natural number.*

Axiom A1.2.3. *0 is not the successor of any natural number; i.e., we have $n++ \neq 0$ for every natural number n.*

Proposition A1.2.1.6. *4 is not equal to 0.*

Axiom A1.2.4. *Different natural numbers must have different successors; i.e., if n, m are natural numbers and $n \neq m$, then $n++ \neq m++$. Equivalently, if $n++ = m++$, then we must have $n = m$.*

Proposition A1.2.1.8. *6 is not equal to 2.*

Axiom A1.2.5. (Principle of mathematical induction) *Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number n.*

Proposition A1.2.1.11. (Example of proof by induction) *A certain property $P(n)$ is true for every natural number n.*

Assumption A1.2.6. *(Informal) There exists a number system $\mathbb{N}$, whose elements we will call natural numbers, for which Axioms 2.1-2.5 are true.*

Proposition A1.2.1.16. (Recursive definitions) *Suppose for each natural number n, we have some functions $f_n : \mathbb{N} \to \mathbb{N}$ from the natural numbers to the natural numbers. Let c be a natural number. Then we can assign a unique natural number a_n to each natural number n, such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number n.*

2.2 Addition

Definition A1.2.2.1. (Addition of natural numbers) Let m be a natural number. To add zero to m, we define $0 + m := m$. Now suppose inductively that we have defined how to add n to m. Then we can add $n{+}{+}$ to m by defining $(n{+}{+}) + m := (n + m){+}{+}$.

Lemma A1.2.2.2. *For any natural number n, $n + 0 = n$.*

Lemma A1.2.2.3. *For any natural numbers n and m, $n + (m{+}{+}) = (n + m){+}{+}$.*

Proposition A1.2.2.4. (Addition is commutative) *For any natural numbers n and m, $n + m = m + n$.*

Proposition A1.2.2.5. (Addition is associative) *For any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.*

Proposition A1.2.2.6. (Cancellation law) *Let a, b, c be natural numbers such that $a + b = a + c$. Then we have $b = c$.*

Definition A1.2.2.7. (Positive natural numbers) A natural number n is said to be *positive* iff it is not equal to 0.

Proposition A1.2.2.8. *If a is positive and b is a natural number, then $a + b$ is positive (and hence $b + a$ is also, by Proposition 2.2.4).*

Corollary A1.2.2.9. If a and b are natural numbers such that $a + b = 0$, then $a = 0$ and $b = 0$.

Lemma A1.2.2.10. *Let a be a positive number. Then there exists exactly one natural number b such that $b{+}{+} = a$.*

Definition A1.2.2.11. (Ordering of the natural numbers) Let n and m be natural numbers. we say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, iff we have $n = m+a$ for some natural number a. We say that n is *strictly greater than* m and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

Proposition A1.2.2.12. (Basic properties of order for natural numbers) *Let a, b, c be natural numbers. Then*

- *(Order is reflexive) $a \geq a$.*
- *(Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$.*
- *(Order is anti-symmetric) If $a \geq b$ and $b \geq a$, then $a = b$.*
- *(Addition preserves order) $a \geq b$ iff $a + c \geq b + c$*
- *$a < b$ iff $a{+}{+} \leq b$*

- *$a < b$ iff $b = a + d$ for some positive number d.*

Proposition A1.2.2.13. (Trichotomy of order for natural numbers). *Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b$, $a = b$, or $a > b$.*

Proposition A1.2.2.14. (Strong principle of induction) *Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.*

2.3 Multiplication

Definition A1.2.3.1. (Multiplication of natural numbers) Let m be a natural number. To multiply zero to m, we define $0 \times x; = 0$. Now suppose inductively that we have defined how to multiply n to m. Then we can multiply $n{+}{+}$ to m by defining $(n{+}{+}) \times m := (n \times m) + m$.

Thus for instance $0 \times m = 0$, $1 \times m = 0 + m$, $2 \times m = 0 + m + m$, etc.. By induction one can easily verify that the product of two natural numbers is a natural number.

Lemma A1.2.3.2. (Multiplication is commutative) *Let n, m be natural numbers. Then $n \times m = m \times n$.*

Lemma A1.2.3.3. (Positive natural numbers have no zero divisors) *Let n, m be natural numbers. Then $n \times m = 0$ if and only if at least one of n, m is equal to zero. In particular, if n and m are both positive, then nm is also positive.*

Lemma A1.2.3.4. (Distributive law) *For any natural numbers a, b, c, we have $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.*

Lemma A1.2.3.5. (Multiplication is associative) *For any natural numbers a, b, c, we have $(a \times b) \times c = a \times (b \times c)$.*

Proposition A1.2.3.6. (Multiplication preserves order) *If a, b are natural numbers such that $a < b$, and c is positive, then $ac < bc$.*

Corollary A1.2.3.7. (Cancellation law) *Let a, b, c be natural numbers such that $ac = bc$ and c is non-zero. Then $a = b$.*

Proposition A1.2.3.9. (Euclidean algorithm) *Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r, such that $0 \leq r < q$ and $n = mq + r$.*

Definition A1.2.3.11. (Exponentiation for natural numbers) Let m be a natural number. TO raise m to the power 0, we define $m^0 := 1$; in particular, we define $0^0 := 1$. Now suppose recursively that m^n has been defined for some natural number n, then we define $m^{n{+}{+}} := m^n \times m$.

3 Set Theory

3.1 Fundamentals

Definition A1.3.1.1. (Informal) We define a *set* A to be any unordered collection of objects, e.g., $\{3, 8, 5, 2\}$ is a set. If x is an object, we say that x is *an element* of A or $x \in A$ if x lies in the collection; otherwise we say $x \notin A$. For instance, $3 \in \{1, 2, 3, 4, 5\}$ but $7 \notin \{1, 2, 3, 4, 5\}$.

Axiom A1.3.1. (Sets are objects) *If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.*

Axiom A1.3.2. (Equality of sets) *Two sets A and B are equal, $A = B$, iff every element of A is an element of B and vice versa. To put it another way, $A = B$ iff every element x of A belongs also to B and every element y of B belongs also to A.*

Axiom A1.3.3. (Empty set) *There exists a set $\emptyset$, known as the empty set, which contains no elements, i.e., for every object x we have $x \notin \emptyset$.*

Lemma A1.3.1.5. (Single choice) *Let A be a non-empty set. Then there exists an object x such that $x \in A$.*

Axiom A1.3.4. (Singleton sets and pair sets) *If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e., for every object y, we have $y \in \{a\}$ iff $y = a$; we refer to $\{a\}$ as the singleton set whose element is a. Furthermore, if a and b are objects, then there exists a set $\{a, b\}$ whose only elements are a and b; i.e., for every object y,we have $y \in \{a, b\}$ iff $y = a$ or $y = b$; we refer to this set as the pair set formed by a and b.*

Axiom A1.3.5. (Pairwise union) *Given any two sets A, B, there exists a set $A \cup B$, called the union of A and B, which consists of all the elements which belong to A, B, which consists of all the elements which belong to A or B or both. In other words, for any object x,*

$$x \in A \cup B \iff (x \in A \text{ or } x \in B)$$

Lemma A1.3.1.12. *If a and b are objects, then $\{a, b\} = \{a\} \cup \{b\}$. If A, B, C are sets, then the union operation is commutative (i.e., $A \cup B = B \cup A$) and associative (i.e., $(A \cup B) \cup C = A \cup (B \cup C)$). Also, we have $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.*

Definition A1.3.1.14. (Subsets) Let A, B be sets. We say that A is a *subset* of B, denoted $A \subseteq B$, iff every element of A is also an element of B, i.e.

$$\text{For any object x, } x \in A \Rightarrow x \in B.$$

We say that A is a *proper subset* of B, denoted $A \subset B$, if $A \subseteq B$ and $A \neq B$.

Axiom A1.3.6. (Axiom of specification) *Let A be a set, and for each $x \in A$, let $P(x)$ be a property pertaining to x (i.e., $P(x)$ is either a true statement or a false statement). Then there exists a set, called $\{x \in A : P(x) \text{ is true}\}$ (or simply $\{x \in A : P(x)\}$) for short, whose elements are precisely the elements x in A for which $P(x)$ is true. In other words, for any object y,*

$$y \in \{x \in A : P(x) \text{ is true}\} \iff (y \in A \text{ and } P(y) \text{ is true}).$$

Definition A1.3.1.22. (Intersections) The *intersection*

$$S_1 \cap S_2 := \{x \in S_1 : x \in S_2\}.$$

In other words, $S_1 \cap S_2$ consists of all the elements which belong to both S_1 and S_2. Thus, for all objects x,

$$x \in S_1 \cap S_2 \iff x \in S_1 \text{ and } x \in S_2.$$

Two sets A, B are said to be *disjoint* if $A \cap B = \emptyset$. Note that this is not the same concept as being *distinct*, $A \neq B$. For instance, the sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$ are distinct (there are elements of one set which are not elements of the other) but not disjoint (because their intersection is non-empty). Meanwhile, the sets $\emptyset$ and $\emptyset$ are disjoint but not distinct.

Definition A1.3.1.26. (Difference sets) Given two sets A and B, we define the set $A - B$ or $A \backslash B$ to be the set A with any elements of B removed:

$$A \backslash B := \{x \in A : x \notin B\};$$

for instance, $\{1, 2, 3, 4\} \backslash \{2, 4, 6\} = \{1, 3\}$. In many cases B will be a subset of A, but not necessarily.

Proposition A1.3.1.27. (Sets form a boolean algebra) *Let A, B, C be sets, and let X be a set containing A, B, C as subsets.*

1. *(Minimal element) We have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.*
2. *(Maximal element) We have $A \cup X = X$ and $A \cap X = A$.*
3. *(Identity) We have $A \cap A = A$ and $A \cup A = A$.*
4. *(Commutativity) We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$*
5. *(Associativity) We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.*
6. *(Distributivity) We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*
7. *(Partition) We have $A \cup (X \backslash A) = X$ and $A \cap (X \backslash A) = \emptyset$.*
8. *(De Morgan's laws) We have $X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B)$ and $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$.*

Axiom A1.3.7. (Replacement) *Let A be a set. For any object $x \in A$, and any object y, suppose we have a statement $P(x, y)$ pertaining to x and y, such that for each $x \in A$ there is at most one y for which $P(x, y)$ is true. Then there exists a set $\{y : P(x, y)\ is\ true\ for\ some\ x \in A\}$, such that for any object z,*

$$z \in \{y : P(x, y)\ is\ true\ for\ some\ x \in A\} \iff P(x, z)\ is\ true\ for\ some\ x \in A.$$

Axiom A1.3.8. (Infinity) *There exists a set $\mathbb{N}$ whose elements are called natural numbers, as well as an object 0 in $\mathbb{N}$, and an object $n++$ assigned to every natural number $n \in \mathbb{N}$, such that the Peano axioms 2.1-2.5 are satisfied.*

3.2 Russell's Paradox (Optional)

Axiom A1.3.9. (Universal specification) *(Dangerous!) Suppose for every object x we have a property $P(x)$ pertaining to x (so that for every x, $P(x)$ is either a true statement or a false statement). Then there exists a set $\{x : P(x)\ is\ true\}$, such that for any object y,*

$$y \in \{x : P(x)\ is\ true\} \iff P(y)\ is\ true.$$

Axiom A1.3.10. (Regularity) *If A is a non-empty set, then there is at least one elemtn x of A which is either not a set, or is disjoint from A.*

3.3 Functions

Definition A1.3.3.1. (Functions) Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$, such that for every $x \in X$ there is exactly one $y \in Y$ for which $P(x, y)$ is true. Then we define the *function $f : X \to Y$ defined by P on the domain X and range Y* to be the object which, given any input $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x, f(x))$ is true. Thus, for any $x \in X$ and $y \in Y$,

$$y = f(x) \iff P(x, y) \text{ is true.}$$

Functions are also referred to as *maps* or *transformations*, depending on the context. They are also sometimes called *morphisms*, although to be more precise, a morphism refers to a more general class of object, which may or may not correspond to actual functions, depending on the context.

Definition A1.3.3.7. (Equality of functions) Two functions $f : X \to Y$, $g : X \to Y$ with the same domain and range are said to be *equal*, $f = g$, if and only if $f(x) = g(x)$ for *all* $x \in X$. If $f(x)$ and $g(x)$ agree for some values of x, but no others, then we do not consider f and g to be equal. If two functions f, g have different domains, or different ranges, we also do not consider them to be equal.

Definition A1.3.3.11. (Composition) Let $f : X \to Y$ and $g : Y \to Z$ be two functions, such that the range of f is the same as the domain of g. We then define the *composition* $g \circ f : X \to Z$ of the two functions g and f to be the function defined explicitly by the formula

$$(g \circ f)(x) = g(f(x))$$

If the range of f does not match the domain of g, we leave the composition $g \circ f$ undefined.

Definition A1.3.3.15. (One-to-one functions) A function f is *one-to-one* (or *injective*) if different elements map to different elements:

$$x \neq x' \Rightarrow f(x) \neq f(x').$$

Equivalently, a function is one-to-one if

$$f(x) = f(x') \Rightarrow x = x'.$$

Definition A1.3.3.18. (Onto functions) A function f is *onto* (or *surjective*) if every element in Y comes from applying f to some element in X:

$$\forall y \in Y, \exists x \in X, f(x) = y$$

Definition A1.3.3.21. (Bijective functions) Functions $f : X \to Y$ which are both one-to-one and onto are called *bijective* or *invertible*.

3.4 Images and inverse images

Definition A1.3.4.1. (Images of sets) If $f : X \to Y$ is a function from $XtoY$, and S is a set in X, we define $f(S)$ to be the set

$$f(S) := \{f(x) : x \in S\};$$

this set is a subset of Y, and is sometimes called the *image* of S under the map f. We sometimes call $f(S)$ the *forward image* of S to distinguish it from the concept of the *inverse image* $f^{-1}(S)$ of S, which is defined below.

Note that the set $f(S)$ is well-defined thanks to the axiom of replacement (Axiom 3.7).

Definition A1.3.4.5. (Inverse images) If U is a subset of Y, we define the set $f^{-1}(U)$ to be the set

$$f^{-1}(U) := \{x \in X : f(x) \in U\}.$$

In other words, $f^{-1}(U)$ consists of all the elements of X which map into U:

$$f(x) \in U \iff x \in f^{-1}(U).$$

We call $f^{-1}(U)$ the *inverse image* of U.

Axiom A1.3.11. (Power set axiom) *Let X and Y be sets. Then there exists a set, denoted Y^X, which consists of all the functions from X to Y, thus*

$$f \in Y^X \iff (f \text{ is a function with domain } X \text{ and range } Y).$$

Lemma A1.3.4.10. *Let X be a set. Then the set*

$$\{Y : Y \text{ is a subset of } X\}$$

is a set.

Axiom A1.3.12. (Union) *Let A be a set, all of whose elements are themselves sets. Then there exists a set $\bigcup A$ whose elements are precisely those objects which are elements of the elements of A, thus for all objects x,*

$$x \in \bigcup A \iff (x \in S \text{ for some } S \in A).$$

3.5 Cartesian products

Definition A1.3.5.1. (Ordered pair) If x and y are any objects (possibly equal), we define the *ordered pair* (x, y) to be a new object, consisting of x as its first component and y as its second component. Two ordered pairs (x, y) and (x', y') are considered equal if and only if both their components match, i.e.

$$(x, y) = (x', y') \iff x = x' \text{ and } y = y'.$$

Definition A1.3.5.4. (Cartesian product) If X and Y are sets, then we define the *Cartesian product* $X \times Y$ to be the collection of ordered pairs, whose first component lies in X and whose second component lies in Y, thus

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.$$

or equivalently

$$a \in X \times Y \iff (a = (x, y) \text{ for some } x \in X \text{ and } y \in Y).$$

Definition A1.3.5.7. (Ordered n-tuple and n-fold Cartesian product) Let n be a natural number. An *ordered n-tuple* $(x_i)_{1 \le i \le n}$ (also denoted $(x_1, \ldots, x_n)$) is a collection of objects x_i, one for every natural number i between 1 and n; we refer to x_i as the i^{th} *component* of the ordered n-tuple. Two ordered n-tuples $(x_i)_{1 \le i \le n}$ and $(y_i)_{1 \le i \le n}$ are said to be equal iff $x_i = y_i$ for all $1 \le i \le n$. If $(X_i)_{1 \le i \le n}$ is an ordered n-tuple of sets, we define their *Cartesian product* $\Pi_{1 \le i \le n} X_i$ (also denoted $\Pi_{i=1}^n X_i$ or $X_1 \times \cdots \times X_n$) by

$$\Pi_{1 \le i \le n} X_i := \{(x_i)_{1 \le i \le n} : x_i \in X_i \text{ for all } 1 \le i \le n\}.$$

Lemma A1.3.5.12. (Finite choice) *Let $n \ge 1$ be a natural number, and for each natural number $1 \le i \le n$, let X_i be a non-empty set. Then there exists an n-tuple $(x_i)_{1 \le i \le n}$ such that $x_i \in X_i$ for all $1 \le i \le n$. In other words, if each X_i is non-empty, then the set $\Pi_{1 \le i \le n} X_i$ is also non-empty.*

3.6 Cardinality of sets

Definition A1.3.6.1. (Equal cardinality) We say that two sets X and Y have *equal cardinality*, iff there exists a bijection $f : X \to Y$ from X to Y.

Proposition A1.3.6.4. *Let X, Y, Z be sets. Then X has equal cardinality with X. If X has equal cardinality with Y, then Y has equal cardinality with X. If X has equal cardinality with Y and Y has equal cardinality with Z, then X has equal cardinality with Z.*

Definition A1.3.6.5. Let n be a natural number. A set X is said to have *cardinality n*, iff it has equal cardinality with $\{i \in \mathbb{N} : 1 \leq i \leq n\}$. We also say that X has *n elements* iff it has cardinality n.

Proposition A1.3.6.8. (Uniqueness of cardinality) *Let X be a set with some cardinality n. Then X cannot have any other cardinality, i.e., X cannot have cardinality m for any $m \neq n$.*

Lemma A1.3.6.9. *Suppose that $n \geq 1$, and X has cardinality n. Then X is non-empty, and if x is any element of X, then the set $X - \{x\}$ (i.e., X with the element x removed) has cardinality $n - 1$.*

Definition A1.3.6.10. (Finite sets) A set is *finite*, iff it has cardinality n for some natural number n; otherwise, the set is called *infinite*. If X is a finite set, we use $\#(X)$ to denote the cardinality of X.

Theorem A1.3.6.12. *The set of natural numbers $\mathbb{N}$ is infinite.*

Proposition A1.3.6.14. (Cardinal arithmetic)

- Let X be a finite set, and let x be an object which is not an element of X. Then $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$.
- Let X and Y be finite sets. Then $X \cup Y$ is finite and $\#(X \cup Y) \leq \#(X) + \#(Y)$. If in addition X and Y are disjoint, then $\#(X \cup Y) = \#(X) + \#(Y)$.
- Let X be a finite set, and let Y be a subset of X. Then Y is finite, and $\#(Y) \leq \#(X)$. If in addition $Y \neq X$ (i.e., Y is a proper subset of X), then $\#(Y) < \#(X)$.
- If X is a finite set, and $f : X \to Y$ is a function, then $f(X)$ is a finite set with $\#(f(X)) \leq \#(X)$. If in addition f is one-to-one, then $\#(f(X)) = \#(X)$.
- Let X and Y be finite sets. Then Cartesian product $X \times Y$ is finite, and $\#(X \times Y) = \#(X) \times \#(Y)$.
- Let X and Y be finite sets. Then the set Y^X (defined in Axiom 3.11) is finite and $\#(Y^X) = \#(Y)^{\#(X)}$.

4 Integers and rationals

4.1 The integers

Definition A1.4.1.1. (Integers) An *integer* is an expression of the form $a—b$, where a and b are natural numbers. Two integers are considered to be equal, $a—b = c—d$, if and only if $a + d = c + b$. We let $\mathbb{Z}$ denote the set of all integers.

Definition A1.4.1.2. The sum of two integers, $a—b + c—d$, is defined by the formula

$$(a—b) + (c—d) := (a + c)—(b + d).$$

The product of two integers, $a—b \times c—d$, is defined by

$$(a—b) \times (c—d) := (ac + bd)—(ad + bc).$$

Lemma A1.4.1.3. (Addition and multiplication are well-defined) *Let a, b, a', b', c, d be natural numbers. If $(a—b) = (a'—b')$, then $(a—b) + (c—d) = (a'—b') + (c—d)$* and $(a—b) \times (c—d) = (a'—b') \times (c—d)$, and also $(c—d) + (a—b) = (c—d) + (a'—b')$ and $(c—d) \times (a—b) = (c—d) \times (a'—b')$. Thus addition and multiplication are well-defined operations (equal inputs give equal outputs).

Definition A1.4.1.4. (Negation of integers) If $(a—b)$ is an integer, we define the negation $-(a—b)$ to be the integer $(b—a)$. In particular if $n = n—0$ is a positive natural number, we can define its negation $-n = 0—n$.

Lemma A1.4.1.5. (Trichotomy of integers) *Let x be an integer. Then exactly one of the following statements is true: (a) x is zero; (b) x is equal to a positive natural number n; or (c) x is the negation $-n$ of a positive natural number n.*

Lemma A1.4.1.6. (Laws of algebra for integers) *Let x, y, z be integers. Then we have*

$$\begin{aligned}
x + y &= y + x,\\
(x + y) + z &= x + (y + z),\\
x + 0 = 0 + x &= x,\\
x + (-x) = (-x) + x &= 0,\\
xy &= yx,\\
(xy)z &= x(yz),\\
x1 = 1x &= x,\\
x(y + z) = xy + xz &= xy + xz,\\
(y - z)x &= yx + zx.
\end{aligned}$$

Proposition A1.4.1.8. (Integers have no zero divisors) *Let a and b be integers such that $ab = 0$. Then either $a = 0$ or $b = 0$ (or both).*

Corollary A1.4.1.9. (Cancellation law for integers) *If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.*

Definition A1.4.1.10. (Ordering of the integers) Let n and m be integers. We say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, if we have $n = m + a$ for some natural number a. We say that n is *strictly greater than* m, and write $n > m$ or $m < n$, if $n \geq m$ and $n \neq m$.

Lemma A1.4.1.11. (Properties of order) *Let a, b, c be integers.*

- *$a > b$ if and only if $a - b$ is a positive natural number.*
- *(Addition preserves order) If $a > b$, then $a + c > b + c$.*
- *(Positive multiplication preserves order) If $a > b$ and $c > 0$, then $ac > bc$.*
- *(Negation reverses order) If $a > b$, then $-a < -b$.*
- *(Order is transitive) If $a > b$ and $b > c$, then $a > c$.*
- *(Order trichotomy) Exactly one of the statements $a > b$, $a = b$, or $a < b$ is true.*

4.2 The rationals

Definition A1.4.2.1. (Rationals) A *rational number* is an expression of the form $a//b$, where a and b are integers and b is non-zero; $a//0$ is not considered to be a rational number. Two rational numbers are considered to be equal, $a//b = c//d$, if and only if $ad = bc$. The set of all rational numbers is denoted $\mathbb{Q}$.

Definition A1.4.2.2. If $a//b$ and $c//d$ are rational numbers, we define the sum

$$(a//b) + (c//d) := (ad + bc)//(bd)$$

and the product

$$(a//b) \times (c//d) := (ac)//(bd)$$

and the negation

$$-(a//b) := (-a)//b.$$

Lemma A1.4.2.3. *The sum, production, and negation operations on rational numbers are well-defined, in the sense that if one replaces $a//b$ with another rational number $a'//b'$ which is equal to $a//b$, then the output of the above operations remains unchanged, and similarly for $c//d$.*

Proposition A1.4.2.4. (Laws of algebra for rationals) *Let x, y, z be rationals. Then the following laws of algebra hold:*

$$\begin{aligned}
x + y &= y + x, \\
(x + y) + z &= x + (y + z), \\
x + 0 = 0 + x &= x, \\
x + (-x) = (-x) + x &= 0, \\
xy &= yx, \\
(xy)z &= x(yz), \\
x1 = 1x &= x, \\
x(y + z) = xy + xz &= xy + xz, \\
(y + z)x &= yx + zx.
\end{aligned}$$

If x is non-zero, we also have

$$xx^{-1} = x^{-1}x = 1.$$

Definition A1.4.2.6. A rational number x is said to be *positive* iff we have $x = a/b$ for some positive integers a, b. It is said to be *negative* iff we have $x = -y$ for some positive rational number y (i.e., $x = (-a)/b$ for some positive integers a and b).

Lemma A1.4.2.7. (Ordering of the rationals). Let x and y be rational numbers. We say that $x > y$ iff $x - y$ is a positive rational number, and $x < y$ iff $x - y$ is a negative rational number. We write $x \geq y$ iff either $x > y$ or $x = y$, and similarly define $x \leq y$.

Proposition A1.4.2.9. (Basic properties of order on the rationals) *Let x, y, z be rational numbers. Then the following properties hold:*

- *(Order trichotomy) Exactly one of the statements $x = y$, $x < y$, or $x > y$ is true.*
- *(Order is anti-symmetric) One has $x < y$ if and only if $y > x$.*
- *(Order is transitive) If $x < y$ and $y < z$, then $x < z$.*
- *(Addition preserves order) If $x < y$, then $x + z < y + z$.*
- *(Positive multiplication preserves order) If $x < y$ and z is positive, then $xz < yz$.*

4.3 Absolute value and exponentiation