Final Project

CS 241 Spring 2021

April 2021

1 Cutting Edge Research

For this final project, you will work in groups of yourself and up to 2 other students of your choosing (solo work is fine) to look into some interesting open problem in the field of computer science or mathematics. For this problem I would like you to write

1. A background section detailing in simple terms the field and necessary information to understand your results
2. A results section detailing in simple terms the prior proofs/experiments performed in the past
3. A summary section detailing potential consequences of a proof of this problem, and how it could impact a sector of computer science

These sections do not need to be long, as they are just so that I can sufficiently understand the following section. I do not have a minimum required length, however note that if you have each of these sections as one single sentence such that it appears you have only a surface level understanding on the material, I will grade accordingly. These three sections will be worth 50% of your final exam grade, and will be graded on how well you describe the material to me, as well as how well I believe you understand the material.

Next I ask that you take the results and document the process of you "playing" with the results. Most research is gained through insight into other's cutting edge research, and through new perspectives that the original researchers did not consider. Playing with the results can be, but are not limited to, any of the following

- Expanding on the results of a paper (better error bounds, considering different cases, etc.)
- Applying new research to a problem
- Using your class knowledge to look at a problem in a new way
- Applying techniques from another field to a problem

Numerically computing large amounts of examples will not be good enough this semester. For example if you are working on Twin Primes you cannot ust calculate the first 100 twin primes.

This section will be also 50% of your final exam grade, and will consist of how well you articulate your thought process while attempting to expand on the results of the paper. You can get negative results (ie the stuff you came up with failed / is uninteresting) but so long as I can tell you made a clear attempt to attack the problem in a way you found interesting, and legitimately tried, I will consider that sufficient.

All research papers should be done in real life using LaTeX, and while I cannot force you to use Latex (without being an asshole at least) I can offer that if you write your research paper in Latex I will give it +5 extra credit points.

If your team has more than 1 member, each team member must have their own section in the paper dedicated to their attempt to play with the problem. If your team is attempting something very difficult and long, it will be fine to have several members work on the same section, but it should be challenging enough to warrant more than one person's attempt. Only 1 background/results/summary section is needed per paper.

2 Potential Problems

In this section I will write out some problems that are simple enough for undergraduate students to understand (or in the case of the millenium problems, different formulations y'all might get)

2.1 Weak Sendov's Conjecture

Field: Calculus, Analysis

Suppose we have a polynomial $p(x)$ such that all the roots of the polynomial p exist on the interval $[-1, 1]$. Let $c \in [-1, 1]$ be a critical point of p, where a critical point is defined to be a point at which $p'(c) = 0$. Is it true that every critical point of p is at least of distance 1 away from at least one root of p?

Formulating this in more mathy lingo. Let the set of roots and critical points be defined as follows for some given polynomial p

$$\begin{aligned} R(p) &= \{x|p(x) = 0\} \\ C(p) &= \{x|p'(x) = 0\} \end{aligned} \tag{1}$$

where $R(p) \subset [-1, 1]$. $\forall c \in C(p)$ does $\exists x \in R(p)$ such that $|c - x| \leq 1$?

2.2 Twin Prime Conjecture

Field: Number Theory

We call a pair of prime numbers that have a difference of two "twin primes". Examples of twin primes include

$$(3, 5), (5, 7), (11, 13), (17, 19), \ldots. \tag{2}$$

Are there an infinite number of twin prime pairs?

Writing this out in a mathy way. Define the set of twin primes to be

$$T = \{(p, p+2) | p, p+2 \text{ are prime}\}. \tag{3}$$

Is $|T|$ infinite?

2.3 Collatz Conjecture

Field: Number Theory

Start with any number $n \in \mathbb{N}$ and apply the following process: if n is even, divide it by 2, if n is odd multiply by 3 and add 1. Repeat this process indefinitely. As an equation this would be if we start with the sequence term $s_0, s_1, s_2, \ldots$ then we would say that

$$s_{n+1} = \begin{cases} \frac{s_n}{2} & \text{n even} \\ 3s_n + 1 & \text{n odd} \end{cases}. \tag{4}$$

Will this process always eventually end up at 1 for any initial input?

2.4 P vs. NP and Polynomial Classification

The problem of P vs. NP is fairly famous in CS, so I'll spare the details of it, but I will provide an interesting reformulation of the problem that uses calculus that y'all might enjoy working on.

Given a polynomial $p(\vec{x})$ with $\vec{x} \in \mathbb{R}^n$, can you devise an algorithm that in polynomial time will return true if $\exists \vec{x} \in [0, 1]^n$ (n-dimensional cube of length 1), and false if otherwise?

p also has some special properties. First it is at most a trinomial, which means that the sum of the powers of each term cannot be more than 3. Also, p is multilinear, which means that you cannot have any individual term be to a power higher than 1. For example, here are some acceptable terms of the polynomial

$$x_1x_2x_3,\ 5x_1x_4,\ -2x_6,\ -x_8x_{11}x_{12} \tag{5}$$

where these would be invalid terms for the polynomial

$$x_1^2x_2,\ x_2x_3^3x_4,\ 5x_5^2 \tag{6}$$

Notice that this also means that for every $1 \leq j \leq n$ that $\frac{\partial^2 p}{\partial x_j^2} = 0$. In case you are still having trouble visualizing, in $\mathbb{R}^3$, every polynomial p would be of the form

$$p(\vec{x}) = c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_1x_2 + c_5x_2x_3 + c_6x_1x_3 + c_7x_1x_2x_3 \tag{7}$$

where $c_i \in \mathbb{R}$ are arbitrary coefficients.

It can be proven that the minima of p on the interval $[0, 1]^n$ must be on a corner point (input where everything is either $0, 1$), however as n increases, the number of corner points scales as 2^n which makes checking each one unfeasible.

2.5 The Riemann Hypothesis and Robin's Inequality

Again, the Riemann Hypothesis is super famous so I'll spare the details, but here is a fun reformulation that doesn't require too much extra knowledge.

Choose some natural number $n > 5040$ and write it as its prime number decomposition: $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$. Is the following inequality always true:

$$\prod_{j=1}^{m} \frac{p_j^{k_j+1} - 1}{p_j^{k_j}(p_j - 1)} \leq e^{\gamma} \log\left(\log\left(\prod_{j=1}^{m} p_j^{k_j}\right)\right) \tag{8}$$

where γ is the Euler-Mascheroni constant ($\approx .577\dots$) and log is the natural logarithm?

The following properties are true about this inequality for numbers > 5040

- Robin's Inequality is true for every odd number
- If Robin's Inequality is true for $p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, then it is true for $P_1^{k_1} P_2^{k_2} \dots P_m^{k_m}$ where $p_j \leq P_j$
- Robin's Inequality is true for any $p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ if $\forall k_j < 5$
- Robin's Inequality is true if $m < 4$
- The left hand side is bounded, but the right hand side is not, so for sufficiently large prime products, all larger exponents are true.

Showing that Robin's Inequality is true $\forall n > 5040$ is equivalent to solving the Riemann Hypothesis.

2.6 Irrationality of $\pi + e$

It is know that both π and e are irrational numbers, however it is still unknown if $\pi + e$ is also irrational.

Question, can we find a $a, b \in \mathbb{N}$ where $\pi + e = \frac{a}{b}$?

2.7 Normality of π

You know how everyone says that since π is infinite you can find any sequence of digits in it (like your phone number)? Well we actually don't know if this is true because we don't know if π is a normal number. Normality means that the digits of π are truly randomly distributed and don't depend on previous numbers. We also know that normal numbers are uncountable so most are, but we can't find any

Is π normal?

2.8 Mersenne Primes

Any prime of the form $2^k - 1$ is called a Mersenne Prime. For example, $2^5 - 1 = 31$ is a Mersenne Prime, but $2^4 - 1 = 15$ is not. Are there an infinite number of Mersenne Primes?

2.9 Subarray Sum

Given a list of positive integers L and target number T, can you write an algorithm that **runs in polynomial time** and returns true if there exists a set of numbers in L that sum to T and false if no such set exists?

2.10 Polynomial Identity Testing

Given a n dimensional polynomial, can you write an algorithm that runs in **polynomial time** that returns true if the polynomial is identically equal to 0 and false if not?

2.11 MOBA Wave Management

See the attached document