Abstract Algebra presentation on Tensor Product, Eric Fay
Rutgers University.

For a digital copy for to see source visit github.com/PersonaErazed/tensors

0.1 A brief reminder of what Modules are

Definition 0.1. Module. Let R be a ring. A set M with a binary operation $+ : M \times M \to M$ and a map $. : R \times M \to M$ is a *left R-module* iff

1. $(M, +)$ is an Abelian group
2. $. : R \times M \to M$ defines a "ring action of R on M" (scalar multiplication) i.e. for all $r, s \in R$ and for all $m, n \in M$ that
   (a) $(r + s).m = r.m + s.m$
   (b) $(r \cdot s).m = r.(s.m)$
   (c) $r.(m + n) = r.m + r.n$
3. if R is a ring with identity then $1.m = m$ for all $m \in M$

Example 0.1. Any ring

$$r.s = r \cdot s$$

Example 0.2. Abelian groups are $\mathbb{Z}$-modules

$$n.a = \begin{cases} a + \cdots + a & (n-1) \text{ addition operations} & : n > 0 \\ 0 & & : n = 0 \\ -a + \cdots + -a & (-n-1) \text{ addition operations} & : n < 0 \end{cases}$$

Example 0.3. Vector spaces are R-modules where R is a field.

Example 0.4. Collection of maps from a set X to a R-module M

$$M^X = \{f : X \to M\}$$

where for all $f, g \in M^X$ and for all $x \in X$ and for all $r \in R$

$$(f + g)(x) = f(x) + g(x) \qquad (r.f)(x) = r.f(x)$$

Example 0.5. X is a finite set e.g. $X = [n] := \{1, 2, \cdots, n\}$ for some natural number n.

Then $M^X = M^{[n]}$ is a R-module for some M R-module i.e. direct product of R-modules

$$M^n = M \times \cdots \times M$$

Example 0.6. Smooth functions on a real smooth n-manifold X such as $\mathbb{R}^n$

$$\mathcal{C}^\infty(X) = \{f : X \to \mathbb{R} | f \text{ is a smooth map } \} \subset \mathbb{R}^X$$

smooth map[1]

Example 0.7. The smooth vector fields on a manifold X is a $\mathcal{C}^\infty(X)$-module

$$\Gamma(TX) = \{v : X \to TX| \text{ smooth and } \pi \circ v = \mathrm{id}_X\}$$

where $\pi : TX \to X$ is the projection from the tangent bundle $TX = \amalg_{x\in X} T_x X$, the disjoint union of the tangent spaces, onto X.

$$(f.v)(x) = f(x).v_x$$

1 Tensor Product

1.1 Special Case: Extension of Scalar Ring

Let us consider the motivating problem presented by Dummit and Foote. Given a R-module such that the ring R is a subring of another ring S, what is the most general S-module which the given R-module can be embedded? i.e. when can a R-module to extended to a S-module? This is know as extension of scalars.

Let us call our R-module to be extended N. Recall the key elements of the definition for a $\mathcal{R}$-module, a Abelian group structure and an action of the ring $\mathcal{R}$ on the Abelian group. It is therefore natural to consider the free Abelian group $F(S \times N)$ as a candidate Abelian group to define a S ring action on and which N can be embedded.

Definition 1.1. For any ring $\mathcal{R}$, a $\mathcal{R}$-module F is **free** iff for some set X, there exists unique nonzero elements $r_1, \dots, r_k \in \mathcal{R}$ and unique elements $x_1, \dots, x_k \in X$ for each element $f \in F$, such that $f = r_1.x_1 + \cdots + r_k.x_k$

Example 1.1. Any ring $\mathcal{R}$ and any set X let

$$F(X) = \{f : X \to \mathcal{R} \mid X \setminus f^{-1}[0] \text{ finite}\} \subset \mathcal{R}^X$$

i.e. maps $X \to \mathcal{R}$ with finte support where support is defined as $\mathrm{supp}(f) := \{x \in X : f(x) \neq 0\}$. The $\mathcal{R}$-module structure is defined componentwise.

To show $F(X)$ is free define the injection $\iota : X \to F(X) : x \mapsto \delta_x$ such that

$$\delta_x(y) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}$$

It is easy to see for any $f \in F(X)$ there exists unique nonzero elements $r_1, \dots, r_k \in \mathcal{R}$ and unqiue elements $x_1, \dots, x_k \in X$ such that

$$f = r_1.\delta_{x_1} + \cdots + r_k.\delta_{x_k}$$

[1] f is a smooth map $\iff$ for all smooth charts (ϕ, U) that the following is a smooth function

$$\phi^{-1} \circ f : \phi(U \cap V) \subset \mathbb{R}^n \to \mathbb{R}$$

Remark 1.1. $\mathcal{R} = \mathbb{Z}$ then F a free $\mathbb{Z}$-module is called a free Abelian group.

With $F(S \times N)$ as our Abelian group let us define a ring action from S as for any $f \in F(S \times N)$ and $s \in S$

$$\begin{aligned} s.f &:= s.(s_1, n_1) + \cdots + s.(s_k, n_k) \\ &= (s \cdot s_1, n_1) + \cdots + (s \cdot s_k, n_k) \end{aligned}$$

Now considering the details of the definition of a general $\mathcal{R}$-module M the ring action must preserve both the ring's and Abelian group's structures. i.e.

$$\begin{aligned} &(r_1 + r_2).m = r_1.m + r_2.m \qquad &(r_1 \cdot r_2).m = r_1.(r_2.m) \\ &r.(m_1 + m_2) = r.m_1 + r.m_2 \end{aligned}$$

for any $r, r_1, r_2 \in \mathcal{R}$ and any $m, m_1, m_2 \in M$.

This leads us to define the equivalence relation $\sim$ on $F(S \times N)$ as follows

$$\begin{aligned} &(s_1 + s_2, n) \sim (s_1, n) + (s_2, n) \qquad &(s \cdot r, n) \sim (s, r.n) \\ &(s, n_1 + n_2) \sim (s, n_1) + (s, n_2) \end{aligned}$$

for any $s, s_1, s_2 \in S$, any $r \in R$, and any $n, n_1, n_2 \in N$. Note the subtle difference in the compatibility relation this is the case since N has a R-module structure.

Since equivalence relations give us a normal subgroup i.e. $a \sim b \iff a - b \in H$. Let us define the S-module we have constructed as

$$S \otimes_R N := F(S \times N)/\sim$$

let us call the cosets, tensors, the following is an example of a simple tensor

$$s \otimes n := \{f \in F(S \times N) : f \sim (s, n)\}$$

The binary Abelian operation $+$ is defined via the natural surjective group homomorphism $\pi : F(S \times N) \to F(S \times N)/\sim$

$$\begin{aligned} (s \otimes n) + (s' \otimes n') &= \pi((s, n)) + \pi((s', n')) \\ &= \pi((s, n) + (s', n')) \\ &= \{f \in F(S \times N) : f \sim (s, n) + (s', n')\} \end{aligned}$$

This yeilds the following relations

$$\begin{aligned} &(s_1 + s_2) \otimes n = s_1 \otimes n + s_2 \otimes n \qquad &(s \cdot r) \otimes n = s.(r.n) \\ &s \otimes (n_1 + n_2) = s \otimes n_1 + s \otimes n_2 \end{aligned}$$

Let us define the S ring action as

$$s_\circ \left(\sum_{i \in [n]} s_i \otimes n_i \right) = \sum_{i \in [n]} (s \cdot s_i) \otimes n_i$$

Exercise check if this is well defined, note these are cosets and check for simple tensors $s \otimes n$ that this makes $S \otimes_R N$ a S-module.

The following theorem shows this is the most general S-module to embedded the R-module N into. First we need a lemma the Universal property of free modules.

Lemma 1.1. *Given any set X, any left R-module M, and any map $\psi : X \to M$. There exists a unique R-module homomorphism from the free R-module generated by X to M*

$$\Psi : F(X) \to M \qquad s.t. \quad \psi(x) = \Psi(x) \forall x \in X$$

The proof follows since X is a generating set of $F(X)$ and each element of $F(X)$ has a unique representation of the formal form $r_1x_1 + \cdots + r_kx_k$ see example 1.1

The map is

$$\Psi : \sum_{i\in[k]} r_i x_i \mapsto \sum_{i\in[k]} r_i \psi(x_i)$$

Theorem 1.1. *Let $R \subset S$ be a subring, N be a R-module, and $\iota : N \to S \otimes_R N$ be the R-module homomorphism $n \mapsto 1 \otimes n$.*

For any left S-module L and R-module homomorphism $\phi : N \to L$ there exists a unique S-module homomorphism $\Phi : S \otimes_R N \to L$ such that $\phi = \Phi \circ \iota$

$$\begin{tikzcd} N \arrow[r, "\iota"] \arrow[dr, "\phi"'] & S \otimes_R N \arrow[d, dotted, "! \; \Phi"] \\ & L \end{tikzcd}$$

Proof. Since L is a left S-module let us define the map

$$\psi : S \times N \to L : (s, n) \mapsto s.\phi(n)$$

By the universal property of free modules, see Lemma 1.1, there exist unqiue $\mathbb{Z}$-module homomorphism from the free Abelian group to the left S-module L

$$\Psi : F(S \times N) \to L$$

Specifically

$$\begin{aligned} \Psi\left(\sum_{i\in[k]} a_i(s_i, n_i)\right) &= \sum_{i\in[k]} a_i \psi((s_i, n_i)) \\ &= \sum_{i\in[k]} \sum_{j=1}^{a_i} \psi((s_i, n_i)) \\ &= \sum_{i\in[k]} \sum_{j=1}^{a_i} s_i.\phi(n_i) \end{aligned}$$

Since ϕ is an R-module homomorphism i.e.

$$\phi(x+y) = \phi(x) + \phi(y) \qquad \text{and} \qquad \phi(r.x) = r.\phi(x)$$

Then the generators for the subgroup from the equivalence relation $\sim$ discussed previously map to zero in L. e.g. $(s_1 + s_2, n) \sim (s_1, n) + (s_2, n)$ gives a generatoring element $(s_1 + s_2, n) - (s_1, n) - (s_2, n)$ which maps to

$$\begin{aligned}
\Psi((s_1 + s_2, n) - (s_1, n) - (s_2, n)) &= \psi((s_1 + s_2, n)) - \psi((s_1, n)) - \psi((s_2, n)) \\
&= (s_1 + s_2).\phi(n) - s_1.\phi(n) - s_2.\phi(n) \\
&= s_1.\phi(n) + s_2.\phi(n) - s_1.\phi(n) - s_2.\phi(n) \\
&= 0
\end{aligned}$$

Similarly for $(s \cdot r, n) \sim (s, r \cdot n)$ and $(s, n_1 + n_2) \sim (s, n_1) + (s, n_2)$ Therefore there exists a well defined $\mathbb{Z}$-module homomorphism

$$\Phi : F(S \times N)/ \sim = S \otimes_R N \to L$$

where $\Phi(\sum s_i \otimes n_i) = \sum s_i.\phi(n_i) \in L$. Since L is a left S-module

$$\begin{aligned}
s.\Phi\left(\sum s_i \otimes n_i\right) &= s.\left(\sum s_i.\phi(n_i)\right) \\
&= \sum s.(s_i.\phi(n_i)) \\
&= \sum (s \cdot s_i).\phi(n_i) \\
&= \Phi\left(\sum (s \cdot s_i) \otimes n_i\right) \\
&= \Phi\left(\sum s.(s_i \otimes n_i)\right)
\end{aligned}$$

From the above calculation we see Φ is a S-module homomorphism. Note any S-module homomorphism is uniquely determined by the values on elements of the generating set, e.g. $1 \otimes n$ and since $\Phi(1 \otimes n) = \phi(n)$ then Φ is uniquely determined by ϕ. □

Remark 1.2. The converse of this theorem is true. If $\Phi : S \otimes_R N \to L$ is a S-module homomorphism then $\phi = \Phi \circ \iota : N \to L$ is a R-module homomorphism.

Corollary 1.1. *$\frac{N}{ker\iota}$ is the unique largest quotient of N that can be embedded in any S-module.*

1.2 General Tensor

Let us consider only the necessary properties from the construction of the most general module for a given scalar extension and apply it to a construct of a space in which the product of elements from two spaces can have a group structure and given sufficient conditions have a module structure.

Observing the relations used in the extenision of scalars when note the requirements of the constituent spaces of the construct needs an abelian group structure. By the relation $(x.r, y) \sim (x, r.y)$ the constituent spaces need a right and left ring action respectivaly. This leads us to the construction of the abelian group known as the tensor product.

Definition 1.2. Tensor Product
Let M be any right module-R and N any left R-module. Consider the equivalance relation $\sim$ on $F(M \times N)$ free Abelian group generated from $M \times N$ where

$$\begin{aligned} &(m_1 + m_2, n) \sim (m_1, n) + (m_2, n) \qquad && (m.r, n) \sim (s, r.n) \\ &(m, n_1 + n_2) \sim (m, n_1) + (m, n_2) && \end{aligned}$$

The abelian group $M \otimes_R N = F(M \times N)/\sim$ is called the **tensor product of M and N over R**

Remark 1.3. In the above definition of tensor product if the right module-R was a bimodule or has a left ring action by another ring S then the tensor product has a natural module structure. Defining it on a simple tensor as $s.(m \otimes n) := (s.m) \otimes n$

If R is a commutative ring we define $m.r := r.m$ for a right module-R M which can be shown that M is a R-bimodule. Note we can see R must be commutative since

$$(r \cdot r').m = m.(r \cdot r') = (m.r).r' = r'.(r.m) = (r' \cdot r).m$$

In the special case in extension of scalars we considered R-module homomorphisms to any S-module where S contained R. In the general case we consider a more general type of map, one which still sends the generators of kernel of $\pi : F(M \times N) \to M \otimes_R N$ e.g. $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$.

Definition 1.3. R-balanced map
Let M be any right module-R, N any left R-module, and L any Abelian group.

$\phi : M \times N \to L$ is R-balanced iff

$$\begin{aligned} &\phi(m_1 + m_2, n) = \phi(m_1, n) + \phi(m_2, n) \qquad && \phi(m, r.n) = \phi(m.r, n) \\ &\phi(m, n_1 + n_2) = \phi(m, n_1) + \phi(m, n_2) && \end{aligned}$$

Theorem 1.2. ***Universal Property of Tensor Product***
Let R be any ring with 1, M right module-R, N left R-module, and L any Abelian group.

If $\phi : M \times N \to L$ is a R-balanced map then there exists a unique group homomorphism $\Phi : M \otimes_R N \to L$ such that $\phi = \Phi \circ \iota$.

If $\Phi : M \otimes_R N \to L$ any group homomorphism then $\phi = \Phi \circ \iota : M \times N \to L$ is a R-balanced map.

Since the product of sets is associative it seems likely that the tensor product is also associative. The next theorem shows this is the case.

Theorem 1.3. ***Associativity of the Tensor Product***
Let M be any right S-module-R, N any R-module-T, and L any left T-module.

There exists a unique S-module isomorphism

$$(M \otimes_R N) \otimes_T L \cong M \otimes_R (N \otimes_T L)$$

$m \otimes (n \otimes l) \mapsto (m \otimes n) \otimes l$

Proof. Note that $M \otimes_R N$ is S-module-T and $N \otimes_T L$ left R-module. For each $l \in L$ define $\phi_l^L : M \times N \to M \otimes_R (N \otimes_T L)$

$$\phi_l^L(m, n) = m \otimes (n \otimes l)$$

$$\begin{aligned} \phi_l^L(m.r, n) &= m.r \otimes (n \otimes l) \\ &= m \otimes r.(n \otimes l) \\ &= m \otimes (r.n \otimes l) \\ &= \phi_l^L(m, r.n) \end{aligned}$$

The above calculation shows ϕ_l^L is R-balanced. By the universal property of tensor product there exist a unique homomorphism

$$\Phi_l^L : M \otimes_R N \to M \otimes_R (N \otimes_T L)$$

and therefore the following map is well defined

$$\begin{aligned} \Phi^L &: (M \otimes_R N) \times L \to M \otimes_R (N \otimes_T L) \\ &: (m \otimes n, l) \mapsto \Phi_l^L(m \otimes n) \end{aligned}$$

A simple calculatin shows that Φ^L is R-balanced and therefore by the universal property of tensor product there exist a unique homomorphism $\Phi : (M \otimes_R N) \otimes_T L \to M \otimes_R (N \otimes_T L)$. Repeat with ψ_m^M maps to get the inverse homomorphism.

To complete the proof note that S action is compatible with the T action

$$s.((m \otimes n).t) = s.(m \otimes n.t) = s.m \otimes n.t = (s.m \otimes n).t = (s.(m \otimes n)).t$$

so $M \otimes_R N$ is a S-module-T making $(M \otimes_R N) \otimes_T L$ a S-module. By the isomorphism constructed above $\Phi(s.(m \otimes n) \otimes l) = \Phi((s.m \otimes n) \otimes l) = s.m \otimes (n \otimes l) = s.(m \otimes (n \otimes l))$ □

Most spaces of interest for which we construct a tensor product have a have action by a commutative ring and therefore are bimodules. Such as vector spaces. In this case the R-balanced maps are multilinear maps.

Definition 1.4. Multilinear Map
Consider R commutative ring and $M_1, \ldots, M_n$ and L be R-modules. Then a map $\phi : M_1 \times \cdots \times M_n \to L$ is a multilinear map if and only if for each component $i \in [n]$ the map where the other entries are fixed is a R-module homomorphism.

$$\phi(m_1, \ldots, m_{i-1}, r.m_i + r'.m_i', m_{i+1} \ldots, m_n) =$$

$$r.\phi(m_1, \ldots, m_{i-1}, m_i, m_{i+1} \ldots, m_n) + r'.\phi(m_1, \ldots, m_{i-1}, m_i', m_{i+1} \ldots, m_n)$$

Corollary 1.2. *Let R be a commutative ring and $M_1, \cdots, M_n$ be R-modules. Let $M_1 \otimes \cdots \otimes M_n$ be any bracketing of the tensor product of the aforementioned modules. Let*

$$\begin{aligned} \iota &: M_1 \times \cdots \times M_n \to M_1 \otimes \cdots \otimes M_n \\ &: (m_1, \ldots, m_n) \mapsto m_1 \otimes \cdots \otimes m_n \end{aligned}$$

For any R-module L there is a bijection between multilinear maps and R-module homomorphism

$$\left\{ \underset{\textit{mulitlinear}}{\phi : M_1 \times \cdots \times M_n \to L} \right\} \left\{ \underset{\textit{R-module homomorphism}}{\Phi : M_1 \otimes \cdots \otimes M_n \to L} \right\}$$

such that the following diagram commutes

$$\begin{tikzcd} M_1 \times \cdots \times M_n \arrow[r, "\iota"] \arrow[dr, "\phi"'] & M_1 \otimes \cdots \otimes M_n \arrow[d, dotted, "! \ \Phi"] \\ & L \end{tikzcd}$$

Theorem 1.4. ***Tensor product commutes with Direct Sum***
Let M,M' be any right modules-R and N, N' be any left R-modules.

There exists a unique group isomorphisms

$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes N)$$

$$M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$$

such that $(m, m') \otimes n \mapsto (m \otimes n, m' \otimes n)$ and $m \otimes (n, n') \mapsto (m \otimes n, m \otimes n')$ respectively.

And if M, M' S-module-R then there exists a S-module isomorphism.