Formalising Euler Method in Isabelle

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Formal Analysis of ODE

- The paper *Numerical Analysis of Ordinary Differential Equations in Isabelle/HOL* aims to formalise initial value problems (IVPs) of ODEs and prove the Picard-Lindelöf theorem.
- They use a generic one-step methods for numerical approximation of the solution and give an executable specification of the Euler method as an instance of one-step methods.

Picard-Lindelöf theorem

Given an ODE of the form

$$\frac{du}{dt} = f(t, u(t)), \quad u(t_0) = x_0$$

if f is *Lipschitz* continuous in u and continuous in t then $\frac{du}{dt}$ has a unique solution.
A fn. is called *Lipschitz* if it's derivative is bounded. Intutively, the fn. is limited in how fast it can change.

Proof

Integrating the ODE gives

$$u(t) - u(t_0) = \int_{t_0}^{t} f(s, u(s))ds$$

Set

$$\phi_0(t) = y_0$$

and

$$\phi_{k+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_k(s))ds$$

Using Banach fixed point theorem, it can be shown that the sequence ϕ_k is *convergent* and the limit is the solution. [1]

One-step Method

One-step method approximates a function (the solution) in discrete steps, each step operating exclusively on the results of one previous step. For one-step methods in general, one can give assumptions under which the method works correctly [2]

- **Consistency**:- If the error in one step goes to zero with the step size, the one-step method is called consistent
- **Convergence**:- The global error, the error after a series of steps, goes to zero with the step size
- **Stability**:- For efficiency reasons, we want to limit the precision of our calculations which causes rounding errors.The error between the ideal and the perturbed function goes to zero with the step size.

One-step Method

- It can be proved that a consistent one-step methods are convergent and stable.For detailed proof refer [2]
- We show that Euler method is consistent and can therefore be used to approximate IVPs.
- We show that the Euler method is convergent which proves the Picard-Lindelöf theorem

$$u'(t) = f(t, u(t))$$
$$gf_j \approx u(t_j)$$
$$gf_{j+1} = L(gf_j, h, t_j)$$
$$L(gf_j, h, t_j) = gf_j + h * f(t_j, gf_j) \quad (h = t_{j+1} - t_j)$$
$$\text{Local Error} = ||u(t_{j+1}) - L(gf_j, h, t_j)||$$

A one-step method is considered consistent with u of order p if the local error is in $\mathcal{O}(h^{p+1})$ (i.e) $\rightarrow$ ① [3]

$$||u(t_{j+1}) - L(gf_j, h, t_j)|| \leq B.h^{p+1}$$

where B is a constant

Using the Taylor series expansion

$$f(x) = f(a) + (x-a) * f'(a) + \frac{(x-a)^2}{2} * f''(a) + \cdots$$

$$\implies f(x) - (f(a) + (x-a) * f'(a)) = \mathcal{O}(h^2)$$

Euler Method:

$$u(t+h) = u(t) + h * (u'(t))$$

$\implies$ The Local Error of Euler Method is in $\mathcal{O}(h^2)$.
From ① we say that,

Euler Method is consistent and therefore convergent and stable

References

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