Section 3.1

Introducing the Derivative

In this section we introduce the concept of the derivative. Suppose the function f represents a quantity of interest, for example the height in feet of a ball above the ground as a function of time or the volume of water in gallons in a tub after t minutes. Then the derivative, denoted f', represents the slope of the curve $y = f(x)$ as it changes with respect to x. Equivalently, $f'(x)$ represents the instantaneous rate of change of $f(x)$ with respect to x.

Recall that if $s(t)$ represents the position of an object as a function of time t, then the average velocity of the object over the time interval $[a, t]$ is given by

$$v_{av} = \frac{s(t) - s(a)}{t - a}.$$

The instantaneous velocity at time $t = a$ is the limit of the average velocity as $t \to a$:

$$v_{inst} = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

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Definition. Rates of Change and the Slope of the Tangent Line
The **average rate of change** in f on the interval $[a, x]$ is the slope of the corresponding secant line,

$$m_{sec} = \frac{f(x) - f(a)}{x - a}.$$

The **instantaneous rate of change** in f at a is

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

which is the **slope of the tangent line** at $(a, f(a))$, provided this limit exists. The equation of the tangent line at $x = a$ is

$$y - f(a) = m_{tan}(x - a)$$

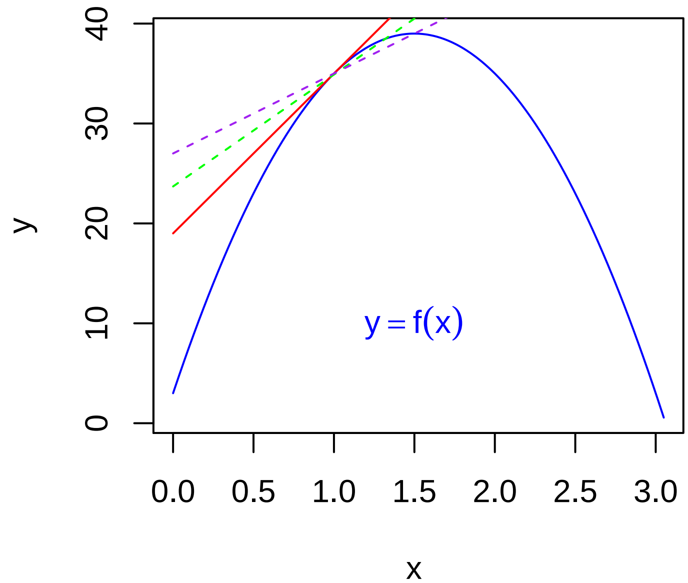

Figure 1: The secant lines (dashed lines) approach the tangent line (in solid red) as x approaches 1

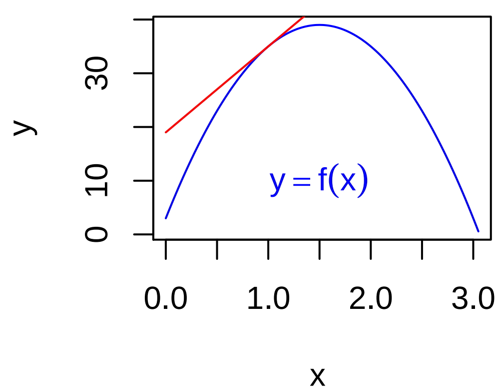

Figure 2: The red line is the tangent line to the blue curve at x=1

Thus, as x approaches a, the slopes of the secant lines approach the slope of the tangent line.

Problem Find the equation of the tangent line to the graph of $f(x) = \frac{2}{x}$ at $x = 1$

Slope:

$$m_{tan} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

Definition. Alternative Definition of Rates of Change and the Slope of the Tangent Line

The **average rate of change** in f on the interval $[a, a+h]$ is the slope of the corresponding secant line,

$$m_{sec} = \frac{f(a+h) - f(a)}{h}.$$

The **instantaneous rate of change** in f at a is

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

which is the **slope of the tangent line** at $(a, f(a))$, provided this limit exists.

The Derivative
The derivative of a function f at a point a is the instantaneous rate of change of f at a.

Definition. The Derivative of a Function at a Point
The **derivative of f at** a, denoted $f'(a)$, is given by either of the two following limits, provided the limits exist and a is in the domain of f,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \text{or} \qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If $f'(a)$ exists, we say that f is **differentiable** at a.

Problem Let $f(x) = 2x^2$.

(A) Find $f'(3)$

(B) Find the equation of the line tangent to the graph f at $x = 3$.

Problem Let $f(t) = \frac{1}{t+2}$.

(A) Find $f'(2)$

(B) Find the equation of the line tangent to the graph f at $t = 2$.

Problem (exercise 3.1.48)
Find the derivative of the function at the given point and interpret the physical meaning of this quantity. Include units in your answer:
When a faucet is turned on to fill a bathtub, the volume of water in gallons in the tub after t minutes is $V(t) = 3t$. Find $V'(12)$.

References

[1] Briggs, W.; Cochran, L.; Gillett, B.; Schulz, E., Calculus, Early Transcendentals. Third edition, Pearson Education, Inc., 2019

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