CS754 Assignment 2

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Question 1.

Claim: If δ_{2s} of $\Phi = 1$, then $2s$ columns of Φ may be linearly dependent.

Proof:

$\delta_{2s} = 1 \Rightarrow (1-1)\|x\|_2^2 \leq \|\Phi x\|_2^2 (1+1)$

$\Rightarrow 0 \leq \|\Phi x\|_2^2$ where x is a $2s$-sparse vector

If there exists a $2s$ sparse vector x_1 such that

$$\|\Phi x_1\|_2^2 = 0$$

Then, $\|\sum_{i \in S} \Phi_i x_{1i}\| = 0$ where S is the support set of x_1 & $|S| = 2s$

This means that a linear combination of some $2s$ columns of Φ is 0 [$\forall i \in S \; x_{1i} \neq 0$].

Hence, if $\delta_{2s} = 1$, then $2s$ columns of Φ may be linearly dependent

Q1)

b) $\| \Phi(x^* - x) \|_{\ell_2} = \| \Phi(x^*) - y + y - \phi(x) \|_{\ell_2}$

As "ℓ_2" norm is similar to distance between two points

$\Rightarrow$ we can apply "triangle inequality" here.

$\therefore \| (\Phi(x^*) - y) + (y - \Phi(x)) \|_{\ell_2} \leq \| \Phi x^* - y \| + \| y - \Phi x \|$

$\Rightarrow \| \Phi(x^* - x) \|_{\ell_2} \leq \| \Phi x^* - y \|_{\ell_2} + \| y - \Phi x \|_{\ell_2}$

$\Rightarrow$ and as we know $\| y - \Phi x \|_{\ell_2} \leq \varepsilon$ for both 'x', 'x^*'

as 'x' is the true image, and x^* is estimated output by above restriction.

$\therefore \| y - \Phi x \|_{\ell_2} \leq \varepsilon$ and $\| y - \Phi x^* \|_{\ell_2} \leq \varepsilon$

$\Rightarrow \| y - \Phi x \|_{\ell_2} + \| y - \Phi^* x \|_{\ell_2} \leq 2\varepsilon$

$\therefore \| \phi(x^* - x) \|_{\ell_2} \leq \| \Phi x^* - y \|_{\ell_2} + \| \Phi x - y \|_{\ell_2} \leq 2\varepsilon$

Equation-1

[Note:- $\| \Phi x - y \|_{\ell_2} = \| y - \Phi x \|_{\ell_2}$]

3) We need to show

$$\|h_{T_j}\|_{\ell_2} \le s^{1/2} \|h_{T_j}\|_{\ell_\infty} \le s^{-1/2} \|h_{T_{j-1}}\|_{\ell_1} \quad \text{— (Equation 2)}$$

<u>Case - 1)</u>:

$$\|h_{T_j}\|_{\ell_\infty} = \lim_{t\to\infty} \left(\sum_{k=0}^{S-1} |a_k^t| \right)^{1/t}$$

here, each $a_i \; \forall i \in \{0, 1, \ldots, S-1\}$ are non zero coefficient in "h_{T_j}".

Let $|a_m| = \underset{\forall i \in \{0,\ldots,S-1\}}{\text{maximum}} |a_i|$. i.e $\forall i \in \{0,\ldots,S-1\}$ $|a_i| \le |a_m|$

$$\therefore \lim_{t\to\infty} \left(\sum_{k=0}^{S-1} |a_k^t| \right)^{1/t} = \lim_{t\to\infty} |a_m| \cdot \left(1 + \sum_{k=0, k\ne m}^{S-1} \left| \left(\frac{a_k}{a_m} \right)^t \right| \right)^{1/t}$$

let $y = \lim_{t\to\infty} \left(1 + \sum_{k=0, k\ne m}^{S-1} \left| \left(\frac{a_k}{a_m} \right)^t \right| \right)^{1/t}$

here $\lim_{t\to\infty} \left| \left(\frac{a_k}{a_m} \right)^t \right| = 0 \quad \forall k \in \{0, 1, \ldots, S-1\} - \{m\}$ as $\left| \frac{a_k}{a_m} \right| < 1$

and as $t \to \infty \Rightarrow 1/t \to 0$

$$\Rightarrow \log(y) = \lim_{t\to\infty} \frac{1}{t} \cdot \log\left(1 + \sum_{k=0, k\ne m}^{S-1} \left| \left(\frac{a_k}{a_m} \right)^t \right| \right)$$

as $t \to \infty$ $\sum_{k=0, k\neq m}^{s-1} \left| \left(\frac{a_k}{a_m}\right)^t \right| = 0$

and $\log\left(1 + \sum_{k=0, k\neq m}^{s-1} \left| \left(\frac{a_k}{a_m}\right)^t \right| \right) \to 0.$

$\therefore$ $\log y = 0$ $\left\{ \text{as both } \frac{1}{t} \to 0 \text{ and } \log\left(1+\sum_{k=0,k\neq m}^{s-1} \left|\left(\frac{a_k}{a_m}\right)^t\right|\right) \text{ tend to "0"} \right]$

$\Rightarrow$ $y = 1$

$\therefore$ $\|h_{T_j}\|_{\ell_\infty} = |a_m| \cdot 1 = |a_m|$

Now, $\|h_{T_j}\|_{\ell_2} = \left(\sum_{k=0}^{s-1} |a_k|^2\right)^{1/2}$

as we know $|a_i| \leq |a_m|$ $\forall$ $i \in \{0, 1, \ldots, s-1\}$

$\Rightarrow$ $\|h_{T_j}\|_{\ell_2} \leq \left(\sum_{k=0}^{s-1} |a_m|^2\right)^{1/2}$

$\leq s^{1/2} |a_m|$

$\Rightarrow$ $\boxed{\|h_{T_j}\|_{\ell_2} \leq s^{1/2} \|h_{T_j}\|_{\ell_\infty}}$ (as $|a_m| = \|h_{T_j}\|_{\ell_\infty}$)

Case-ii)

we need to show $s^{1/2} \|h_{T_j}\|_{\ell_\infty} \leq s^{-1/2} \|h_{T_{j-1}}\|_{\ell_1}$

$\Rightarrow$ $s \|h_{T_j}\|_{\ell_\infty} \leq \|h_{T_{j-1}}\|_{\ell_1}$

Let, $\{b_0, b_2, \ldots, b_s\}$ be set of nonzero elements

in "$h_{T_{j-1}}$".

we know from defination that

$$\boxed{\forall\, m, n \quad |b_m| \geq |a_n|} \quad m \in \{0,1,\ldots,s\},\ n \in \{0,1,\ldots,s-1\}$$

(as we choose "s" largest elements in $h_{T_{j-1}}$ and then from h_{T_j}.)

$\Rightarrow \quad \forall\, i \in \{0,1,\ldots,s-1\} \quad |b_i| \geq |a_m|$ [where $|a_m|$ = maximum among $\{|a_1|, |a_2|, \ldots, |a_{s-1}|$, $|a_0|\}$]

$$\therefore \quad \|h_{T_{j-1}}\|_{\ell_1} = \sum_{i=0}^{s-1} |b_i|$$

$$\geq \sum_{i=0}^{s-1} |a_m|$$

$$\geq \sum_{i=0}^{s-1} \|h_{T_j}\|_{\ell_\infty} \quad \left(\text{as we know } \|h_{T_j}\|_{\ell_\infty} = |a_m| \right)$$

$$\Rightarrow \quad \boxed{\|h_{T_{j-1}}\|_{\ell_1} \geq s \cdot \|h_{T_j}\|_{\ell_\infty}}$$

$$\Rightarrow \quad \boxed{s^{-1/2} \|h_{T_{j-1}}\|_{\ell_1} \geq s^{1/2} \|h_{T_j}\|_{\ell_\infty}}$$

$$\therefore \quad \boxed{\|h_{T_j}\|_{\ell_2} \leq s^{1/2} \|h_{T_j}\|_{\ell_\infty} \leq s^{-1/2} \|h_{T_{j-1}}\|_{\ell_1}} \text{ — equation - ②}$$

4) as $\forall j$ $\|h_{T_j}\|_{\ell_2} \leq s^{-1/2} \|h_{T_{j-1}}\|_{\ell_1}$

$$\Rightarrow \boxed{\begin{aligned} \sum_{j\geq 2} \|h_{T_j}\|_{\ell_2} &\leq s^{-1/2} \sum_{j\geq 2} \|h_{T_{j-1}}\|_{\ell_1} \\ &\leq s^{-1/2} \left(\|h_{T_1}\|_{\ell_1} + \|h_{T_2}\|_{\ell_2} + \cdots \right) \end{aligned}} \text{ - Equation (3)}$$

as $h = h_{T_0} + h_{T_1} + h_{T_2} + \cdots$ from defination.

~~$h_{T_1} + h_{T_2} + \cdots$~~

$$\Rightarrow \|h_{T_0}\|_{\ell_1} + \|h_{T_1}\|_{\ell_2} + \|h_{T_2}\|_{\ell_1} + \cdots = \|h\|_{\ell_1}$$

(as we choose "s" different locations for each T_j)

$$\Rightarrow \sum_{j\geq 1} \|h_{T_j}\| = \|h\|_{\ell_1} - \|h_{T_0}\|_{\ell_1}$$

$\Rightarrow$ From triangle inequality $\Rightarrow$ $|a| - |b| \leq |a-b|$

$$\Rightarrow \|h\|_{\ell_1} - \|h_{T_0}\|_{\ell_1} \leq \|h - h_{T_0}\|_{\ell_1}$$

$$\Rightarrow \|h\|_{\ell_1} - \|h_{T_0}\|_{\ell_1} \leq \|h_{T_0^c}\|_{\ell_1}$$

$$\therefore \boxed{\sum_{j\geq 1} \|h_{T_j}\| \leq \|h_{T_0^c}\|_{\ell_1}}$$

$$\boxed{\therefore \sum_{j\geq 2} \|h_{T_j}\|_{\ell_2} \leq s^{-1/2} \left(\|h_{T_1}\|_{\ell_1} + \|h_{T_2}\| + \cdots \right) \leq \|h_{T_0^c}\|_{\ell_1}}$$

5)

$$h_{(T_0 \cup T_1)^c} = \sum_{j\geq 2} h_{T_j}$$

$$\| h_{(T_0 \cup T_1)^c} \|_{\ell_2} = \| \sum_{j\geq 2} h_{T_j} \|_{\ell_2}$$

$$\| \sum_{j\geq 2} h_{T_j} \|_{\ell_2} \leq \sum_{j\geq 2} \| h_{T_j} \|_{\ell_2}$$

(from extended triangle inequality . i.e $|a+b+c+\ldots| \leq |a|+|b|+|c|+\ldots$)

and from equation ③ i.e $\sum_{j\geq 2} \| h_{T_j} \|_{\ell_2} \leq s^{-1/2} \| h_{T_0^c} \|_{\ell_1}$

$$\underbrace{\| \sum_{j\geq 2} h_{T_j} \|_{\ell_2}}_{\| h_{(T_0 \cup T_1)^c} \|_{\ell_2}} \leq \sum_{j\geq 2} \| h_{T_j} \|_{\ell_2} \leq s^{-1/2} \| h_{T_0^c} \|_{\ell_1} \quad \text{- equation ④}$$

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6) As we know $\| x^* \|_{\ell_1} \leq \| x \|_{\ell_1}$ (x = true image, x^* = computed / estimated image)

$$\Rightarrow \quad \| x + h \|_{\ell_1} \leq \| x \|_{\ell_1} \quad (x^* - x = h)$$

$$\Rightarrow \quad \sum_{i\in T_0} |x_i + h_i| + \sum_{i \in T_0^c} |x_i + h_i| \leq \| x \|_{\ell_1}$$

Now, $\sum_{i\in T_0} |x_i + h_i| \geq \sum_{i\in T_0} |x_i| - |h_i|$

and

$$\sum_{i\in T_0^c} |x_i + h_i| \geq \sum_{i\in T_0^c} |h_i| - |x_i|$$

(by triangle inequality . i.e $|a+b| \geq |a| - |b|$ and $|a+b| \geq |b| - |a|$)

$$\Rightarrow \sum_{i\in T_0} (|x_i| - |h_i|) + \sum_{i\in T_0^c} (|h_i| - |x_i|) \leq \|x\|_{\ell_1}$$

$$\Rightarrow \|x_{T_0}\|_{\ell_1} - \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1} - \|x_{T_0^c}\|_{\ell_1} \leq \|x\|_{\ell_1}$$

$$\therefore \boxed{\|x\|_{\ell_1} \geq \|x_{T_0}\|_{\ell_1} - \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1} - \|x_{T_0^c}\|_{\ell_1}} \text{ - equation (5)}$$

7) as $\|x\|_{\ell_1} = \sum_i |x_i| = \sum_{i\in T_0} |x_i| + \sum_{i\in T_0^c} |x_i| = \|x_{T_0}\|_{\ell_1} + \|x_{T_0^c}\|_{\ell_1}$,

$\Rightarrow$ from equation - (5)

$$\cancel{\|x_{T_0}\|_{\ell_1}} + \|x_{T_0^c}\|_{\ell_1} \geq \cancel{\|x_{T_0}\|_{\ell_1}} - \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1} - \|x_{T_0^c}\|_{\ell_1}$$

$$\Rightarrow \boxed{\|h_{T_0^c}\|_{\ell_1} \leq 2\|x_{T_0^c}\|_{\ell_1} + \|h_{T_0}\|_{\ell_1}} \text{ - equation - (6)}$$

8) From Cauchy-Schwarz's inequality $\|h_{T_0}\|_{\ell_1} \leq s^{1/2} \|h_{T_0}\|_{\ell_2}$.

<u>Proof</u>: take as 's' sparse vector "A" such that the position's where 'h_{T_0}' has non-zero values, vector "A" has 1

(i.e if $h_{T_0} = (0,1,2,0,3)^T \Rightarrow A = (0,1,1,0,1)^T$)

$\Rightarrow$ from Cauchy-Schwarz's $\Rightarrow |\langle h_{T_0} A\rangle| \leq \|h_{T_0}\|_{\ell_2} \|A\|_{\ell_2}$

here $|\langle h_{T_0}, A\rangle| = \|h_{T_0}\|_{\ell_1}$ and $\|A\|_{\ell_2} = s^{1/2}$;

$$\therefore \|h_{T_0}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_2} \cdot s^{1/2}.$$

$\Rightarrow \quad \|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1} \leq s^{1/2}\|h_{T_0}\|_{\ell_2} + 2\|x_{T_0^c}\|_{\ell_1}$

but from equation-6 i.e. $\|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1} \geq \|h_{T_0^c}\|_{\ell_1}$

$\Rightarrow \quad \|h_{T_0^c}\|_{\ell_1} \leq s^{1/2}\|h_{T_0}\|_{\ell_2} + 2\|x_{T_0^c}\|_{\ell_1}$

$\Rightarrow \quad s^{-1/2}\|h_{T_0^c}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_2} + 2s^{-1/2}\|x_{T_0^c}\|_{\ell_1}$

now from equation-4 . i.e $\|h_{(T_0 \cup T_1)^c}\|_{\ell_2} \leq s^{-1/2}\|h_{T_0^c}\|_{\ell_1}$

$\Rightarrow \quad \|h_{(T_0 \cup T_1)^c}\|_{\ell_2} \leq \|h_{T_0}\|_{\ell_2} + 2s^{-1/2}\|x_{T_0^c}\|_{\ell_1}$

Let $e_0 = s^{-1/2}\|x_{T_0^c}\|_{\ell_1}$

$\Rightarrow \quad \boxed{\|h_{(T_0 \cup T_1)^c}\|_{\ell_2} \leq \|h_{T_0}\|_{\ell_2} + 2e_0}$ - equation-7

9) $|\langle \Phi h_{(T_0 \cup T_1)}, \Phi h \rangle| \leq \|\Phi h_{(T_0 \cup T_1)}\|_{\ell_2}\|\Phi h\|_{\ell_2}$

(from Cauchy-Schwartz inequality)

as $\|\Phi(x - x^*)\|_{\ell_2} \leq 2\varepsilon$ from equation-1

$\Rightarrow \quad \|\Phi h\|_{\ell_2} \leq 2\varepsilon$

from RIP: here $h = x - x^*$ $\Rightarrow$ 'h' is atmost '$2s$' sparse

$\Rightarrow$ $h_{(T_0 \cup T_1)}$ is atmost '$2s$' sparse

$\Rightarrow \quad \|\Phi h_{(T_0 \cup T_1)}\|_{\ell_2}^2 \leq (1+\delta_{2s})\|h_{(T_0 \cup T_1)}\|_{\ell_2}^2$

$\Rightarrow \quad \|\Phi h_{(T_0 \cup T_1)}\|_{\ell_2} \leq \sqrt{1+\delta_{2s}}\,\|h_{(T_0 \cup T_1)}\|_{\ell_2}$

From above two equation's we can say,

$$\|\Phi h_{(T_0 \cup T_1)}\|_{\ell_2} \|\Phi h\|_{\ell_2} \le 2\varepsilon \sqrt{1+\delta_{2s}} \, \|h_{(T_0 \cup T_1)}\|_{\ell_2}$$

$$\Rightarrow \boxed{|\langle \Phi h_{(T_0 \cup T_1)}, \Phi h \rangle| \le \|\Phi h_{(T_0 \cup T_1)}\|_{\ell_2} \|\Phi h\|_{\ell_2} \le 2\varepsilon \sqrt{1+\delta_{2s}} \, \|h_{(T_0 \cup T_1)}\|_{\ell_2}}$$

– (equation-⑧)

10) from $|\langle \Phi x, \Phi x' \rangle| \le \delta_{s+s'} \|x\|_{\ell_2} \|x'\|_{\ell_2}$

where x, x' are disjoint set's : s, s' are sparsity of x, x'.

$\Rightarrow$ Now, $x = h_{(T_0)}$ and $x' = h_{(T_j)}$: both h_{T_0}, h_{T_j} are 's'-sparse

from defination of h_{T_1} and h_{T_0} they are disjoint set's

$$\Rightarrow \boxed{|\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| \le \delta_{2s} \|h_{T_0}\|_{\ell_2} \|h_{T_j}\|}$$

– (equation-⑨)

11) as h_{T_0} and h_{T_1} are disjoint

$$\|h_{(T_0 \cup T_1)}\|_{\ell_2}^2 = \|h_{T_0}\|_{\ell_2}^2 + \|h_{T_1}\|_{\ell_2}^2$$

$\Rightarrow$ and $\dfrac{\|h_{T_0}\|_{\ell_2}^2 + \|h_{T_1}\|_{\ell_2}^2}{2} \ge \|h_{T_0}\|_{\ell_2} \|h_{T_1}\|_{\ell_2}$

(from AM-GM)

$$\Rightarrow (\|h_{T_0}\|_{\ell_2} - \|h_{T_1}\|_{\ell_2})^2 \ge 0$$

$$\Rightarrow \|h_{T_0}\|_{\ell_2}^2 + \|h_{T_1}\|_{\ell_2}^2 \ge 2 \|h_{T_0}\|_{\ell_2} \cdot \|h_{T_1}\|_{\ell_2}$$

$$\Rightarrow 2\|h_{T_0}\|_{\ell_2}^2 + 2\|h_{T_1}\|_{\ell_2}^2 \ge \|h_{T_0}\|_{\ell_2}^2 + \|h_{T_1}\|_{\ell_2}^2 + 2$$

We can see that from $(\|h_{T_0}\|_{\ell_2} - \|h_{T_1}\|_{\ell_2})^2 \geq 0$

$$\Rightarrow \quad 2(\|h_{T_0}\|_{\ell_2}^2 + \|h_{T_1}\|_{\ell_2}^2) \geq (\|h_{T_0}\|_{\ell_2} + \|h_{T_1}\|_{\ell_2})^2$$

$$\left(\text{as } \|h_{(T_0 \cup T_1)}\|^2 = \|h_{T_0}\|_{\ell_2}^2 + \|h_{T_1}\|_{\ell_2}^2 \right)$$

$$\Rightarrow \quad 2(\|h_{(T_0 \cup T_1)}\|_{\ell_2}^2) \geq (\|h_{T_0}\|_{\ell_2} + \|h_{T_1}\|_{\ell_2})^2$$

$$\Rightarrow \quad \boxed{\sqrt{2}\, \|h_{(T_0 \cup T_1)}\|_{\ell_2} \geq \|h_{T_0}\|_{\ell_2} + \|h_{T_1}\|_{\ell_2}}$$

– (equation - 10)

12) as $\|\Phi h_{(T_0 \cup T_1)}\|_{\ell_2}^2 = \langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle - \langle \Phi h_{(T_0 \cup T_1)}, \sum_{j \geq 2} \Phi h_{T_j} \rangle$

$$\Rightarrow \quad \langle \Phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \Phi h_{T_j} \rangle = \langle \Phi h_{T_0}, \sum_{j \geq 2} \Phi h_{T_j} \rangle + \langle \Phi h_{T_1}, \sum_{j \geq 2} \Phi h_{T_j} \rangle$$

$$\left[\begin{array}{l} \text{as } \Phi h_{T_0 \cup T_1} = \Phi h_{T_0} + \Phi h_{T_1} \\ \text{as } h_{T_0}, h_{T_1} \text{ are disjoint} \end{array} \right]$$

$$\Rightarrow \quad \langle \Phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \Phi h_{T_j} \rangle = \sum_{j \geq 2} (\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle + \langle \Phi h_{T_1}, \Phi h_{T_j} \rangle)$$

now from equation - 9 . i.e $\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle = \delta_{2S} \|h_{T_0}\|_{\ell_2} \|h_{T_j}\|_{\ell_2}$

$$\Rightarrow \quad \langle \Phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \Phi h_{T_j} \rangle = \sum_{j \geq 2} \delta_{2S} (\|h_{T_0}\|_{\ell_2} \cdot \|h_{T_j}\|_{\ell_2} + \|h_{T_1}\|_{\ell_2} \cdot \|h_{T_j}\|_{\ell_2})$$

$$\Rightarrow \langle \Phi h_{(T_0 \cup T_1)}, \sum_{j\geq 2} \Phi h_{T_j} \rangle \leq \delta_{2s} \left(\|h_{T_0}\|_{\ell_2} + \|h_{T_1}\|_{\ell_2} \right) \cdot \left(\sum_{j\geq 2} \|h_{T_j}\|_{\ell_2} \right)$$

now from equation - (10) . i.e $\|h_{T_0}\|_{\ell_2} + \|h_{T_1}\|_{\ell_2} \leq \sqrt{2} \|h_{T_0 \cup T_1}\|_{\ell_2}$

$$\boxed{\therefore \langle \Phi h_{T_0 \cup T_1}, \sum_{j\geq 2} \Phi h_{T_j} \rangle \leq \sqrt{2}\delta_{2s} \|h_{T_0 \cup T_1}\|_{\ell_2} \left(\sum_{j\geq 2} \|h_{T_j}\|_{\ell_2} \right)}$$

$\hookrightarrow$ equation-(11)

and from equation - (8) $|\langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle| \leq 2\varepsilon\sqrt{1+\delta_{2s}} \|h_{T_0 \cup T_1}\|_{\ell_2}$

as $\|\Phi h_{T_0 \cup T_1}\|^2_{\ell_2} \leq |\langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle - \langle \Phi h_{T_0 \cup T_1}, \sum_{j\geq 2} \Phi h_{T_j} \rangle|$

$$\leq |\langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle| + |\langle \Phi h_{T_0 \cup T_1}, \sum_{j\geq 2} \Phi h_{T_j} \rangle|$$

- (from triangle inequality)

and from the above equation (8) and equation-(11)

$$\Rightarrow \|\Phi h_{T_0 \cup T_1}\|^2_{\ell_2} \leq 2\varepsilon\sqrt{1+\delta_{2s}} \|h_{T_0 \cup T_1}\|_{\ell_2} + \sqrt{2}\delta_{2s} \|h_{T_0 \cup T_1}\|_{\ell_2} \cdot \left(\sum_{j\geq 2} \|h_{T_j}\|_{\ell_2} \right)$$

$\Rightarrow$ and from RIP $\Rightarrow$ $(1-\delta_{2s}) \|h_{T_0 \cup T_1}\|^2_{\ell_2} \leq \|\Phi h_{T_0 \cup T_1}\|^2$

[as $h_{T_0 \cup T_1}$ is '2s - sparse']

From above two we can say

$$\boxed{(1-\delta_{2s})\|h_{T_{OUT,1}}\|_{\ell_2}^2 \le \|\Phi h_{T_{OUT,1}}\|_{\ell_2}^2 \le \|h_{T_{OUT,1}}\|_{\ell_2}\left(2\epsilon\sqrt{1+\delta_{2s}}+\sqrt{2}\,\delta_{2s}\sum_{j\ge 2}\|h_{T_j}\|_{\ell_2}\right)}$$

equation - (12)

13) From this equation - (12).

$\Rightarrow$ d.

$$(1-\delta_{2s})\|h_{T_{OUT,1}}\|_{\ell_2} \le 2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s}\left(\sum_{j\ge 2}\|h_{T_j}\|_{\ell_2}\right)$$

and from equation - (4) that $\sum_{j\ge 2}\|h_{T_j}\|_{\ell_2} \le s^{-1/2}\|h_{T_0^c}\|_{\ell_1}$

$$\Rightarrow \boxed{\|h_{T_{OUT,1}}\|_{\ell_2} \le \alpha\epsilon + \rho s^{-1/2}\|h_{T_0^c}\|_{\ell_1}} \quad \text{- equation - (13)}$$

where $\alpha = \dfrac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}$, $\rho = \dfrac{\sqrt{2}\,\delta_{2s}}{1-\delta_{2s}}$

14) from equation - (6) . i.e $\|h_{T_0^c}\|_{\ell_1} \le \|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}$

and equation - (13) we can say

$$\|h_{T_{OUT,1}}\|_{\ell_2} \le \alpha\epsilon + \rho s^{-1/2}\|h_{T_0}\|_{\ell_1} + 2\rho s^{-1/2}\|x_{T_0^c}\|_{\ell_1}$$

Now, we know $\|h_{T_0}\|_{\ell_1} < \|h_{T_0 \cup T_1}\|_1$

(as T_0, T_1 are disjont and it's L_1 norm)

$$\Rightarrow \boxed{\|h_{T_0 \cup T_1}\|_{\ell_2} \le \alpha \varepsilon + \rho \|h_{T_0 \cup T_1}\|_2 + 2\rho e_0} \to \boxed{\text{equation-14}}$$

where $e_0 = s^{-1/2} \|x_{T_0^c}\|_{\ell_1}$.

$$\Rightarrow \boxed{\|h_{T_0 \cup T_1}\|_{\ell_2} \le (1-\rho)^{-1} (\alpha \varepsilon + 2\rho e_0)} \to \boxed{\text{equation-15}}$$

15) $\|h\|_{\ell_2} \le \|h_{T_0 \cup T_1}\|_{\ell_2} + \|h_{(T_0 \cup T_1)^c}\|_{\ell_2}$

from equation-7. i.e $\|h_{(T_0 \cup T_1)^c}\|_{\ell_2} \le \|h_{T_0}\|_{\ell_2} + 2e_0$

and. $\|h_{T_0}\|_{\ell_2} \le \|h_{T_0 \cup T_1}\|_{\ell_2}$

$$\Rightarrow \|h\|_{\ell_2} \le 2\|h_{T_0 \cup T_1}\|_{\ell_2} + 2e_0$$

and from equation-15 i.e $\|h_{T_0 \cup T_1}\|_{\ell_2} \le (1-\rho)^{-1} (\alpha \varepsilon + 2\rho e_0)$

$$\Rightarrow \|h\|_{\ell_2} \le 2(1-\rho)^{-1} (\alpha \varepsilon + 2\rho e_0) + 2e_0$$

$$\Rightarrow \|h\|_{\ell_2} \le 2(1-\rho)^{-1} \left(\alpha \varepsilon + 2\rho e_0 + (1-\rho) e_0 \right)$$

$$\Rightarrow \boxed{\|h\|_{\ell_2} \le 2(1-\rho)^{-1} (\alpha \varepsilon + (1+\rho) e_0)} - \boxed{\text{equation-16}}$$

16)

As $\boxed{\|h_{T_0}\|_{\ell_1} \le \rho \|h_{T_0^c}\|_{\ell_1} \quad \text{when } \varepsilon = 0}$ - equation (17)

and $\|h_{T_0^c}\|_{\ell_1} \le \|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}$

$\Rightarrow \|h_{T_0^c}\|_{\ell_1} \le \rho\|h_{T_0^c}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}$

$\Rightarrow \boxed{\|h_{T_0^c}\|_{\ell_1} \le 2(1-\rho)^{-1} \|x_{T_0^c}\|_{\ell_1}}$ - equation - (18)

and as $\|h\|_{\ell_1} = \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1}$

$\Rightarrow \|h\|_{\ell_1} \le \|h_{T_0^c}\|_{\ell_1} + \rho\|h_{T_0^c}\|_{\ell_1}$

$\Rightarrow \boxed{\|h\|_{\ell_1} \le (1+\rho)\|h_{T_0^c}\|_{\ell_1}}$ - equation (19)

and from equation - (18)

$\Rightarrow \boxed{\|h\|_{\ell_1} \le 2(1+\rho)(1-\rho)^{-1} \|x_{T_0^c}\|_{\ell_1}}$

equation - (20)

Question 2.

Code for a, b, d parts is present in q2_d.m and c part is in q3.m

a) Noised Image Reconstructed Image

RMSE = 5.3617e-04 using alpha = 2* max eigenvalue = 2, Lambda = 1, epsilon = 1

b) Original Image Reconstructed Image

RMSE = 0.4064; using epsilon = 0.05 and lambda = 1

The alpha value we choose was, alpha = 3* Maximum EigenValue of [A*A_t(transpose)] .

We got an RMSE of 0.4064 but it could have been reduced by running the code for a longer time with better parameters.

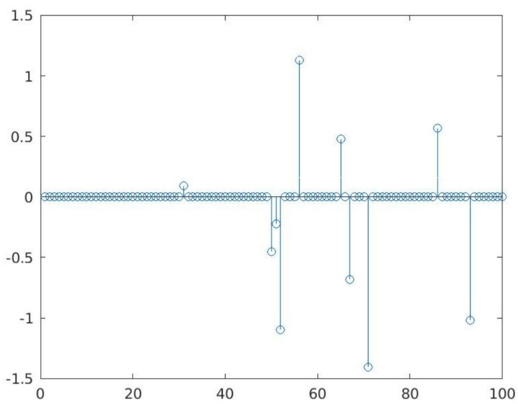
c) Original Image

Reconstructed Image

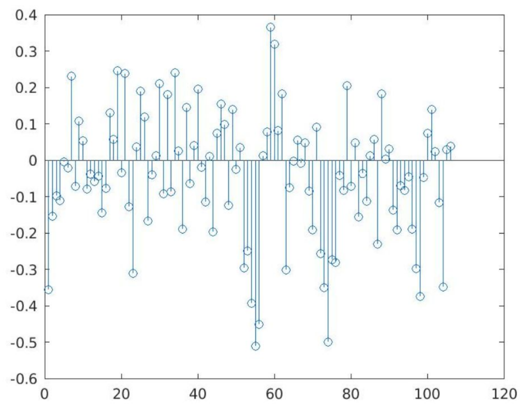
RMSE = 0.5054

The code was taking a long time to run. We reduced the number of iterations in the ISTA algorithm to reduce the computation time. Our computers were not able to run the codes for so long. Hence, we decided to get an output with low computation costs albeit with a higher error.

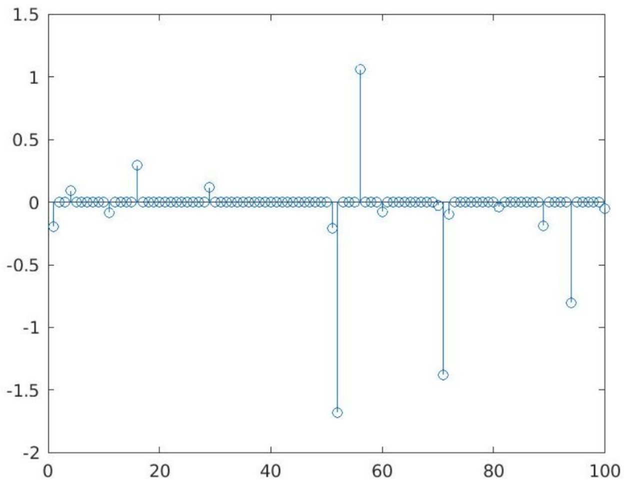
d) a)



b)



c) Using alpha = max eigenvalue of A, epsilon = 10^-7, lambda = 0.1



a) Original vector
b) Noisy vector
c) Reconstructed vector

Question 3.

For this problem we shall consider, the dimensions of $y = m \times 1$, $\phi = m \times n$, $x = n \times 1$

(a) We know,

$$y = \Phi x + \eta = \Phi_S x_S + \eta$$

where, Φ_S represents the columns corresponding to the indices in S and x_S represents a $|S| \times 1$ vector with only the non-zero values of x. Therefore,

$$y = \Phi_S x_S + \eta \implies y = \Phi_S \tilde{x}_S \implies \Phi_S^T y = \Phi_S^T \Phi_S \tilde{x}_S \implies (\Phi_S^T \Phi_S)^{-1} \Phi_S^T y = \tilde{x}_S$$

$$\tilde{x} = 0_{n\times 1};\ \tilde{x_S} = \Phi_S^{\dagger} y$$

(b) Substituting $y = \Phi_S x_S + \eta$ and using $\Phi_S^{\dagger}\Phi_S = (\Phi_S^T\Phi_S)^{-1}(\Phi_S^T\Phi_S) = I$ in the above equation, we get

$$\|\tilde{\mathbf{x}} - \mathbf{x}\|_2 = \|\tilde{\mathbf{x}}_\mathbf{S} - \mathbf{x}_\mathbf{S}\|_2 = \left\|\phi_\mathbf{S}^{\dagger}(\phi_\mathbf{S}\mathbf{x}_\mathbf{S} + \eta) - \mathbf{x}\right\|_2 = \left\|\phi_\mathbf{S}^{\dagger}\phi_\mathbf{S}\mathbf{x} + \phi_\mathbf{S}^{\dagger}\eta - \mathbf{x}\right\|_2 = \left\|\phi_\mathbf{S}^{\dagger}\eta\right\|$$

Let $\left\|\phi_\mathbf{S}^{\dagger}\right\|_2$ be the largest Singular value of $\Phi_S^{\dagger}$. By definition of singular values, for any vector x,

$$\left\|\phi_\mathbf{S}^{\dagger}\mathbf{x}\right\|_2 \leq \|\mathbf{x}\|_2 \left\|\phi_\mathbf{S}^{\dagger}\right\|_2$$

Hence,

$$\|\tilde{\mathbf{x}} - \mathbf{x}\|_2 = \left\|\phi_\mathbf{S}^{\dagger}\eta\right\|_2 \leq \left\|\phi_\mathbf{S}^{\dagger}\right\|_2 \|\eta\|_2$$

(c) δ_{2k} is defined as the minimum δ such that

$$(1-\delta)\|\theta\|^2 \leq \|\phi\theta\|^2 < (1+\delta)\|\theta\|^2$$

(c) To find an upper bound on the maximum singular value of $\phi_S^{\dagger}$ $\quad \Phi_S \to m\times s$

Firstly, if $\phi_s = USV^T$ where, U is a $m\times m$ orthonormal matrix
V is a $n \times n$ orthonormal matrix
S is a diagonal matrix with diagonal elements = singular values of Φ_s

then, $\phi_S^{\dagger} = V S_0^{-1} U^T$ $\quad S_0^{-1}$ is $s\times m$ matrix

where $S_0^{-1}(i,j) = 1/S(i,j)$ if $i=j$ & $S(i,j) \neq 0$
$= 0$ otherwise

Hence, we can say if singular values of ϕ are bounded by (a, b) then singular values of $\phi^{\dagger}$ are bounded by $(1/b, 1/a)$

Now, the product $\phi_s y$ for any $y \neq 0 \to s\times 1$ vector can be written as,

$$\phi_s y = \phi x \quad \text{where } x_S = y \text{ and } x_{S^c} = 0$$

x is a s-sparse vector.

Now, we know that

$$\sqrt{1-\delta_K} \leq \frac{\|\phi x\|}{\|x\|} \leq \sqrt{1+\delta_K}$$

Also, for any $s<t$, $\delta_s \leq \delta_t$ (Proved in Q4)

Hence, $\delta_K \leq \delta_{2K}$ and

$$\sqrt{1-\delta_{2K}} \leq \sqrt{1-\delta_K} \leq \frac{\|\phi x\|}{\|x\|} \leq \sqrt{1+\delta_K} \leq \sqrt{1+\delta_{2K}}$$

$$\Rightarrow \quad \sqrt{1-\delta_{2K}} \leq \frac{\|\phi_S y\|}{\|y\|} \leq \sqrt{1+\delta_{2K}} \qquad -①$$

[$\|x\| = \|y\|$ because $x_S = y$ and $x_{S^c} = 0$]

$\therefore$ For any $s\times 1$ vector y, equation ① holds true

Therefore, singular values of ϕ_s are bounded by $(\sqrt{1-\delta_{2K}}, \sqrt{1+\delta_{2K}})$

This implies,

singular values of ϕ_s^{+} are bounded by $\left(\frac{1}{\sqrt{1+\delta_{2K}}}, \frac{1}{\sqrt{1-\delta_{2K}}}\right)$

$(\delta_{2K}<1)$

Therefore, $$\frac{1}{\sqrt{1+\delta_{2K}}} \leq \|\phi_s^{+}\|_2 \leq \frac{1}{\sqrt{1-\delta_{2K}}}$$

(d) As $|\eta| \leq \epsilon$, we have from (b) & (c)

$$\frac{\epsilon}{\sqrt{1+\delta_{2K}}} \leq \|x - \tilde{x}\|_2 \leq \frac{\epsilon}{\sqrt{1-\delta_{2K}}} \qquad -①$$

Theorem 3 states that,

Let the solution to the problem P_1 be x^*

i.e., $\min_x \|x\|_1$ s.t. $\|y - \phi x\| \leq \epsilon$

If RIC δ_{2K} of ϕ $< \sqrt{2}-1$ then,

$$\|x^* - x\|_2 \leq C_0 \|x - x_s\| + C_1 \epsilon$$

x is s-sparse, Hence $C_0\|x - x_s\| = 0$

Hence, $0 \leq \|x^* - x\|_2 \leq C_1\epsilon$ $\qquad C_1 = \dfrac{4\sqrt{1+\delta_{2K}}}{1-\delta_{2K}(\sqrt{2}+1)}$

We have $\epsilon \leq \sqrt{1+\delta_{2K}}\, \|x - \tilde{x}\|_2$ from ①

So, $\|x^* - x\|_2 \leq C_1 \sqrt{1+\delta_{2K}}\, \|x - \tilde{x}\|_2$

$$\|x^* - x\|_2 \leq \frac{4(1+\delta_{2K})}{(1-\delta_{2K})(\sqrt{2}+1)} \|x - \tilde{x}\|_2$$

Hence, the oracular solution and the solution given by Theorem 3 differ by a constant term.

Question 4.

Consider the set of s-sparse vectors, Θ_s and the set of t-sparse vectors, Θ_t in the domain $\mathbb{R}^n$, $n > t, s$. Now, as $s < t$, $\Theta_s \subseteq \Theta_t$. This is because, Θ_t has vectors which have at most t non-zero values. This also includes the vectors which have at most s non-zero values, which is nothing but Θ_s.

δ_t is defined as the minimum δ such that

$$(1-\delta)\|\theta\|^2 \leq \|\mathbf{A}\theta\|^2 < (1+\delta)\|\theta\|^2$$

where $\theta \in \Theta_t$. Suppose, $\delta_t < \delta_s$. This means that

$$(1-\delta_t)\|\theta\|^2 \leq \|\mathbf{A}\theta\|^2 < (1+\delta_t)\|\theta\|^2$$

for $\theta \in \Theta_s \subseteq \Theta_t$. Since, δ_s is defined as the smallest number that satisfies the equation of the above form, $\delta_s \leq \delta_t$.

This is a contradiction. Hence, $\delta_s \leq \delta_t$.

Q5) **Link : http://science.fau.edu/docs/data-sci/dsaai-3-paper-35.pdf**

Question 5.

(1) The title of the paper is *Group Testingand Compressed Sensing for COVID-19 Using ddPCR*

Link : http://science.fau.edu/docs/data-sci/dsaai-3-paper-35.pdf

(2) The key objective function being minimised is

$$F(x) = \|\phi\mathbf{x} - \mathbf{y}\|_2^2 + \tau\|\mathbf{x}\|_1$$

Here, $\phi \in \mathbb{R}^{M\times N}$ is the pooling/sensing matrix given by

$$\Phi_{ij} = \frac{1}{G}\ j \in \text{Pool } i \text{ and } 0 \text{ otherwise}$$

$x \in (\mathbb{R}_{\geq 0})^N$ is the signal being measured and $y \in (\mathbb{Z}_{\geq 0})^M$ is the measured vector. y is estimated as a poisson random variable as $y \sim \text{Poisson}(\phi x)$. $\tau > 0$ is a regularization parameter.

The pooling matrices have an l_1 RIP

$$(1-\varepsilon)\|\mathbf{x}\|_1 \leq \|\phi\mathbf{x}\|_1 \leq \|\mathbf{x}\|_1$$

for some small ε.

When y is with Poisson noise, the function being minimised is

$$G(x) = \sum_{i=1}^{M} y_i - y_i \log[\phi x]_i + \tau a(x)$$

where, $x \in [\phi x]_i$ represents the ith entry of the concentrations of pools calculated assuming sample concentrations x and $a(x)$ is a penalty function. The penalty function is chosen to penalize sets of viral concentrations with higher prevalence.

$$a(x) = \|\mathbf{x}\|_{\frac{1}{2}} := \sum_{i=1}^{N} \sqrt{x}$$

(3) The tapestry pooling paper mainly focuses on tailoring the LASSO problem to obtain the signal vector. This paper optimises a slightly different function ($G(x)$) with Poisson noise.

This paper considers various penalty functions $a(x)$ such as $\|\mathbf{x}\|_1$ which basically minimises the sparsity of x. It also explores $a(x^*) = \frac{G}{L}\sum_{i=1}^{M} y_i$ for group sparsity. Although, with this penalty function, this paper considers equally sized groups where as the Tapestry pooling paper did not have any restrictions on group sizes.

This paper considers the ddPCR testing for Covid -19 and Tapestry pooling paper considers RT-PCR testing.

The pooling matrices are derived from Kirkman Triples, Reed-Solomon and random Bernoulli matrices are used. The tapestry pooling paper explored various other pooling matrices such as Expander matrices that obey RIP-1. There are many algorithms used in the Tapestry pooling paper such as COMP, NN-OMP and Sparse Bayesian Learning(EM).

Question 6.

The problem $P1$ is given by the following expression

$$min_x \|\mathbf{x}\|_1 \text{ s.t. } \|\mathbf{y}-\phi\mathbf{x}\|_2 \leq \varepsilon$$

The `LASSO` problem is given by

$$J(x) = \|\mathbf{y}-\phi\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1$$

Now, $\mathbf{x}$ is a minimizer of $J(.)$ for some $\lambda > 0$. Consider $\varepsilon = \|\mathbf{y}-\phi\mathbf{x}\|_2$. For the sake of contradiction, let us say there exists $\mathbf{x}' \neq \mathbf{x}$ such that $\mathbf{x}'$ is the minimizer of $P1$. $\|\mathbf{y}-\phi\mathbf{x}'\|_2 \leq \varepsilon = \|\mathbf{y}-\phi\mathbf{x}\|_2$ because $\mathbf{x}$ is minimizer of $P1$. Now,

$$\|\mathbf{x}'\|_1 < \|\mathbf{x}\|_1 \text{ and } \|\mathbf{y}-\phi\mathbf{x}'\|_2^2 \leq \|\mathbf{y}-\phi\mathbf{x}\|_2^2 \implies \|\mathbf{y}-\phi\mathbf{x}'\|_2^2 + \lambda\|\mathbf{x}'\|_1 \leq \|\mathbf{y}-\phi\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}'\|_1$$

$$\|\mathbf{y}-\phi\mathbf{x}'\|_2^2 + \lambda\|\mathbf{x}'\|_1 < \|\mathbf{y}-\phi\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1 \implies J(\mathbf{x}') < J(\mathbf{x})$$

This is a contradiction. Hence, minimizer $\mathbf{x}$ of $J(.)$ is also a minimizer of $P1$ problem.