Section 2.7

Precise Definitions of Limits

In this section we discuss the more precise definition of limits by transforming the concept of limit into solid mathematical foundation.

Assume the function f is defined for all x near a except possible at a. We will now define mathematically, what we mean with $\lim_{x\to a} f(x) = L$ for a real number L.

The distance between $f(x)$ and L is given by $|f(x) - L|$ and the distance between x and a is given by $|x - a|$.

Therefore we can write $\lim_{x\to a} f(x) = L$ if we can make $|f(x) - L|$ arbitrary small for any x, distinct from a with $|x - a|$ sufficiently small.

For example, if we want $|f(x) - L|$ to be less than 0.01, then we must find a number $\delta > 0$ such that

$$|f(x) - L| < 0.01 \quad \text{whenever} \quad |x - a| < \delta \quad \text{and} \quad x \neq a.$$

For example, if we instead want $|f(x) - L|$ to be less than 0.001, then we must find another number $\delta > 0$ such that

$$|f(x) - L| < 0.001 \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

For the limit to exist, then for any $\epsilon > 0$, we can always find a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Problem
Given the function $f(x) = \frac{(0.5x+1)(x-4)}{(x-4)}$. We are interested in the limit $\lim_{x\to 4} f(x) = 3$. For each value of $\epsilon > 0$, determine a value of $\delta > 0$ satisfying the statement

$$|f(x) - 3| < \epsilon \quad \text{whenever} \quad 0 < |x - 4| < \delta$$

for

(A) $\epsilon = 1$
(B) $\epsilon = \frac{1}{2}$.

Definition. Limit of a Function
Assume $f(x)$ is defined for all x in some open interval containing a, except possibly at a. We say the **limit of** $f(x)$ **as** x **approaches** a **is** L, written

$$\lim_{x \to a} f(x) = L$$

if for any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Steps for proving that $\lim_{x \to a} f(x) = L$

1. Find δ. Let ϵ be an arbitrary positive number. Use the inequality $|f(x) - L| < \epsilon$ to find a condition of the form $|x - a| < \delta$, where δ depends only on the value of ϵ.
2. Write a proof. For any $\epsilon > 0$, assume $0 < |x - a| < \delta$ and use the relationship between ϵ and δ in Step 1 to prove that $|f(x) - L| < \epsilon$.

Problem Prove that $\lim_{x \to 3}(2x + 1) = 7$ using the precise definition of a limit.

Problem (exercise 2.7.33 modified)

Prove that $\lim_{x \to 4} \frac{1}{x} = \frac{1}{4}$ using the precise definition of a limit.

Definition. Two-sided Infinite Limit

The infinite limit $\lim_{x \to a} f(x) = \infty$ means that for any positive number N, there exists a corresponding number $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Steps for proving that $\lim_{x \to a} f(x) = \infty$

1. Find δ. Let N be an arbitrary positive number. Use the statement $f(x) > N$ to find a condition of the form $|x - a| < \delta$, where δ depends only N.

2. Write a proof. For any $N > 0$, assume $0 < |x - a| < \delta$ and use the relationship between N and δ in Step 1 to prove that $f(x) > N$.

Problem Let $f(x) = \frac{1}{(x-3)^2}$. Prove that $\lim_{x \to 3} f(x) = \infty$.