Q3. 1)

$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

Since $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ only 2 columns are linearly independent

$\therefore$ rank $(A) = \boxed{2}$

2) $\sigma_1 = \sqrt{10+\sqrt{97}}$, $\sigma_2 = \sqrt{10-\sqrt{97}}$

$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$, $A^TA = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 3 \\ 8 & 13 & 5 \\ 3 & 5 & 2 \end{bmatrix}$

$\det(A^TA - \lambda I) \Rightarrow \det \begin{bmatrix} 5-\lambda & 8 & 3 \\ 8 & 13-\lambda & 5 \\ 3 & 5 & 2-\lambda \end{bmatrix}$

$\therefore (5-\lambda)\big((13-\lambda)(2-\lambda) - 25\big) - 8\big(8(2-\lambda)+15\big) + 3\big(40 - 3(13-\lambda)\big) =$

$= -\lambda(\lambda^2 - 20\lambda + 3)$

$\therefore \lambda_1 = 10+\sqrt{97}$, $\lambda_2 = 10-\sqrt{97}$

$\therefore \boxed{\sigma_1 = \sqrt{10+\sqrt{97}}}$, $\boxed{\sigma_2 = \sqrt{10-\sqrt{97}}}$

3) For $\lambda = 10+\sqrt{97}$:

$$\begin{bmatrix} -5-\sqrt{97} & 8 & 3 \\ 8 & 3-\sqrt{97} & 5 \\ 3 & 5 & -8-\sqrt{97} \end{bmatrix} \Rightarrow \begin{bmatrix} -5-\sqrt{97} & 8 & 3 \\ 0 & \frac{10+\sqrt{97}}{3} & \frac{-69-7\sqrt{97}}{8} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$L = \sqrt{a^2+b^2+c^2} \Rightarrow 2\sqrt{97+7\sqrt{97}}$ $\therefore v_1 = \begin{bmatrix} \frac{3+\sqrt{97}}{2\sqrt{97+7\sqrt{97}}} \\ \frac{11+\sqrt{97}}{2\sqrt{97+7\sqrt{97}}} \\ \frac{4}{\sqrt{97+7\sqrt{97}}} \end{bmatrix}$

For $\lambda = 10-\sqrt{97}$

$$[\cdots] \Rightarrow \begin{bmatrix} -5+\sqrt{97} & 8 & 3 \\ 8 & 3+\sqrt{97} & 5 \\ 3 & 5 & -8+\sqrt{97} \end{bmatrix} \Rightarrow \begin{bmatrix} 8 & 3+\sqrt{97} & 5 \\ 0 & \frac{\sqrt{97}-9}{4} & \frac{49-5\sqrt{97}}{8} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \\ v_2 \\ \\ \end{bmatrix} = 0$$

$$L = 2\sqrt{97-7\sqrt{97}} \quad \therefore v_2 = \begin{bmatrix} \frac{3-\sqrt{97}}{2\sqrt{97-7\sqrt{97}}} \\ \frac{11-\sqrt{97}}{2\sqrt{97-7\sqrt{97}}} \\ \frac{4}{\sqrt{97-7\sqrt{97}}} \end{bmatrix}$$

for $\lambda = 0$, we need a vector perpendicular to v_1 & v_2.
we need $b=-a$. and $v_2^T v_3 = 0$

$$\begin{bmatrix} \frac{3-\sqrt{97}}{2\sqrt{97-7\sqrt{97}}} & \frac{11-\sqrt{97}}{2\sqrt{97-7\sqrt{97}}} & \frac{8}{2\sqrt{97-7\sqrt{97}}} \end{bmatrix} \begin{bmatrix} a \\ -a \\ c \end{bmatrix} = 0.$$

$(3-\sqrt{97})a - a(11-\sqrt{97}) + 8c = 0$

$8a = 8c$, $c = a$

Normalize it, we have $a = \frac{\sqrt{3}}{3}$, $v_3 = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}$

$$u_i = \frac{1}{\sigma_i} A v, \quad u_1 = \frac{1}{\sqrt{10+\sqrt{97}}} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{3+\sqrt{97}}{2\sqrt{97+7\sqrt{97}}} \\ \frac{11+\sqrt{97}}{2\sqrt{97+7\sqrt{97}}} \\ \frac{4}{\sqrt{97+7\sqrt{97}}} \end{bmatrix} = \begin{bmatrix} \frac{33+3\sqrt{97}}{2\sqrt{1649+167\sqrt{97}}} \\ \frac{47+5\sqrt{97}}{2\sqrt{1649+167\sqrt{97}}} \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{10-\sqrt{97}}} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{3-\sqrt{97}}{2\sqrt{97-7\sqrt{97}}} \\ \frac{11-\sqrt{97}}{2\sqrt{97-7\sqrt{97}}} \\ \frac{4}{\sqrt{97-7\sqrt{97}}} \end{bmatrix} = \begin{pmatrix} \frac{33-3\sqrt{97}}{2\sqrt{1649-167\sqrt{97}}} \\ \frac{47-5\sqrt{97}}{2\sqrt{1649-169\sqrt{97}}} \end{pmatrix}$$

$$\therefore U = \begin{bmatrix} \frac{33+3\sqrt{97}}{2\sqrt{1649+167\sqrt{97}}} & \frac{33-3\sqrt{97}}{2\sqrt{1649-167\sqrt{97}}} \\ \frac{47+5\sqrt{97}}{2\sqrt{1649+167\sqrt{97}}} & \frac{47-5\sqrt{97}}{2\sqrt{1649-169\sqrt{97}}} \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{3+\sqrt{97}}{2\sqrt{97+7\sqrt{97}}} & \frac{3-\sqrt{97}}{2\sqrt{97-7\sqrt{97}}} & \frac{\sqrt{3}}{3} \\ \frac{11+\sqrt{97}}{2\sqrt{97+7\sqrt{97}}} & \frac{11-\sqrt{97}}{2\sqrt{97-7\sqrt{97}}} & -\frac{\sqrt{3}}{3} \\ \frac{4}{\sqrt{97+7\sqrt{97}}} & \frac{4}{\sqrt{97-7\sqrt{97}}} & \frac{\sqrt{3}}{3} \end{bmatrix}$$

Q2) Consider 6 points in $\mathbb{R}$ as follows:

$x^{(1)} = 1$, $x^{(2)} = 2$, $x^{(3)} = 3$, $x^{(4)} = 4$, $x^{(5)} = 5$, $x^{(6)} = 6$

choose $k=2$, if random initialization gives two centroid as $x^{(2)}$ and $x^{(6)}$

① ⇒ Cluster 1: $x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}$

Cluster 2: $x^{(5)}, x^{(6)}$

② ⇒ new centroid 1: $x = 2.5$

new centroid 2: $x = 5.5$

③ ⇒ Clusters remain the same. As we can see, the two clusters contain different numbers of element.

As a result, k-means might be not global optimal.