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## 0.1 A brief reminder of what Modules are

**Definition 0.1.** Module. Let  $R$  be a ring. A set  $M$  with a binary operation  $+: M \times M \rightarrow M$  and a map  $\cdot: R \times M \rightarrow M$  is a *left  $R$ -module* iff

1.  $(M, +)$  is an Abelian group
2.  $\cdot: R \times M \rightarrow M$  defines a “ring action of  $R$  on  $M$ ” (scalar multiplication)  
i.e. for all  $r, s \in R$  and for all  $m, n \in M$  that
  - (a)  $(r + s) \cdot m = r \cdot m + s \cdot m$
  - (b)  $(r \cdot s) \cdot m = r \cdot (s \cdot m)$
  - (c)  $r \cdot (m + n) = r \cdot m + r \cdot n$
3. if  $R$  is a ring with identity then  $1 \cdot m = m$  for all  $m \in M$

*Example 0.1.* Any ring

$$r \cdot s = r \cdot s$$

*Example 0.2.* Abelian groups are  $\mathbb{Z}$ -modules

$$n \cdot a = \begin{cases} a + \cdots + a & (n-1) \text{ addition operations} & : n > 0 \\ 0 & & : n = 0 \\ -a + \cdots + -a & (-n-1) \text{ addition operations} & : n < 0 \end{cases}$$

*Example 0.3.* Vector spaces are  $R$ -modules where  $R$  is a field.

*Example 0.4.* Collection of maps from a set  $X$  to a  $R$ -module  $M$

$$M^X = \{f: X \rightarrow M\}$$

where for all  $f, g \in M^X$  and for all  $x \in X$  and for all  $r \in R$

$$(f + g)(x) = f(x) + g(x) \quad (r \cdot f)(x) = r \cdot f(x)$$

*Example 0.5.*  $X$  is a finite set e.g.  $X = [n] := \{1, 2, \dots, n\}$  for some natural number  $n$ .

Then  $M^X = M^{[n]}$  is a  $R$ -module for some  $M$   $R$ -module i.e. direct product of  $R$ -modules

$$M^n = M \times \cdots \times M$$

*Example 0.6.* Smooth functions on a real smooth  $n$ -manifold  $X$  such as  $\mathbb{R}^n$

$$\mathcal{C}^\infty(X) = \{f: X \rightarrow \mathbb{R} | f \text{ is a smooth map} \} \subset \mathbb{R}^X$$

smooth map<sup>1</sup>

*Example 0.7.* The smooth vector fields on a manifold  $X$  is a  $\mathcal{C}^\infty(X)$ -module

$$\Gamma(TX) = \{v : X \rightarrow TX \mid \text{smooth and } \pi \circ v = \text{id}_X\}$$

where  $\pi : TX \rightarrow X$  is the projection from the tangent bundle  $TX = \coprod_{x \in X} T_x X$ , the disjoint union of the tangent spaces, onto  $X$ .

$$(f.v)(x) = f(x).v_x$$

## 1 Tensor Product

### 1.1 Special Case: Extension of Scalar Ring

Let us consider the motivating problem presented by Dummit and Foote. Given a  $R$ -module such that the ring  $R$  is a subring of another ring  $S$ , what is the most general  $S$ -module which the given  $R$ -module can be embedded? i.e. when can a  $R$ -module be extended to a  $S$ -module? This is known as extension of scalars.

Let us call our  $R$ -module to be extended  $N$ . Recall the key elements of the definition for a  $\mathcal{R}$ -module, a Abelian group structure and an action of the ring  $\mathcal{R}$  on the Abelian group. It is therefore natural to consider the free Abelian group  $F(S \times N)$  as a candidate Abelian group to define a  $S$  ring action on and which  $N$  can be embedded.

**Definition 1.1.** For any ring  $\mathcal{R}$ , a  $\mathcal{R}$ -module  $F$  is **free** iff for some set  $X$ , there exists unique nonzero elements  $r_1, \dots, r_k \in \mathcal{R}$  and unique elements  $x_1, \dots, x_k \in X$  for each element  $f \in F$ , such that  $f = r_1.x_1 + \dots + r_k.x_k$

*Example 1.1.* Any ring  $\mathcal{R}$  and any set  $X$  let

$$F(X) = \{f : X \rightarrow \mathcal{R} \mid X \setminus f^{-1}[0] \text{ finite}\} \subset \mathcal{R}^X$$

i.e. maps  $X \rightarrow \mathcal{R}$  with finite support where support is defined as  $\text{supp}(f) := \{x \in X : f(x) \neq 0\}$ . The  $\mathcal{R}$ -module structure is defined componentwise.

To show  $F(X)$  is free define the injection  $\iota : X \rightarrow F(X) : x \mapsto \delta_x$  such that

$$\delta_x(y) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}$$

It is easy to see for any  $f \in F(X)$  there exists unique nonzero elements  $r_1, \dots, r_k \in \mathcal{R}$  and unique elements  $x_1, \dots, x_k \in X$  such that

$$f = r_1.\delta_{x_1} + \dots + r_k.\delta_{x_k}$$

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<sup>1</sup> $f$  is a smooth map  $\iff$  for all smooth charts  $(\phi, U)$  that the following is a smooth function

$$\phi^{-1} \circ f : \phi(U \cap V) \subset \mathbb{R}^n \rightarrow \mathbb{R}$$

*Remark 1.1.*  $\mathcal{R} = \mathbb{Z}$  then  $F$  a free  $\mathbb{Z}$ -module is called a free Abelian group.

With  $F(S \times N)$  as our Abelian group let us define a ring action from  $S$  as for any  $f \in F(S \times N)$  and  $s \in S$

$$\begin{aligned} s.f &:= s.(s_1, n_1) + \cdots + s.(s_k, n_k) \\ &= (s \cdot s_1, n_1) + \cdots + (s \cdot s_k, n_k) \end{aligned}$$

Now considering the details of the definition of a general  $\mathcal{R}$ -module  $M$  the ring action must preserve both the ring's and Abelian group's structures. i.e.

$$\begin{aligned} (r_1 + r_2).m &= r_1.m + r_2.m & (r_1 \cdot r_2).m &= r_1.(r_2.m) \\ r.(m_1 + m_2) &= r.m_1 + r.m_2 \end{aligned}$$

for any  $r, r_1, r_2 \in \mathcal{R}$  and any  $m, m_1, m_2 \in M$ .

This leads us to define the equivalence relation  $\sim$  on  $F(S \times N)$  as follows

$$\begin{aligned} (s_1 + s_2, n) &\sim (s_1, n) + (s_2, n) & (s \cdot r, n) &\sim (s, r.n) \\ (s, n_1 + n_2) &\sim (s, n_1) + (s, n_2) \end{aligned}$$

for any  $s, s_1, s_2 \in S$ , any  $r \in R$ , and any  $n, n_1, n_2 \in N$ . Note the subtle difference in the compatibility relation this is the case since  $N$  has a  $R$ -module structure.

Since equivalence relations give us a normal subgroup i.e.  $a \sim b \iff a - b \in H$ . Let us define the  $S$ -module we have constructed as

$$S \otimes_R N := F(S \times N) / \sim$$

let us call the cosets, tensors, the following is an example of a simple tensor

$$s \otimes n := \{f \in F(S \times N) : f \sim (s, n)\}$$

The binary Abelian operation  $+$  is defined via the natural surjective group homomorphism  $\pi : F(S \times N) \rightarrow F(S \times N) / \sim$

$$\begin{aligned} (s \otimes n) + (s' \otimes n') &= \pi((s, n)) + \pi((s', n')) \\ &= \pi((s, n) + (s', n')) \\ &= \{f \in F(S \times N) : f \sim (s, n) + (s', n')\} \end{aligned}$$

This yeilds the following relations

$$\begin{aligned} (s_1 + s_2) \otimes n &= s_1 \otimes n + s_2 \otimes n & (s \cdot r) \otimes n &= s.(r.n) \\ s \otimes (n_1 + n_2) &= s \otimes n_1 + s \otimes n_2 \end{aligned}$$

Let us define the  $S$  ring action as

$$s \circ \left( \sum_{i \in [n]} s_i \otimes n_i \right) = \sum_{i \in [n]} (s \cdot s_i) \otimes n_i$$

Exercise check if this is well defined, note these are cosets and check for simple tensors  $s \otimes n$  that this makes  $S \otimes_R N$  a  $S$ -module.

The following theorem shows this is the most general  $S$ -module to embed the  $R$ -module  $N$  into. First we need a lemma the Universal property of free modules.

**Lemma 1.1.** *Given any set  $X$ , any left  $R$ -module  $M$ , and any map  $\psi : X \rightarrow M$ . There exists a unique  $R$ -module homomorphism from the free  $R$ -module generated by  $X$  to  $M$*

$$\Psi : F(X) \rightarrow M \quad \text{s.t.} \quad \psi(x) = \Psi(x) \forall x \in X$$

The proof follows since  $X$  is a generating set of  $F(X)$  and each element of  $F(X)$  has a unique representation of the formal form  $r_1x_1 + \dots + r_kx_k$  see example 1.1

The map is

$$\Psi : \sum_{i \in [k]} r_i x_i \mapsto \sum_{i \in [k]} r_i \psi(x_i)$$

**Theorem 1.1.** *Let  $R \subset S$  be a subring,  $N$  be a  $R$ -module, and  $\iota : N \rightarrow S \otimes_R N$  be the  $R$ -module homomorphism  $n \mapsto 1 \otimes n$ .*

*For any left  $S$ -module  $L$  and  $R$ -module homomorphism  $\phi : N \rightarrow L$  there exists a unique  $S$ -module homomorphism  $\Phi : S \otimes_R N \rightarrow L$  such that  $\phi = \Phi \circ \iota$*

$$\begin{array}{ccc} N & \xrightarrow{\iota} & S \otimes_R N \\ & \searrow \phi & \downarrow \Phi \\ & & L \end{array}$$

*Proof.* Since  $L$  is a left  $S$ -module let us define the map

$$\psi : S \times N \rightarrow L : (s, n) \mapsto s \cdot \phi(n)$$

By the universal property of free modules, see Lemma 1.1, there exist unique  $\mathbb{Z}$ -module homomorphism from the free Abelian group to the left  $S$ -module  $L$

$$\Psi : F(S \times N) \rightarrow L$$

Specifically

$$\begin{aligned} \Psi \left( \sum_{i \in [k]} a_i (s_i, n_i) \right) &= \sum_{i \in [k]} a_i \psi((s_i, n_i)) \\ &= \sum_{i \in [k]} \sum_{j=1}^{a_i} \psi((s_i, n_i)) \\ &= \sum_{i \in [k]} \sum_{j=1}^{a_i} s_i \cdot \phi(n_i) \end{aligned}$$

Since  $\phi$  is an  $R$ -module homomorphism i.e.

$$\phi(x + y) = \phi(x) + \phi(y) \quad \text{and} \quad \phi(r.x) = r.\phi(x)$$

Then the generators for the subgroup from the equivalence relation  $\sim$  discussed previously map to zero in  $L$ . e.g.  $(s_1 + s_2, n) \sim (s_1, n) + (s_2, n)$  gives a generating element  $(s_1 + s_2, n) - (s_1, n) - (s_2, n)$  which maps to

$$\begin{aligned} \Psi((s_1 + s_2, n) - (s_1, n) - (s_2, n)) &= \psi((s_1 + s_2, n)) - \psi((s_1, n)) - \psi((s_2, n)) \\ &= (s_1 + s_2).\phi(n) - s_1.\phi(n) - s_2.\phi(n) \\ &= s_1.\phi(n) + s_2.\phi(n) - s_1.\phi(n) - s_2.\phi(n) \\ &= 0 \end{aligned}$$

Similarly for  $(s \cdot r, n) \sim (s, r \cdot n)$  and  $(s, n_1 + n_2) \sim (s, n_1) + (s, n_2)$  Therefore there exists a well defined  $\mathbb{Z}$ -module homomorphism

$$\Phi : F(S \times N) / \sim = S \otimes_R N \rightarrow L$$

where  $\Phi(\sum s_i \otimes n_i) = \sum s_i.\phi(n_i) \in L$ . Since  $L$  is a left  $S$ -module

$$\begin{aligned} s.\Phi\left(\sum s_i \otimes n_i\right) &= s.\left(\sum s_i.\phi(n_i)\right) \\ &= \sum s.(s_i.\phi(n_i)) \\ &= \sum (s \cdot s_i).\phi(n_i) \\ &= \Phi\left(\sum (s \cdot s_i) \otimes n_i\right) \\ &= \Phi\left(\sum s.(s_i \otimes n_i)\right) \end{aligned}$$

From the above calculation we see  $\Phi$  is a  $S$ -module homomorphism. Note any  $S$ -module homomorphism is uniquely determined by the values on elements of the generating set, e.g.  $1 \otimes n$  and since  $\Phi(1 \otimes n) = \phi(n)$  then  $\Phi$  is uniquely determined by  $\phi$ .  $\square$

*Remark 1.2.* The converse of this theorem is true. If  $\Phi : S \otimes_R N \rightarrow L$  is a  $S$ -module homomorphism then  $\phi = \Phi \circ \iota : N \rightarrow L$  is a  $R$ -module homomorphism.

**Corollary 1.1.**  $\frac{N}{\ker \iota}$  is the unique largest quotient of  $N$  that can be embedded in any  $S$ -module.

## 1.2 General Tensor

**Definition 1.2.** Let  $M$  be any right module- $R$  and  $N$  any left  $R$ -module. Consider the equivalence relation  $\sim$  on  $F(M \times N)$  free Abelian group generated from  $M \times N$  where

$$\begin{aligned} (m_1 + m_2, n) &\sim (m_1, n) + (m_2, n) & (m.r, n) &\sim (s, r.n) \\ (m, n_1 + n_2) &\sim (m, n_1) + (m, n_2) \end{aligned}$$

The abelian group  $M \otimes_R N = F(M \times N) / \sim$  is called the **tensor product of  $M$  and  $N$  over  $R$**

**Definition 1.3.** Let  $M$  be any right module- $R$ ,  $N$  any left  $R$ -module, and  $L$  any Abelian group.

$\phi : M \times N \rightarrow L$  is  $R$ -balanced iff

$$\begin{aligned}\phi(m_1 + m_2, n) &= \phi(m_1, n) + \phi(m_2, n) & \phi(m, r.n) &= \phi(m.r, n) \\ \phi(m, n_1 + n_2) &= \phi(m, n_1) + \phi(m, n_2)\end{aligned}$$

**Theorem 1.2.** Let  $R$  be any ring with 1,  $M$  right module- $R$ ,  $N$  left  $R$ -module, and  $L$  any Abelian group.

If  $\phi : M \times N \rightarrow L$  is a  $R$ -balanced map then there exists a unique group homomorphism  $\Phi : M \otimes_R N \rightarrow L$  such that  $\phi = \Phi \circ \iota$ .

If  $\Phi : M \otimes_R N \rightarrow L$  any group homomorphism then  $\phi = \Phi \circ \iota : M \times N \rightarrow L$  is a  $R$ -balanced map.

**Theorem 1.3. Associativity of the Tensor Product**

Let  $M$  be any right  $S$ -module- $R$ ,  $N$  any  $R$ -module- $T$ , and  $L$  any left  $T$ -module.

There exists a unique  $S$ -module isomorphism

$$(M \otimes_R N) \otimes_T L \cong M \otimes_R (N \otimes_T L)$$

$$m \otimes (n \otimes l) \mapsto (m \otimes n) \otimes l$$

*Proof.* Note that  $M \otimes_R N$  is  $S$ -module- $T$  and  $N \otimes_T L$  left  $R$ -module. For each  $l \in L$  define  $\phi_l^L : M \times N \rightarrow M \otimes_R (N \otimes_T L)$

$$\phi_l^L(m, n) = m \otimes (n \otimes l)$$

$$\begin{aligned}\phi_l^L(m.r, n) &= m.r \otimes (n \otimes l) \\ &= m \otimes r.(n \otimes l) \\ &= m \otimes (r.n \otimes l) \\ &= \phi_l^L(m, r.n)\end{aligned}$$

The above calculation shows  $\phi_l^L$  is  $R$ -balanced. By the universal property of tensor product there exist a unique homomorphism

$$\Phi_l^L : M \otimes_R N \rightarrow M \otimes_R (N \otimes_T L)$$

and therefore the following map is well defined

$$\begin{aligned}\Phi^L : (M \otimes_R N) \times L &\rightarrow M \otimes_R (N \otimes_T L) \\ (m \otimes n, l) &\mapsto \Phi_l^L(m \otimes n)\end{aligned}$$

A simple calculation shows that  $\Phi^L$  is  $R$ -balanced and therefore by the universal property of tensor product there exist a unique homomorphism  $\Phi : (M \otimes_R N) \otimes_T L \rightarrow M \otimes_R (N \otimes_T L)$ . Repeat with  $\psi_m^M$  maps to get the inverse homomorphism.

To complete the proof note that  $S$  action is compatible with the  $T$  action

$$s.((m \otimes n).t) = s.(m \otimes n.t) = s.m \otimes n.t = (s.m \otimes n).t = (s.(m \otimes n)).t$$

so  $M \otimes_R N$  is a  $S$ -module- $T$  making  $(M \otimes_R N) \otimes_T L$  a  $S$ -module. By the isomorphism constructed above  $\Phi(s.(m \otimes n) \otimes l) = \Phi((s.m \otimes n) \otimes l) = s.m \otimes (n \otimes l) = s.(m \otimes (n \otimes l))$   $\square$

**Theorem 1.4. Tensor product commutes with Direct Sum**

Let  $M, M'$  be any right modules- $R$  and  $N, N'$  be any left  $R$ -modules.

There exists a unique group isomorphisms

$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$$

$$M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$$

such that  $(m, m') \otimes n \mapsto (m \otimes n, m' \otimes n)$  and  $m \otimes (n, n') \mapsto (m \otimes n, m \otimes n')$  respectively.

And if  $M, M'$   $S$ -module- $R$  then there exists a  $S$ -module isomorphism.