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2 Natural Numbers

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Axiom A1.2.1. *0 is a natural number.*

Axiom A1.2.2. *If n is a natural number, then $n++$ is also a natural number.*

Definition A1.2.1.3. We define 1 to be the number $0++$, 2 to be the number $(0++)++$, etc.

Proposition A1.2.1.4. *3 is a natural number.*

Axiom A1.2.3. *0 is not the successor of any natural number; i.e., we have $n++ \neq 0$ for every natural number n.*

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Axiom A1.2.4. *Different natural numbers must have different successors; i.e., if n, m are natural numbers and $n \neq m$, then $n++ \neq m++$. Equivalently, if $n++ = m++$, then we must have $n = m$.*

Proposition A1.2.1.8. *6 is not equal to 2.*

Axiom A1.2.5. (Principle of mathematical induction) *Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number n.*

Proposition A1.2.1.11. (Example of proof by induction) *A certain property $P(n)$ is true for every natural number n.*

Assumption A1.2.6. *(Informal) There exists a number system $\mathbb{N}$, whose elements we will call natural numbers, for which Axioms 2.1-2.5 are true.*

Proposition A1.2.1.16. (Recursive definitions) *Suppose for each natural number n, we have some functions $f_n : \mathbb{N} \to \mathbb{N}$ from the natural numbers to the natural numbers. Let c be a natural number. Then we can assign a unique natural number a_n to each natural number n, such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number n.*

2.2 Addition

Definition A1.2.2.1. (Addition of natural numbers) Let m be a natural number. To add zero to m, we define $0+m := m$. Now suppose inductively that we have defined how to add n to m. Then we can add $n++$ to m by defining $(n++)+m := (n+m)++$.

Lemma A1.2.2.2. *For any natural number n, $n+0=n$.*

Lemma A1.2.2.3. *For any natural numbers n and m, $n+(m++) = (n+m)++$.*

Proposition A1.2.2.4. (Addition is commutative) *For any natural numbers n and m, $n+m=m+n$.*

Proposition A1.2.2.5. (Addition is associative) *For any natural numbers a, b, c, we have $(a+b)+c = a+(b+c)$.*

Proposition A1.2.2.6. (Cancellation law) *Let a, b, c be natural numbers such that $a+b = a+c$. Then we have $b=c$.*

Definition A1.2.2.7. (Positive natural numbers) A natural number n is said to be *positive* iff it is not equal to 0.

Proposition A1.2.2.8. *If a is positive and b is a natural number, then $a+b$ is positive (and hence $b+a$ is also, by Proposition 2.2.4).*

Corollary A1.2.2.9. If a and b are natural numbers such that $a+b=0$, then $a=0$ and $b=0$.

Lemma A1.2.2.10. *Let a be a positive number. Then there exists exactly one natural number b such that $b++=a$.*

Definition A1.2.2.11. (Ordering of the natural numbers) Let n and m be natural numbers. we say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, iff we have $n = m+a$ for some natural number a. We say that n is *strictly greater than* m and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

Proposition A1.2.2.12. (Basic properties of order for natural numbers) *Let a, b, c be natural numbers. Then*

1. *(Order is reflexive) $a \geq a$.*
2. *(Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$.*
3. *(Order is anti-symmetric) If $a \geq b$ and $b \geq a$, then $a = b$.*
4. *(Addition preserves order) $a \geq b$ iff $a+c \geq b+c$*
5. *$a < b$ iff $a++ \leq b$*

6. $a < b$ iff $b = a + d$ for some positive number d.

Proposition A1.2.2.13. (Trichotomy of order for natural numbers). *Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b$, $a = b$, or $a > b$.*

Proposition A1.2.2.14. (Strong principle of induction) *Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.*

2.3 Multiplication

Definition A1.2.3.1. (Multiplication of natural numbers) Let m be a natural number. To multiply zero to m, we define $0 \times x ;= 0$. Now suppose inductively that we have defined how to multiply n to m. Then we can multiply $n{+}{+}$ to m by defining $(n{+}{+}) \times m := (n \times m) + m$.

Thus for instance $0 \times m = 0$, $1 \times m = 0 + m$, $2 \times m = 0 + m + m$, etc.. By induction one can easily verify that the product of two natural numbers is a natural number.

Lemma A1.2.3.2. (Multiplication is commutative) *Let n, m be natural numbers. Then $n \times m = m \times n$.*

Lemma A1.2.3.3. (Positive natural numbers have no zero divisors) *Let n, m be natural numbers. Then $n \times m = 0$ if and only if at least one of n, m is equal to zero. In particular, if n and m are both positive, then nm is also positive.*

Lemma A1.2.3.4. (Distributive law) *For any natural numbers a, b, c, we have $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.*

Lemma A1.2.3.5. (Multiplication is associative) *For any natural numbers a, b, c, we have $(a \times b) \times c = a \times (b \times c)$.*

Proposition A1.2.3.6. (Multiplication preserves order) *If a, b are natural numbers such that $a < b$, and c is positive, then $ac < bc$.*

Corollary A1.2.3.7. (Cancellation law) *Let a, b, c be natural numbers such that $ac = bc$ and c is non-zero. Then $a = b$.*

Proposition A1.2.3.9. (Euclidean algorithm) *Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r, such that $0 \leq r < q$ and $n = mq + r$.*

Definition A1.2.3.11. (Exponentiation for natural numbers) Let m be a natural number. TO raise m to the power 0, we define $m^0 := 1$; in particular, we define $0^0 := 1$. Now suppose recursively that m^n has been defined for some natural number n, then we define $m^{n{+}{+}} := m^n \times m$.

3 Set Theory

3.1 Fundamentals

Definition A1.3.1.1. (Informal) We define a *set* A to be any unordered collection of objects, e.g., $\{3, 8, 5, 2\}$ is a set. If x is an object, we say that x is *an element* of A or $x \in A$ if x lies in the collection; otherwise we say $x \notin A$. For instance, $3 \in \{1, 2, 3, 4, 5\}$ but $7 \notin \{1, 2, 3, 4, 5\}$.

Axiom A1.3.1. (Sets are objects) *If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.*

Axiom A1.3.2. (Equality of sets) *Two sets A and B are equal, $A = B$, iff every element of A is an element of B and vice versa. To put it another way, $A = B$ iff every element x of A belongs also to B and every element y of B belongs also to A.*

Axiom A1.3.3. (Empty set) *There exists a set $\emptyset$, known as the empty set, which contains no elements, i.e., for every object x we have $x \notin \emptyset$.*

Lemma A1.3.1.5. (Single choice) *Let A be a non-empty set. Then there exists an object x such that $x \in A$.*

Axiom A1.3.4. (Singleton sets and pair sets) *If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e., for every object y, we have $y \in \{a\}$ iff $y = a$; we refer to $\{a\}$ as the singleton set whose element is a. Furthermore, if a and b are objects, then there exists a set $\{a, b\}$ whose only elements are a and b; i.e., for every object y,we have $y \in \{a, b\}$ iff $y = a$ or $y = b$; we refer to this set as the pair set formed by a and b.*

Axiom A1.3.5. (Pairwise union) *Given any two sets A, B, there exists a set $A \cup B$, called the union of A and B, which consists of all the elements which belong to A, B, which consists of all the elements which belong to A or B or both. In other words, for any object x,*

$$x \in A \cup B \iff (x \in A \text{ or } x \in B)$$

Lemma A1.3.1.12. *If a and b are objects, then $\{a, b\} = \{a\} \cup \{b\}$. If A, B, C are sets, then the union operation is commutative (i.e., $A \cup B = B \cup A$) and associative (i.e., $(A \cup B) \cup C = A \cup (B \cup C)$). Also, we have $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.*

Definition A1.3.1.14. (Subsets) Let A, B be sets. We say that A is a *subset* of B, denoted $A \subseteq B$, iff every element of A is also an element of B, i.e.

$$\text{For any object x, } x \in A \Rightarrow x \in B.$$

We say that A is a *proper subset* of B, denoted $A \subset B$, if $A \subseteq B$ and $A \neq B$.

Axiom A1.3.6. (Axiom of specification) *Let A be a set, and for each $x \in A$, let $P(x)$ be a property pertaining to x (i.e., $P(x)$ is either a true statement or a false statement). Then there exists a set, called $\{x \in A : P(x) \text{ is true}\}$ (or simply $\{x \in A : P(x)\}$) for short, whose elements are precisely the elements x in A for which $P(x)$ is true. In other words, for any object y,*

$$y \in \{x \in A : P(x) \text{ is true}\} \iff (y \in A \text{ and } P(y) \text{ is true}).$$

Definition A1.3.1.22. (Intersections) The *intersection*

$$S_1 \cap S_2 := \{x \in S_1 : x \in S_2\}.$$

In other words, $S_1 \cap S_2$ consists of all the elements which belong to both S_1 and S_2. Thus, for all objects x,

$$x \in S_1 \cap S_2 \iff x \in S_1 \text{ and } x \in S_2.$$

Two sets A, B are said to be *disjoint* if $A \cap B = \emptyset$. Note that this is not the same concept as being *distinct*, $A \neq B$. For instance, the sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$ are distinct (there are elements of one set which are not elements of the other) but not disjoint (because their intersection is non-empty). Meanwhile, the sets $\emptyset$ and $\emptyset$ are disjoint but not distinct.

Definition A1.3.1.26. (Difference sets) Given two sets A and B, we define the set $A - B$ or $A \backslash B$ to be the set A with any elements of B removed:

$$A \backslash B := \{x \in A : x \notin B\};$$

for instance, $\{1, 2, 3, 4\} \backslash \{2, 4, 6\} = \{1, 3\}$. In many cases B will be a subset of A, but not necessarily.

Proposition A1.3.1.27. (Sets form a boolean algebra) *Let A, B, C be sets, and let X be a set containing A, B, C as subsets.*

1. *(Minimal element) We have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.*
2. *(Maximal element) We have $A \cup X = X$ and $A \cap X = A$.*
3. *(Identity) We have $A \cap A = A$ and $A \cup A = A$.*
4. *(Commutativity) We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$*
5. *(Associativity) We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.*
6. *(Distributivity) We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*
7. *(Partition) We have $A \cup (X \backslash A) = X$ and $A \cap (X \backslash A) = \emptyset$.*
8. *(De Morgan's laws) We have $X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B)$ and $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$.*

Axiom A1.3.7. (Replacement) *Let A be a set. For any object $x \in A$, and any object y, suppose we have a statement $P(x, y)$ pertaining to x and y, such that for each $x \in A$ there is at most one y for which $P(x, y)$ is true. Then there exists a set $\{y : P(x, y) \text{ is true for some } x \in A\}$, such that for any object z,*

$$z \in \{y : P(x, y) \textit{ is true for some } x \in A\} \\ \iff P(x, z) \textit{ is true for some } x \in A.$$

Axiom A1.3.8. (Infinity) *There exists a set $\mathbb{N}$ whose elements are called natural numbers, as well as an object 0 in $\mathbb{N}$, and an object $n{+}{+}$ assigned to every natural number $n \in \mathbb{N}$, such that the Peano axioms 2.1-2.5 are satisfied.*

3.2 Russell's Paradox (Optional)

Axiom A1.3.9. (Universal specification) *(Dangerous!) Suppose for every object x we have a property $P(x)$ pertaining to x (so that for every x, $P(x)$ is either a true statement or a false statement). Then there exists a set $\{x : P(x) \text{ is true}\}$, such that for any object y,*

$$y \in \{x : P(x) \textit{ is true}\} \iff P(y) \textit{ is true.}$$

Axiom A1.3.10. (Regularity) *If A is a non-empty set, then there is at least one elemtn x of A which is either not a set, or is disjoint from A.*

3.3 Functions

Definition A1.3.3.1. (Functions) Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$, such that for every $x \in X$ there is exactly one $y \in Y$ for which $P(x, y)$ is true. Then we define the *function $f : X \to Y$ defined by P on the domain X and range Y* to be the object which, given any input $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x, f(x))$ is true. Thus, for any $x \in X$ and $y \in Y$,

$$y = f(x) \iff P(x, y) \text{ is true.}$$

Functions are also referred to as *maps* or *transformations*, depending on the context. They are also sometimes called *morphisms*, although to be more precise, a morphism refers to a more general class of object, which may or may not correspond to actual functions, depending on the context.

Definition A1.3.3.7. (Equality of functions) Two functions $f : X \to Y$, $g : X \to Y$ with the same domain and range are said to be *equal*, $f = g$, if and only if $f(x) = g(x)$ for *all* $x \in X$. If $f(x)$ and $g(x)$ agree for some values of x, but no others, then we do not consider f and g to be equal. If two functions f, g have different domains, or different ranges, we also do not consider them to be equal.

Definition A1.3.3.11. (Composition) Let $f : X \to Y$ and $g : Y \to Z$ be two functions, such that the range of f is the same as the domain of g. We then define the *composition* $g \circ f : X \to Z$ of the two functions g and f to be the function defined explicitly by the formula

$$(g \circ f)(x) = g(f(x))$$

If the range of f does not match the domain of g, we leave the composition $g \circ f$ undefined.

Definition A1.3.3.15. (One-to-one functions) A function f is *one-to-one* (or *injective*) if different elements map to different elements:

$$x \neq x' \Rightarrow f(x) \neq f(x').$$

Equivalently, a function is one-to-one if

$$f(x) = f(x') \Rightarrow x = x'.$$

Definition A1.3.3.18. (Onto functions) A function f is *onto* (or *surjective*) if every element in Y comes from applying f to some element in X:

$$\forall y \in Y, \exists x \in X, f(x) = y$$

Definition A1.3.3.21. (Bijective functions) Functions $f : X \to Y$ which are both one-to-one and onto are called *bijective* or *invertible*.

3.4 Images and inverse images

Definition A1.3.4.1. (Images of sets) If $f : X \to Y$ is a function from $X to Y$, and S is a set in X, we define $f(S)$ to be the set

$$f(S) := \{f(x) : x \in S\};$$

this set is a subset of Y, and is sometimes called the *image* of S under the map f. We sometimes call $f(S)$ the *forward image* of S to distinguish it from the concept of the *inverse image* $f^{-1}(S)$ of S, which is defined below.

Note that the set $f(S)$ is well-defined thanks to the axiom of replacement (Axiom 3.7).

Definition A1.3.4.5. (Inverse images) If U is a subset of Y, we define the set $f^{-1}(U)$ to be the set

$$f^{-1}(U) := \{x \in X : f(x) \in U\}.$$

In other words, $f^{-1}(U)$ consists of all the elements of X which map into U:

$$f(x) \in U \iff x \in f^{-1}(U).$$

We call $f^{-1}(U)$ the *inverse image* of U.

Axiom A1.3.11. (Power set axiom) *Let X and Y be sets. Then there exists a set, denoted Y^X, which consists of all the functions from X to Y, thus*

$$f \in Y^X \iff (f \textit{ is a function with domain } X \textit{ and range } Y).$$

Lemma A1.3.4.10. *Let X be a set. Then the set*

$$\{Y : Y \textit{ is a subset of } X\}$$

is a set.

Axiom A1.3.12. (Union) *Let A be a set, all of whose elements are themselves sets. Then there exists a set $\bigcup A$ whose elements are precisely those objects which are elements of the elements of A, thus for all objects x,*

$$x \in \bigcup A \iff (x \in S \textit{ for some } S \in A).$$

3.5 Cartesian products

Definition A1.3.5.1. (Ordered pair) If x and y are any objects (possibly equal), we define the *ordered pair* (x, y) to be a new object, consisting of x as its first component and y as its second component. Two ordered pairs (x, y) and (x', y') are considered equal if and only if both their components match, i.e.

$$(x, y) = (x', y') \iff x = x' \text{ and } y = y'.$$

Definition A1.3.5.4. (Cartesian product) If X and Y are sets, then we define the *Cartesian product* $X \times Y$ to be the collection of ordered pairs, whose first component lies in X and whose second component lies in Y, thus

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.$$

or equivalently

$$a \in X \times Y \iff (a = (x, y) \text{ for some } x \in X \text{ and } y \in Y).$$

Definition A1.3.5.7. (Ordered n-tuple and n-fold Cartesian product) Let n be a natural number. An *ordered n-tuple* $(x_i)_{1 \le i \le n}$ (also denoted $(x_1, \ldots, x_n)$) is a collection of objects x_i, one for every natural number i between 1 and n; we refer to x_i as the i^{th} *component* of the ordered n-tuple. Two ordered n-tuples $(x_i)_{1 \le i \le n}$ and $(y_i)_{1 \le i \le n}$ are said to be equal iff $x_i = y_i$ for all $1 \le i \le n$. If $(X_i)_{1 \le i \le n}$ is an ordered n-tuple of sets, we define their *Cartesian product* $\Pi_{1 \le i \le n} X_i$ (also denoted $\Pi_{i=1}^n X_i$ or $X_1 \times \cdots \times X_n$) by

$$\Pi_{1 \le i \le n} X_i := \{(x_i)_{1 \le i \le n} : x_i \in X_i \text{ for all } 1 \le i \le n\}.$$

Lemma A1.3.5.12. (Finite choice) *Let $n \ge 1$ be a natural number, and for each natural number $1 \le i \le n$, let X_i be a non-empty set. Then there exists an n-tuple $(x_i)_{1 \le i \le n}$ such that $x_i \in X_i$ for all $1 \le i \le n$. In other words, if each X_i is non-empty, then the set $\Pi_{1 \le i \le n} X_i$ is also non-empty.*

3.6 Cardinality of sets

Definition A1.3.6.1. (Equal cardinality) We say that two sets X and Y have *equal cardinality*, iff there exists a bijection $f : X \to Y$ from X to Y.

Proposition A1.3.6.4. *Let X, Y, Z be sets. Then X has equal cardinality with X. If X has equal cardinality with Y, then Y has equal cardinality with X. If X has equal cardinality with Y and Y has equal cardinality with Z, then X has equal cardinality with Z.*

Definition A1.3.6.5. Let n be a natural number. A set X is said to have *cardinality n*, iff it has equal cardinality with $\{i \in \mathbb{N} : 1 \leq i \leq n\}$. We also say that X has *n elements* iff it has cardinality n.

Proposition A1.3.6.8. (Uniqueness of cardinality) *Let X be a set with some cardinality n. Then X cannot have any other cardinality, i.e., X cannot have cardinality m for any $m \neq n$.*

Lemma A1.3.6.9. *Suppose that $n \geq 1$, and X has cardinality n. Then X is non-empty, and if x is any element of X, then the set $X - \{x\}$ (i.e., X with the element x removed) has cardinality $n - 1$.*

Definition A1.3.6.10. (Finite sets) A set is *finite*, iff it has cardinality n for some natural number n; otherwise, the set is called *infinite*. If X is a finite set, we use $\#(X)$ to denote the cardinality of X.

Theorem A1.3.6.12. *The set of natural numbers $\mathbb{N}$ is infinite.*

Proposition A1.3.6.14. (Cardinal arithmetic)

1. Let X be a finite set, and let x be an object which is not an element of X. Then $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$.

2. Let X and Y be finite sets. Then $X \cup Y$ is finite and $\#(X \cup Y) \leq \#(X) + \#(Y)$. If in addition X and Y are disjoint, then $\#(X \cup Y) = \#(X) + \#(Y)$.

3. Let X be a finite set, and let Y be a subset of X. Then Y is finite, and $\#(Y) \leq \#(X)$. If in addition $Y \neq X$ (i.e., Y is a proper subset of X), then $\#(Y) < \#(X)$.

4. If X is a finite set, and $f : X \to Y$ is a function, then $f(X)$ is a finite set with $\#(f(X)) \leq \#(X)$. If in addition f is one-to-one, then $\#(f(X)) = \#(X)$.

5. Let X and Y be finite sets. Then Cartesian product $X \times Y$ is finite, and $\#(X \times Y) = \#(X) \times \#(Y)$.

6. Let X and Y be finite sets. Then the set Y^X (defined in Axiom 3.11) is finite and $\#(Y^X) = \#(Y)^{\#(X)}$.

4 Integers and rationals

4.1 The integers

Definition A1.4.1.1. (Integers) An *integer* is an expression of the form $a—b$, where a and b are natural numbers. Two integers are considered to be equal, $a—b = c—d$, if and only if $a + d = c + b$. We let $\mathbb{Z}$ denote the set of all integers.

Definition A1.4.1.2. The sum of two integers, $a—b + c—d$, is defined by the formula

$$(a—b) + (c—d) := (a + c)—(b + d).$$

The product of two integers, $a—b \times c—d$, is defined by

$$(a—b) \times (c—d) := (ac + bd)—(ad + bc).$$

Lemma A1.4.1.3. (Addition and multiplication are well-defined) *Let a, b, a', b', c, d be natural numbers. If $(a—b) = (a'—b')$, then $(a—b) + (c—d) = (a'—b') + (c—d)$* and $(a—b) \times (c—d) = (a'—b') \times (c—d)$, and also $(c—d) + (a—b) = (c—d) + (a'—b')$ and $(c—d) \times (a—b) = (c—d) \times (a'—b')$. Thus addition and multiplication are well-defined operations (equal inputs give equal outputs).

Definition A1.4.1.4. (Negation of integers) If $(a—b)$ is an integer, we define the negation $-(a—b)$ to be the integer $(b—a)$. In particular if $n = n—0$ is a positive natural number, we can define its negation $-n = 0—n$.

Lemma A1.4.1.5. (Trichotomy of integers) *Let x be an integer. Then exactly one of the following statements is true: (a) x is zero; (b) x is equal to a positive natural number n; or (c) x is the negation $-n$ of a positive natural number n.*

Lemma A1.4.1.6. (Laws of algebra for integers) *Let x, y, z be integers. Then we have*

$$\begin{aligned}
x + y &= y + x, \\
(x + y) + z &= x + (y + z), \\
x + 0 = 0 + x &= x, \\
x + (-x) = (-x) + x &= 0, \\
xy &= yx, \\
(xy)z &= x(yz), \\
x1 = 1x &= x, \\
x(y + z) = xy + xz &= xy + xz, \\
(y - z)x &= yx + zx.
\end{aligned}$$

Proposition A1.4.1.8. (Integers have no zero divisors) *Let a and b be integers such that $ab = 0$. Then either $a = 0$ or $b = 0$ (or both).*

Corollary A1.4.1.9. (Cancellation law for integers) *If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.*

Definition A1.4.1.10. (Ordering of the integers) Let n and m be integers. We say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, if we have $n = m + a$ for some natural number a. We say that n is *strictly greater than* m, and write $n > m$ or $m < n$, if $n \geq m$ and $n \neq m$.

Lemma A1.4.1.11. (Properties of order) *Let a, b, c be integers.*

1. *$a > b$ if and only if $a - b$ is a positive natural number.*
2. *(Addition preserves order) If $a > b$, then $a + c > b + c$.*
3. *(Positive multiplication preserves order) If $a > b$ and $c > 0$, then $ac > bc$.*
4. *(Negation reverses order) If $a > b$, then $-a < -b$.*
5. *(Order is transitive) If $a > b$ and $b > c$, then $a > c$.*
6. *(Order trichotomy) Exactly one of the statements $a > b$, $a = b$, or $a < b$ is true.*

4.2 The rationals

Definition A1.4.2.1. (Rationals) A *rational number* is an expression of the form $a//b$, where a and b are integers and b is non-zero; $a//0$ is not considered to be a rational number. Two rational numbers are considered to be equal, $a//b = c//d$, if and only if $ad = bc$. The set of all rational numbers is denoted $\mathbb{Q}$.

Definition A1.4.2.2. If $a//b$ and $c//d$ are rational numbers, we define the sum

$$(a//b) + (c//d) := (ad + bc)//(bd)$$

and the product

$$(a//b) \times (c//d) := (ac)//(bd)$$

and the negation

$$-(a//b) := (-a)//b.$$

Lemma A1.4.2.3. *The sum, production, and negation operations on rational numbers are well-defined, in the sense that if one replaces $a//b$ with another rational number $a'//b'$ which is equal to $a//b$, then the output of the above operations remains unchanged, and similarly for $c//d$.*

Proposition A1.4.2.4. (Laws of algebra for rationals) *Let x, y, z be rationals. Then the following laws of algebra hold:*

$$\begin{aligned} x + y &= y + x, \\ (x + y) + z &= x + (y + z), \\ x + 0 = 0 + x &= x, \\ x + (-x) = (-x) + x &= 0, \\ xy &= yx, \\ (xy)z &= x(yz), \\ x1 = 1x &= x, \\ x(y + z) = xy + xz &= xy + xz, \\ (y + z)x &= yx + zx. \end{aligned}$$

If x is non-zero, we also have

$$xx^{-1} = x^{-1}x = 1.$$

Definition A1.4.2.6. A rational number x is said to be *positive* iff we have $x = a/b$ for some positive integers a, b. It is said to be *negative* iff we have $x = -y$ for some positive rational number y (i.e., $x = (-a)/b$ for some positive integers a and b).

Lemma A1.4.2.7. (Ordering of the rationals). Let x and y be rational numbers. We say that $x > y$ iff $x - y$ is a positive rational number, and $x < y$ iff $x - y$ is a negative rational number. We write $x \geq y$ iff either $x > y$ or $x = y$, and similarly define $x \leq y$.

Proposition A1.4.2.9. (Basic properties of order on the rationals) *Let x, y, z be rational numbers. Then the following properties hold:*

1. *(Order trichotomy) Exactly one of the statements $x = y$, $x < y$, or $x > y$ is true.*
2. *(Order is anti-symmetric) One has $x < y$ if and only if $y > x$.*
3. *(Order is transitive) If $x < y$ and $y < z$, then $x < z$.*
4. *(Addition preserves order) If $x < y$, then $x + z < y + z$.*
5. *(Positive multiplication preserves order) If $x < y$ and z is positive, then $xz < yz$.*

4.3 Absolute value and exponentiation

Definition A1.4.3.1. (Absolute value) If x is a rational number, the *absolute value* $|x|$ of x is defined as follows. If x is positive, then $|x| := x$. If x is negative, then $|x| := -x$. If x is zero, then $|x| := 0$.

Definition A1.4.3.2. (Distance) Let x and y be rational numbers. The quantity $|x - y|$ is called the *distance between x and y* and is sometimes denoted $d(x, y)$, thus $d(x, y) = |x - y|$. For instance, $d(3, 5) = 2$.

Proposition A1.4.3.3. (Basic properties of absolute value and distance) *Let x, y, z be rational numbers.*

1. *(Non-degeneracy of absolute value) We have $|x| \geq 0$. Also, $|x| = 0$ if and only if $x = 0$.*
2. *(Triangle inequality for absolute value) We have $|x + y| \leq |x| + |y|$.*
3. *We have inequalities $-y \leq x \leq y$ if and only if $y \geq |x|$. In particular, we have $-|x| \leq x \leq |x|$.*
4. *(Multiplicativity of absolute value) We have $|xy| = |x||y|$. In particular, $|-x| = |x|$.*
5. *(Non-degeneracy of distance) We have $d(x, y) \geq 0$. Also, $d(x, y) = 0$ if and only if $x = y$.*
6. *(Symmetry of distance) We have $d(x, y) = d(y, x)$.*
7. *(Triangle inequality for distance) We have $d(x, z) \leq d(x, y) + d(y, z)$.*

Definition A1.4.3.4. (ϵ -closeness). Let $\epsilon > 0$ be a rational number, and let x, y be rational numbers. We say that y is ϵ *-close* to x iff we have $d(x, y) < \epsilon$.

Proposition A1.4.3.7. *Let x, y, z be rational numbers.*

1. *If $x = y$, then x is ϵ-close to y for every $\epsilon > 0$. Conversely, if x is ϵ-close to y for every $\epsilon > 0$, then $x = y$.*
2. *Let epsilon > 0. If x is ϵ-close to y, then y is ϵ-close to x.*
3. *Let $\epsilon, \delta > 0$. If x is ϵ-close to y and y is δ-close to z, then x and z are $(\epsilon + \delta)$-close.*
4. *Let $\epsilon, \delta > 0$. If x and y are ϵ-close, and z and w are δ-close, then $x + z$ and $y + w$ are $(\epsilon + \delta)$-close, and $x - z$ and $y - w$ are also $(\epsilon + \delta)$-close.*
5. *Let $\epsilon > 0$. If x and y are ϵ-close, they are also ϵ'-close for every $\epsilon' > \epsilon$.*
6. *Let $\epsilon > 0$. If y and z are both ϵ-close to x, and w is between y and z (i.e., $y \leq w \leq z$ or $z \leq w \leq y$), then w is also ϵ-close to x.*
7. *Let $\epsilon > 0$. If x and y are ϵ-close, and z is non-zero, then xz and yz are ϵz-close.*
8. *Let $\epsilon, \delta > 0$. If x and y are ϵ-close, and z and w are δ-close, then xz and yw are $(\epsilon|z| + \delta|w| + \epsilon\delta)$-close.*

Definition A1.4.3.9. (Exponentiation to a natural number) Let x be a rational number. To raise x to the power 0, we define $x^0 := 1$; in particular we define $0^0 := 1$. Now suppose inductively that we have defined how to raise x to the power n. Then we can raise x to the power $n{+}{+}$ by defining $x^{n++} := x^n \times x$.

Proposition A1.4.3.10. (Properties of exponentiation, I). Let x, y be rational numbers, and let n, m be natural numbers.

1. We have $x^n x^m = x^{n+m}, (x^n)^m = x^{nm}$, and $(xy)^n = x^n y^n$.
2. Suppose $n > 0$. Then we have $x^n = 0$ if and only if $x = 0$.
3. If $x \geq y \geq 0$, then $x^n \geq y^n \geq 0$. If $x > y \geq 0$ and $n > 0$, then $x^n > y^n \geq 0$.
4. We have $|x^n| = |x|^n$.

Definition A1.4.3.11. (Exponentiation to a negative number) Let x be a non-zero rational number. Then for any negative integer $-n$, we define $x^{-n} := 1/x^n$.

Proposition A1.4.3.12. (Properties of exponentiation, II) *Let x, y be rational numbers, and let n, m be integers.*

1. *We have $x^n x^m = x^{n+m}, (x^n)^m = x^{nm}$, and $(xy)^n = x^n y^n$.*
2. *If $x \geq y > 0$, then $x^n \geq y^n > 0$ if n is positive, and $0 < x^n \leq y^n$ if n is negative.*
3. *If $x, y > 0, n \neq 0$, and $x^n = y^n$, then $x = y$.*
4. *We have $|x^n| = |x|^n$*

4.4 Gaps in the rational numbers

Proposition A1.4.4.1. (Interspersing of integers by rationals) *Let x be a rational number. Then there exists an integer n such that $n \leq x < n+1$. In fact, this integer is unique (i.e., for each x there is only one n for which $n \leq x < n+1$). In particular, there exists a natural number N such that $N > x$ (i.e., there is no such thing as a rational number which is larger than all the natural numbers).*

Proposition A1.4.4.3. (Interspersing of rationals by rations) *If x and y are two rationals such that $x < y$, then there exists a third rational number z such that $x < z < y$.*

Proposition A1.4.4.4. *There does not exist a rational number x such that $x^2 = 2$.*

Proposition A1.4.4.5. *For every rational number $\epsilon > 0$, there exists a non-negative rational number x such that $x^2 < 2 < (x+\epsilon)^2$.*

5 The real numbers

5.1 Cauchy sequences

Definition A1.5.1.1. (Sequences) Let m be an integer. A *sequence $(a_n)_{n=m}^{\infty}$ of rational numbers* is any function from the set $\{n \in \mathbb{Z} : n \geq m\}$ to Q, i.e., a mapping which assigns to each integer n greater than or equal to m, a rational number a_n. More informally, a sequence $(a_n)_{n=0}^{\infty}$ of rational numbers is a collection of rationals $a_m, a_{m+1}, a_{m+2}, \ldots$.

Definition A1.5.1.3. (ϵ-steadiness) Let $\epsilon > 0$ be a positive rational number. A sequence $(a_n)_{n=0}^{\infty}$ is said to be *ϵ-steady* iff each pair a_j, a_k of sequence elements is ϵ-close for every natural number j, k. In other words, the sequence $a_0, a_1, a_2, \ldots$ is ϵ-steady iff $|a_j - a_k| < \epsilon$ for all $j, k \geq m$.

Definition A1.5.1.6. (Eventual ϵ-steadiness) Let $\epsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be *eventually ϵ-steady* iff the sequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is ϵ-steady for some natural number $N \geq 0$. In other words, the sequence $a_0, a_1, a_2, \ldots$ is eventually ϵ-steady iff there exists an $N \geq 0$ such that $|a_j - a_k| \leq \epsilon$ for all $j, k \geq N$.

Definition A1.5.1.8. (Cauchy sequences) A sequence $(a_n)_{n=0}^{\infty}$ of rational numbers is said to be a *Cauchy sequence* iff for every positive rational number $\epsilon > 0$, the sequence $(a_n)_{n=0}^{\infty}$ is eventually ϵ-steady. In other words, the sequence $a_0, a_1, a_2, \ldots$ is a Cauchy sequence iff for every $\epsilon > 0$, there exists an $N \geq 0$ such that $d(a_j, a_k) \leq \epsilon$ for all $j, k \geq N$.

Proposition A1.5.1.11. The sequence $a_1, a_2, a_3, \ldots$ defined by $a_n := 1/n$ (i.e., the sequence $1, 1/2, 1/3, \ldots$)is a Cauchy sequence.

Definition A1.5.1.12. (Bounded sequences) Let $M \geq 0$ be rational. A finite sequence $a_1, a_2, a_3, \ldots, a_n$ is *bounded by M* iff $|a_i| \leq M$ for all $1 \leq i \leq n$. An infinite sequence $(a_n)_{n=1}^{\infty}$ is *bounded by M* iff $|a_i| \leq M$ for all $i \geq 1$. A sequence is said to be *bounded* iff it is bounded by some $M \geq 0$.

Lemma A1.5.1.14. (Finite sequences are bounded) *Every finite sequence $a_1, a_2, a_3, \ldots, a_n$ is bounded.*

Lemma A1.5.1.15. (Cauchy sequences are bounded) *Every Cauchy sequence $(a_n)_{n=1}^{\infty}$ is bounded.*

5.2 Equivalent Cauchy sequences

Definition A1.5.2.1. ($\epsilon - close sequences$) Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be two sequences, and let $\epsilon > 0$. We say that the sequence $(a_n)_{n=0}^{\infty}$ is *ϵ-close* to $(b_n)_{n=0}^{\infty}$ iff a_n is ϵ-close to b_n for each $n \in \mathbb{N}$. In other words, $a_0, a_1, a_2, \ldots$ is ϵ-close to $b_0, b_1, b_2, \ldots$ iff $|a_n - b_n| \leq \epsilon$ for all $n = 0, 1, 2, \ldots$.

Definition A1.5.2.3. (Eventually ϵ-close sequences) Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be two sequences. We say that the sequence $(a_n)_{n=0}^{\infty}$ is *eventually ϵ-close* to $(b_n)_{n=0}^{\infty}$ iff there exists an $N \geq 0$ such that the sequences $(a_n)_{n=N}^{\infty}$ and $(b_n)_{n=N}^{\infty}$ are ϵ-close. In other words, $a_0, a_1, a_2, \ldots$ is eventually ϵ-close to $b_0, b_1, b_2, \ldots$ iff there exists an $N \geq 0$ such that $|a_n - b_n| \leq \epsilon$ for all $n \geq N$.

Definition A1.5.2.6. (Equivalent sequences) Two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are *equivalent* iff for each rational $\epsilon > 0$, the sequences are eventually ϵ-close. In other words, $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are equivalent iff for every $\epsilon > 0$, there exists an $N \geq 0$ such that $|a_n - b_n| \leq \epsilon$ for all $n \geq N$.

Proposition A1.5.2.8. *Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be the sequences $a_n = 1 + 10^{-n}$ and $b_n = 1 - 10^{-n}$. Then the sequences a_n, b_n are equivalent.*

5.3 The construction of the real numbers

Definition A1.5.3.1. (Real numbers) A *real number* is defined to be an object of the form $\mathrm{LIM}_{n\to\infty}a_n$, where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Two real numbers $\mathrm{LIM}_{n\to\infty}a_n$ and $\mathrm{LIM}_{n\to\infty}b_n$ are said to be equal iff $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences. The set of all real numbers is denoted $\mathbb{R}$.

Proposition A1.5.3.2. (Formal limits are well-defined) *Let $x = LIM_{n\to\infty}a_n, y = LIM_{n\to\infty}b_n$ and $z = LIM_{n\to\infty}c_n$ be real numbers. Then, with the above definition of equality, we have $x = x$. Also, if $x = y$, then $y = x$. Finally, if $x = y$ and $y = z$, then $x = z$.*

Definition A1.5.3.4. (Addition of reals) Let $x = \mathrm{LIM}_{n\to\infty}a_n$ and $y = \mathrm{LIM}_{n\to\infty}b_n$ be real numbers. Then we define the sum $x + y$ to be $x + y := \mathrm{LIM}_{n\to\infty}a_n + b_n$.

Lemma A1.5.3.6. (Sum of Cauchy sequences is Cauchy) *Let $x = LIM_{n\to\infty}a_n$ and $y = LIM_{n\to\infty}b_n$ be real numbers. Then $x+y$ is also a real number (i.e. $(a_n+b_n)_{n=1}^{\infty}$ is a Cauchy sequence of rationals).*

Lemma A1.5.3.7. (Sums of equivalent Cauchy sequences are equivalent) *Let $x = LIM_{n\to\infty}a_n$, $y = LIM_{n\to\infty}b_n$, and $x' = LIM_{n\to\infty}a'_n$ be real numbers. Suppose that $x = x'$. Then we have $x + y = x' + y$.*

Definition A1.5.3.9. (Multiplication of reals) Let $x = \mathrm{LIM}_{n\to\infty}a_n$ and $y = \mathrm{LIM}_{n\to\infty}b_n$ be real numbers. Then we define the product xy to be $xy := \mathrm{LIM}_{n\to\infty}a_nb_n$.

Lemma A1.5.3.10. (Multiplication is well defined) *Let $x = LIM_{n\to\infty}a_n$, $y = LIM_{n\to\infty}b_n$, and $x' = LIM_{n\to\infty}a'_n$, be real numbers. Then xy is also a real number. Furthermore, if $x = x'$, then $xy = x'y$.*

Proposition A1.5.3.11. *All the laws of algebra from Proposition 4.1.6 hold not only for the integers, but for the reals as well.*

Definition A1.5.3.12. (Sequences bounded away from zero) A sequence $(a_n)_{n=1}^{\infty}$ of rational numbers is said to be *bounded away from zero* iff there exists a rational number $c > 0$ such that $|a_n| \geq c$ for all $n \geq 1$.

Lemma A1.5.3.14. *Let x be a non-zero real number. Then $x = LIM_{n\to\infty}a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is bounded away from zero.*

Lemma A1.5.3.15. *Suppose that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence which is bounded away from zero. Then the sequence $(a_n^{-1})_{n=1}^{\infty}$ is also a Cauchy sequence.*

Definition A1.5.3.16. (Reciprocal of real numbers) Let x be a non-zero real number. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence bounded away from zero such that $x = \mathrm{LIM}_{n\to\infty}a_n$ (such a sequence exists by Lemma 5.3.14). Then we define the reciprocal $1/x$ by the formula $x^-1 := \mathrm{LIM}_{n\to\infty}a_n^-1$. (From Lemma 5.3.15 we know that x^{-1} is a real number.)

Lemma A1.5.3.17. (Reciprocation is well defined) *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two Cauchy sequences bounded away from zero such that $LIM_{n\to\infty}a_n = LIM_{n\to\infty}b_n$. Then $LIM_{n\to\infty}a_n^{-1} = LIM_{n\to\infty}b_n^{-1}$.*

5.4 Ordering the reals

Definition A1.5.4.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of rationals. We say that this seqeunce is *positively bounded away from zero* iff we have a positive rational $c > 0$ such that $a_n \geq c$ for all $n \geq 1$ (in particular, the sequence is entirely positive). The sequence is *negatively bounded away from zero* iff we have a negative rational $c < 0$ such that $a_n \leq c$ for all $n \geq 1$ (in particular, the sequence is entirely negative).

Definition A1.5.4.2. A real number x is said to be *positive* iff it can be written as $x = \text{LIM}_{n\to\infty} a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is positively bounded away from zero. x is said to be *negative* iff it can be written as $x = \text{LIM}_{n\to\infty} a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is negatively bounded away from zero.

Proposition A1.5.4.4. (Basic properties of positive reals) *For every real number x, exactly one of the following three statements is true: (a) x is zero; (b) x is positive; (c) x is negative. A real number x is negative iff $-x$ is positive. If x and y are positive, then so are $x + y$ and xy.*

Definition A1.5.4.5. (Absolute value) Let x be a real number. We define the *absolute value* $|x|$ of x to equal x if x is positive, $-x$ if x is negative, and 0 when x is zero.

Definition A1.5.4.6. (Ordering of the real numbers) Let x and y be real numbers. We say that x is *greater than* y, and write $x > y$, iff $x - y$ is a positive real number, and $x < y$ iff $x - y$ is a negative real number. We define $x \geq y$ iff $x > y$ or $x = y$, and similarly define $x \leq y$.

Proposition A1.5.4.7. *All the claims in Proposition 4.2.9 which held for rationals, continue to hold for real numbers.*

Proposition A1.5.4.8. *Let x be a positive real number. Then x^{-1} is also positive. Also, if y is another positive number and $x > y$, then $x^{-1} < y^{-1}$.*

Proposition A1.5.4.9. (The non-negative reals are closed) *Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of non-negative rational numbers. Then $LIM_{n\to\infty} a_n$ is a non-negative real number.*

Corollary A1.5.4.10. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be Cauchy sequences of rationals such that $a_n \geq b_n$ for all $n \geq 1$. Then $LIM_{n\to\infty} a_n \geq LIM_{n\to\infty} b_n$.*

Proposition A1.5.4.12. (Bounding of reals by rationals) *Let x be a positive real number. Then there exists a positive rational number q such that $q < x$, and there exists a positive integer N such that $x \leq N$.*

Corollary A1.5.4.13. (Archimedean property) *Let x and ϵ be any positive real numbers. Then there exists a positive integer M such that $M\epsilon \geq x$.*

Proposition A1.5.4.14. *Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.*

5.5 The least upper bound property

Definition A1.5.5.1. (Upper bound) Let E be a subset of $\mathbb{R}$, and let M be a real number. We say that M is an *upper bound* for E iff we have $x \leq M$ for every $x \in E$.

Definition A1.5.5.5. (Least upper bound) Let E be a subset of $\mathbb{R}$, and M be a real number. We say that M is a *least upper bound* for E iff (a) M is an upper bound for E, and also (b) any other upper bound M' for E must be larger than or equal to M.

Proposition A1.5.5.8. (Uniqueness of least upper bound) *Let R be a subset of $\mathbb{R}$. Then E can have at most one least upper bound.*

Theorem A1.5.5.9. (Existence of least upper bound) *Let E be a non-empty subset of $\mathbb{R}$. If E has an upper bound, (i.e., E has some upper bound M), then it must have exactly one least upper bound.*

Definition A1.5.5.10. (Supremum) Let E be a subset of the real numbers. If E is non-empty and has some upper bound, then we define $\sup(E)$ to the least upper bound of E (this is well-defined by Theorem 5.5.9). We introduce two additional symbols, $+\infty$ and $-\infty$. If E is non-empty and has no upper bound, we set $\sup(E) := +\infty$; if E is empty, we set $\sup(E) := -\infty$. We refer to $\sup(E)$ as the *supremum* of E.

Proposition A1.5.5.12. *There exists a positive real number x such that $x^2 = 2$.*

5.6 Real exponentiation, part I

Definition A1.5.6.1. (Exponentiating a real by a natural number) Let x be a real number. To raise x to the power 0, we define $x^0 := 1$. Now suppose recursively that x^n has been defined for some natural number n, then we define $x^{n+1} := x^n \times x$.

Definition A1.5.6.2. (Exponentiation a real by an integer) Let x be a non-zero real number. Then for any negative integer $-n$, we define $x^{-n} := 1/x^n$.

Proposition A1.5.6.3. *All the properties in Propositions 4.3.10 and 4.3.12 remain valid if x and y are assumed to e real numbers instead of rational numbers.*

Definition A1.5.6.4. Let $x \geq 0$ be a non-negative real, and let $n \geq 1$ be a positive integer. We define $x^{1/n}$, also known as the n^{th} *root* of x, by the formula

$$x^{1/n} := \sup(\{y \in \mathbb{R} : y \geq 0 \text{ and } y^n \leq x\}).$$

Lemma A1.5.6.5. (Existence of n^{th} roots) *Let $x \geq 0$ be a non-negative real, and let $n \geq 1$ be a positive integer. Then the set $E := \{y \in \mathbb{R} : y \geq 0 \text{ and } y^n \leq x\}$ is non-empty and bounded above. In particular, $x^{1/n}$ is a real number.*

Lemma A1.5.6.6. *Let $x, y > 0$ be non-negative reals, and let $n, m \geq 1$ be positive integers.*

1. *If $y = x^{1/n}$, then $y^n = x$.*

2. *Conversely, if* $y^n = x$, *then* $y = x^{1/n}$.
3. $x^{1/n}$ *is a non-negative real number, and is positive iff* x *is positive.*
4. *We have* $x > y$ *if and only if* $x^{1/n} > y^{1/n}$.
5. *If* $x > 1$, *then* $x^{1/k}$ *is a decreasing function of* k. *If* $x < 1$, *then* $x^{1/k}$ *is an increasing function of* k. *If* $x = 1$, *then* $x^{1/k} = 1$ *for all* k.
6. *We have* $(xy)^{1/n} = x^{1/n}y^{1/n}$.
7. *We have* $(x^{1/n})^m = x^{m/n}$.

Definition A1.5.6.7. Let $x > 0$ be a positive real number, and let q be a rational number. To define x^q, we write $q = a/b$ for some integer a and positive integer b, and define

$$x^q := (x^{1/b})^a$$

Lemma A1.5.6.8. *Let* a, a' *be integers and* b, b' *be positive integers such that* $a/b = a'/b'$ *, and let* x *be a positive real number. Then we have* $(x^{1/b'})^{a'} = (x^{1/b})^a$.

Lemma A1.5.6.9. *Let* $x, y > 0$ *be positive reals, and let* q, r *be rationals.*

1. x^q *is a positive real.*
2. $x^{q+r} = x^q x^r$ *and* $(x^q)^r = x^{qr}$
3. $x^{-q} = 1/x^q$
4. *If* $q > 0$, *then* $x > y$ *if and only if* $x^q > y^q$
5. *If* $x > 1$, *then* $x^q > x^r$ *if and only if* $q > r$. *If* $x < 1$, *then* $x^q > x^r$ *if and only if* $q < r$.

6 Limits of sequences

6.1 Convergence and limit laws

Definition A6.1.1. (Distance between two real numbers) Given two real numbers x and y, we define their distance $d(x, y)$ to be $d(x, y) := |x - y|$.

Definition A6.1.2. (ϵ-close real numbers) Let $\epsilon > 0$ be a real number. We say that two real numbers x, y are *ϵ-close* iff we have $d(x, y) \leq \epsilon$.

Definition A1.6.1.3. (Cauchy sequences of reals) Let $\epsilon > 0$ be a real number. A sequence $(a_n)_{n=N}^{\infty}$ starting at some integer N is said to be *ϵ-steady* iff a_j and a_k are *epsilon*-close for every $j, k \geq N$. A sequence $(a_n)_{n=m}^{\infty}$ is said to be *eventually ϵ-steady* iff there exists an $N \geq m$ such that $(a_n)_{n=N}^{\infty}$ is ϵ-steady. We say that $(a_n)_{n=m}^{\infty}$ is a *Cauchy sequence* iff it is eventually ϵ-steady for every $\epsilon > 0$.

Proposition A1.6.1.4. *Let $(a_n)_{n=m}^{\infty}$ be a sequence of rational numbers starting at some integer m. Then $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of Definition 5.1.8 if and only if it is a Cauchy sequence in the sense of Defitnition 6.1.3.*

Definition A1.6.1.5. (Convergence of sequences) Let $\epsilon > 0$ be a real number, and let L be a real number. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers is said to be ϵ-close to L iff a_n is *epsilon*-close to L for every $n \geq N$, i.e., we have $|a_n - L| \leq \epsilon$ for every $n \geq N$. We say that a sequence $(a_n)_{n=m}^{\infty}$ is *eventually ϵ-close* to L iff there exists an $N \geq m$ such that $(a_n)_{n=N}^{\infty}$ is *epsilon*-close to L. We say that a sequence $(a_n)_{n=m}^{\infty}$ *converges* to L iff it is eventually *epsilon*-close to L for every $\epsilon > 0$.

Proposition A1.6.1.7. (Uniqueness of limits) *Let $(a_n)_{n=m}^{\infty}$ be a real sequence starting at some integer index m, and let $L \neq L'$ be two distinct real numbers. THen it is not possible for $(a_n)_{n=m}^{\infty}$ to converge to L while also converging to L'.*

Definition A1.6.1.8. (Limits of sequences) If a sequence $(a_n)_{n=m}^{\infty}$ converges to some real number L, we say that $(a_n)_{n=m}^{\infty}$ is *convergent* and that its *limit* is L; we write

$$L = \lim_{n\to\infty} a_n$$

to denote this fact. If a sequence $(a_n)_{n=m}^{\infty}$ is not converging to any real number L, we say that the sequence $(a_n)_{n=m}^{\infty}$ is *divergent* and we leave $\lim_{n\to\infty} a_n$ undefined.

Proposition A1.6.1.10. *We have $lim_{n\to\infty}(1/n) = 0$.*

Proposition A1.6.1.12. (Convergent sequences are Cauchy) *Suppose that $(a_n)_{n=m}^{\infty}$ is a convergent sequence of real numbers. Then $(a_n)_{n=m}^{\infty}$ is also a Cauchy sequence.*

Proposition A1.6.1.15. (Formal limits are genuine limits) *Suppose that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Then $(a_n)_{n=1}^{\infty}$ converges to $LIM_{n\to\infty}(a_n)$, i.e.*

$$LIM_{n\to\infty} a_n = lim_{n\to\infty} a_n.$$

Definition A1.6.1.16. (Bounded sequences) A sequence $(a_n)_{n=m}^{\infty}$ of real numbers is *bounded* by a real number M iff we have $|a_n| \leq M$ for all $n \geq m$. We say that $(a_n)_{n=m}^{\infty}$ is *bounded* iff it is bounded for some real number $M > 0$.

Corollary A1.6.1.17. *Every convergent sequence of real numbers is bounded.*

Theorem A1.6.1.19. (Limit Laws) Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be convergent sequences of real numbers, and let x, y be the real numbers $x := \lim_{n\to\infty} a_n$ and $y := \lim_{n\to\infty} b_n$.

1. The sequences $(a_n + b_n)_{n=m}^{\infty}$ converges to $x + y$; in other words,

$$\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

2. The sequence $(a_n b_n)_{n=m}^{\infty}$ converges to xy; in other words,

$$\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$$

3. For any real number c, the sequence $(ca_n)_{n=m}^{\infty}$ converges to cx; in other words,

$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$$

4. The sequence $(a_n - b_n)_{n=m}^{\infty}$ converges to $x - y$; in other words,

$$\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

5. Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(b_n^{-1})_{n=m}^{\infty}$ converges to y^{-1}; in other words,

$$\lim_{n\to\infty} b_n^{-1} = (\lim_{n\to\infty} b_n)^{-1}$$

6. Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(a_n/b_n)_{n=m}^{\infty}$ converges to x/y; in other words,

$$\lim_{n\to\infty} a_n/b_n = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

7. The sequence $(\max(a_n, b_n))_{n=m}^{\infty}$ converges to $\max(x, y)$; in other words,

$$\lim_{n\to\infty} \max(a_n/b_n) = \max(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n)$$

8. The sequence $(\min(a_n, b_n))_{n=m}^{\infty}$ converges to $\min(x, y)$; in other words,

$$\lim_{n\to\infty} \min(a_n/b_n) = \min(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n)$$

6.2 The extended real number system

Definition A1.6.2.1. (Extended real number system) The *extended real number system* $\mathbb{R}^*$ is the real line $\mathbb{R}$ with two additional elements attached, called $+\infty$ and $-\infty$. These elements are distinct from each other and also distinct from every real number. An extended real number x is called *finite* if it is a real number, and *infinite* iff it is equal to $+\infty$ or $-\infty$. (This definition is not directly related to the notion of finite and infinite sets in Section 3.6, though it is of course similar in spirit.)

Definition A1.6.2.2. (Negation of extended reals) The operation of negation $x \mapsto -x$ on $\mathbb{R}$, we now extend to $\mathbb{R}^*$ by defining $-(+\infty) := -\infty$ and $-(-\infty) := +\infty$.

Thus every extended real number x has a negation, and $-(-x)$ is always equal to x.

Definition A1.6.2.3. (Ordering of extended reals) Let x and y be extended real numbers. We say that $x \leq y$, i.e., x is less than or equal to y, iff one of the following three statements is true:

1. x and y are real numbers, and $x \leq y$ as real numbers

2. $y = +\infty$

3. $x = -\infty$

We say that $x < y$ if we have $x \leq y$ and $x \neq y$. We sometimes write $x < y$ as $y > x$, and $x \leq y$ as $y \geq x$.

Proposition A1.6.2.5. *Let x, y, z be extended real numbers. Then the following statements are true:*

1. *(Reflexivity) We have $x \leq x$*
2. *(Trichotomy) Exactly one of the statements $x < y$, $x = y$, or $x > y$ is true*
3. *(Transitivity) If $x \leq y$ and $y \leq z$, then $x \leq z$*
4. *(Negation reverses order) If $x \leq y$, then $-y \leq -x$*

Definition A1.6.2.6. (Supremum of sets of extended reals) Let E be a subset of $\mathbb{R}^*$. Then we define the *supremum* $\sup(E)$ or *least upper bound* of E by the following rule.

1. If E is contained in $\mathbb{R}$ (i.e., $+\infty$ and $-\infty$ are not elements of E), then we let $\sup(E)$ be as defined in Definition 5.5.10
2. If E contains $+\infty$, then we set $\sup(E) := +\infty$
3. If E does not contain $+\infty$, but does contain $-\infty$, then we set $\sup(E) := \sup(E \setminus \{-\infty\})$ (which is a subset of $\mathbb{R}$ and thus falls under case (a))

We also define the *infimum* $\inf(E)$ of E (also known as the *greatest lower bound* of E) by the formula

$$\inf(E) := -\sup(-E)$$

where $-E$ is the set $\{-x : x \in E\}$.

Theorem A1.6.2.11. *Let E be a subset of $\mathbb{R}^*$. Then the following statements are true.*

1. *For every $x \in E$, we have $x \leq \sup(E)$ and $x \geq \inf(E)$*
2. *Suppose that $M \in \mathbb{R}^*$ is an upper bound for E, i.e., $x \leq M$ for all $x \in E$. Then we have $\sup(E) \leq M$.*
3. *Suppose that $m \in \mathbb{R}^*$ is a lower bound for E, i.e., $x \geq m$ for all $x \in E$. Then we have $\inf(E) \geq m$.*

6.3 Suprema and Infima of sequences

Definition A1.6.3.1. (Sup and inf of sequences) Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Then we define $\sup(a_n)_{n=m}^{\infty}$ to be the supremum of the set $\{a_n : n \geq m\}$, and $\inf(a_n)_{n=m}^{\infty}$ to be the infimum of the same set $\{a_n : n \geq m\}$.

Proposition A1.6.3.6. (Least upper bound property) *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be the extended real number $x := sup(a_n)_{n=m}^{\infty}$. Then we have $a_n \leq x$ for all $n \geq m$. Also, whenever $M \in \mathbb{R}^*$ is an upper bound for a_n (i.e., $a_n \leq M$ for all $n \geq m$), we have $x \leq M$. Finally, for every extended real number y for which $y < x$, there exists at least one $n \geq m$ for which $y < a_n \leq x$.*

Proposition A1.6.3.8. (Monotone bounded sequences converge) *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which has some finite upper bound $M \in \mathbb{R}$, and which is also increasing (i.e., $a_{n+1} \geq a_n$ for all $n \geq m$). Then the sequence $(a_n)_{n=m}^{\infty}$ converges to some real number L. Then $(a_n)_{n=m}^{\infty}$ is convergent, and in fact*

$$lim_{n\to\infty} a_n = sup(a_n)_{n=m}^{\infty} \leq M$$

Proposition A1.6.3.10. *Let $0 < x < 1$. Then we have $lim_{n\to\infty} x^n = 0$.*

6.4 Limsup, Liminf, and limit points

Definition A1.6.4.1. (Limit points) Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let x be a real number, and let $\epsilon > 0$ be a real number. We say that x is *ϵ-adherent* to $(a_n)_{n=m}^{\infty}$ iff there exists an $n \geq m$ such that a_n is ϵ-close to x. We say that x is a *limit point* or *adherent point* of $(a_n)_{n=m}^{\infty}$ iff it is continually ϵ-adherent to $(a_n)_{n=m}^{\infty}$ for every $\epsilon > 0$.

Proposition A1.6.4.5. (Limits are limit points) *Let $(a_n)_{n=m}^{\infty}$ be a sequence which converges to a real number c. Then c is a limit point of $(a_n)_{n=m}^{\infty}$, and in fact, c is the only limit point of $(a_n)_{n=m}^{\infty}$.*

Definition A1.6.4.6. (Limit superior and limit inferior) Suppose that $(a_n)_{n=m}^{\infty}$ is a sequence. We define a new sequence $(a_N^+)_{N=m}^{\infty}$ by the formula

$$a_N^+ := \sup(a_n)_{n=N}^{\infty}$$

More informally, a_N^+ is the supremum of all the elements of the sequence from a_N onwards. We then define the *limit superior* of the sequence $(a_n)_{n=m}^{\infty}$, denoted $\lim\sup_{n\to\infty} a_n$, by the formula

$$\lim\sup_{n\to\infty} a_n := \inf(a_N^+)_{N=m}^{\infty}$$

Similarly, we can define

$$a_N^- := \inf(a_n)_{n=N}^{\infty}$$

and define the *limit inferior* of the sequence $(a_n)_{n=m}^{\infty}$, denoted $\lim\inf_{n\to\infty} a_n$, by the formula

$$\lim\inf_{n\to\infty} a_n := \sup(a_N^-)_{N=m}^{\infty}$$

Proposition A1.6.4.12. *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let L^+ be the limit superior of this sequence, and let L^- be the limit inferior of this sequence (thus L^+ and L^- are extended real numbers).*

1. *For every $x > L+$, there exists an $N \geq m$ such that $a_n < x$ for all $n \geq N$. (In other words, for every $x > L^+$, the elements of the sequence $(a_n)_{n=m}^{\infty}$ are eventually less than x.) Similarly, for every $y < L^-$ there exists an $N \geq m$ such that $a_n > y$ for all $n \geq N$.*

2. *For every $x < L^+$, and every $N \geq m$, there exists an $n \geq N$ such that $a_n > x$. (In other words, for every $x < L^+$, the elements of the sequence $(a_n)_{n=m}^{\infty}$ exceed x infinitely often.) Similarly, for every $y > L^-$, and every $N \geq m$, there exists an $n \geq N$ such that $a_n < y$.*

3. *We have $inf(a_n)_{n=m}^{\infty} \leq L^- \leq L^+ \leq sup(a_n)_{n=m}^{\infty}$.*

4. *If c is any limit point of $(a_n)_{n=m}^{\infty}$, then we have $L^- \leq c \leq L^+$.*

5. *If L^+ is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$. Similarly, if L^- is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$.*

6. *Let c be a real number. If $(a_n)_{n=m}^{\infty}$ converges to c , then we must have $L^+ = L^- = c$. Conversely, if $L^+ = L^- = c$, then $(a_n)_{n=m}^{\infty}$ converges to c.*

Lemma A1.6.4.13. (Comparison principle) *Suppose that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are two sequences of real numbers such that $a_n \leq b_n$ for all $n \geq m$. Then we have the inequalities*

$$\begin{gathered} sup(a_n)_{n=m}^{\infty} \leq sup(b_n)_{n=m}^{\infty} \\ inf(a_n)_{n=m}^{\infty} \leq inf(b_n)_{n=m}^{\infty} \\ lim\ sup_{n\to\infty} a_n \leq lim\ sup_{n\to\infty} b_n \\ lim\ inf_{n\to\infty} a_n \leq lim\ inf_{n\to\infty} b_n \end{gathered}$$

Corollary A1.6.4.14. (Squeeze test) *Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$, and $(c_n)_{n=m}^{\infty}$ be sequences of real numbers such that*

$$a_n \leq b_n \leq c_n$$

for all $n \geq m$. Suppose also that $(a_n)_{n=m}^{\infty}$ and $(c_n)_{n=m}^{\infty}$ both converge to the same limit L. Then $(b_n)_{n=m}^{\infty}$ is also convergent to L.

Corollary A1.6.4.17. (Zero test for sequences) *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Then the limit $lim_{n\to\infty} a_n$ exists and is equal to 0 if and only if the limit $lim_{n\to\infty} |a_n|$ exists and is equal to 0.*

Theorem A1.6.4.18. (Completeness of the reals) *A sequence $(a_n)_{n=m}^{\infty}$ of real numbers is a Cauchy sequence if and only if it is convergent.*

6.5 Some standard limits

Corollary A1.6.5.1. *We have* $lim_{n\to\infty} 1/n^{1/k} = 0$ *for every integer* $k \geq 1$.

Lemma A1.6.5.2. *Let* x *be a real number. Then we have* $lim_{n\to\infty} x^n$ *exists and is equal to zero when* $|x| < 1$, *exists and is equal to* 1 *when* $x = 1$, *and diverges when* $x < -1$ *or when* $|x| > 1$.

Lemma A1.6.5.3. *For any* $x > 0$, *we have* $lim_{n\to\infty} x^{1/n} = 1$

6.6 Subsequences

Definition A1.6.6.1. (Subsequences) Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=N}^{\infty}$ be sequences of real numbers. We say that $(b_n)_{n=N}^{\infty}$ is a *subsequence* of $(a_n)_{n=m}^{\infty}$ iff there exists a function $f : \mathbb{N} \to \mathbb{N}$ which is strictly increasing (i.e., $f(n+1) > f(n)$ for all $n \in \mathbb{N}$) such that

$$b_n = a_{f(n)} \text{for all} n \in \mathbb{N}.$$

Lemma A1.6.6.4. *Let* $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=N}^{\infty}$, *and* $(c_n)_{n=P}^{\infty}$ *be sequences of real numbers. Then* $(a_n)_{n=m}^{\infty}$ *is the subsequence of* $(a_n)_{n=m}^{\infty}$. *Furthermore, if* $(b_n)_{n=N}^{\infty}$ *is a subsequence of* $(a_n)_{n=m}^{\infty}$, *and* $(c_n)_{n=P}^{\infty}$ *is a subsequence of* $(b_n)_{n=N}^{\infty}$, *then* $(c_n)_{n=P}^{\infty}$ *is a subsequence of* $(a_n)_{n=m}^{\infty}$.

Proposition A1.6.6.5. (Subsequences related to limits) *Let* $(a_n)_{n=m}^{\infty}$ *be a sequence of real numbers, and let* L *be a real number. Then the following two statements are logically equivalent (each one implies the other):*

1. *The sequence* $(a_n)_{n=m}^{\infty}$ *converges to* L.
2. *Every subsequence of* $(a_n)_{n=m}^{\infty}$ *converges to* L.

Proposition A1.6.6.6. (Subsequences related to limit points) *Let* $(a_n)_{n=m}^{\infty}$ *be a sequence of real numbers, and let* L *be a real number. Then the following two statements are logically equivalent.*

1. L *is a limit point of* $(a_n)_{n=m}^{\infty}$.
2. *There exists a subsequence of* $(a_n)_{n=m}^{\infty}$ *which converges to* L.

Theorem A1.6.6.8. (Bolzano-Weierstrass theorem) *Let* $(a_n)_{n=0}^{\infty}$ *be a bounded sequence (i.e., there exists a real number* $M > 0$ *such that* $|a_n| \leq M$ *for all* $n \in \mathbb{N}$*). Then there is at least one subsequence of* $(a_n)_{n=0}^{\infty}$ *which converges.*

6.7 Real exponentiation, part II

Lemma A1.6.7.1. (Continuity of exponentiation) *Let $x > 0$, and let α be a real number. Let $(q_n)_{n=0}^{\infty}$ be a sequence of rational numbers converging to α. Then $(x^{q_n})_{n=0}^{\infty}$ is also a convergent sequence. Furthermore, if $(q'_n)_{n=0}^{\infty}$ is any other sequence of rational numbers converging to α, then $(x^{q'_n})_{n=0}^{\infty}$ has the same limit as $(x^{q_n})_{n=0}^{\infty}$.*

Definition A1.6.7.2. (Exponentiation to a real exponent) Let $x > 0$ be real, and let α be a real number. We define the quantity x^{α} by the formula $x^{\alpha} := \lim_{n\to\infty} x^{q_n}$, where $(q_n)_{n=0}^{\infty}$ is any sequence of rational numbers converging to α.

Proposition A1.6.7.3. *All the results of Lemma 5.6.9, which held for rational numbers q and r, continue to hold for real numbers q and r.*