✱ Countable Sets :-

① Denumerable sets :-

An infinite set is said to be denumerable if its element can be put in one - one correspondence with the set of all natural numbers.
i.e. if $\exists$ at least one function $f : A \to \mathbb{N}$ $\ni$ f is one-one and onto then set A is said to be denumurable.

② Countable sets:-

A set which is either empty or finite or denumerable then it is called countable set.

✱ Uncountable set :-

A set which is non-denumerable is called uncountable set.

Ex:- ① Let $A = \phi$ then A is countable set with $|\phi| = 0$

② $A = \{a, b, c, d\}$ then A is countable set with $|A| = 4$

③ The set of all natural number $\mathbb{N}$ is countable.

④ The set of all integers $\mathbb{Z}$ is also countable

⑤ The set of all rational numbers $\mathbb{Q}$ is also countable.

⑥ The set of all real numbers $\mathbb{R}$ is uncountable

Ex:- Prove that the set of rational numbers in $[0,1]$ is countable.

Sol:- Here, we need to show that set of rational number in $[0,1]$ i.e. $[0,1] \cap \mathbb{Q}$ is countable, we have to show that $\exists$ at least one function $f : [0,1] \cap \mathbb{Q} \to \mathbb{N}$ such that f is one-one & onto

We arrange the rational numbers of the interval according to increasing denominators as, $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5},$ ---- then the one-one correspondence can be indicated as,

$1 \leftrightarrow 0$ $\qquad$ $7 \longleftrightarrow \frac{3}{4}$

$\vdots$ $\qquad$ $\vdots$

$1 \leftrightarrow 0$

$2 \leftrightarrow 1$

$3 \leftrightarrow \frac{1}{2}$

$4 \leftrightarrow \frac{1}{3}$

$5 \leftrightarrow \frac{2}{3}$

$6 \leftrightarrow \frac{1}{4}$

$7 \leftrightarrow \frac{3}{4}$

$8 \leftrightarrow \frac{1}{5}$ ----

So, there is one-one correspondence between $[0,1] \cap \mathbb{Q}$ to $\mathbb{N}$

$\therefore$ The given set is countable.

✯ Properties of countable sets and Uncountable sets :-

① A subset of countable set is always countable

② A superset of Uncountable set is always uncountable.
i.e. if $A \subset B$ and A is uncountable then B is also uncountable

③ Union of countable sets is countable. i.e. if A and B are countable sets then $A \cup B$ is also countable.

④ Cartesian product of countable sets is also countable.
i.e. if A and B are countable then $A \times B$ is also countable.

⑤ Union of countable collection of countable set is also countable.
i.e. $\bigcup_{i=1}^{n} A_i$ is countable if each A_i is countable.

Remarks:

① Set of all real numbers in $[0,1]$ is an uncountable set hence super set of $[0,1]$ is also uncountable.

② Any interval (a,b), $[a,b]$, $[a,b)$, $(a,b]$ are uncountable sets

③ Set of all irrational numbers i.e. $\mathbb{Q}'$ is uncountable

④ $P(\mathbb{N})$ is uncountable set.

Ex:- $A = \{ (x,y) \in \mathbb{R}^2 \,/\, x = y \}$

Sol:- Here, define $f : A \to \mathbb{R}$, $f(x,y) = x$ then f is one-one & onto both. So, every element of A are in one-one correspondence with set of all real numbers. But, set of real number is uncountable. Hence, A is uncountable set

Ex - $B = \{ (a,b) \in \mathbb{R}^2 \,/\, a+b \in \mathbb{Q} \}$

Sol:- Here, define a set $A = \{(a,-a) \in \mathbb{R}^2 \mid a \in \mathbb{R}\} \subseteq B$

$f : A \to \mathbb{R}$, $f(a,-a) = a$ then f is one-one and onto.

Hence, A is uncountable set. So, B is also uncountable

Ex:- $X = \{(a,b) \in \mathbb{R}^2 \mid a \cdot b \in \mathbb{Q}\}$

Sol:- Here, define a set $Y = \{(a,0) \in \mathbb{R}^2 \mid a \in \mathbb{R}\} \subseteq X$

$f : Y \to \mathbb{R}$, $f(a,0) = a$ then f is one-one and onto.

Hence, Y is uncountable set so, X is also uncountable.

Ex:- ① $A = \{(a,b) \in \mathbb{R}^2 \mid a+b \in \mathbb{N}\}$ ② $X = \{(a,b) \in \mathbb{R}^2 \mid a \cdot b \in \mathbb{N}\}$

✱ Pigeon Hole Principle:-

Pigeon hole principle says that if there are many 'pigeons' and 'a few' pigeon holes, then there must be some pigeon holes occupied by two or more pigeons

Let A and B be finite sets and $|A| > |B|$ then for any function $f : A \to B$, $\exists\, a_1, a_2 \in A \ni f(a_1) = f(a_2)$

Sol:- ① Let $X = \{(a, 1-a) \mid a \in \mathbb{R}\} \subseteq A$

$f : X \to \mathbb{R}$, $f(a, 1-a) = a$, f is one-one and onto.

So, X is in one-one correspondence with $\mathbb{R}$, so X is uncountable. Hence, A is also uncountable

② Let $Y = \{(a, \frac{1}{a}) \mid a \in \mathbb{R}-\{0\}\} \subseteq B$

$f : Y \to \mathbb{R}-\{0\}$, $f(a, \frac{1}{a}) = a$, f is one-one and onto.

So, Y is in one-one correspondence with $\mathbb{R}-\{0\}$. So, Y is uncountable. Hence, B is also uncountable.

Ex:- If 11 shoes are selected from 10 pairs of shoes, there must be a pair of matched shoes among the selection

[Note:- Pigeon hole principal is also known as the Shoe box argument]

✱ The extended pigeon hole principle :-

If there are m pigeon holes and more than $3m$ pigeons then 4 or more pigeons will occupy atleast one of the pigeon hole. In general if the number of pigeons is much larger than the number of pigeon holes, the pigeon hole principle can be extended as,

Theorem:- If n pigeons are assign to m pigeon holes, then one of the pigeon holes must be occupied by, at least $\left[\frac{n-1}{m}\right]+1$ pigeons.

Ex:- Show that any 30 people are selected, then at least 5 people must have been born on the same day of the week.

Sol:- $n = 30$, $m = 7$. By extended pigeon hole principle, at least $\left[\frac{n-1}{m}\right]+1$ are born on the same day

$\Rightarrow \left[\frac{30-1}{7}\right]+1 = 4+1 = 5$ people

Ex:- Show that 7 colours are used to paint 50 bicycles, at least 8 bicycles will be of same colour.

Sol:- $n = 50$, $m = 7$. By extended pigeon hole principle, at least $\left[\frac{n-1}{m}\right]+1$ bicycles have same colour.

$\Rightarrow \left[\frac{50-1}{7}\right]+1 = 7+1 = 8$

Ex:- Show that if seven distinct numbers from 1 to 12 are chosen then two of them will add upto 13.

Sol:- Let $A_1 = \{1, 12\}$, $A_2 = \{2, 11\}$, $A_3 = \{3, 10\}$, $A_4 = \{4, 9\}$, $A_5 = \{5, 8\}$, $A_6 = \{6, 7\}$ are the six different set containing two distinct numbers such that the sum of two numbers is 13. Now, there are only 6 such sets, hence by pigeon hole principle two of the chosen numbers must belong to the same set.

✱ Permutation and Combination :-

⇒ The selection of object without consideration of the order of their selection is called combination.

⇒ The selection of object with consideration of the order of their selection is called permutation.

⇒ permutation is an arrangement of objects and combination is selection of objects

Ex:- Four students → A, B, C, D

Three students selected from given four students

ABC, ABD, ACD, BCD. i.e. 4 different selections are possible.

Let the first group ABC

$\overset{1}{A}\overset{2}{B}\overset{3}{C}$, $\overset{1}{A}\overset{2}{C}\overset{3}{B}$, $\overset{1}{B}\overset{2}{A}\overset{3}{C}$, $\overset{1}{B}\overset{2}{C}\overset{3}{A}$, $\overset{1}{C}\overset{2}{A}\overset{3}{B}$, $\overset{1}{C}\overset{2}{B}\overset{3}{A}$ i.e 6 different ways

similarly, we can give rank for ABD, ACD, BCD and that will have 6 different ways for giving rank 1, 2 and 3

The total number of ways to give rank $= 6 \times 4 = 24$

✱ Rule of Product:-

If one experiment has m possible outcomes and another experiment has n possible outcomes, then there are $m \times n$ possible outcomes when both of these experiment take place.

✱ Rule of sum:-

If one experiment has m possible outcomes and another experiment has n possible outcomes, then there are $m+n$ possible outcomes when exactly one of these experiment take place.

Ex:- Suppose we have 5 students and 20 chairs. Students wish to sit on these chairs

1^{st} student can have ${}^{20}C_1 = 20$ choices to sit on the chair. Now one chair is occupied

2^{nd} student can have ${}^{19}C_1 = 19$ choices to sit on the chair.

This way 3^{rd} student can have 18 choices, 4^{th} student have 17 choices and 5^{th} student have 16 choices

Now 1^{st}, 2^{nd}, 3^{rd}, 4^{th} and 6^{th} (i.e. all 5 students are sitting) the total no of distinct ways to sit on chairs

$$= 20 \times 19 \times 18 \times 17 \times 16$$

\# Defn (Permutation) :-

An arrangement in seqn. of elements of set is called a permutation of the elements

Essentially, there are three types of arrangement of elements to be considered

Type-1 :- Let $0 \le r \le n$. The number of ways to have an ordered sequence of n distinct elements, taken r at a time is called as an r-permutation of n elements and it is denoted by $P(n,r)$ or ${}^{n}P_{r}$.

First place in a seqn. can be filled up in n-ways, then second place in a seqn. can be filled up in $(n-1)$ ways and proceeding in this matter the r^{th} place can be filled up in $n-(r-1) = n-r+1$ ways.

Hence, $P(n,r)$ or ${}^{n}P_{r} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$

$$= \frac{n!}{(n-r)!}, \quad 0 \le r \le n$$

Ex:- Compute the permutation on the set $\{1,2,3,4,5\}$

Sol:- The no. of permutation of the set $\{1,2,3,4,5\}$

$$\text{is} = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5!$$

Ex:- Find the no. of permutations of A taken r at a time.

$A = \{a, b, c, d, e, f\}$, $r = 2$

Sol:- The no of permutations of A taken r at a time $= {}^{6}P_{2}$

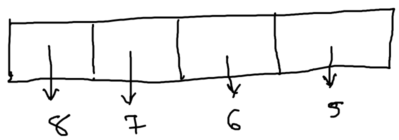
$$= \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6 \times 5 \times 4!}{4!} = 30$$

Ex:- How many 4 digit numbers can be formed out of the digits

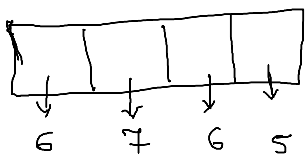
— 1, 2, 3, 4, 5, 7, 8, 9 if (i) No repetation is permitted
(ii) How many of these will be greater than 3000 ?

sol:- (i) out of 8 digits, 4 digits number can be formed in

$= 8 \times 7 \times 6 \times 5 = 1680$



(ii)

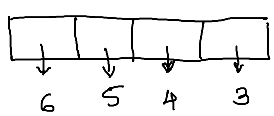


The total no. of 4 digit number, greater than 3000 can be formed in $6 \times 7 \times 6 \times 5$ different ways

$= 1260$ ways

Ex:- (i) Suppose repetations are not permitted, how many four digit numbers can be formed from six digits 1, 2, 3, 5, 7, 8 ?
(ii) How many of such numbers are less than 4000 ?
(iii) How many in (i) are even ?
(iv) How many in (ii) are odd ?
(v) How many in (i) contains both 3 and 5 ?
(vi) How many in (i) are divisible by 10 ?

sol:- (i)

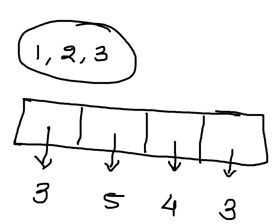


1, 2, 3, 5, 7, 8

out of 6 numbers, 4 digit numbers can be formed in

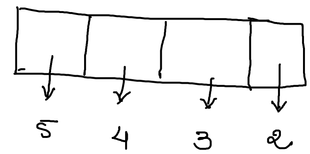
$$6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 6 \times 5 \times 4 \times 3 = 360$$

(ii) The four digit numbers which are less than 4000 are the numbers in which first digit is 1, 2 or 3. i.e. 1st digit can be chosen in 3 ways, 2nd digit can be any one of the remaining 5



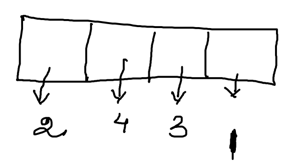
digits. 3^{rd} digit can be any one of the remaining 4 digits and the 4^{th} digit can be any one of the remaining 3 digits. Hence, the total no. of ways $= 3 \times 5 \times 4 \times 3 = 180$

(iii) Those numbers ending in 2 or 8 are even numbers. Hence, the last digit can be chosen in 2 ways (the number 2 or 8). The first digit can be fill by any one of the remaining 5 digits. 2^{nd} digit can be fill by any one of the remaining 4 digits and the 3^{rd} digit can be fill by any one of the remaining 3 digits.



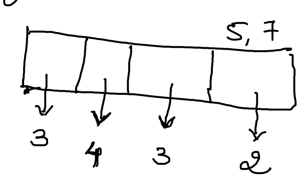
$\therefore$ The total no of ways $= 5 \times 4 \times 3 \times 2 = 120$

(iv) The numbers less than 4000 and are odd.
The numbers ending with 1, 3, 5 or 7 are odd.
The four digit number ending in 1 and less than 4000 should begin with either 2 or 3.



Then there are $2 \times 4 \times 3 \times 1 = 24$ such numbers. Similarly if the unit digit is 3 then 24 such numbers can be form. If the unit digit of the number is either 5 or 7 then the total no of ways

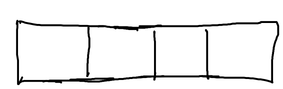
$$= 3 \times 4 \times 3 \times 2 = 72$$



Hence, the total number which are odd and less than 4000 are

$$= 24 + 24 + 72 = 120$$

(v) The digit 3 can occupy any of the 4 positions and the remaining 3 positions will be occupied by the digit 5. Hence the number of ways in which two positions are occupied by 3 and 5 will be $3 \times 4 = 12$



Now the remaining two positions will be filled by the remaining 4 numbers 1, 2, 7 and 8. Hence out of remaining two positions one can be occupied by 4 ways and second can be occupied by 3 ways.

Hence, total number of 4 digits in which both 3 and 5 are present $= 12 \times 4 \times 3 = 144$

(vi) Not even a single number is divisible by 10 as there is no zero at unit's place.

Ex:- Out of 9 cabins in office, in how many ways 4 cabins can be assigned to 4 officers.

Sol:- The first officer may choose cabin in 9 ways
second officer may choose cabin in 8 ways
Third officer may choose cabin in 7 ways and the fourth officer in 6 ways.
Hence, the total no. of distinct ways

$$= 9 \times 8 \times 7 \times 6 = 3024$$

Ex:- A menu card in a restaurant displays four soups, five main courses, three desserts and 5 beverages. How many different menus can be customer select if:

(i) He select one item from each group without omission
(ii) He chooses to omit the beverages, but select one each from the other groups
(iii) He chooses to omit the desserts but decides to take beverages and one item each from the remaining groups

Sol:- (i) The customer can select the soup in 4 ways, the main course in 5 ways, the dessert in 3 ways and beverges in 5 ways

Total no. of ways $= 4 \times 5 \times 3 \times 5 = 300$ ways

(ii) The number of ways in which he omit beverges.

$$= 4 \times 5 \times 3 = 60 \text{ ways}$$

(iii) The number of ways in which he ommits desserts but he takes all other items $= 4 \times 5 \times 5 = 100$ ways

Ex:- Suppose that repetations are not permitted, how many five digits decimal numbers can be formed

Sol:- Out of five digits, the first digit can be any of the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9. 9 9 8 7 6
the second digit can be any of the remaining 8 digits or zero

Third digit can be any of the remaining 8 digits. Fourth digit can be any of the 7 remaining and the last digit can be any of the 6 remaining

Hence, the total no. of five digits can be formed in $9 \times 9 \times 8 \times 7 \times 6$ ways

Ex:- In how many ways can three examination be scheduled within a five days period so that no two examinations are scheduled on the same day?

Sol:- The first examination can be scheduled on any of the five days. Hence, there are 5 ways to schedule first examination. Second examination can be scheduled on remaining four days. So, there are 4 ways to schedule second examination and similarly, there are 3 ways to schedule third examination.

Hence, the total no. of ways $= 5 \times 4 \times 3 = 60$ ways

Ex:- 2 mathematics papers and 5 other papers are to be arranged at an examination. Find the total no. of ways if

(i) Mathematics papers are consecutive

(ii) Mathematics papers are not consecutive

Sol:- (i) Both mathematics papers are together so, let us consider it as a single paper.

Now, 6 papers can be arranged in $6!$ ways and also both mathematics paper can be arrange in $2!$ ways

Hence, the total no. of arrangement $= 6! \times 2!$

(ii) If M_1 and M_2 are not consecutive then they are arranged between the 4 gaps or at the 2 ends.

M_1 or M_2 $\square$ M_1 or M_2 $\square$ M_1 or M_2 $\square$ M_1 or M_2 $\square$ M_1 or M_2 $\square$ M_1 or M_2

Hence, there are 6 places where mathematics papers can be arranged.

Therefore, 2 mathematics papers can be arranged in 6 places in $^6P_2 = \frac{6!}{4!} = 6 \times 5$ ways.

Five other papers can be arranged among themselves in $5!$ ways.

Therefore, total no. of arrangements

$$= 6 \times 5 \times 5! = 30 \times 120 = 3600$$

Type-II :- The general formula for the no. of ways to place r coloured balls in n number boxes, where m_1 of these of one colour, m_2 of them are of a second colour and m_r of them are of a r^{th} colour.

Here the placement of the r balls is not changed by rearranging the m_1 balls of the same colour among the boxes in which they are placed or rearranging the m_2 balls of the same colour among the boxes in which they are placed and so on.

on the other hand, if the r balls of distinct colours are arranged then it will yield a different placement.

It follows that each way to place the r not completely distinct coloured bales corresponds to $m_1!\, m_2! \ldots m_r!$ ways to place r distinct coloured balls since there are $P(n,r)$ ways to place r distinctly coloured balls in n numbered boxes, the total number of ways to place r coloured balls in n no. of boxes are

$$\frac{P(n,r)}{m_1! \cdot m_2! \ldots m_r!} = \frac{{}^nP_r}{m_1!\, m_2! \ldots m_r!}$$

Ex:- Find the number of ways to paint 15 offices so that 4 of them will be blue, 3 of them green, 5 of them yellow and remaining all are white?

Sol:-

$$n = 15, \quad r = 15$$

$$m_1 = 4, \quad m_2 = 3, \quad m_3 = 5, \quad m_4 = 3$$

$$\text{Total no of ways} = \frac{{}^nP_r}{m_1!\, m_2!\, m_3!\, m_4!} = \frac{15!}{4!\; 3! \cdot 5! \cdot 3!}$$

Ex:- Find the number of permutations obtained by arranging all letters of the word 'COMBINATION'.

Sol:- $n = 11$, $r = 11$, $m_1 = 2$, $m_2 = 2$, $m_3 = 2$

out of 11 letters, O, I and N are the letters repeated 2 times.

Total no of permutations are

$$= \frac{11!}{2! \cdot 2! \cdot 2!} = \frac{11!}{8}$$

Ex:- Find the no. of distinct permutations that can be formed from all the letters of each word (i) RADAR (ii) UNUSUAL

Sol:- (i) Total no. of letters in RADAR are 5, in which A & R are repeated twice.

Total no of permutations $= \frac{5!}{2! \cdot 2!} = 30$

(ii) Total no of letters in UNUSUAL are 7, in which U is repeated 3 times

Total no of permutations $= \frac{7!}{3!} = 7 \times 6 \times 5 \times 4 = 840$

Ex:- Determine the no. of ways in which letters in the word PIONEER be arranged so that two E's are always together

Sol:- Let us consider two E's as a one letter. so, in the word PIONEER, we have 6 letters. $n = 6$, $r = 6$, $m = 0$

Total no. of arrangement $= \frac{{}^nP_r}{m!} = \frac{{}^6P_6}{0!} = \frac{6!}{1} = 6!$

$= 720$

Type-iii:- The number of permutations of n elements, r at a time, when each element may be repeated once, twice, --- upto r times in any arrangement

In this case, the first place may be filled up in n ways, then Second element can also be filled up in n ways, and so on... Proceeding in this manner, the number of ways in which the r places can be filled up is n^r.

Ex:- A bit is either 0 or 1 A byte is a sequence of 8 bits Then find (i) Number of bytes (ii) Number of bytes that begin with 11 and end with 11

sol:- (i) Total number of bytes is

$$= 2 \times 2 \times 2 \times \_\_\_ \times 2 \ (8-\text{times})$$

$$= 2^8 = 256$$

$$\underset{1}{\downarrow}\ \underset{1}{\downarrow}\ \underset{2}{\downarrow}\ \underset{2}{\downarrow}\ \underset{2}{\downarrow}\ \underset{2}{\downarrow}\ \underset{1}{\downarrow}\ \underset{1}{\downarrow}$$

(ii) Since the first two and last two bits are fixed, i.e. 11, the remaining bits in the sequence are either 0 or 1.

Hence, the total no. of bytes $= 1 \times 1 \times 2 \times 2 \times 2 \times 2 \times 1 \times 1$

$$= 2^4 = 16$$

Ex:- How many auto license plates can be made if each identified by 2 letters followed by 4 digits?

sol:- The first two positions can be occupied by $26 \times 26 = 26^2$ ways.

$$\underset{26}{\downarrow}\ \underset{26}{\downarrow}\ \underset{10}{\downarrow}\ \underset{10}{\downarrow}\ \underset{10}{\downarrow}\ \underset{10}{\downarrow}$$

The next 4 positions can be occupied in $10 \times 10 \times 10 \times 10 = 10^4$ ways

Hence, the total no. of auto license plates that can be made is

$$= 26^2 \times 10^4$$

Ex:- A die is rolled four times, find the number of faces that can be appear on top

sol:- If the die is rolled once, the faces appearing on the top can be any one of the six faces 1 to 6. when it is rolled second time, again there are 6 choices for the face appearing on top and same situation for the another two cases.

$$\underset{6}{\downarrow}\ \underset{6}{\downarrow}\ \underset{6}{\downarrow}\ \underset{6}{\downarrow}$$

Hence, the total no. of ways of a face appearing on top is

$$= 6 \times 6 \times 6 \times 6 = 6^4$$

Ex:- There are 9 switches on a fuse box. How many different arrangements are there?

Sol:- Each switch has 2 possible positions on or off. Placing a 2 in each of the 9 positions we have $2^9 = 512$ different arrangements

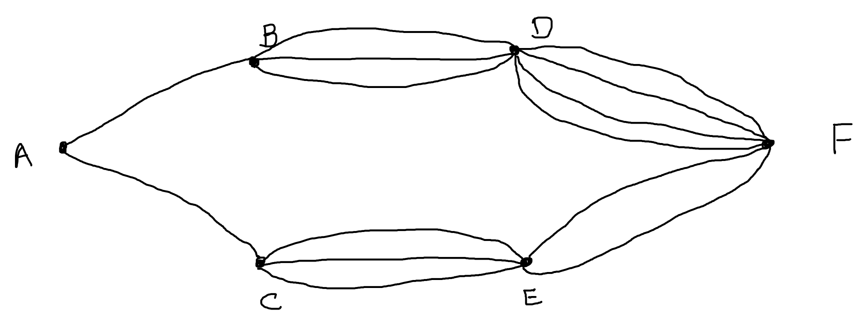
Ex:- There are 10 true/false questions on a test and 5 MCQ with 4 possible answers. How many different choices for answering the 15 questions?

Sol:- If the student attempts to answer all the questions, 5 mcq can be answered in $4\times4\times4\times4\times4 = 4^5$ ways and 10 true/false questions can be answered in $2\times2\times \cdots \times2$ (10-times) $= 2^{10}$ ways

Hence, the total number of choices for answering all 15 questions

$$= 4^5 \times 2^{10}$$

Ex:- The number of routes between few cities A, B, C, D, E and F are shown in the below figure.



Find (i) In how many ways can you go from city A to city F via city B?

(ii) In how many ways can you go from city A to city F via city C?

(iii) Find the total number of ways you can go from city A to city F.

Sol:- (i) We can go from A to B in only one way, from B to D in 3 ways and from D to F in 4 ways

Hence, total no of ways from A to F via B $= 1\times3\times4 = 12$

(ii) We can go from A to C in one way, from C to E in 3 ways and from E to F in 2 ways.

Hence, total no. of ways from A to F via C = $1 \times 3 \times 2 = 6$

(iii) Total no. of ways from A to F = $12 + 6 = 18$

✱ Combinations :-

The counting method in which order does not matter is known as combination

Selection of a set of r elements from a set of n distinct elements is called a combination $(0 \le r \le n)$. The notation for combination is ${}^nC_r$ and it is define as

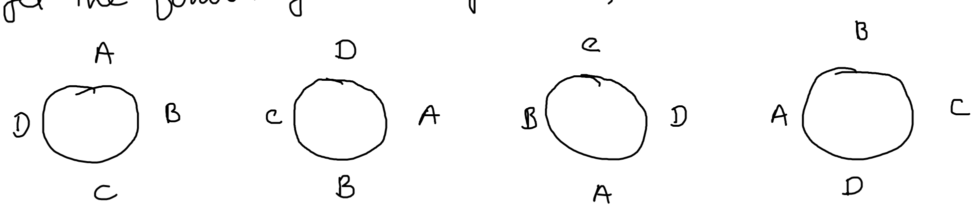
$$ {}^nC_r = \frac{n!}{r!\,(n-r)!} \qquad \underline{OR} $$

$$ {}^nC_r = \frac{{}^nP_r}{r!} $$ (Relation between permutation ${}^nP_r$ and combination ${}^nC_r$)

$$ \Rightarrow \quad {}^nP_r = r!\,({}^nC_r) $$

✱ Circular Arrangements :-

Let's consider that 4 persons A, B, C and D are sitting around a round table. Shifting A, B, C, D one position clockwise direction we get the following arrangements,



Thus, we use that if 4 persons are sitting at a round table, then they can be shifted four times, but these arrangements will be the same, because the seq^n. of A, B, C, D is same. But if A, B, C, D are sitting in a row and they are shifted, then the four linear arrangement will be different

A D C B

Shifted, [illegible]

A	D	C	B
B	A	D	C
C	B	A	D
D	C	B	A

we can see that in circular permutation these four arrangements have reduced to one arrangement. Likewise, in circular permutation of n things, a set of n linear arrangements reduce into one arrangement

Hence, the number of permutation in circular arrangement

$$= \frac{\text{No. of Permutations in linear arrangement}}{n}$$

$$= \frac{n!}{n} = \frac{n(n-1)!}{n} = (n-1)!$$

Ex:- A committee of 12 students consist of 3 representatives from first year, 4 students from second year and 5 students from third year. out of these 12 students, 3 are to be excluded from the committee by drawing lots. what is the chance that:

(i) 3 students belongs to 3 different classes
(ii) 2 belong to one class and 1 belong to another class
(iii) 3 belongs to same class.

Sol:- Among 12 students if 3 are excluded,

(i) They can be chosen as 1 from first year, 1 from second year and 1 from third year

Hence, chances $= {}^3C_1 \times {}^4C_1 \times {}^5C_1 = \frac{3!}{1!(3-1)!} \times \frac{4!}{1!(4-1)!} \times \frac{5!}{1!(5-1)!}$

$= 3 \times 4 \times 5 = 60$ ways

(ii) 2 students belong to same class and 1 from another class

$2 \rightarrow 1^{st}$ year and $1 \rightarrow$ second year or $1 \rightarrow$ Third year
$2 \rightarrow 2^{nd}$ year and $1 \rightarrow$ first year or $1 \rightarrow$ Third year } or,

$2 \to 1^{st}$ year and $1 \to$ second year or $1 \to$ Third year
$2 \to 2^{nd}$ year and $1 \to$ first year or $1 \to$ Third year } or
$2 \to 3^{rd}$ year and $1 \to$ first year or $1 \to$ Second year

$$= {}^3C_2 \cdot \left({}^4C_1 + {}^5C_1\right) + {}^4C_2 \cdot \left({}^3C_1 + {}^5C_1\right) + {}^5C_2 \cdot \left({}^3C_1 + {}^4C_1\right)$$

$$= 3(4+5) + 6(3+5) + 10(3+4)$$

$$= 27 + 48 + 70 = 145$$

(iii) 3 belong to same class then either all 3 from 1^{st} year or all 3 from 2^{nd} year or all 3 from 3^{rd} year

$$= {}^3C_3 + {}^4C_3 + {}^5C_3 = 1 + 4 + 10 = 15 \text{ ways}$$

Ex:- How many automobile license plates can be made if each plate contains two different letters followed by three digits which are different. Solve the problem if first digit cannot be zero

Sol:- 1^{st} position is a letter and that can be selected in 26 ways $= {}^{26}C_1$ ways

$$\begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 26 & 25 & 10 & 9 & 8 \end{array}$$

2^{nd} position is a letter which is different from the first position and that can be selected in 25 ways $= {}^{25}C_1$ ways

Now, in 1^{st} position of digit can be filled up by 10 digits $= {}^{10}C_1$ ways

2^{nd} position of digit can be filled up by 9 digits $= {}^{9}C_1$ ways
3^{rd} position of digit can be filled up by 8 digits $= {}^{8}C_1$ ways

Hence, the total number of license plates

$$= {}^{26}C_1 \times {}^{25}C_1 \times {}^{10}C_1 \times {}^{9}C_1 \times {}^{8}C_1 = 26 \times 25 \times 10 \times 9 \times 8$$

$$= 468000$$

Now, in license plate first digit can not be zero then 1^{st} digit can be filled up by 9 digits $= {}^{9}C_1$ ways

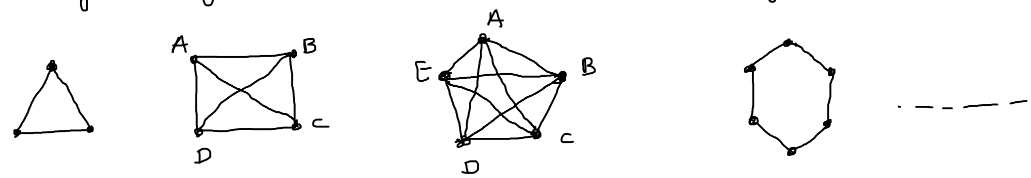
2^{nd} digit can be filled up by 9 digits = $^{7}C_1$ ways

3^{rd} digit can be filled up by 8 digits = $^{8}C_1$ ways

The total no. of license plate $= 26 \times 25 \times 9 \times 9 \times 8 = 421200$

Ex:- How many diagonals does an n sided regular polygon have?

Sol:-



A diagonal is formed by a line joining two vertices of a polygon. Hence, the number of lines joining two vertices of polygon

$= $ no. of selecting 2 vertices out of n vertices

$$= {}^{n}C_2 = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)(n-2)!}{2\,(n-2)!} = \frac{n(n-1)}{2}$$

out of these $\frac{n(n-1)}{2}$ lines, n of them are forming the sides of polygon. Therefore, the no of diagonals

$$= \frac{n(n-1)}{2} - n = \frac{n(n-1) - 2n}{2} = \frac{n(n-1-2)}{2} = \frac{n(n-3)}{2}$$

Ex:- How many rectangles are there in 8×8 chessboard?

Sol:- To make 8×8 chessboard, we need 9 horizontal lines and 9 vertical lines

$\therefore$ The no. of rectangles $= {}^{9}C_2 \times {}^{9}C_2 = \frac{9 \cdot 8}{2} \times \frac{9 \cdot 8}{2} = 1296$

Ex:- In how many ways can you take 5 cards, with atleast 2 aces, out of a well shuffled pack of 52 cards

Sol:- First consider 5 cards with exactly 2 aces. For the two aces we have ${}^{4}C_2$ possibilities and for the three cards we have ${}^{48}C_3$ possibilities

Similarly for 3 aces we have ${}^{4}C_3$ possibilities and for the 2 cards we have $^{48}C_2$ possibilities For 4 aces we have $^4C_4 = 1$ posibility and for 1 card we have $^{48}C_1$ possibilities

Hence, total number of ways

$$= \left({}^4C_2 \cdot {}^{48}C_3\right) + \left({}^4C_3 \cdot {}^{48}C_2\right) + \left({}^4C_4 \cdot {}^{48}C_1\right)$$

Ex:- There are 12 points in a plane of which 5 are colinear. Find the number of triangles that can be formed with vertices at these points

Sol:- since 5 points are collinear it means that remaining 7 points are not collinear. These 7 points among themselves can form $^7C_3 = \frac{7!}{3! \cdot 4!} = 7 \times 5 = 35$ triangles

Taking any two points which are collinear and 1 non collinear point can form one triangle similarly taking 2 non collinear points and 1 collinear point, we can form triangle.

For first case, we can select 2 points which are collinear in $^5C_2$ ways and 1 point which is non collinear in $^7C_1$ ways

Total possible triangles = $^5C_2 \times {}^7C_1 = \frac{5!}{2! \times 3!} \times 7 = 5 \times 2 \times 7 = 70$

For the second case, we can select 1 point which is collinear in $^5C_1$ ways and 2 points which are non collinear in $^7C_2$ ways.

Total possible triangles = $^5C_1 \times {}^7C_2 = 5 \times \frac{7!}{2! \cdot 5!} = 5 \times 7 \times 3 = 105$

Hence, total number of triangles = $70 + 105 + 35 = 210$

Ex:- There are 45 songs and you want to make a mix CD of 18 songs that must include 3 particular songs. How many different selections could you make?

Sol:- out of 45 songs, we have to choose 18 songs. In particular, selected 3 songs must be in the CD, then out of remaining 42 songs, we have to choose 15 songs, which can be selected in

$^{42}C_{15}$ ways

Ex:- A research team of 6 people is to be formed from 10 chemists, 5 politicians, 8 economists and 15 biologists. How many teams have

(i) Atleast 5 chemists?

(ii) Exactly 3 economists?

(iii) 4 chemists, but no economists?

(iv) Atleast 2 biologist?

(v) 4 economists and 2 biologists?

Sol:- (i) A team may have 5 chemists and 1 non-chemist or all 6 chemists. 5 chemists can be select in $^{10}C_5$ ways and 1 non-chemists can be select in $^{28}C_1$ ways.

The no. of ways $= {}^{10}C_5 \cdot {}^{28}C_1$

Team may contain all 6 chemists in $^{10}C_6$ ways

Hence, total no of ways in which there are atleast 5 chemists

$= {}^{10}C_5 \cdot {}^{28}C_1 + {}^{10}C_6$

(ii) 3 economists can be selected out of 8 in $^{8}C_3$ ways and remaining 3 can be selected in $^{30}C_3$ ways which are non-economists

Hence, total no. of ways $= {}^{8}C_3 \; {}^{30}C_3$

(iii) 4 chemists can be selected out of 10 in $^{10}C_4$ ways and remaining

Hence, total no. of ways $= {}^{10}C_4 \cdot {}^{20}C_2$

(iv) $\boxed{{}^{15}C_2 \cdot {}^{23}C_4 + {}^{15}C_3 \cdot {}^{23}C_3 + {}^{15}C_4 \cdot {}^{23}C_2 + {}^{15}C_5 \cdot {}^{23}C_1 + {}^{15}C_6}$

Total no. of cases to select 6 people from 38 people $= {}^{38}C_6$

If no biologist is selected in the team, that means all 6 are selected from remaining 23 people and hence, no. of ways $= {}^{23}C_6$

If one biologist is selected in a team then no. ways $= {}^{15}C_1 \; {}^{23}C_5$

Hence, no. of ways to select atleast 2 biologists

$$= {}^{38}C_6 - \left({}^{23}C_6 + {}^{15}C_1 \cdot {}^{23}C_5 \right)$$

(v) 4 economists can be selected out of 8 in ${}^{8}C_4$ ways and 2 biologists can be selected out of 15 in ${}^{15}C_2$ ways

Hence, total no. of ways $= {}^{8}C_4 \cdot {}^{15}C_2$

Theorem:-

The number of ways to fill r slots from n categories with repetation allowed is $C(n+r-1, r) = C(n+r-1, n-1)$

Ex:- How many ways can one fill a box holding 100 pieces of candy from 30 different types of candy?

Sol:- Here, $r = 100$, $n = 30$

So there are $C(100+30-1, 100) = C(129, 100) = \dfrac{129!}{100! \cdot 29!}$

different ways to fill the box

Ex:- How many ways are there if one must have atleast 1 piece of each type?

Sol:- Now, the slots are reduced to $(100-30)$ as we have to select atleast 1 piece of each type of candy, so there are

$n = 70, r = 30$

$$C(70+30-1, 70) = C(99, 70)$$

$$= \frac{99!}{70! \; 29!}$$ different ways to fill the box

Ex:- How many non-negative solutions are there to the equation $a+b+c+d = 100$

Sol:- Here, we could have 100 a's or 99 a's and 1 b or 98 a's and 2 d's etc. we see that slots are 100 and we are ranging over 4 categories. There fore, 98 a's and 2 d's etc. we see that slots are 100 and we are ranging over 4 categories. There fore, there are $(n=4,\ r=100)$

$$C(100+4-1,\ 4-1) = C(103,3) = \frac{103!}{3!\cdot 100!}$$

Ex:- How many integer solutions are there to:

$a+b+c+d=5$, when $a\geq 3,\ b\geq 0,\ c\geq 2,\ d\geq 1$?

Sol:- Here, no solution is possible for the given equation.

Ex:- How many integer solutions are there to: $a+b+c+d=15$ when $a\geq -3,\ b\geq 0,\ c\geq -2,\ d\geq -1$?

Sol:- In this case, we alter the restriction and equation so that the restrictions go away. To do this, we need each restriction ≥ 0 and balance the number of slots accordingly.

Hence, $a\geq -3+3,\ b\geq 0,\ c\geq -2+2,\ d\geq -1+1$

$$a+b+c+d = 15+3+2+1 = 21$$

Now, $a+b+c+d=21,\quad a\geq 0,\ b\geq 0,\ c\geq 0,\ d\geq 0$

so, $n=4,\ r=21$

There are $C(n+r-1,\ r) = C(4+21-1,\ 21)$

$$= C(24,21)$$

$$= \frac{24!}{21!\cdot 3!} \text{ sol}^n.$$

Theorem :-

The number of integer solutions to $a_1+a_2+\cdots+a_n=r$ when $a_1\geq b_1,\ a_2\geq b_2,\ \ldots,\ a_n\geq b_n$ is

$$C(n+r-1-b_1-b_2-\cdots-b_n,\ r-b_1-b_2-\cdots-b_n).$$

Theorem :-

The number of ways to select r things from n categories with b total restrictions on the r things is

$$C(n+r-1-b,\ r-b).$$

Corollary :-

The number of ways to select r things from n

categories with atleast 1 thing from each category is $C(r-1, r-n)$ $\quad (\because b = n)$

* Some New Results regarding combinations:-

① $^nC_r = {^nC_{n-r}}$

Proof:- R.H.S. $= {^nC_{n-r}} = \frac{n!}{(n-r)! \times (n-n+r)!}$

$$= \frac{n!}{(n-r)! \cdot r!} = \frac{n!}{r!\,(n-r)!} = {^nC_r} = \text{L.H.S.}$$

⇒ Remarks:-

① ^nC_r has a meaning only when $0 \le r \le n$

② $^nC_{n-r}$ has a meaning only when $0 \le n-r \le n$.

③ ^nC_r and $^nC_{n-r}$ are called complementary combinations.

② $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}, \quad 1 \le r \le n$

Proof:- L.H.S. $= {^nC_r} + {^nC_{r-1}} = \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$

$$= \frac{n!}{r(r-1)!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)\,(n-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n-r+1+r}{r\,(n-r+1)}\right] = \frac{(n+1)\,n!}{r\cdot(r-1)!\,(n-r+1)\,(n-r)!}$$

$$= \frac{(n+1)!}{r!\,(n-(r-1))!} = \frac{(n+1)!}{r!\,(n+1-r)!} = {^{n+1}C_r} = \text{R.H.S.}$$

⇒ Binomial Theorem:- For $n \in \mathbb{N}$

$$(a+b)^n = \sum_{r=0}^{n} {^nC_r}\, a^{n-r} b^r$$

$$= {^nC_0}\, a^n b^0 + {^nC_1}\, a^{n-1} b + {^nC_2}\, a^{n-2} b^2 + \dots + {^nC_n}\, a^0 b^n$$

The coefficients $^nC_0, {^nC_1}, {^nC_2}, \dots, {^nC_n}$ are known as binomial coefficients as they are present in expansion of $(a+b)^n$.

③ $^nC_0 + {^nC_1} + {^nC_2} + \dots + {^nC_n} = 2^n$

The sum of binomial coefficients is equal to 2^n

Proof:- Using binomial theorem,

$$(a+b)^n = {^nC_0}\, a^n b^0 + {^nC_1}\, a^{n-1} b + {^nC_2}\, a^{n-2} b^2 + \dots + {^nC_n}\, a^0 b^n$$

substituting $a = 1 = b$

$$(1+1)^n = {^nC_0}(1)(1) + {^nC_1}(1)(1) + {^nC_2}(1)(1) + \dots + {^nC_n}(1)(1)$$

$$\Rightarrow 2^n = {^nC_0} + {^nC_1} + {^nC_2} + \dots + {^nC_n}$$

4. ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots = 2^{n-1}$

Proof:- $(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + \cdots + {}^nC_n\, a^0 b^n$

Let $a = 1$, $b = -1$ then

$(1-1)^n = {}^nC_0 (1)^n (-1)^0 + {}^nC_1 (1)^{n-1} (-1) + {}^nC_2 (1)^{n-2} (-1)^2 + \cdots + {}^nC_n (1)^0 (-1)^n$

$\Rightarrow 0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - \cdots$

$\Rightarrow 0 = -[{}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots] + [{}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots]$

$\Rightarrow {}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots$

We know that ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$

$\Rightarrow [{}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots] + [{}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots] = 2^n$

$\Rightarrow [{}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots] + [{}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots] = 2^n$

$\Rightarrow 2[{}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots] = 2^n$

$\Rightarrow {}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = \frac{2^n}{2} = 2^{n-1}$

$= {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots$

5. ${}^nC_r \cdot {}^rC_m = {}^nC_m \cdot {}^{n-m}C_{r-m}$, $1 \le m \le r \le n$

Proof:- L.H.S. $= {}^nC_r \cdot {}^rC_m = \frac{n!}{r!(n-r)!} \times \frac{r!}{m!(r-m)!}$

$= \frac{n!}{m!(n-m)!} \times \frac{(n-m)!}{(n-r)!(r-m)!}$

$= {}^nC_m \times {}^{n-m}C_{r-m}$ = R.H.S

Ex:- show that ${}^nC_1 + 6({}^nC_2) + 6({}^nC_3) = n^3$

Sol:- ${}^nC_1 + 6({}^nC_2) + 6({}^nC_3) = \frac{n!}{1!(n-1)!} + 6\frac{n!}{2!(n-2)!} + 6\frac{n!}{3!(n-3)!}$

$= \frac{n!}{(n-3)!}\left[\frac{1}{(n-1)(n-2)} + \frac{6 \cdot 3}{2(n-2)} + 1\right]$

$= \frac{n!}{(n-3)!}\left[\frac{1 + 3(n-1) + (n-1)(n-2)}{(n-1)(n-2)}\right]$

$= \frac{n(\cancel{n-1})(\cancel{n-2})\cancel{(n-3)!}}{\cancel{(n-3)!}}\left[\frac{\cancel{1} + 3n - \cancel{3} + n^2 - 3n + \cancel{2}}{(\cancel{n-1})(\cancel{n-2})}\right]$

$= n[n^2] = n^3$

Ex:- If ${}^nC_3 + {}^{n+2}C_3 = {}^nP_3$ then find n. $(n=4)$

Ex:- ${}^{n}C_3 + {}^{n+2}C_3 = {}^{n}P_3$, Find n.

Sol:- ${}^{n}C_3 + {}^{n+2}C_3 = {}^{n}P_3$

$$\Rightarrow \frac{n!}{3!\,(n-3)!} + \frac{(n+2)!}{3!\,(n-1)!} = \frac{n!}{(n-3)!}$$

$$\Rightarrow \frac{n(n-1)(n-2)\cancel{(n-3)!}}{3!\,\cancel{(n-3)!}} + \frac{(n+2)(n+1)\,n\,\cancel{(n-1)!}}{3!\,\cancel{(n-1)!}} = \frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}}$$

$$\Rightarrow \frac{n(n-1)(n-2)}{6} + \frac{(n+2)(n+1)n}{6} = n(n-1)(n-2)$$

$$\Rightarrow (n-1)(n-2) + (n+2)(n+1) = 6(n-1)(n-2)$$

$$\Rightarrow (n+2)(n+1) = 5(n-1)(n-2)$$

$$\Rightarrow n^2+3n+2 = 5(n^2-3n+2)$$

$$\Rightarrow n^2+3n+2 = 5n^2-15n+10$$

$$\Rightarrow 5n^2-15n+10-n^2-3n-2 = 0$$

$$\Rightarrow 4n^2-18n+8=0 \Rightarrow 2n^2-9n+4=0$$

$$an^2+bn+c=0 \Rightarrow n = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{9\pm\sqrt{81-32}}{4} = \frac{9\pm\sqrt{49}}{4} = \frac{9\pm7}{4}$$

$$= 4, \frac{1}{2}$$

Here, $n = 4, \frac{1}{2}$. But $n=\frac{1}{2}$ is not possible

$\boxed{\therefore n=4}$

(2) ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r \ , \ 1 \le r \le n$

Proof:- Let A be the set containing n elements and $b \notin A$.
Let us form a new set $B = A \cup \{b\}$. Then $C(n+1, r)$ is the number of r elements subsets of B. These can be divided into two disjoint classes

(1) subsets of B not containing b

(2) Subsets of B containing b

The subset of class 1 are just r element subsets of set A and those of class 2 consist of a r-1 element subset of set A together with b. Therefore, the no. of subsets of class 1 is ${}^{n}C_r$ and the no. of subsets of class 2 is ${}^{n}C_{r-1}$.

$\therefore$ Total subsets of r elements $= {}^{n}C_r + {}^{n}C_{r-1}$

Hence, ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

(3) ${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \text{---} + {}^{n}C_n = 2^n$

Proof:- Let A be the set of n elements. Then the total number of subsets of $A = 2^n$

Number of subsets of A containing no element (i.e. empty set) is 1 or ${}^{n}C_0$

Number of subsets of A containing 1 element are n or ${}^{n}C_1$

Similarly, the no. of subsets of A containing 2, 3, 4, --, n elements will be ${}^{n}C_2, {}^{n}C_3, {}^{n}C_4, \text{---}, {}^{n}C_n$.

Hence, total number of subset of A,

$${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 + \text{---} + {}^{n}C_n$$

$\therefore$ ${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \text{---} + {}^{n}C_n = 2^n$