Section 3.2

The Derivative as a Function

In this section we extend the concept of the derivative at a point to all points in the domain of a function f to create a new function called the derivative of f.

Definition. The Derivative Function
The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists and x is in the domain of f. If $f'(x)$ exists, we say that f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

Problem
Find the derivative of $f(x) = x^2 + 2$.

Derivative Notation

We have several notations for the derivative. A standard notation for change is the symbol Δ (Greeek letter, delta). We can write change in x by Δx. We can write change in y as $\Delta y = f(x + \Delta x) - f(x)$. Thus, the slope of the secant line through the points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$ is given by

$$m_{sec} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}.$$

By letting $\Delta x \to 0$, the slope of the tangent line at $(x, f(x))$ is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

Thus, we can write $\frac{dy}{dx}$ for $f'(x)$.

Other notation for the derivative are

$$\frac{df}{dx}, \qquad \frac{d}{dx} f(x), \qquad D_x(f(x)), \qquad y'(x).$$

Notation that represent the derivative of f at a are

$$f'(a), \qquad y'(a), \qquad \left.\frac{df}{dx}\right|_{x=a}, \qquad \left.\frac{dy}{dx}\right|_{x=a}.$$

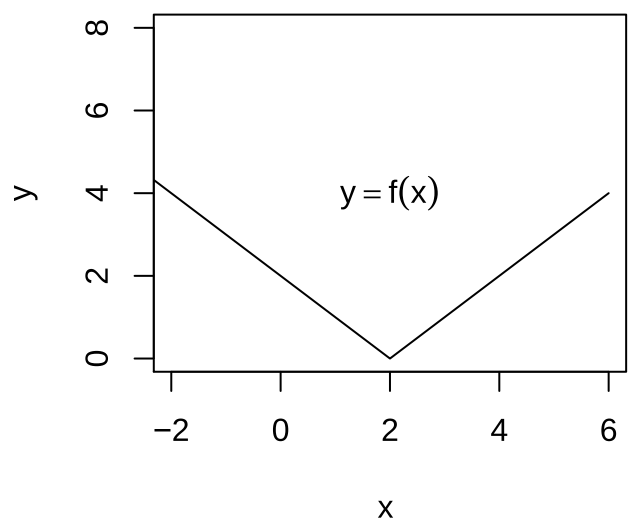
Problem Let $y = f(x) = \frac{1}{x^2}$.
(A) Compute $\frac{dy}{dx}$.
(B) Find the equation of the tangent line to the graph of f at $(1, 1)$.

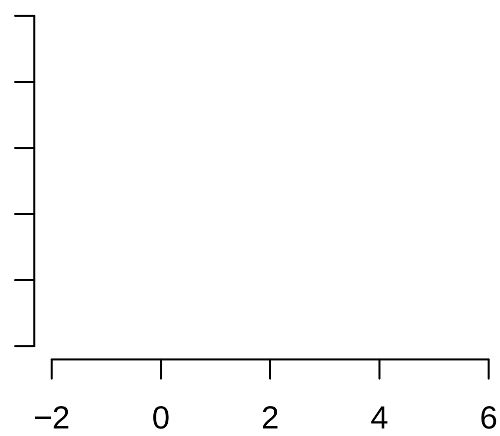
Problem
Let $g(t) = \frac{1}{\sqrt{t}}$. Compute $g'(t)$.

Graphs of Derivatives

Problem

Sketch the graph of f' using the graph of f.





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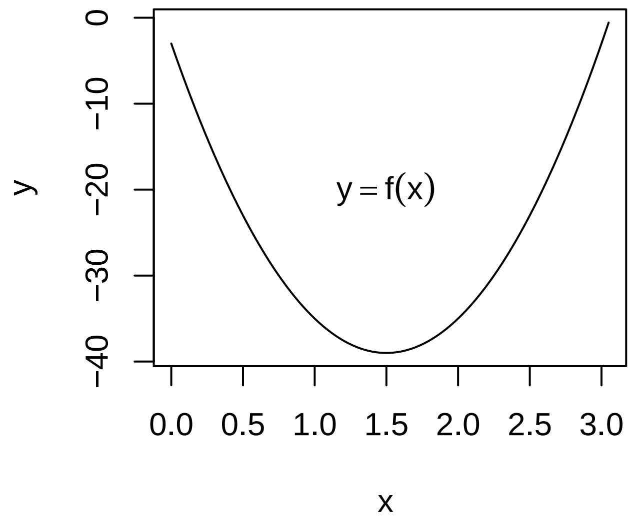

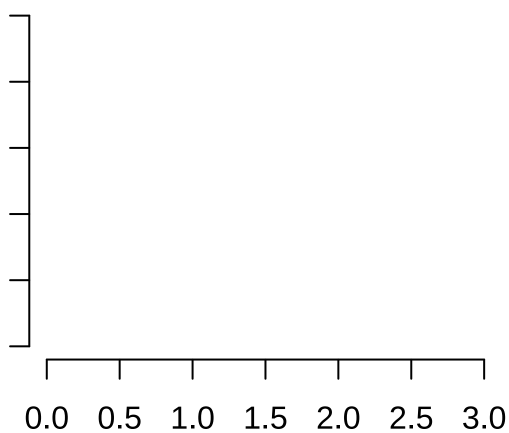

Problem
Sketch the graph of f' using the graph of f for

$$f(x) = \frac{x^2 - 4}{x^2 - 1}.$$

Theorem 0.1. *If f is differentiable at a, then f is continuous at a.*

Theorem 0.2. ***(Alternative version)***
If f is not continuous at a, then f is not differentiable at a.

When is a Function not Differentiable at a Point
A function f is not differentiable at a if at least one of the following conditions hold:

- f is not continuous at a.
- f has a corner at a.
- f has a vertical tangent at a.

Problem (exercise 3.2.56)
Consider the logistic curve in the figure, where $P(t)$ is the population at time $t \geq 0$.
(A) At approximately what time is the rate of growth P' the greatest?
(B) Is P' positive or negative for $t \geq 0$?
(C) Is P' an increasing or decreasing function of time (or neither)?
(D) Sketch the graph of P'.

References

[1] Briggs, W.; Cochran, L.; Gillett, B.; Schulz, E., Calculus, Early Transcendentals. Third edition, Pearson Education, Inc., 2019

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