## 0.1 A brief reminder of what Modules are

**Definition 0.1.** Module. Let $R$ be a ring. A set $M$ with a binary operation $+ : M \times M \to M$ and a map $. : R \times M \to M$ is a *left $R$-module* iff

1. $(M, +)$ is an Abelian group
2. $. : R \times M \to M$ defines a "ring action of $R$ on $M$" (scalar multiplication) i.e. for all $r, s \in R$ and for all $m, n \in M$ that
   (a) $(r + s).m = r.m + s.m$
   (b) $(r \cdot s).m = r.(s.m)$
   (c) $r.(m + n) = r.m + r.n$
3. if $R$ is a ring with identity then $1.m = m$ for all $m \in M$

*Example* 0.1. Any ring

$$r.s = r \cdot s$$

*Example* 0.2. Abelian groups are $\mathbb{Z}$-modules

$$n.a = \begin{cases} a + \cdots + a & (n-1) \text{ addition operations} & : n > 0 \\ 0 & & : n = 0 \\ -a + \cdots + -a & (-n-1) \text{ addition operations} & : n < 0 \end{cases}$$

*Example* 0.3. Vector spaces are $R$-modules where $R$ is a field.

*Example* 0.4. Collection of maps from a set $X$ to a $R$-module $M$

$$M^X = \{f : X \to M\}$$

where for all $f, g \in M^X$ and for all $x \in X$ and for all $r \in R$

$$(f + g)(x) = f(x) + g(x) \qquad (r.f)(x) = r.f(x)$$

*Example* 0.5. $X$ is a finite set e.g. $X = [n] := \{1, 2, \cdots, n\}$ for some natural number $n$.

Then $M^X = M^{[n]}$ is a $R$-module for some $M$ $R$-module i.e. direct product of $R$-modules

$$M^n = M \times \cdots \times M$$

*Example* 0.6. Smooth functions on a real smooth $n$-manifold $X$ such as $\mathbb{R}^n$

$$\mathcal{C}^\infty(X) = \{f : X \to \mathbb{R} | f \text{ is a smooth map } \} \subset \mathbb{R}^X$$

smooth map[1]

[1] $f$ is a smooth map $\iff$ for all smooth charts $(\phi, U)$ that the following is a smooth function

$$\phi^{-1} \circ f : \phi(U \cap V) \subset \mathbb{R}^n \to \mathbb{R}$$

*Example* 0.7. The smooth vector fields on a manifold $X$ is a $\mathcal{C}^\infty(X)$-module

$$\Gamma(TX) = \{v : X \to TX |\text{ smooth and } \pi \circ v = \mathrm{id}_X\}$$

where $\pi : TX \to X$ is the projection from the tangent bundle $TX = \amalg_{x\in X} T_x X$, the disjoint union of the tangent spaces, onto X.

$$(f.v)(x) = f(x).v_x$$

# 1 Tensor Product

## 1.1 Special Case: Extension of Scalar Ring

Let us consider the motivating problem presented by Dummit and Foote. Given a $R$-module such that the ring $R$ is a subring of another ring $S$, what is the most general $S$-module which the given $R$-module can be embedded? i.e. when can a $R$-module to extended to a $S$-module? This is know as extension of scalars.

Let us call our $R$-module to be extended $N$. Recall the key elements of the definition for a $\mathcal{R}$-module, a Abelian group structure and an action of the ring $\mathcal{R}$ on the Abelian group. It is therefore natural to consider the free Abelian group $F(S \times N)$ as a candidate Abelian group to define a $S$ ring action on and which $N$ can be embedded.

**Definition 1.1.** For any ring $\mathcal{R}$, a $\mathcal{R}$-module $F$ is **free** iff for some set $X$, there exists unique nonzero elements $r_1, \ldots, r_k \in \mathcal{R}$ and unique elements $x_1, \ldots, x_k \in X$ for each element $f \in F$, such that $f = r_1.x_1 + \cdots + r_k.x_k$

*Example* 1.1. Any ring $\mathcal{R}$ and any set $X$ let

$$F(X) = \{f : X \to \mathcal{R} \mid X \setminus f^{-1}[0] \text{ finite}\} \subset \mathcal{R}^X$$

i.e. maps $X \to \mathcal{R}$ with finte support where support is defined as $\mathrm{supp}(f) := \{x \in X : f(x) \neq 0\}$. The $\mathcal{R}$-module structure is defined componentwise.

To show $F(X)$ is free define the injection $\iota : X \to F(X) : x \mapsto \delta_x$ such that

$$\delta_x(y) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}$$

It is easy to see for any $f \in F(X)$ there exists unique nonzero elements $r_1, \ldots, r_k \in \mathcal{R}$ and unqiue elements $x_1, \ldots, x_k \in X$ such that

$$f = r_1.\delta_{x_1} + \cdots + r_k.\delta_{x_k}$$

*Remark* 1.1. $\mathcal{R} = \mathbb{Z}$ then $F$ a free $\mathbb{Z}$-module is called a free Abelian group.

With $F(S \times N)$ as our Abelian group let us define a ring action from $S$ as for any $f \in F(S \times N)$ and $s \in S$

$$\begin{aligned} s.f &:= s.(s_1, n_1) + \cdots + s.(s_k, n_k) \\ &= (s \cdot s_1, n_1) + \cdots + (s \cdot s_k, n_k) \end{aligned}$$

Now considering the details of the definition of a general $\mathcal{R}$-module $M$ the ring action must preserve both the ring's and Abelian group's structures. i.e.

$$\begin{aligned} &(r_1 + r_2).m = r_1.m + r_2.m && (r_1 \cdot r_2).m = r_1.(r_2.m) \\ &r.(m_1 + m_2) = r.m_1 + r.m_2 \end{aligned}$$

for any $r, r_1, r_2 \in \mathcal{R}$ and any $m, m_1, m_2 \in M$.

This leads us to define the equivalence relation $\sim$ on $F(S \times N)$ as follows

$$\begin{aligned} &(s_1 + s_2, n) \sim (s_1, n) + (s_2, n) && (s \cdot r, n) \sim (s, r.n) \\ &(s, n_1 + n_2) \sim (s, n_1) + (s, n_2) \end{aligned}$$

for any $s, s_1, s_2 \in S$, any $r \in R$, and any $n, n_1, n_2 \in N$. Note the subtle difference in the compatibility relation this is the case since $N$ has a $R$-module structure.

Since equivalence relations give us a normal subgroup i.e. $a \sim b \iff a - b \in H$. Let us define the $S$-module we have constructed as

$$S \otimes_R N := F(S \times N)/\sim$$

let us call the cosets, tensors, the following is an example of a simple tensor

$$s \otimes n := \{f \in F(S \times N) : f \sim (s, n)\}$$

The binary Abelian operation $+$ is defined via the natural surjective group homomorphism $\pi : F(S \times N) \to F(S \times N)/\sim$

$$\begin{aligned} (s \otimes n) + (s' \otimes n') &= \pi((s, n)) + \pi((s', n')) \\ &= \pi((s, n) + (s', n')) \\ &= \{f \in F(S \times N) : f \sim (s, n) + (s', n')\} \end{aligned}$$

This yeilds the following relations

$$\begin{aligned} &(s_1 + s_2) \otimes n = s_1 \otimes n + s_2 \otimes n && (s \cdot r) \otimes n = s.(r.n) \\ &s \otimes (n_1 + n_2) = s \otimes n_1 + s \otimes n_2 \end{aligned}$$

Let us define the $S$ ring action as

$$s_\circ \left( \sum_{i \in [n]} s_i \otimes n_i \right) = \sum_{i \in [n]} (s \cdot s_i) \otimes n_i$$

Exercise check if this is well defined, note these are cosets and check for simple tensors $s \otimes n$ that this makes $S \otimes_R N$ a $S$-module.

The following theorem shows this is the most general $S$-module to embedded the $R$-module $N$ into. First we need a lemma the Universal property of free modules.

**Lemma 1.1.** *Given any set $X$, any left $R$-module $M$, and any map $\psi : X \to M$. There exists a unique $R$-module homomorphism from the free $R$-module generated by $X$ to $M$*

$$\Psi : F(X) \to M \qquad s.t. \quad \psi(x) = \Psi(x) \forall x \in X$$

The proof follows since $X$ is a generating set of $F(X)$ and each element of $F(X)$ has a unique representation of the formal form $r_1x_1 + \cdots + r_kx_k$ see example 1.1

The map is

$$\Psi : \sum_{i\in[k]} r_i x_i \mapsto \sum_{i\in[k]} r_i \psi(x_i)$$

**Theorem 1.1.** *Let $R \subset S$ be a subring, $N$ be a $R$-module, and $\iota : N \to S \otimes_R N$ be the $R$-module homomorphism $n \mapsto 1 \otimes n$.*

*For any left $S$-module $L$ and $R$-module homomorphism $\phi : N \to L$ there exists a unique $S$-module homomorphism $\Phi : S \otimes_R N \to L$ such that $\phi = \Phi \circ \iota$*

$$\begin{tikzcd}
N \arrow[r, "\iota"] \arrow[dr, "\phi"'] & S \otimes_R N \arrow[d, dotted, "! \ \Phi"] \\
& L
\end{tikzcd}$$

*Proof.* Since $L$ is a left $S$-module let us define the map

$$\psi : S \times N \to L : (s, n) \mapsto s.\phi(n)$$

By the universal property of free modules, see Lemma 1.1, there exist unqiue $\mathbb{Z}$-module homomorphism from the free Abelian group to the left $S$-module $L$

$$\Psi : F(S \times N) \to L$$

Specifically

$$\begin{aligned}
\Psi\left(\sum_{i\in[k]} a_i(s_i, n_i)\right) &= \sum_{i\in[k]} a_i \psi((s_i, n_i)) \\
&= \sum_{i\in[k]} \sum_{j=1}^{a_i} \psi((s_i, n_i)) \\
&= \sum_{i\in[k]} \sum_{j=1}^{a_i} s_i.\phi(n_i)
\end{aligned}$$

Since $\phi$ is an $R$-module homomorphism i.e.

$$\phi(x+y) = \phi(x) + \phi(y) \qquad \text{and} \qquad \phi(r.x) = r.\phi(x)$$

Then the generators for the subgroup from the equivalence relation $\sim$ discussed previously map to zero in $L$. e.g. $(s_1+s_2,n) \sim (s_1,n)+(s_2,n)$ gives a generatoring element $(s_1+s_2,n)-(s_1,n)-(s_2,n)$ which maps to

$$\begin{aligned}\Psi((s_1+s_2,n)-(s_1,n)-(s_2,n)) &= \psi((s_1+s_2,n)) - \psi((s_1,n)) - \psi((s_2,n)) \\ &= (s_1+s_2).\phi(n) - s_1.\phi(n) - s_2.\phi(n) \\ &= s_1.\phi(n) + s_2.\phi(n) - s_1.\phi(n) - s_2.\phi(n) \\ &= 0\end{aligned}$$

Similarly for $(s \cdot r, n) \sim (s, r \cdot n)$ and $(s, n_1+n_2) \sim (s,n_1)+(s,n_2)$ Therefore there exists a well defined $\mathbb{Z}$-module homomorphism

$$\Phi : F(S \times N)/\sim = S \otimes_R N \to L$$

where $\Phi(\sum s_i \otimes n_i) = \sum s_i.\phi(n_i) \in L$. Since $L$ is a left $S$-module

$$\begin{aligned}s.\Phi\left(\sum s_i \otimes n_i\right) &= s.\left(\sum s_i.\phi(n_i)\right) \\ &= \sum s.(s_i.\phi(n_i)) \\ &= \sum (s \cdot s_i).\phi(n_i) \\ &= \Phi\left(\sum (s \cdot s_i) \otimes n_i\right) \\ &= \Phi\left(\sum s.(s_i \otimes n_i)\right)\end{aligned}$$

From the above calculation we see $\Phi$ is a $S$-module homomorphism. Note any $S$-module homomorphism is uniquely determined by the values on elements of the generating set, e.g. $1 \otimes n$ and since $\Phi(1 \otimes n) = \phi(n)$ then $\Phi$ is uniquely determined by $\phi$. $\square$

*Remark* 1.2. The converse of this theorem is true. If $\Phi : S \otimes_R N \to L$ is a $S$-module homomorphism then $\phi = \Phi \circ \iota : N \to L$ is a $R$-module homomorphism.

**Corollary 1.1.** $\frac{N}{ker \iota}$ *is the unique largest quotient of $N$ that can be embedded in any $S$-module.*

## 1.2 General Tensor

**Definition 1.2.** Let $M$ be any right module-$R$ and $N$ any left $R$-module. Consider the equivalance relation $\sim$ on $F(M \times N)$ free Abelian group generated from $M \times N$ where

$$\begin{aligned}&(m_1+m_2,n) \sim (m_1,n)+(m_2,n) \qquad && (m.r,n) \sim (s,r.n) \\ &(m,n_1+n_2) \sim (m,n_1)+(m,n_2)\end{aligned}$$

The abelian group $M \otimes_R N = F(M \times N)/\sim$ is called the **tensor product of $M$ and $N$ over $R$**

**Definition 1.3.** Let $M$ be any right module-$R$, $N$ any left $R$-module, and $L$ any Abelian group.

$\phi : M \times N \to L$ is $R$-balanced iff

$$\phi(m_1 + m_2, n) = \phi(m_1, n) + \phi(m_2, n) \qquad \phi(m, r.n) = \phi(m.r, n)$$
$$\phi(m, n_1 + n_2) = \phi(m, n_1) + \phi(m, n_2)$$

**Theorem 1.2.** *Let $R$ be any ring with 1, $M$ right module-$R$, $N$ left $R$-module, and $L$ any Abelian group.*

*If $\phi : M \times N \to L$ is a $R$-balanced map then there exists a unique group homomorphism $\Phi : M \otimes_R N \to L$ such that $\phi = \Phi \circ \iota$.*

*If $\Phi : M \otimes_R N \to L$ any group homomorphism then $\phi = \Phi \circ \iota : M \times N \to L$ is a $R$-balanced map.*

**Theorem 1.3.** ***Associativity of the Tensor Product***
*Let $M$ be any right $S$-module-$R$, $N$ any $R$-module-$T$, and $L$ any left $T$-module.*

*There exists a unique $S$-module isomorphism*

$$(M \otimes_R N) \otimes_T L \cong M \otimes_R (N \otimes_T L)$$

$m \otimes (n \otimes l) \mapsto (m \otimes n) \otimes l$

*Proof.* Note that $M \otimes_R N$ is $S$-module-$T$ and $N \otimes_T L$ left $R$-module. For each $l \in L$ define $\phi_l^L : M \times N \to M \otimes_R (N \otimes_T L)$

$$\phi_l^L(m, n) = m \otimes (n \otimes l)$$

$$\begin{aligned}
\phi_l^L(m.r, n) &= m.r \otimes (n \otimes l) \\
&= m \otimes r.(n \otimes l) \\
&= m \otimes (r.n \otimes l) \\
&= \phi_l^L(m, r.n)
\end{aligned}$$

The above calculation shows $\phi_l^L$ is $R$-balanced. By the universal property of tensor product there exist a unique homomorphism

$$\Phi_l^L : M \otimes_R N \to M \otimes_R (N \otimes_T L)$$

and therefore the following map is well defined

$$\begin{aligned}
\Phi^L &: (M \otimes_R N) \times L \to M \otimes_R (N \otimes_T L) \\
&: (m \otimes n, l) \mapsto \Phi_l^L(m \otimes n)
\end{aligned}$$

A simple calculatin shows that $\Phi^L$ is $R$-balanced and therefore by the universal property of tensor product there exist a unique homomorphism $\Phi : (M \otimes_R N) \otimes_T L \to M \otimes_R (N \otimes_T L)$. Repeat with $\psi_m^M$ maps to get the inverse homomorphism.

To complete the proof note that $S$ action is compatible with the $T$ action

$$s.((m \otimes n).t) = s.(m \otimes n.t) = s.m \otimes n.t = (s.m \otimes n).t = (s.(m \otimes n)).t$$

so $M \otimes_R N$ is a $S$-module-$T$ making $(M \otimes_R N) \otimes_T L$ a $S$-module. By the isomorphism constructed above $\Phi(s.(m \otimes n) \otimes l) = \Phi((s.m \otimes n) \otimes l) = s.m \otimes (n \otimes l) = s.(m \otimes (n \otimes l))$ $\square$

**Theorem 1.4.** ***Tensor product commutes with Direct Sum***
*Let $M$,$M'$ be any right modules-$R$ and $N, N'$ be any left $R$-modules.*

*There exists a unique group isomorphisms*

$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes N)$$

$$M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$$

*such that $(m, m') \otimes n \mapsto (m \otimes n, m' \otimes n)$ and $m \otimes (n, n') \mapsto (m \otimes n, m \otimes n')$ respectively.*

*And if $M, M'$ $S$-module-$R$ then there exists a $S$-module isomorphism.*