



What every designer & developer should know about

Mathematical Optimization

Intelligent Computational Media, spring 2021

Aalto University

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All course materials and syllabus: <https://github.com/PerttuHamalainen/MediaAI/blob/master/Syllabus.md>

Optimize

Reward Shaping

Predicted score: 0

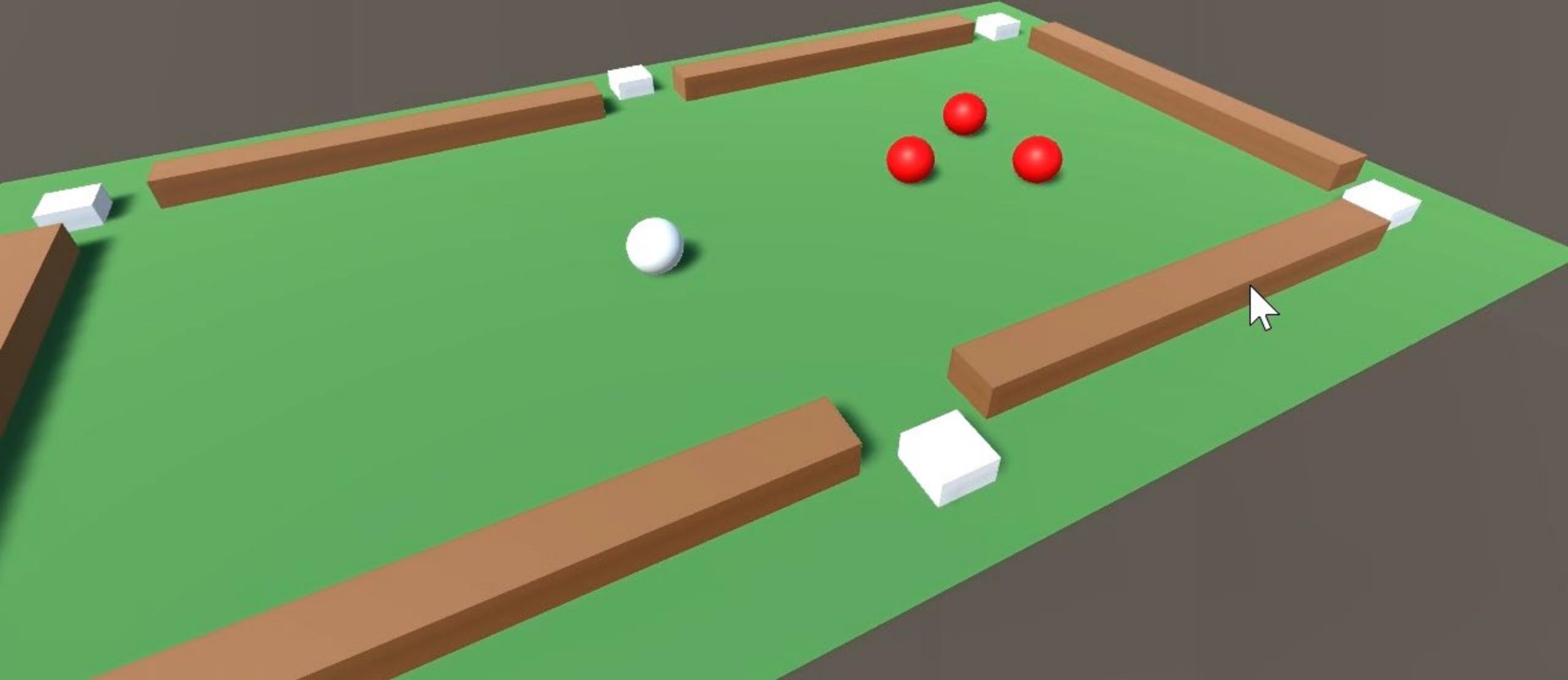
Max Iter:

50

Population size:

16

Discovering billiards trick shots through optimization





Synthesizing abstract art through optimization



Perception Engines: cello, cabbage, hammerhead shark, iron, tick, starfish, binoculars, measuring cup, blow dryer, and jack-o'-lantern

https://medium.com/@tom_25234/synthetic-abstractions-8f0e8f69f390



Contents

- Optimization: the what and why
- Some intuitions from simple random search
- Gradient-based optimization
- Sampling-based optimization
- From continuous-valued to discrete/combinatorial optimization
- Reinforcement learning
- Forward search methods, e.g., Monte Carlo Tree Search
- Appendix: Optional material like neuroevolution

Approach

- This lecture is designed considering the highly multidisciplinary and heterogeneous audience that typically attends the course
- For those with minimal math background (high school level): Visual intuitions that should help in successfully applying optimization, e.g., how results are affected by the choice of optimization method and parameters
- For the more technical audience, e.g., CS students: Some mathematical foundation and pointers for learning more. Feel free to skip if not interested.

Optimization: the what and why



What is mathematical optimization?

- Optimizing code is just one optimization problem
- In general: find parameters \mathbf{x} that minimize or maximize some objective function $f(\mathbf{x})$
- We denote vectors of variables with boldface, i.e., $\mathbf{x}=[x_1, x_2, \dots, x_N]$

Why does it matter?

- Game Design is optimization: For example, maximize *enjoyment(x)*
- A/B testing is a simple optimization method
- Game playing is (usually) optimization => optimization algorithms can be used for game playing and testing
- More generally, AI = problem solving = optimization (e.g., adjust neural network parameters to minimize some loss function, or find a gameplay strategy that maximizes the probability of winning)

Optimizing actions

Interactive emergent movement. No training data.



Optimizing neural network parameters

Epoch 000,000 Learning rate 0.03 Activation Tanh Regularization None Regularization rate 0 Problem type Classification

DATA FEATURES 2 HIDDEN LAYERS OUTPUT

Which dataset do you want to use? Which properties do you want to feed in?

+ - + -

4 neurons 2 neurons

X₁ X₂ X₁₂ X₂₂ X_{1X2}

The outputs are mixed with varying weights, shown by the thickness of the lines.

This is the output from one neuron. Hover to see it larger.

Ratio of training to test data: 50% Noise: 0 Batch size: 10

REGENERATE

sin(X₁) sin(X₂)

Test loss 0.507 Training loss 0.500

-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6

6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6

Colors shows data, neuron and weight values. -1 0 1

Show test data Discretize output

The visualization illustrates a neural network architecture with two hidden layers. The input features are X₁, X₂, X₁₂, X₂₂, and X_{1X2}. The first hidden layer contains 4 neurons, and the second hidden layer contains 2 neurons. The output layer displays a scatter plot of data points, where colors represent data, neuron, and weight values. A note explains that line thickness represents varying weights. The interface includes controls for epoch, learning rate, activation, regularization, regularization rate, and problem type. Data and feature selection dropdowns are also present.



Simple random search

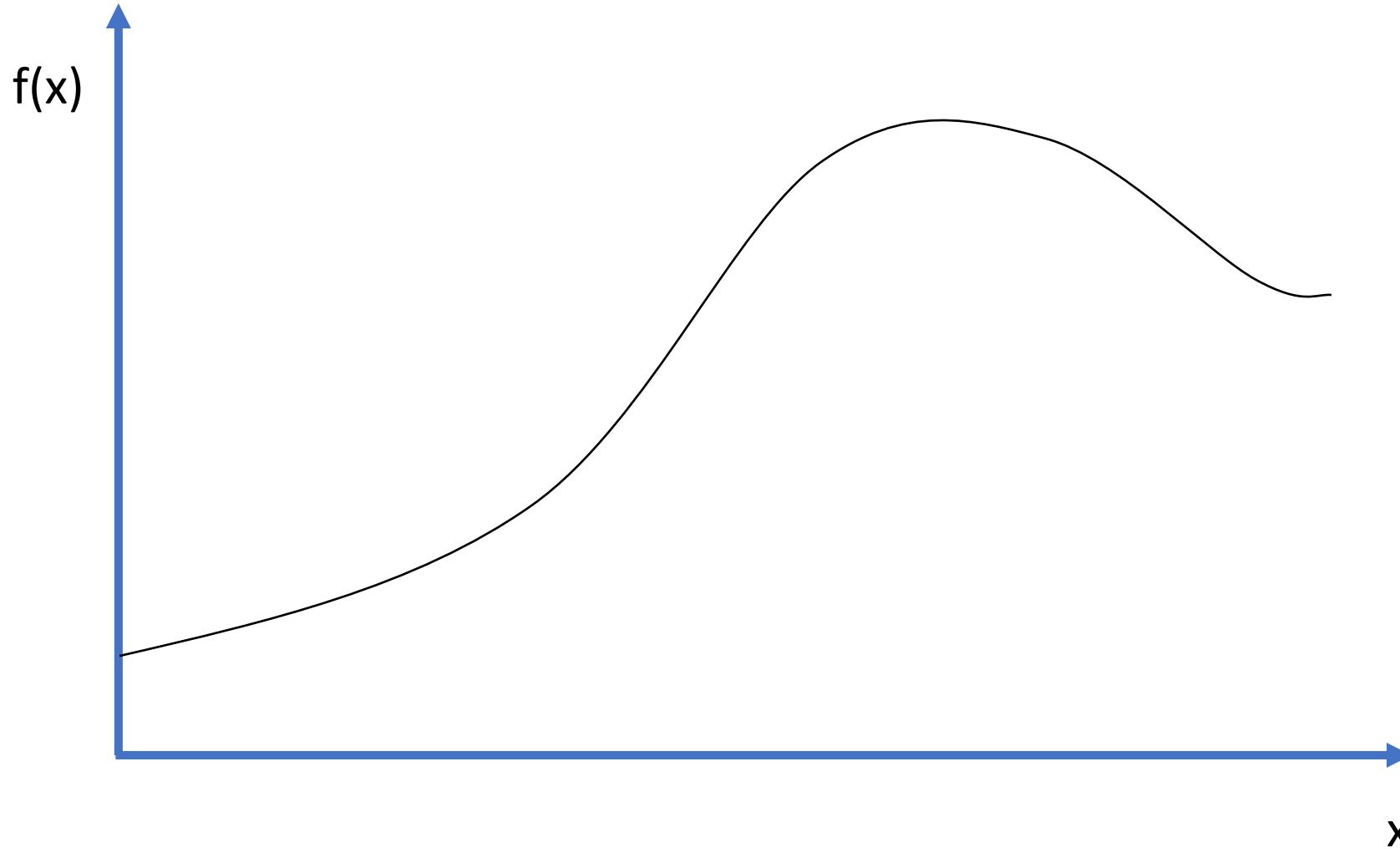
The simplest optimization procedure

1. Initialize \mathbf{x} randomly or based on some initial guess
2. Try some new \mathbf{x}_{new} (typically near the current \mathbf{x})
3. If $f(\mathbf{x}_{\text{new}}) > f(\mathbf{x})$, set $\mathbf{x} = \mathbf{x}_{\text{new}}$ //assuming $f()$ is to be maximized
4. Repeat steps 2 & 3

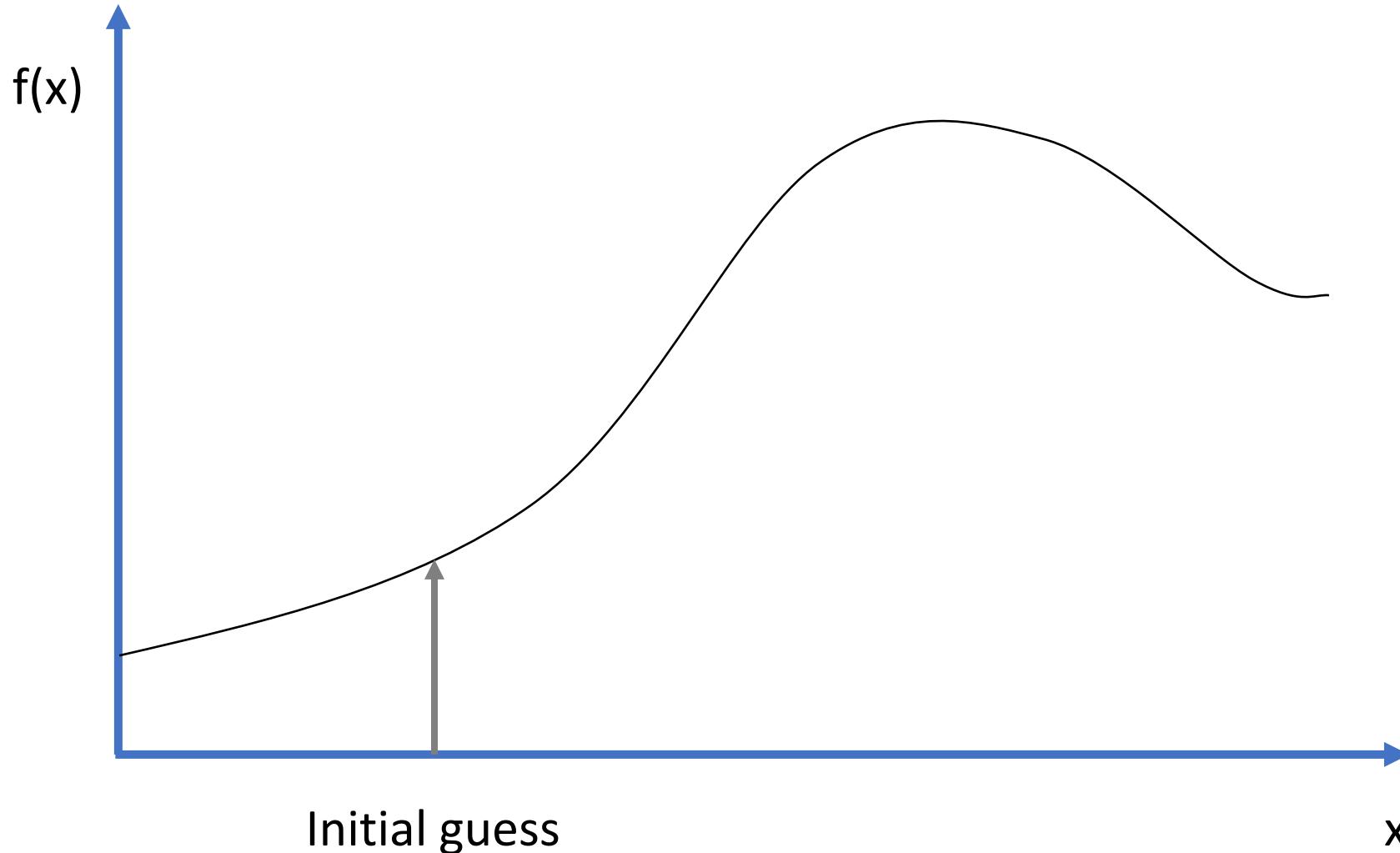
This is similar to *simulated annealing* with specific parameters



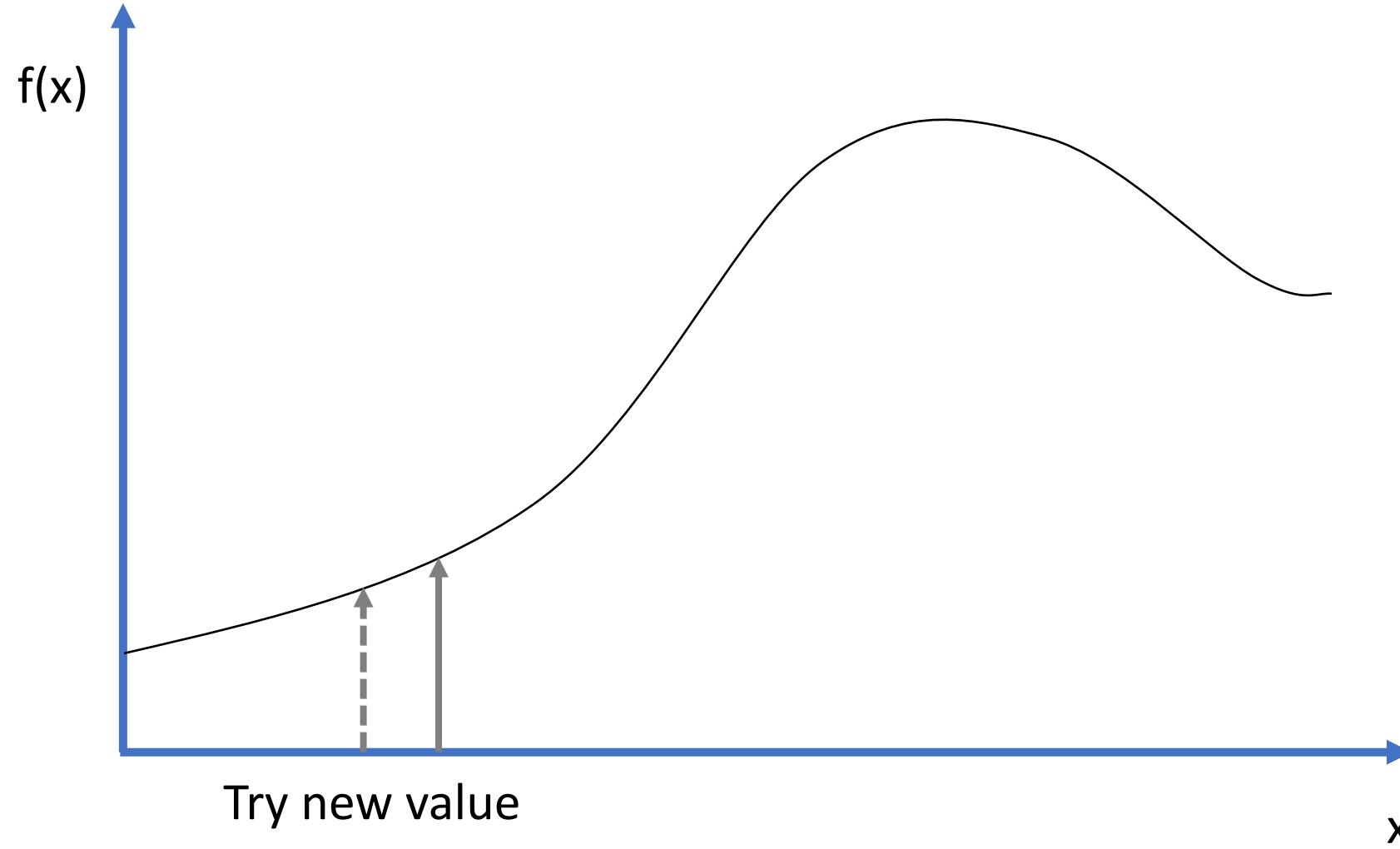
The simplest optimization procedure



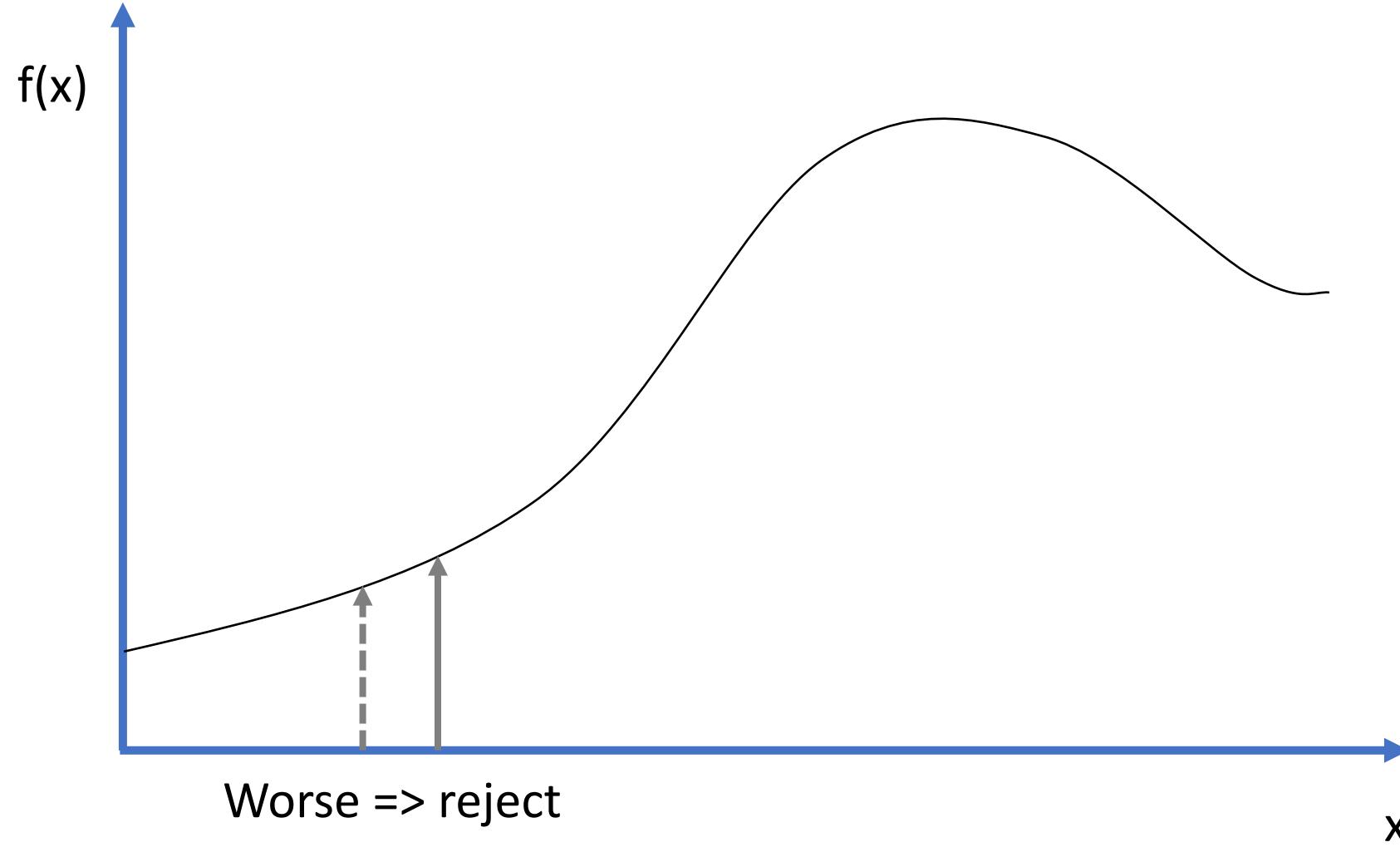
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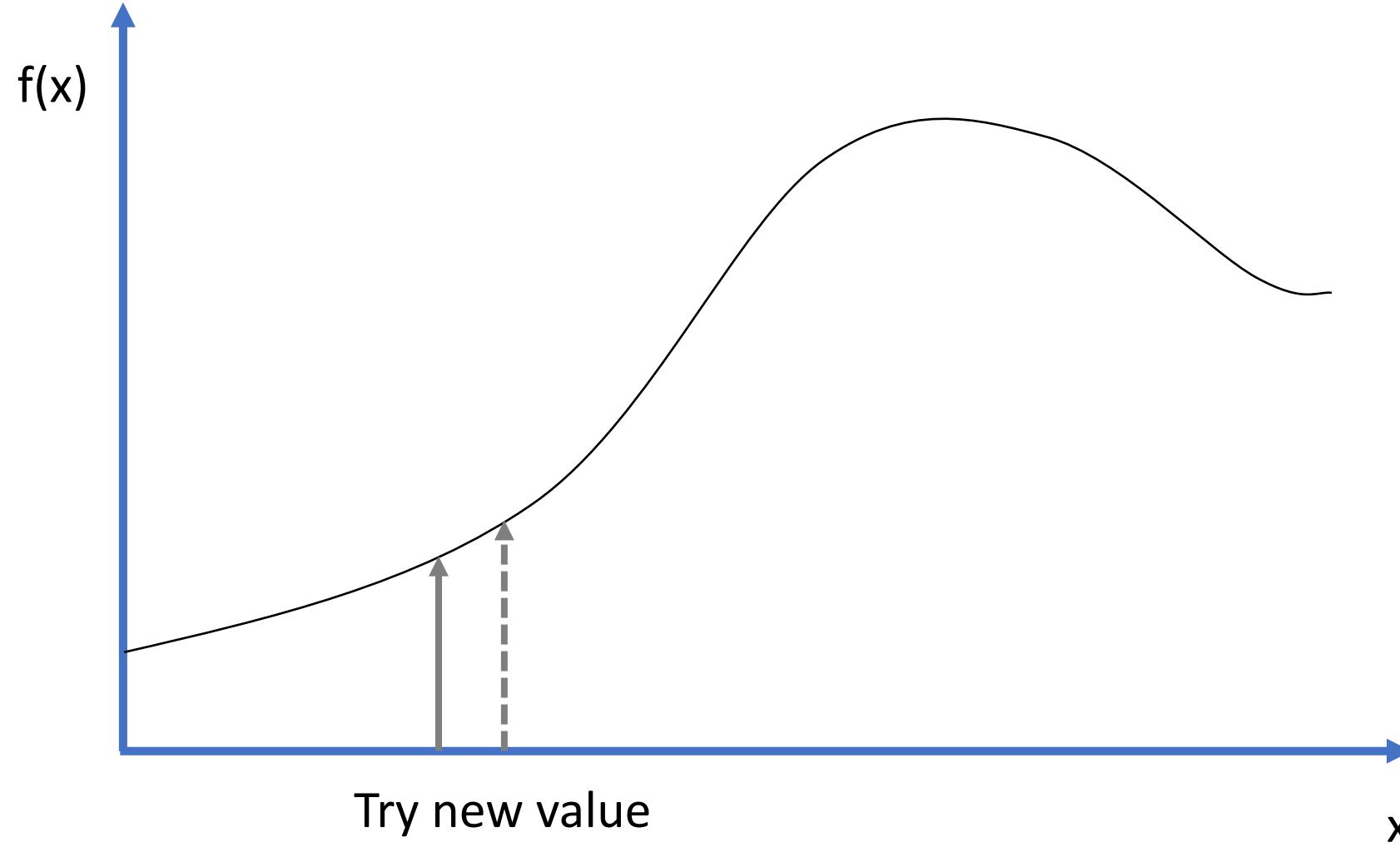
The simplest optimization procedure



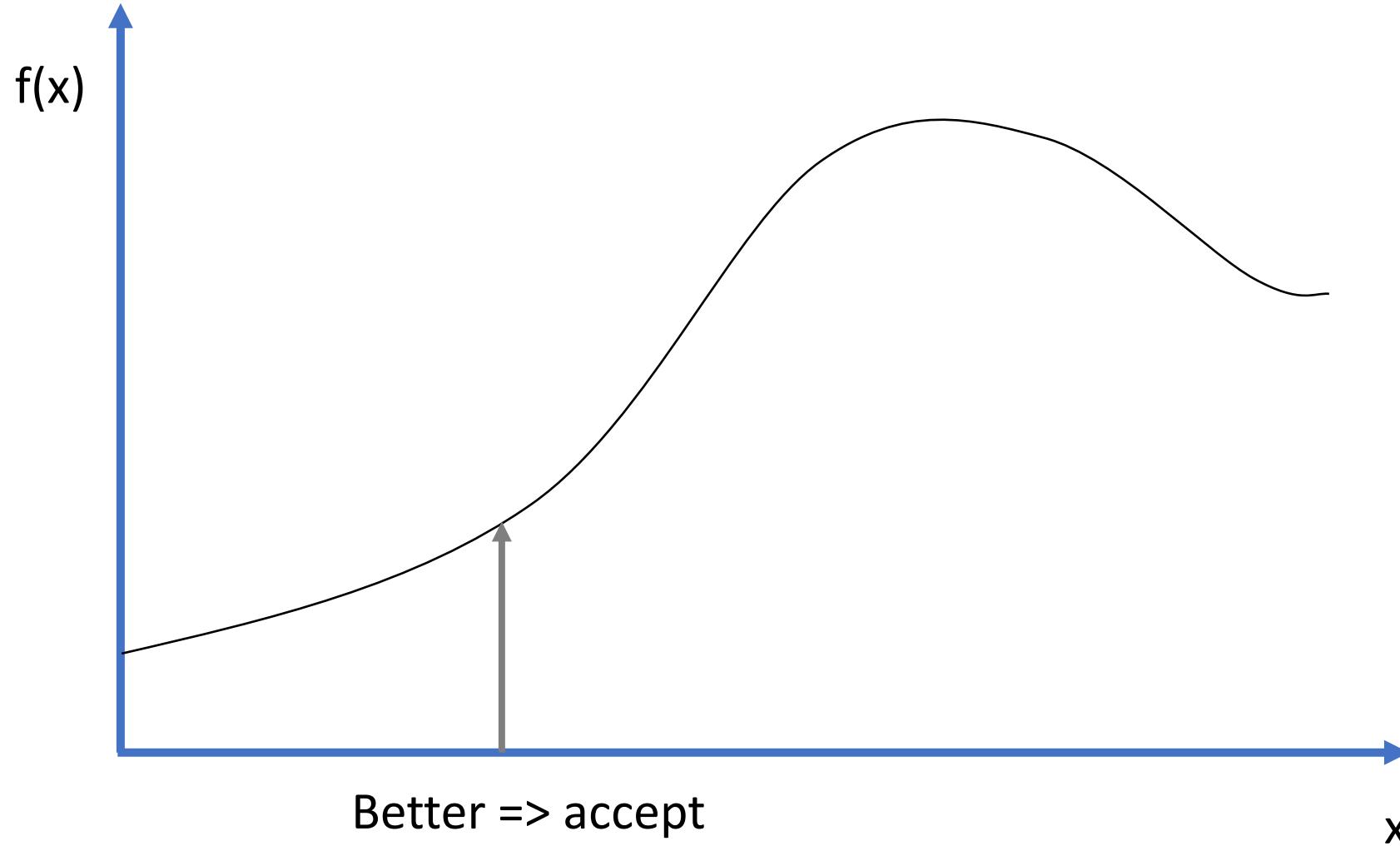
The simplest optimization procedure



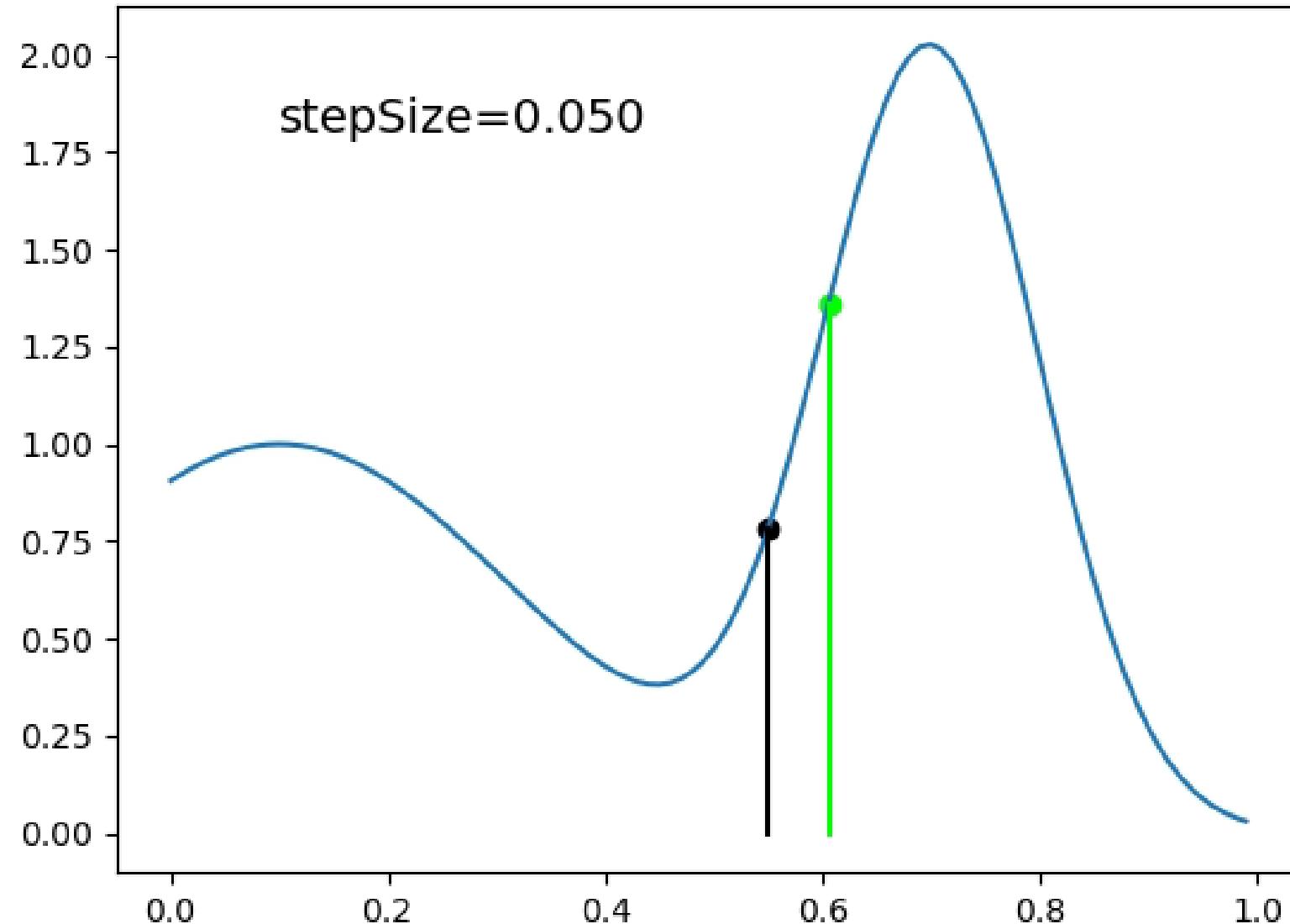
The simplest optimization procedure



The simplest optimization procedure



The simplest optimization procedure



A/B testing

- The simple optimization procedure where
 - x denotes some game design parameters (e.g., monetization strategy)
 - $f(x)$ is evaluated through deploying the game to some players, and measuring impact, e.g., monetization
 - x_{new} is selected by a human designer based on some hypotheses of player preferences and behavior (typically more efficient than algorithms that don't understand player psychology)

The simplest optimization procedure

1. Initialize \mathbf{x} randomly or based on some initial guess
2. Try some new \mathbf{x}_{new} (typically near the current \mathbf{x})
3. If $f(\mathbf{x}_{\text{new}}) > f(\mathbf{x})$, set $\mathbf{x} = \mathbf{x}_{\text{new}}$ //assuming $f()$ is to be maximized
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The simplest optimization procedure

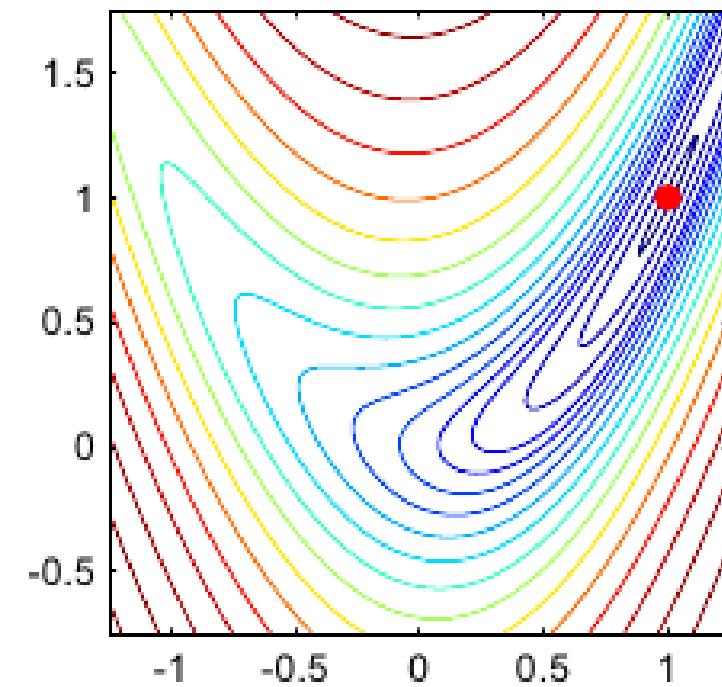
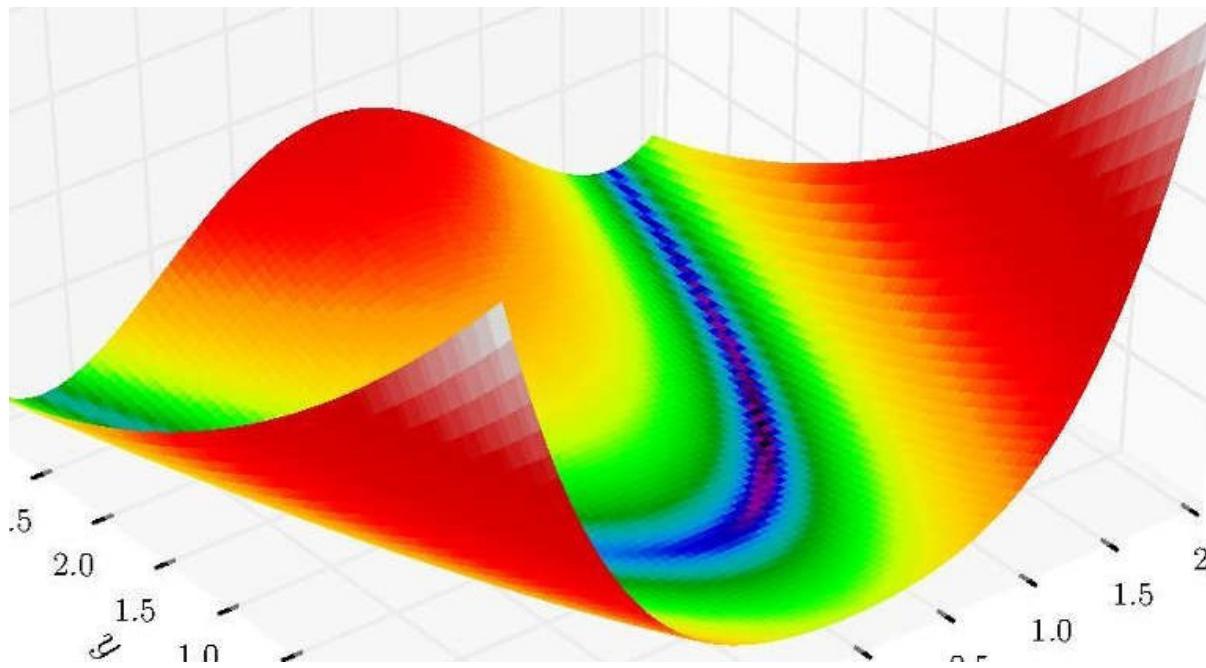
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3. If $f(\mathbf{x}_{\text{new}}) > f(\mathbf{x})$, set $\mathbf{x} = \mathbf{x}_{\text{new}}$ //assuming $f()$ is to be maximized
4. Repeat steps 2 & 3

Modifications:

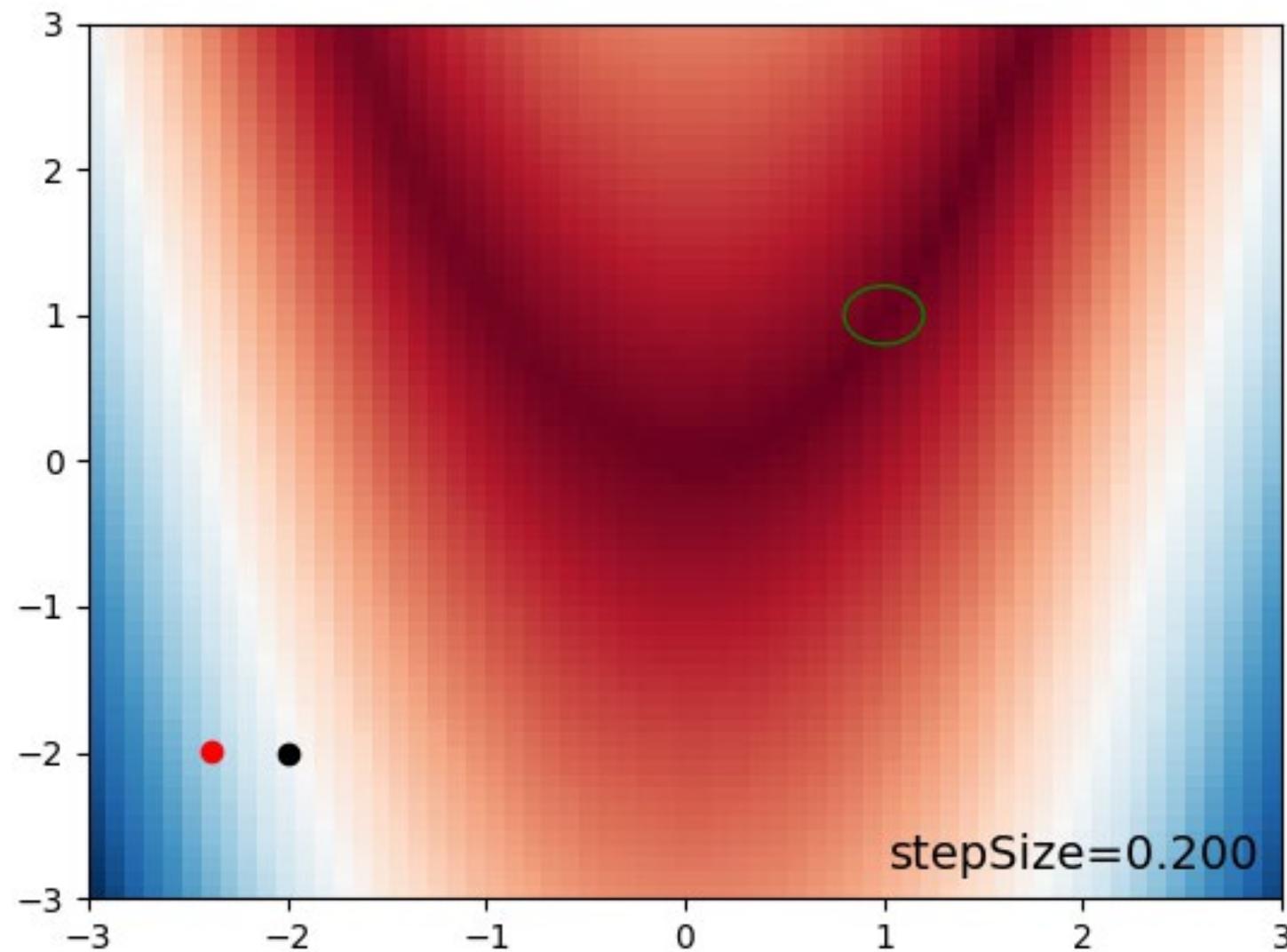
- Generic: How to select \mathbf{x}_{new} ?
- Generic: How to adjust algorithm parameters?
- Problem-specific: How to initialize?
- Problem-specific: How to modify $f(\mathbf{x})$ or parameterize \mathbf{x} such that optimization is easier?

Curse of dimensionality

- Problem: in high-dimensional problems, random search is bad at finding the direction of improvements
- A good test function: Rosenbrock



Progress slows down in the Rosenbrock valley



Curse of dimensionality

- Problem: in high-dimensional problems, random search is bad at finding the direction of improvements
- 2D Rosenbrock: a narrow valley
- 3D Rosenbrock: a narrow “tunnel”
- The search space grows exponentially with dimensionality! (e.g., if one tries even just 2 values for each x_1, x_2, \dots, x_N , there's 2^N possible combinations.)

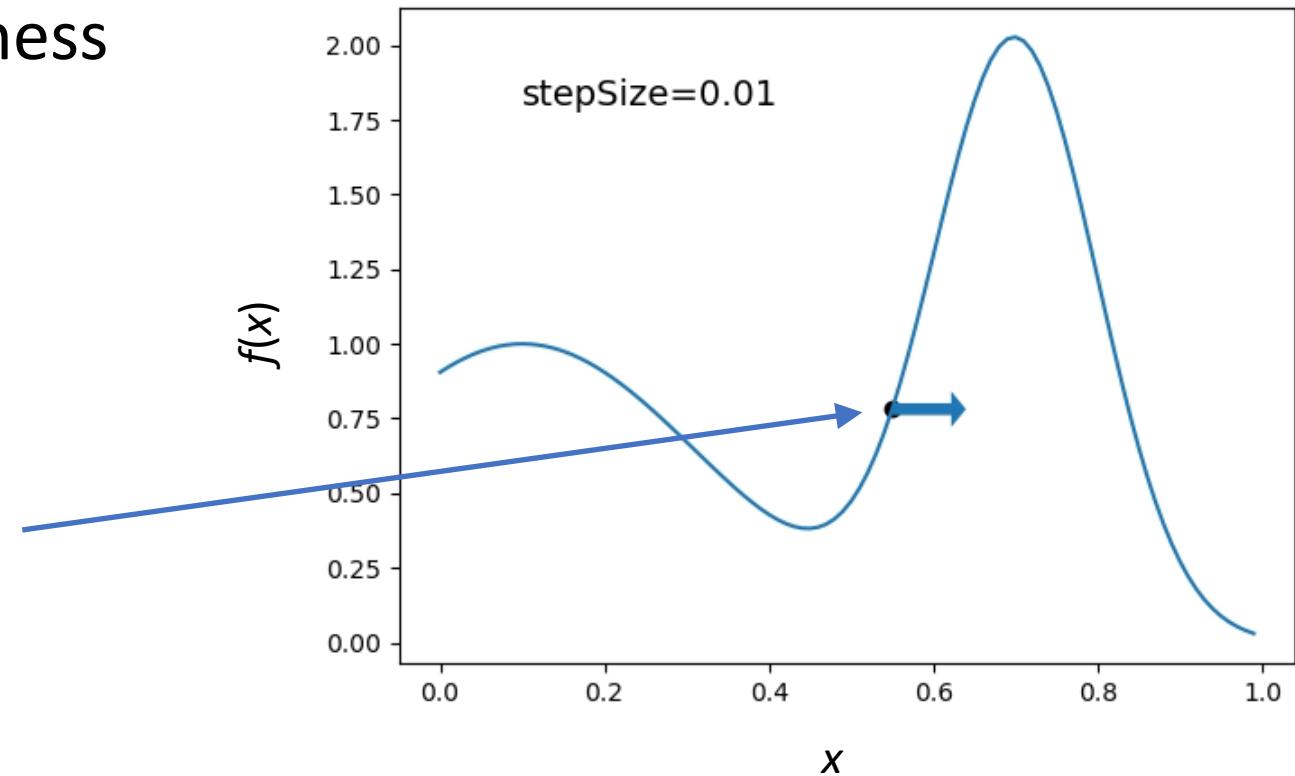
Gradient-based optimization

Utilizing gradient information

- Finding the direction of improvement => use *gradient* information
- Gradient is *the multivariate generalization of the derivative*
- A vector that points to direction of steepest ascent of $f(x)$ in the space of x . Denoted $\nabla f(x)$.
- Length proportional to steepness

Gradient vector:

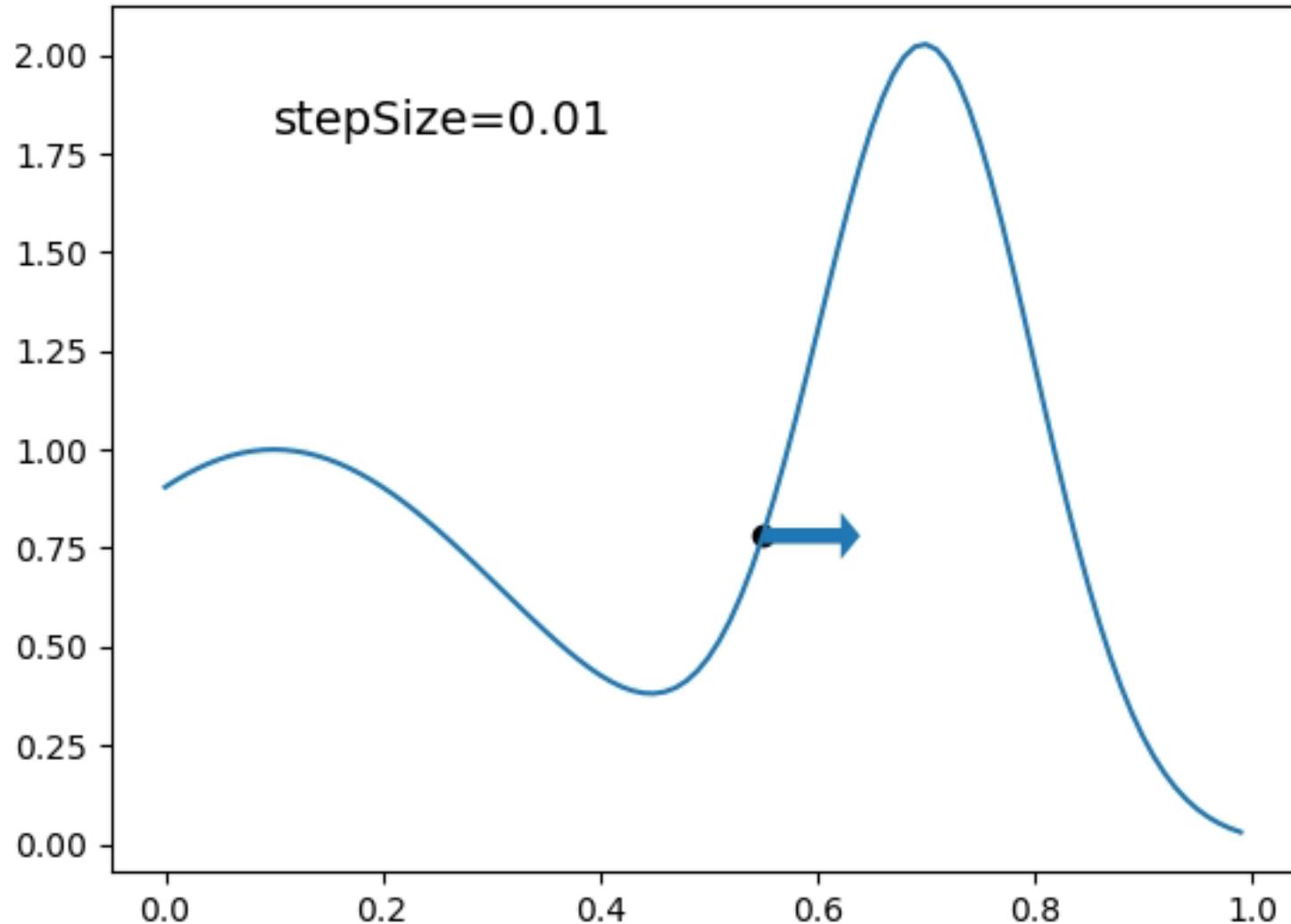
To make $f(x)$ larger,
move x to the right



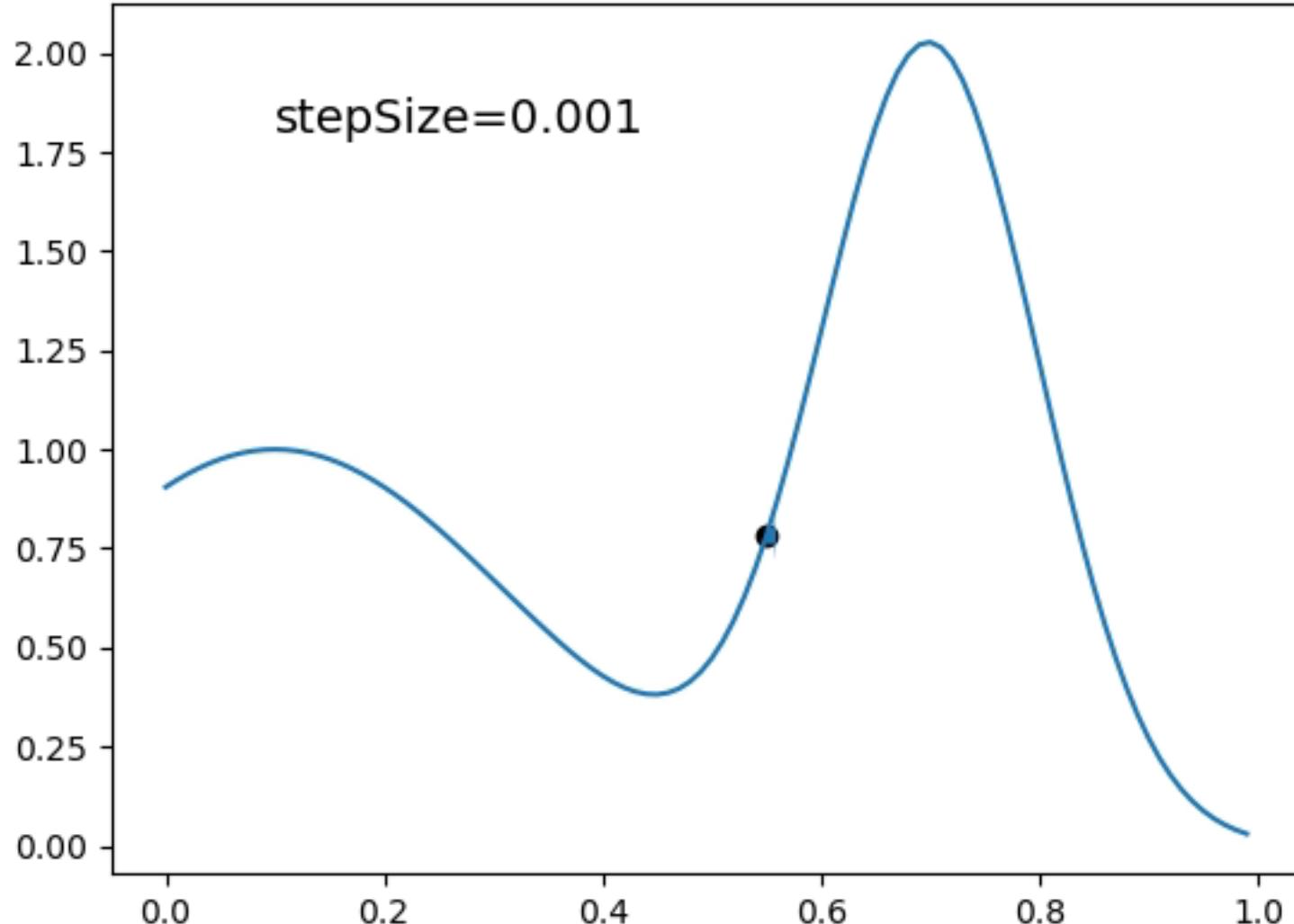
Computing the gradient

- Analytically (TensorFlow, PyTorch etc. do this automatically if your objective function math is implemented using the provided operations)
- Finite differences: measure the change of $f(\mathbf{x})$ when \mathbf{x} perturbed along each coordinate axis
- $g_i(\mathbf{x}) = [f(\mathbf{x}) - f(\mathbf{x} + k\mathbf{u}_i)] / k$, where \mathbf{u}_i is a unit vector pointing along the i :th coordinate axis, k is a small number
- For N dimensions, this requires $N+1$ evaluations of $f(\mathbf{x})$

Gradient ascend: $\mathbf{x}_{\text{new}} = \mathbf{x} + \text{stepSize} * \nabla f(\mathbf{x})$



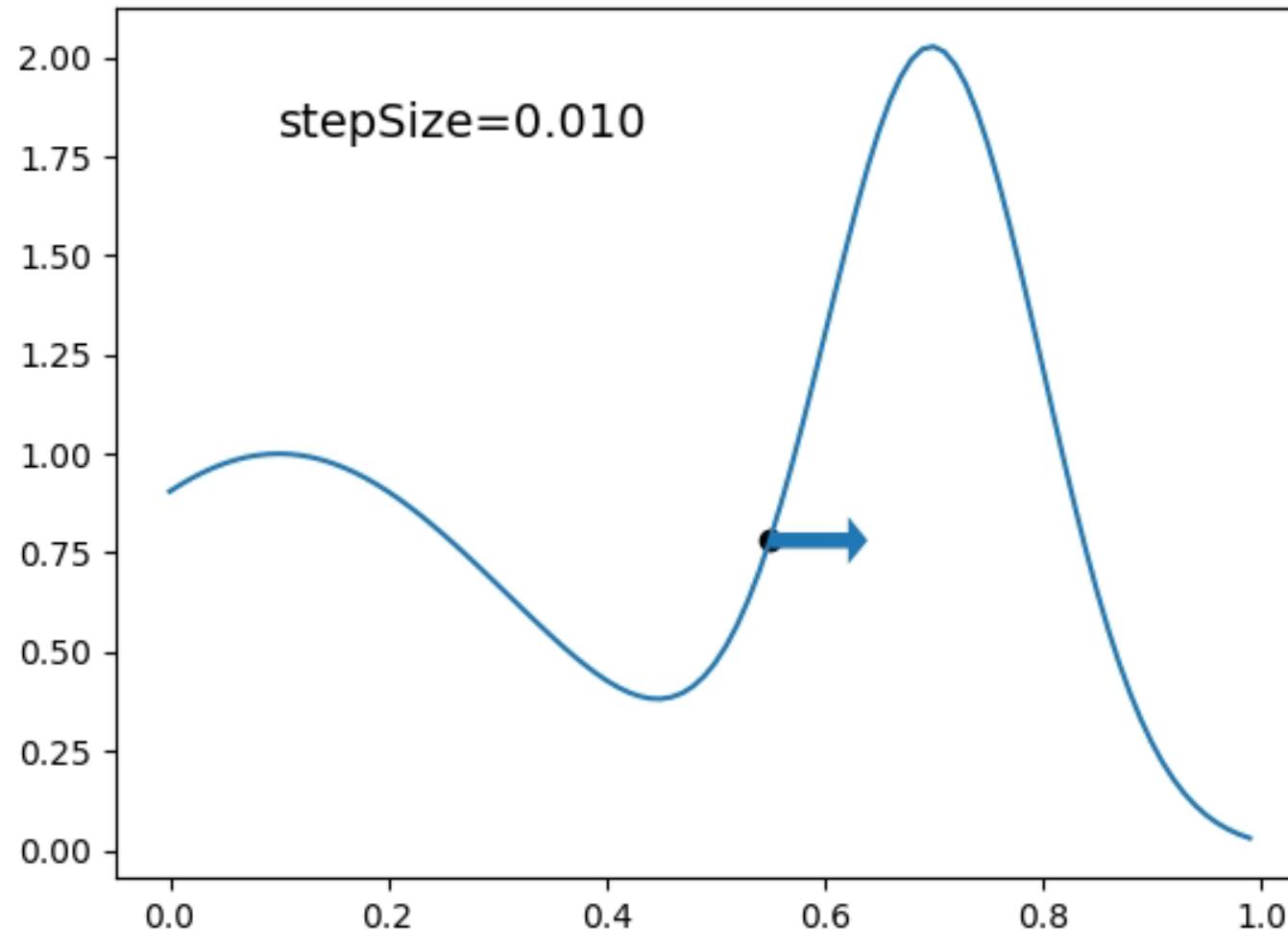
Smaller stepsize: no oscillations, but slow convergence



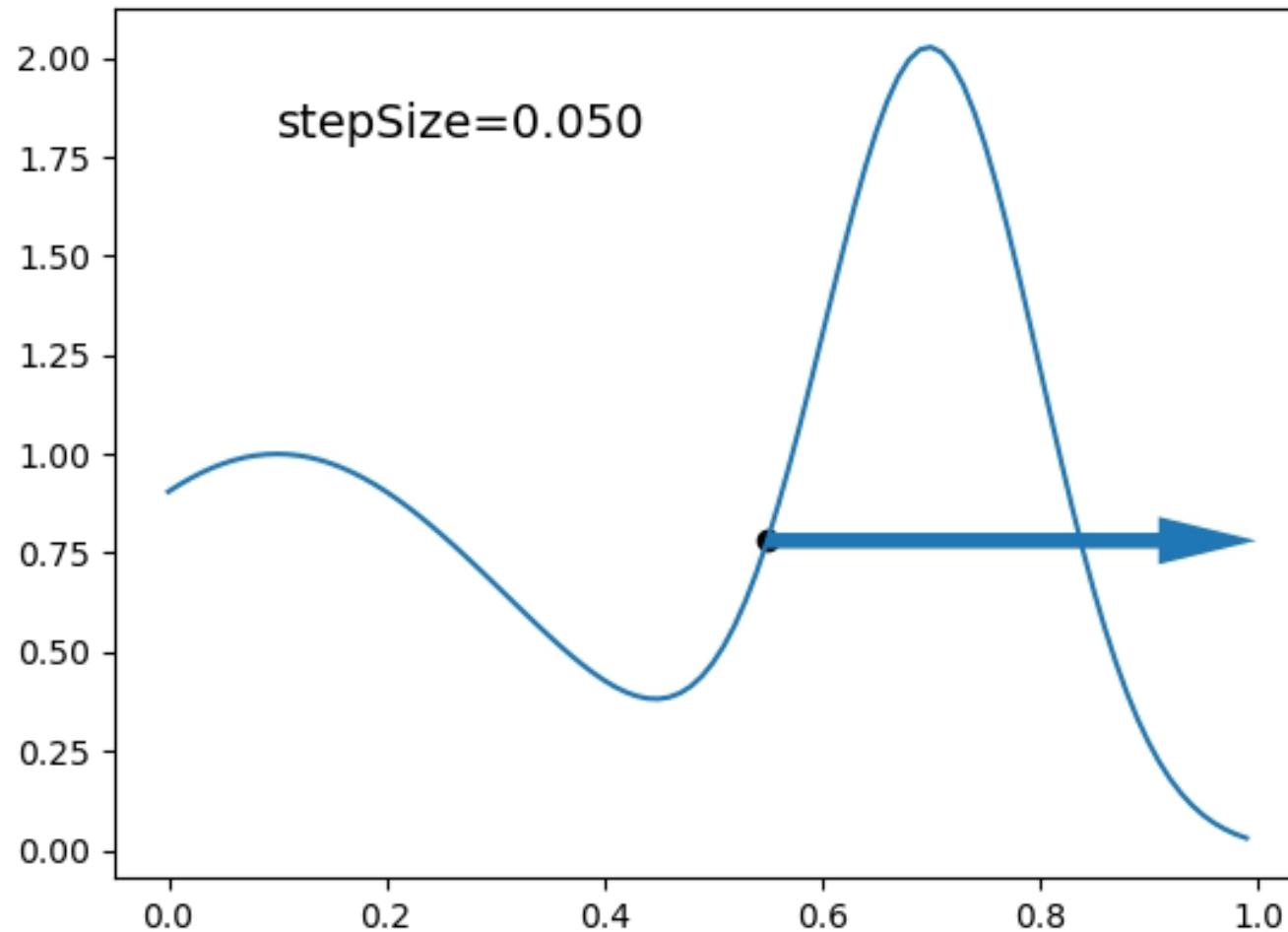
Speeding up convergence

- Step size decay
- Gradient clipping
- Momentum

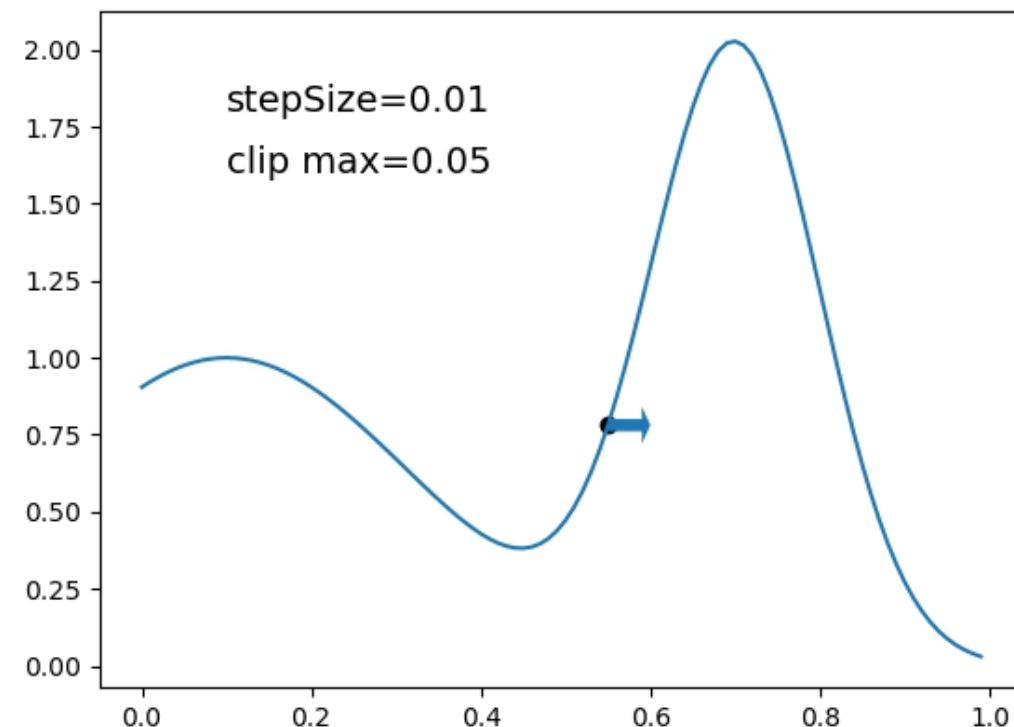
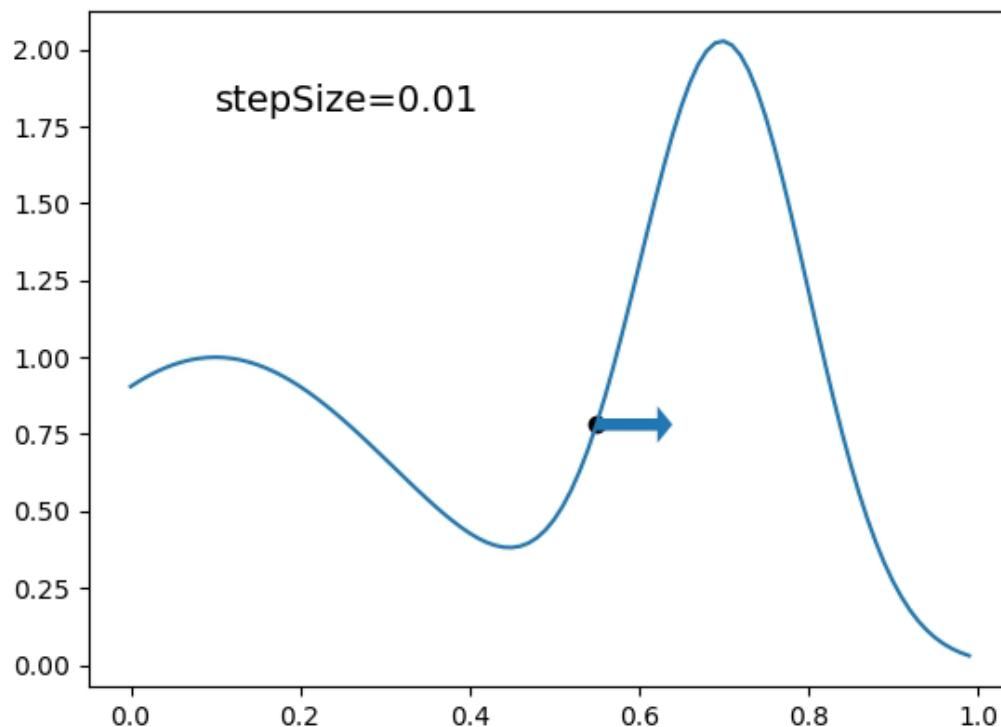
Step size decay



Step size decav



Gradient clipping

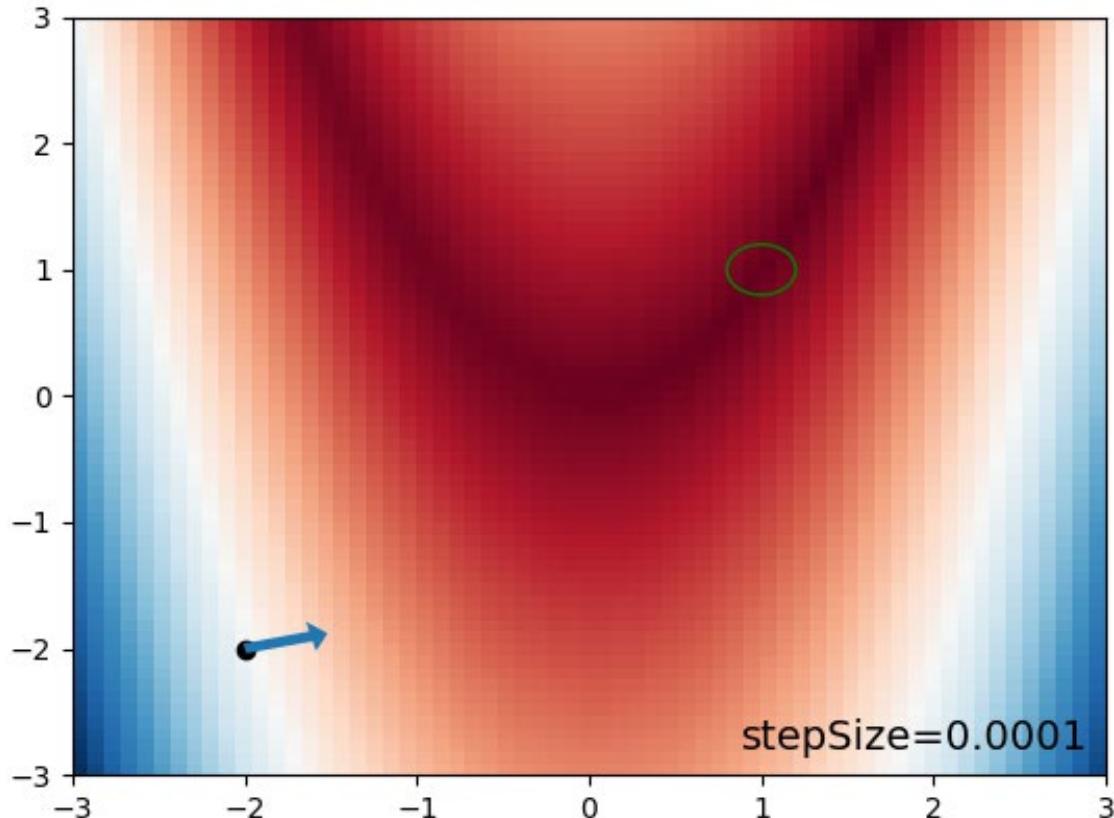


Momentum

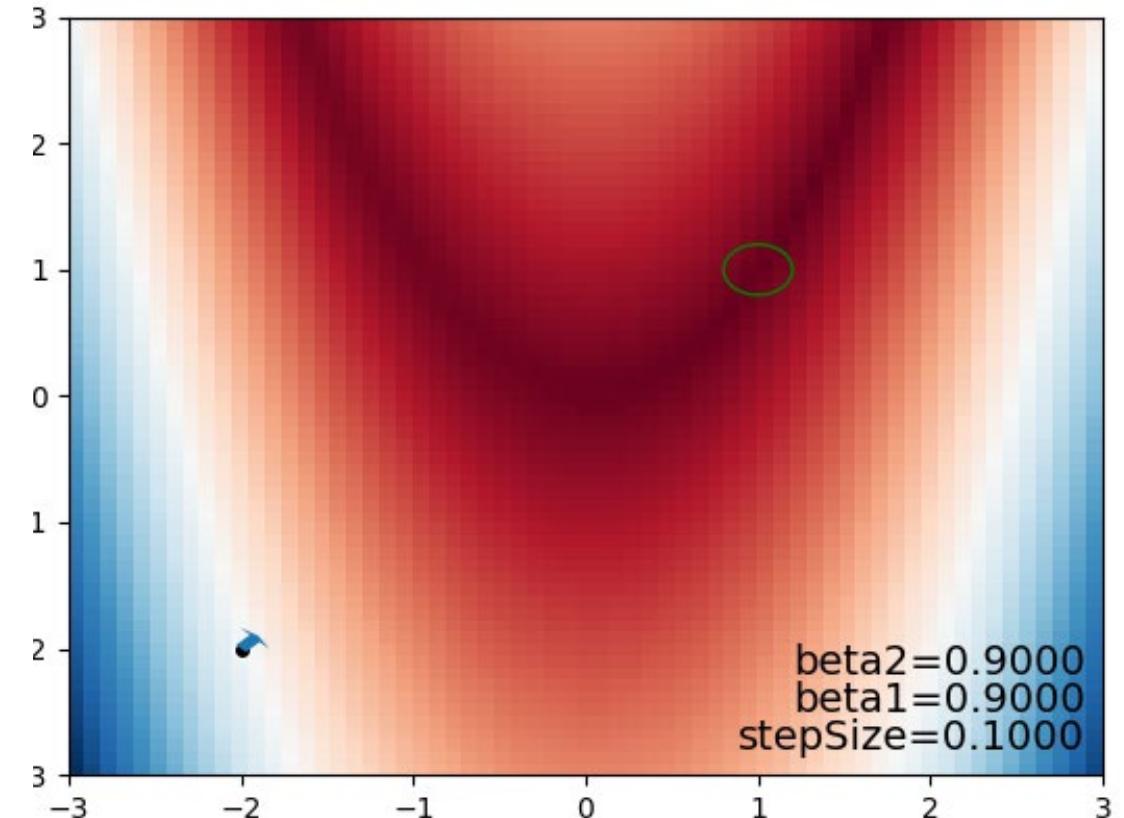
- Gradient interpreted as acceleration instead of velocity
- $v = \text{momentum} * v + \nabla f(x)$
- $x_{\text{new}} = -\text{stepSize} * v$
- 1-momentum = friction, damping of velocity between updates
- Can be used with gradient clipping
- Adam is a modern momentum-based method with some clever additional tricks. A good default choice for optimizing neural networks



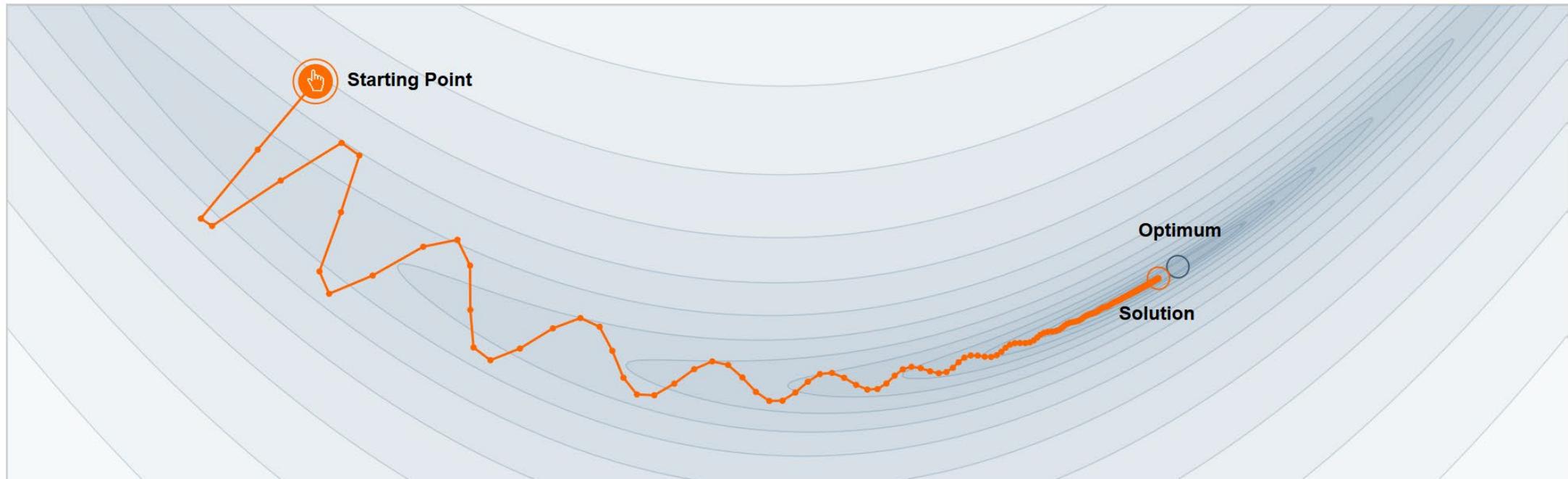
Without momentum



Adam



Why Momentum Really Works



Step-size $\alpha = 0.0015$



Momentum $\beta = 0.88$



We often think of Momentum as a means of dampening oscillations and speeding up the iterations, leading to faster convergence. But it has other interesting behavior. It allows a larger range of step-sizes to be used, and creates its own oscillations. What is going on?



ADAM: A METHOD FOR STOCHASTIC OPTIMIZATION

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ABSTRACT

We introduce *Adam*, an algorithm for first-order gradient-based optimization of stochastic objective functions, based on adaptive estimates of lower-order moments. The method is straightforward to implement, is computationally efficient, has little memory requirements, is invariant to diagonal rescaling of the gradients, and is well suited for problems that are large in terms of data and/or parameters. The method is also appropriate for non-stationary objectives and problems with very noisy and/or sparse gradients. The hyper-parameters have intuitive interpretations and typically require little tuning. Some connections to related algorithms, on which *Adam* was inspired, are discussed. We also analyze the theoretical convergence properties of the algorithm and provide a regret bound on the convergence rate that is comparable to the best known results under the online convex optimization framework. Empirical results demonstrate that Adam works well in practice and compares favorably to other stochastic optimization methods. Finally, we discuss *AdaMax*, a variant of *Adam* based on the infinity norm.

Algorithm 1: Adam, our proposed algorithm for stochastic optimization. See section 2 for details, and for a slightly more efficient (but less clear) order of computation. g_t^2 indicates the elementwise square $g_t \odot g_t$. Good default settings for the tested machine learning problems are $\alpha = 0.001$, $\beta_1 = 0.9$, $\beta_2 = 0.999$ and $\epsilon = 10^{-8}$. All operations on vectors are element-wise. With β_1^t and β_2^t we denote β_1 and β_2 to the power t .

Require: α : Stepsize

Require: $\beta_1, \beta_2 \in [0, 1]$: Exponential decay rates for the moment estimates

Require: $f(\theta)$: Stochastic objective function with parameters θ

Require: θ_0 : Initial parameter vector

$m_0 \leftarrow 0$ (Initialize 1st moment vector)

$v_0 \leftarrow 0$ (Initialize 2nd moment vector)

$t \leftarrow 0$ (Initialize timestep)

while θ_t not converged **do**

$t \leftarrow t + 1$

$g_t \leftarrow \nabla_{\theta} f_t(\theta_{t-1})$ (Get gradients w.r.t. stochastic objective at timestep t)

$m_t \leftarrow \beta_1 \cdot m_{t-1} + (1 - \beta_1) \cdot g_t$ (Update biased first moment estimate)

$v_t \leftarrow \beta_2 \cdot v_{t-1} + (1 - \beta_2) \cdot g_t^2$ (Update biased second raw moment estimate)

$\hat{m}_t \leftarrow m_t / (1 - \beta_1^t)$ (Compute bias-corrected first moment estimate)

$\hat{v}_t \leftarrow v_t / (1 - \beta_2^t)$ (Compute bias-corrected second raw moment estimate)

$\theta_t \leftarrow \theta_{t-1} - \alpha \cdot \hat{m}_t / (\sqrt{\hat{v}_t} + \epsilon)$ (Update parameters)

end while

return θ_t (Resulting parameters)



while θ_t not converged **do**

$$t \leftarrow t + 1$$

$$g_t \leftarrow \nabla_{\theta} f_t(\theta_{t-1})$$

$$m_t \leftarrow \beta_1 \cdot m_{t-1} + (1 - \beta_1) \cdot g_t$$

$$v_t \leftarrow \beta_2 \cdot v_{t-1} + (1 - \beta_2) \cdot g_t^2$$

$$\hat{m}_t \leftarrow m_t / (1 - \beta_1^t)$$

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$$\theta_t \leftarrow \theta_{t-1} - \alpha \cdot \hat{m}_t / (\sqrt{\hat{v}_t} + \epsilon)$$

end while

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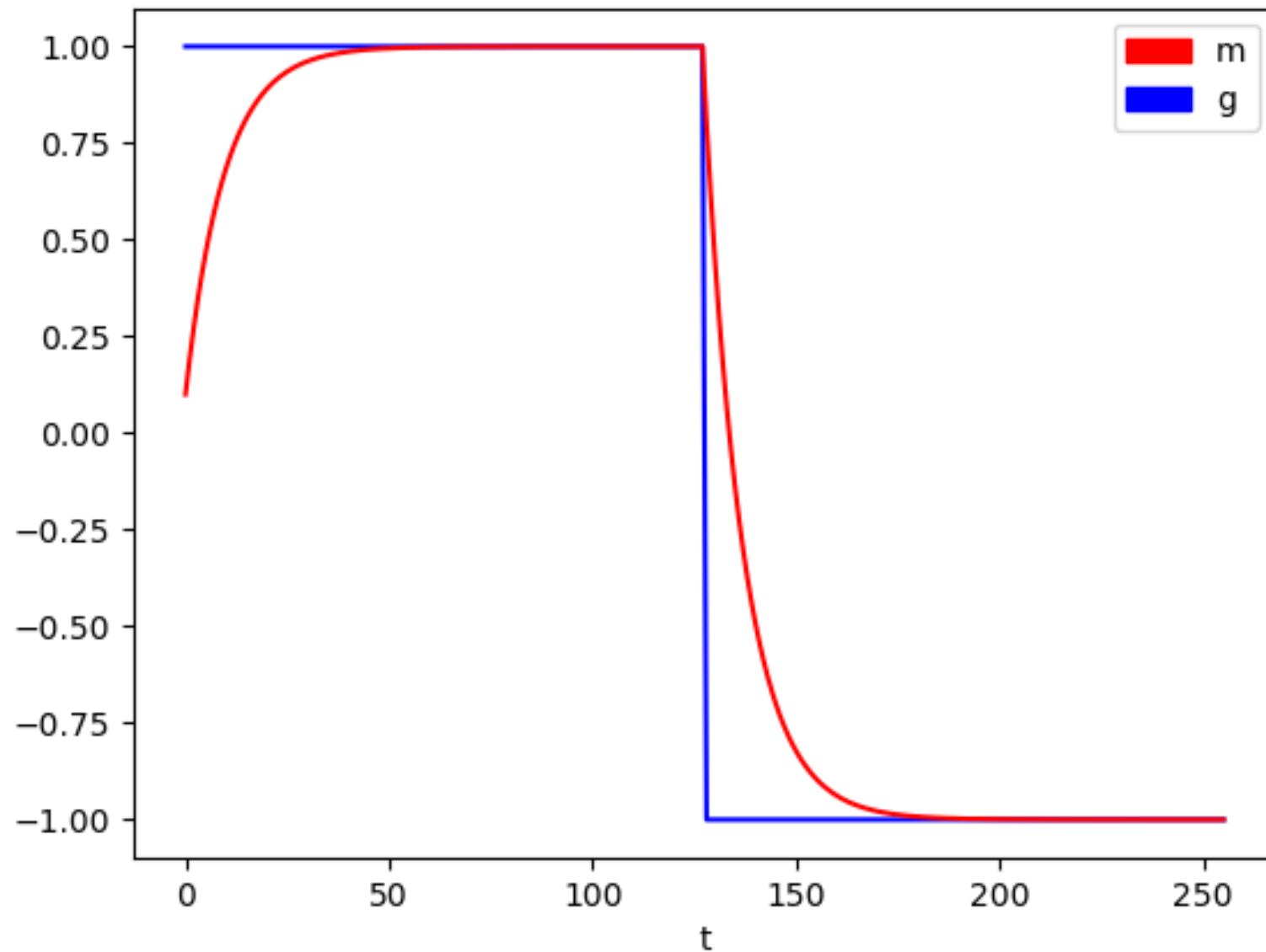
$$\hat{m}_t \leftarrow m_t / (1 - \beta_1^t)$$

$$\hat{v}_t \leftarrow v_t / (1 - \beta_2^t)$$

$$\theta_t \leftarrow \theta_{t-1} - \alpha \cdot \hat{m}_t / (\sqrt{\hat{v}_t} + \epsilon)$$

end while

Exponential smoothing, beta1=0.9



while θ_t not converged **do**

$$t \leftarrow t + 1$$

$$g_t \leftarrow \nabla_{\theta} f_t(\theta_{t-1})$$

$$m_t \leftarrow \beta_1 \cdot m_{t-1} + (1 - \beta_1) \cdot g_t$$

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$$\theta_t \leftarrow \theta_{t-1} - \alpha \cdot \hat{m}_t / (\sqrt{\hat{v}_t} + \epsilon)$$

end while

while θ_t not converged **do**

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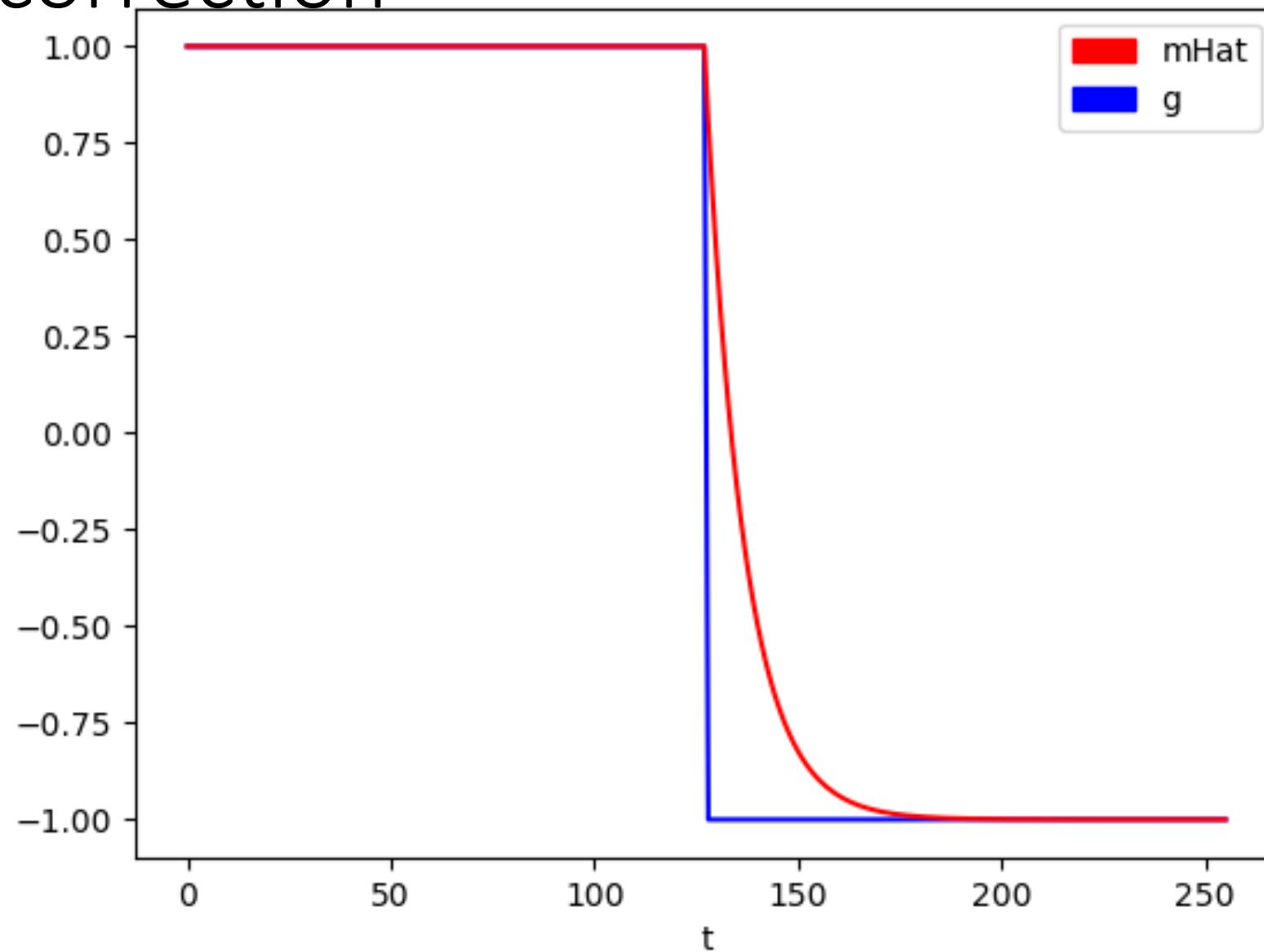
$$\hat{m}_t \leftarrow m_t / (1 - \beta_1^t)$$

$$\hat{v}_t \leftarrow v_t / (1 - \beta_2^t)$$

$$\theta_t \leftarrow \theta_{t-1} - \alpha \cdot \hat{m}_t / (\sqrt{\hat{v}_t} + \epsilon)$$

end while

Bias correction



while θ_t not converged **do**

$$t \leftarrow t + 1$$

$$g_t \leftarrow \nabla_{\theta} f_t(\theta_{t-1})$$

$$m_t \leftarrow \beta_1 \cdot m_{t-1} + (1 - \beta_1) \cdot g_t$$

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$$\hat{m}_t \leftarrow m_t / (1 - \beta_1^t)$$

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$$\theta_t \leftarrow \theta_{t-1} - \alpha \cdot \hat{m}_t / (\sqrt{\hat{v}_t} + \epsilon)$$

end while

while θ_t not converged **do**

Change of parameter values

$$\theta_t \leftarrow \theta_{t-1} - \alpha \cdot \hat{m}_t / (\sqrt{\hat{v}_t} + \epsilon)$$

end while

while θ_t not converged do

Step size

$$\theta_t \leftarrow \theta_{t-1} - \boxed{\alpha} \cdot \hat{m}_t / (\sqrt{\hat{v}_t} + \epsilon)$$

end while

while θ_t not converged **do**

Smoothed gradient

$$\theta_t \leftarrow \theta_{t-1} - \alpha \cdot \boxed{\hat{m}_t} / (\sqrt{\hat{v}_t} + \epsilon)$$

end while

while θ_t not converged **do**

Per-variable scaling

$$\theta_t \leftarrow \theta_{t-1} - \alpha \cdot \hat{m}_t / (\sqrt{\hat{v}_t} + \epsilon)$$

end while

while θ_t not converged **do**

Per-variable scaling

$$\theta_t \leftarrow \theta_{t-1} - \alpha \cdot \hat{m}_t / (\sqrt{\hat{v}_t} + \epsilon)$$

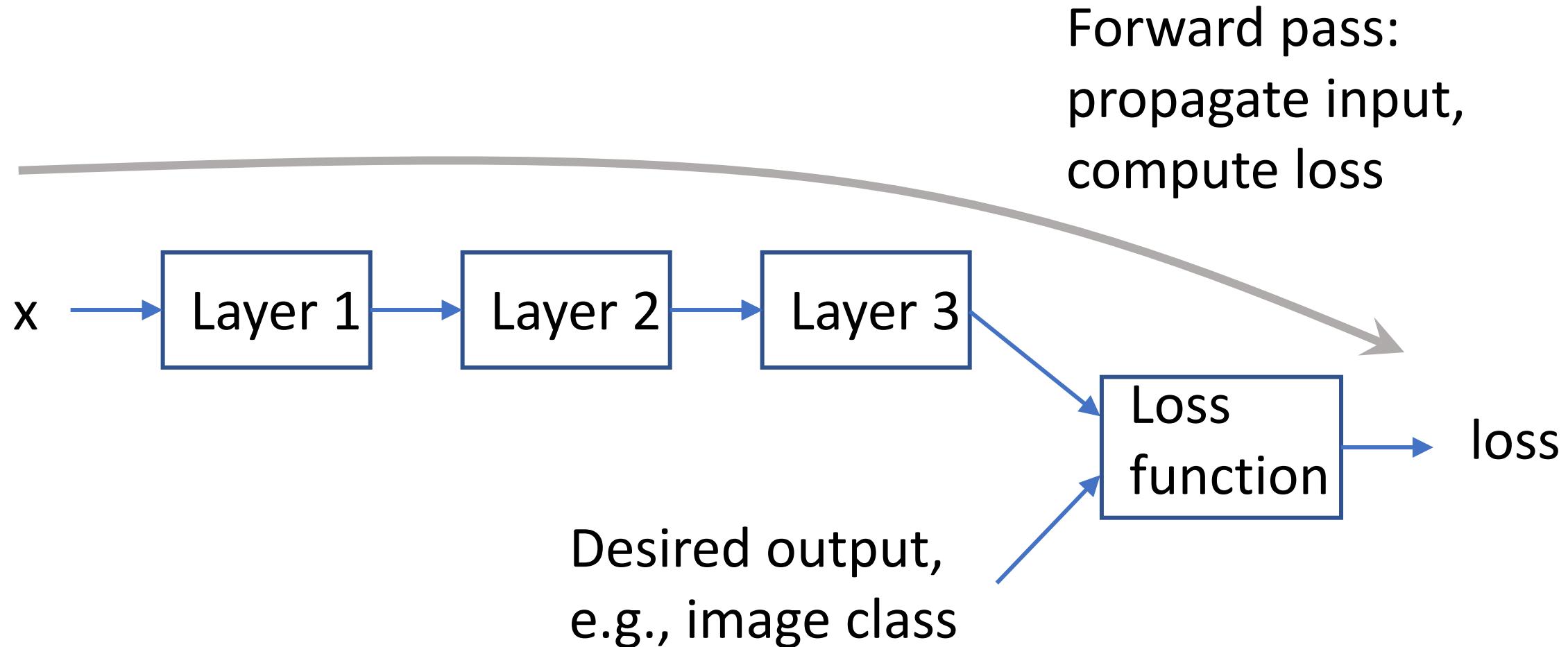
end while



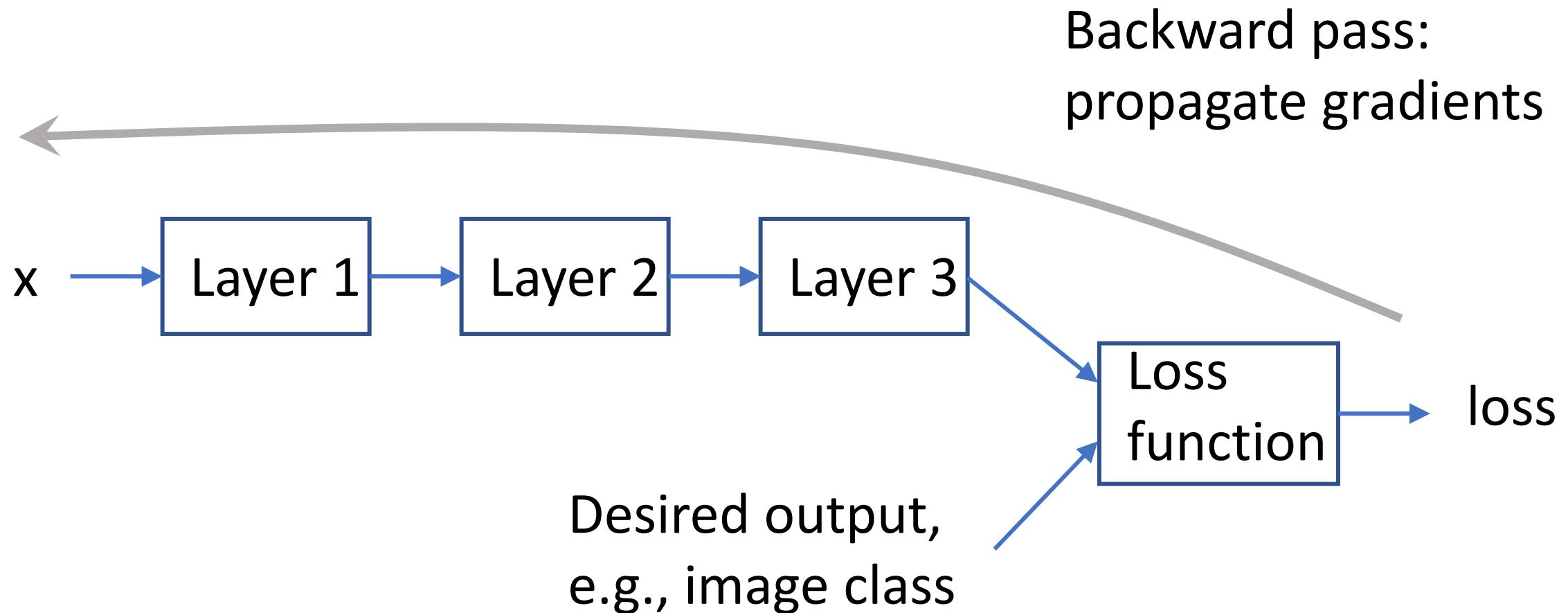
Gradients and compute graphs

- Utilizing the chain rule of calculus, Tensorflow & Pytorch allow one to compute gradients of compute graph outputs (e.g., loss function) with respect to variables (e.g., neural network weights)
- This process is called backpropagation

Backpropagation



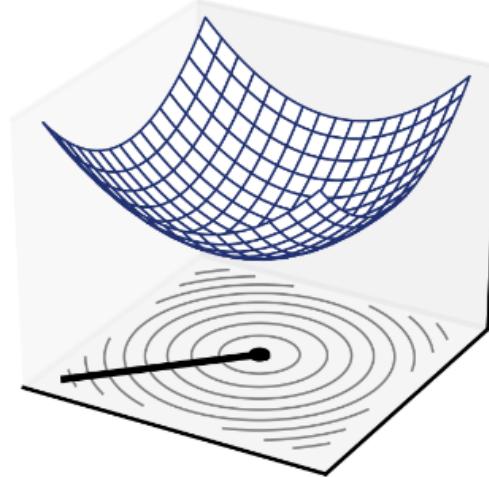
Backpropagation



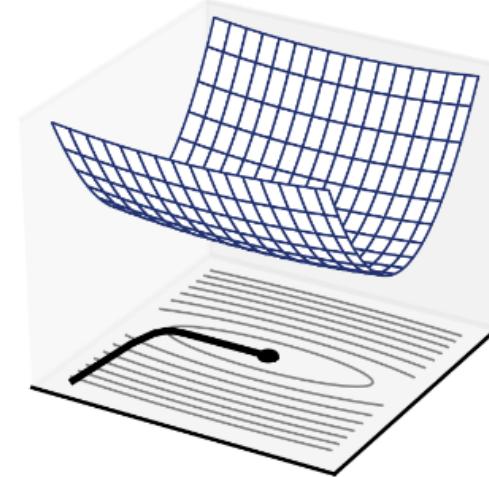


Difficulty of optimization

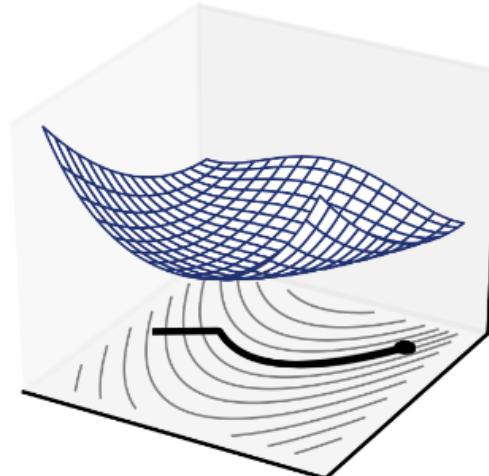
Convex,
well-conditioned



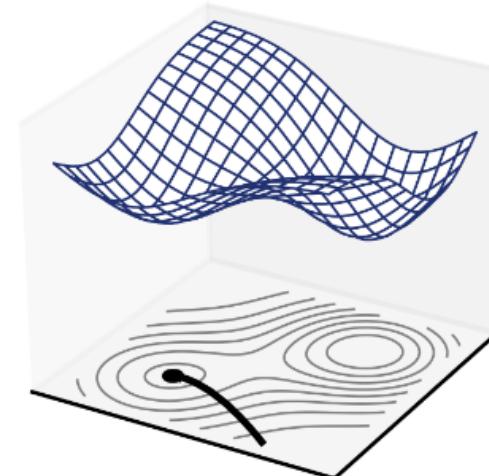
Convex,
ill-conditioned



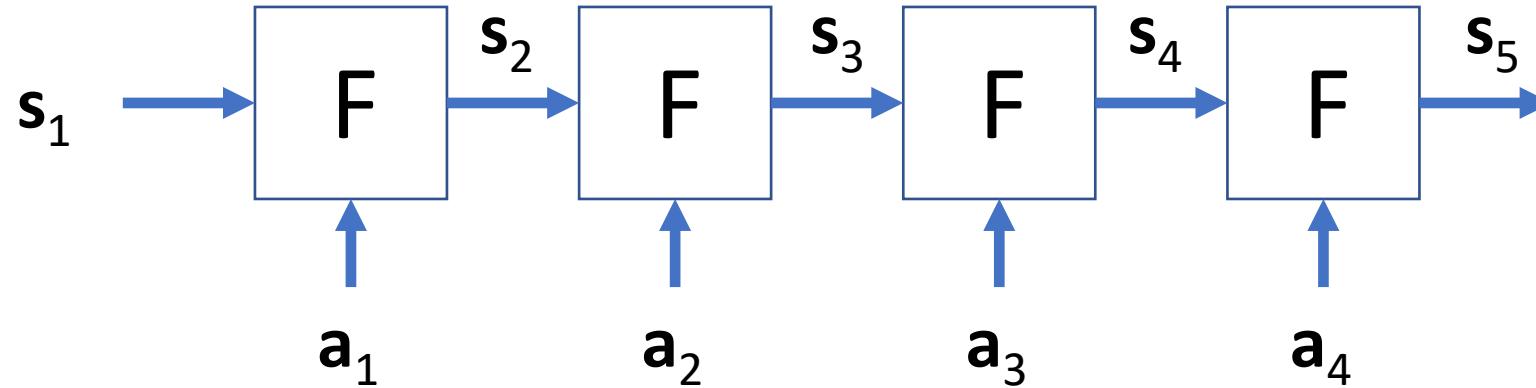
Non-convex,
unimodal



Non-convex,
multimodal



Planning through backpropagation



Subscripts denote simulation time

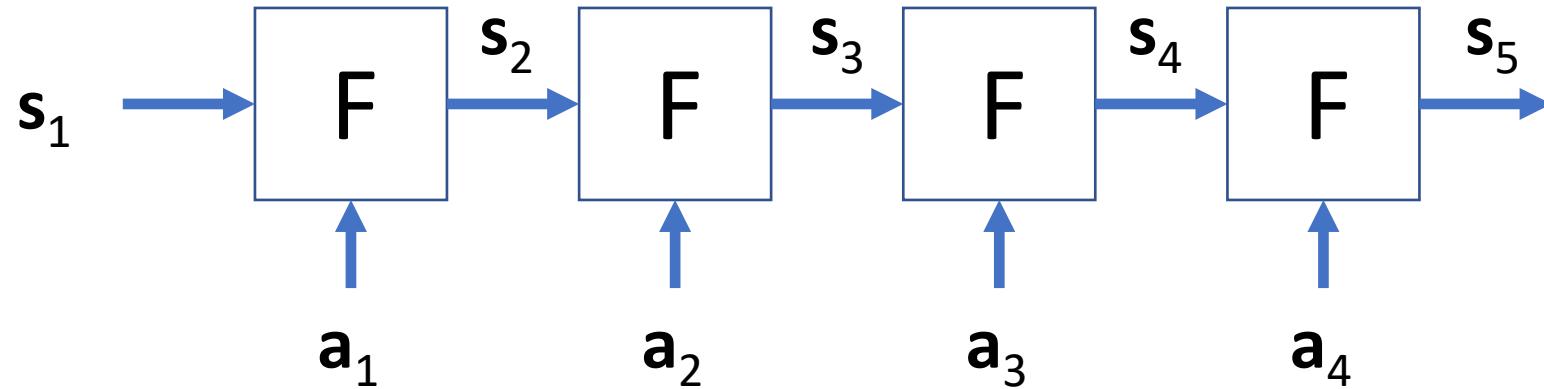
s: state, e.g., the physics simulation parameters of a robot

a: action, e.g., the robot's actuation torques

F: simulation model (code or learned neural network), $s_2 = F(s_1, a_1)$

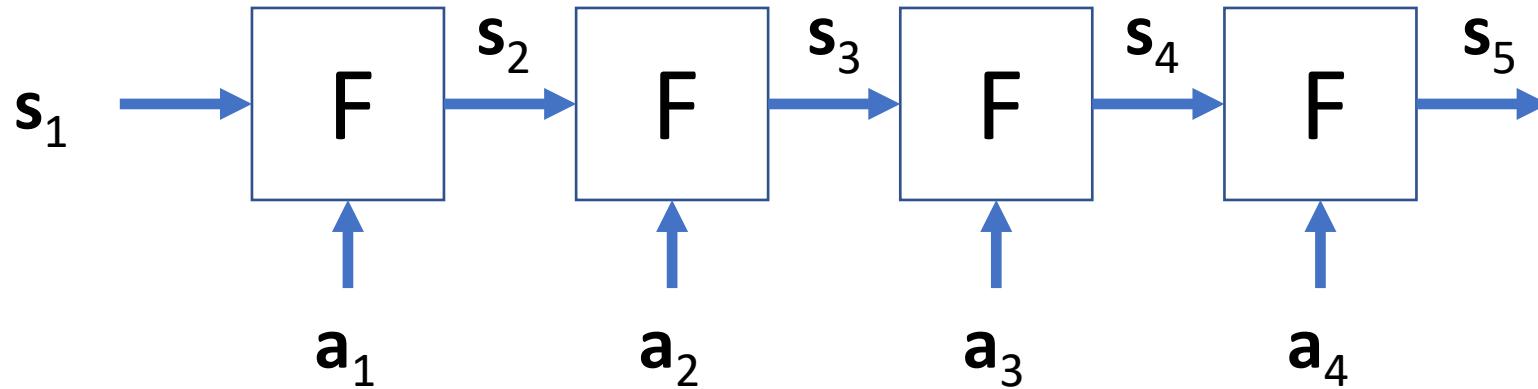
Planning through backpropagation

Objective: Maximize $\sum_i r(\mathbf{a}_i, \mathbf{s}_i)$



Planning through backpropagation

Objective: Maximize $\sum_i r(\mathbf{a}_i, \mathbf{s}_i)$

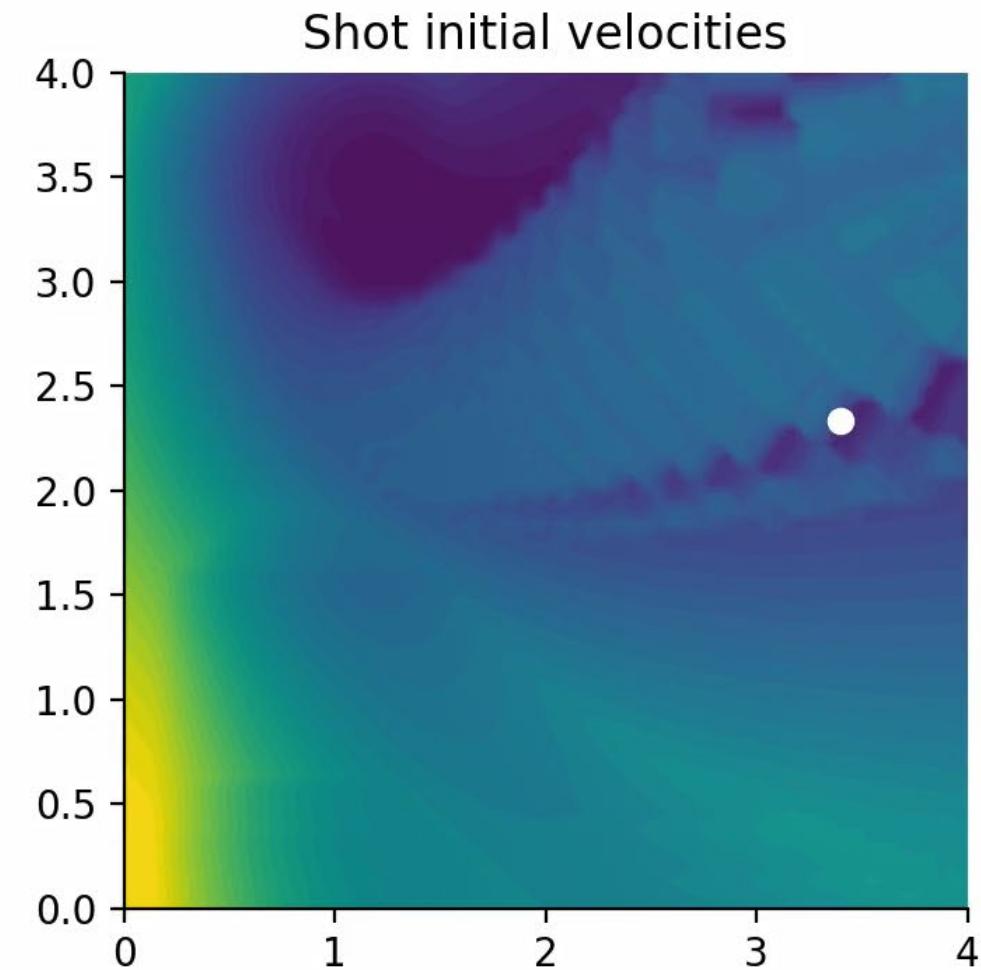
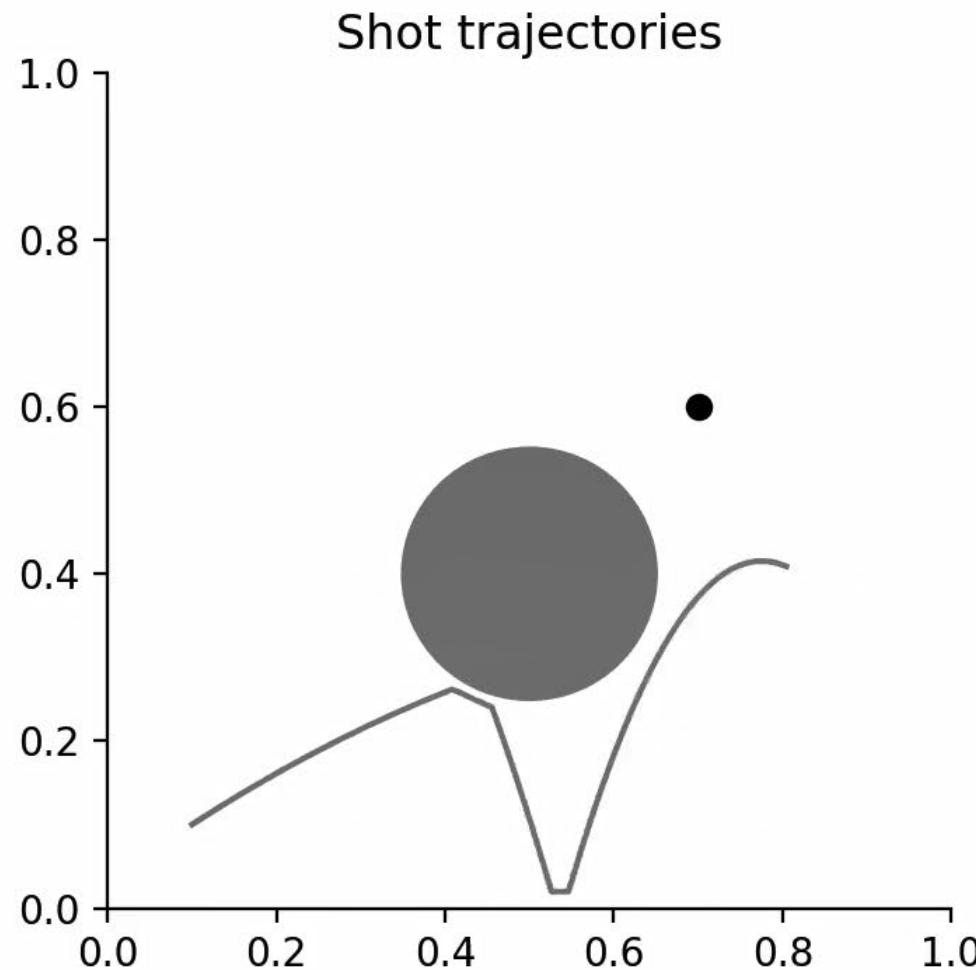


Gradient-based optimization of actions is trivial in PyTorch etc., if both F and $r(\mathbf{a}_i, \mathbf{s}_i)$ are differentiable

Problem: Local optima, discontinuities



Didactic example: Ballistic shot with collisions

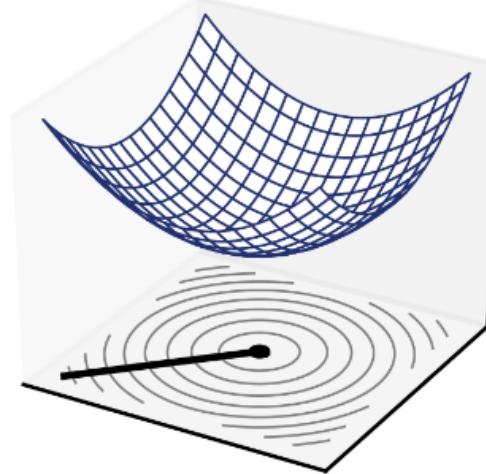


Directly optimizing initial shot vx, vy using Adam (possible because of differentiable dynamics) gets stuck in local optimum

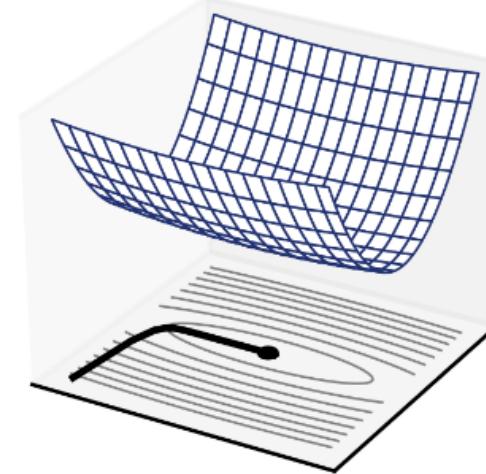


Difficulty of optimization

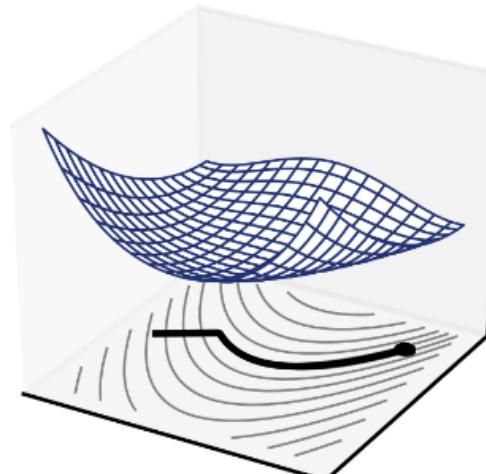
Convex,
well-conditioned



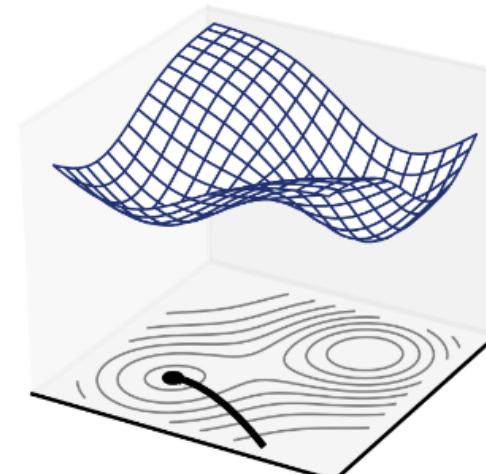
Convex,
ill-conditioned



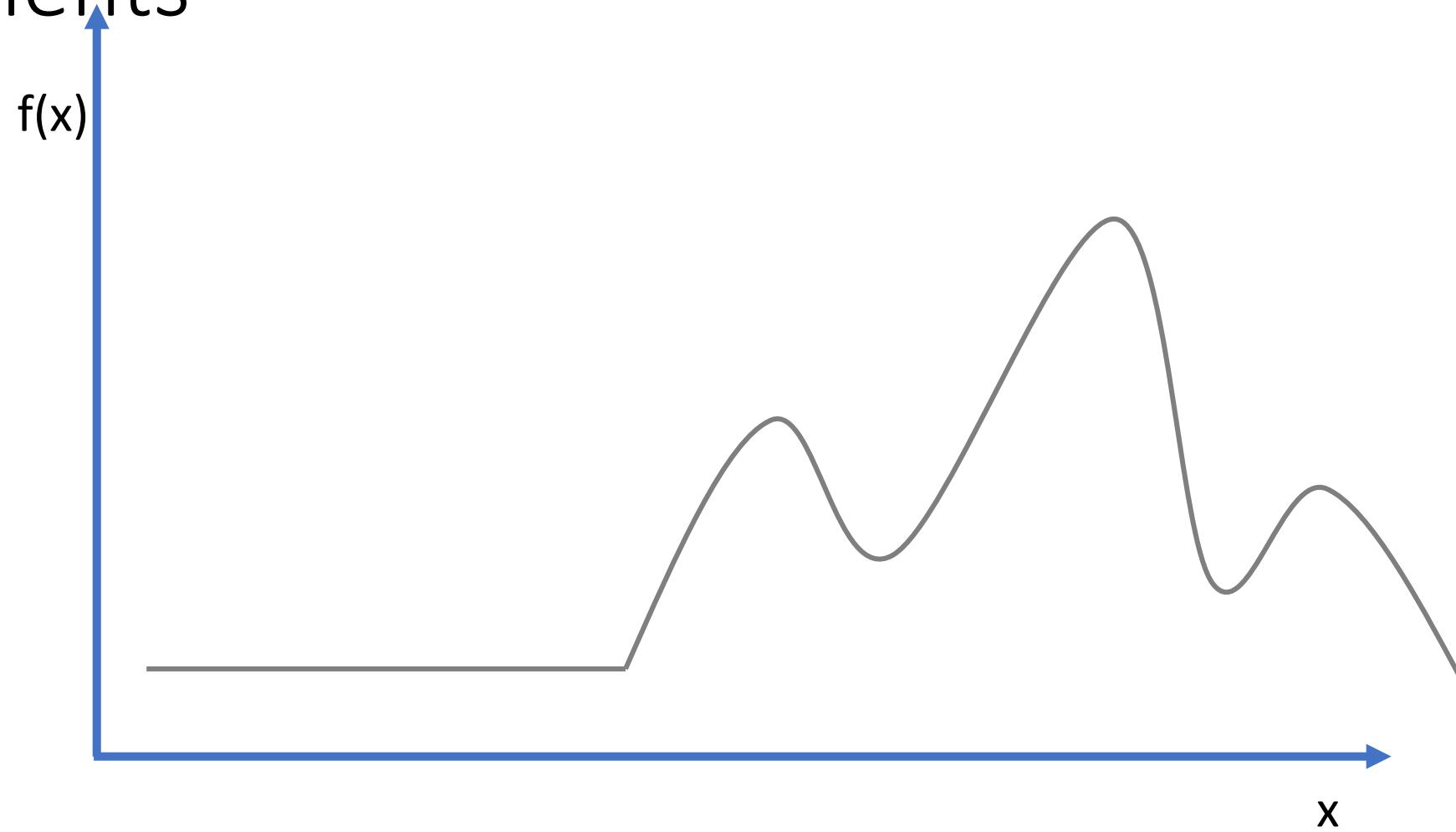
Non-convex,
unimodal



Non-convex,
multimodal

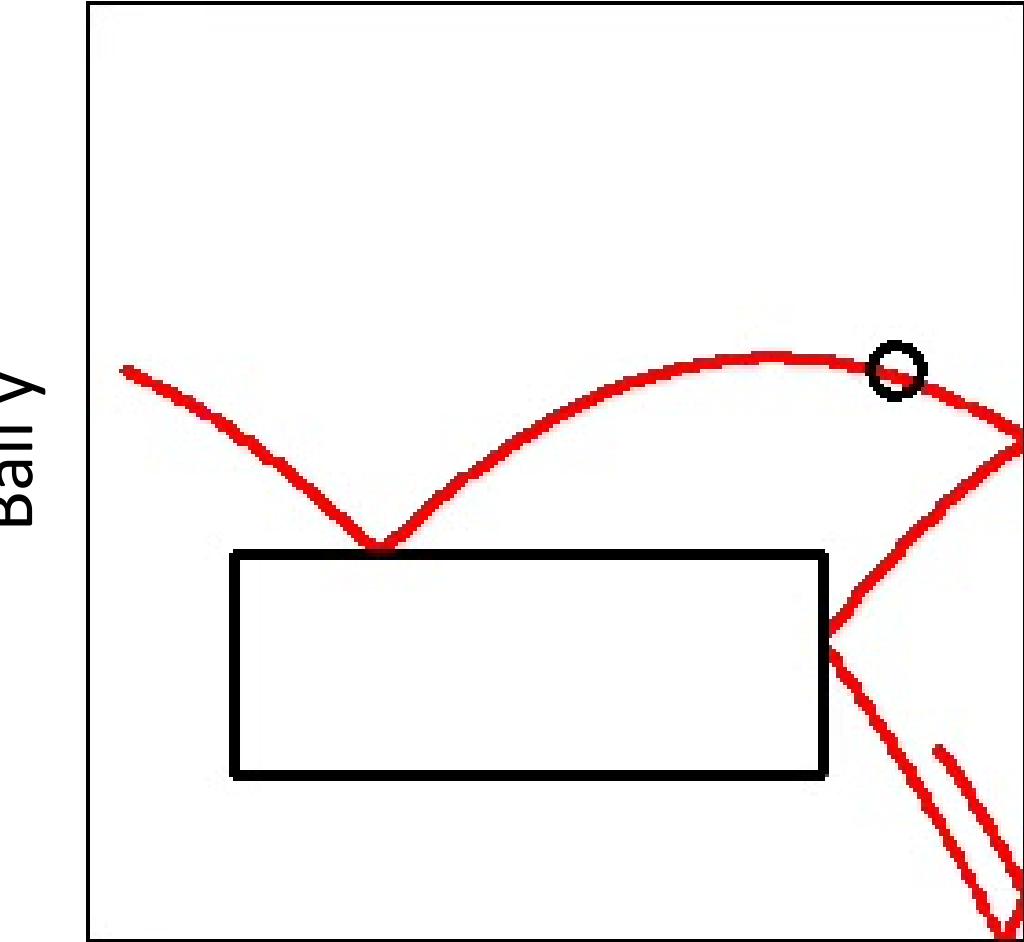


Real-life problems: multimodality, absence of gradients



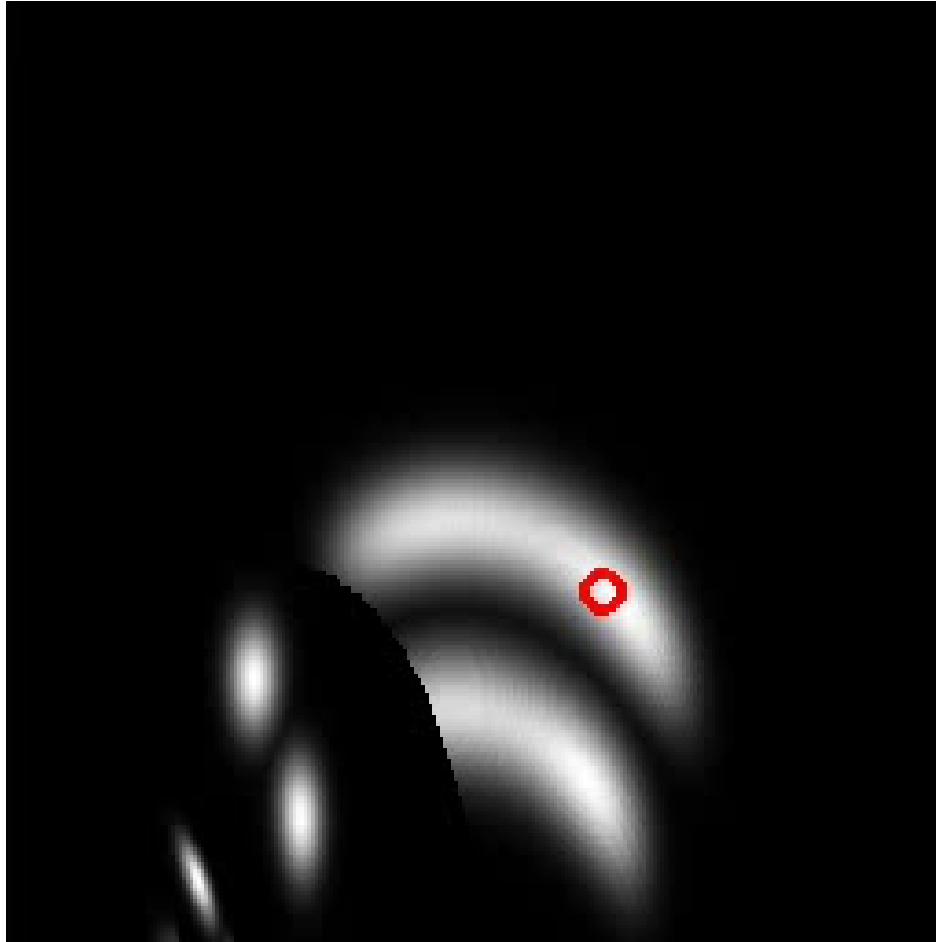
Ball y

Ball trajectories



Ball x

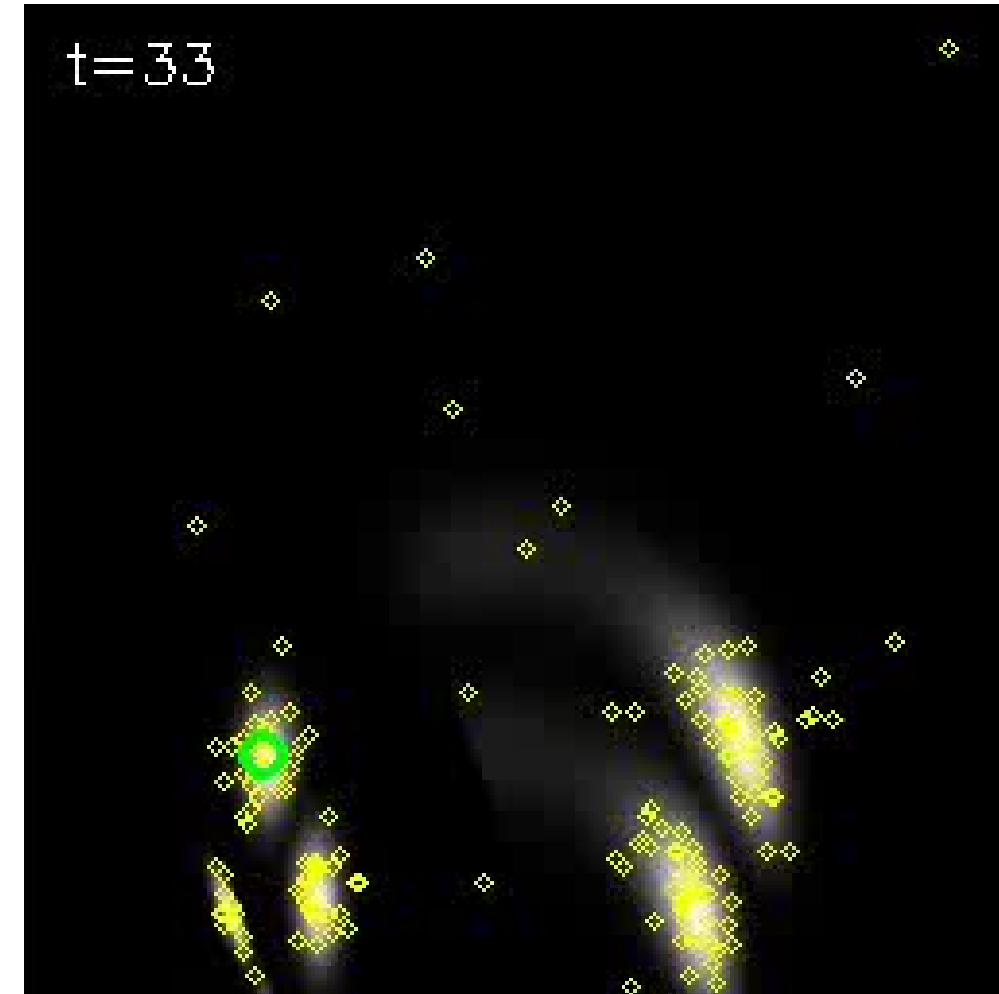
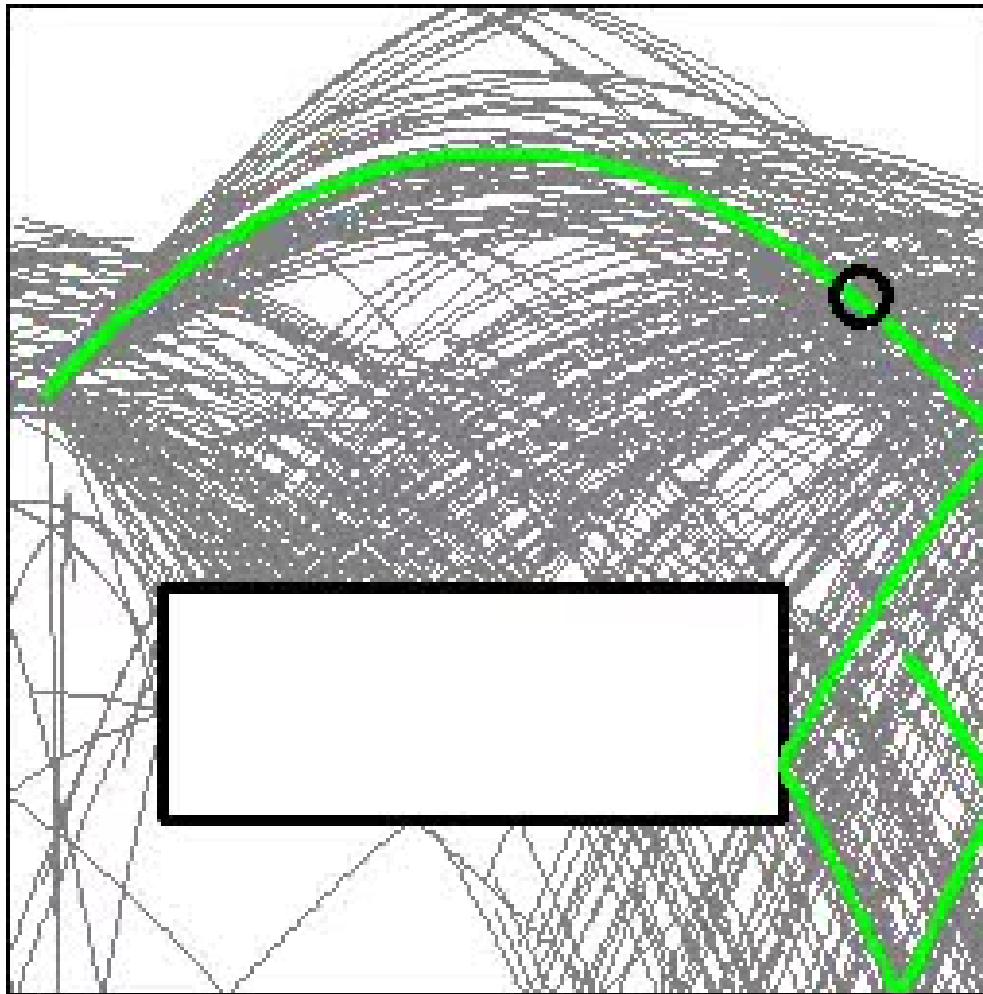
Optimization landscape



Shot angle

Shot velocity

Sampling-based methods to the rescue

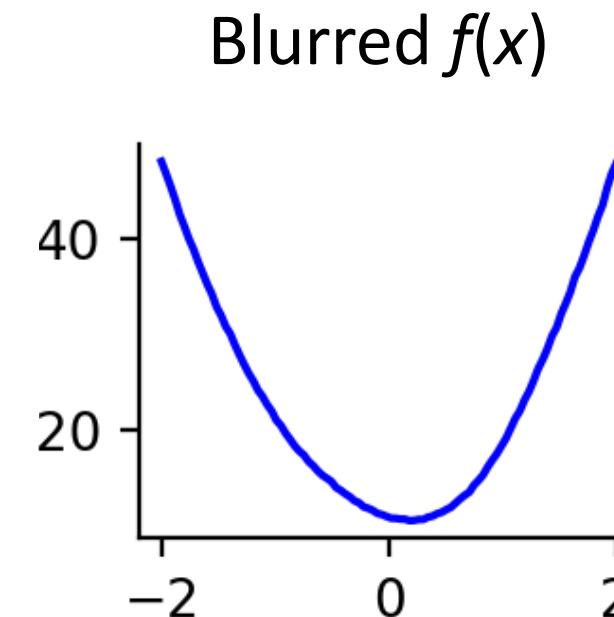
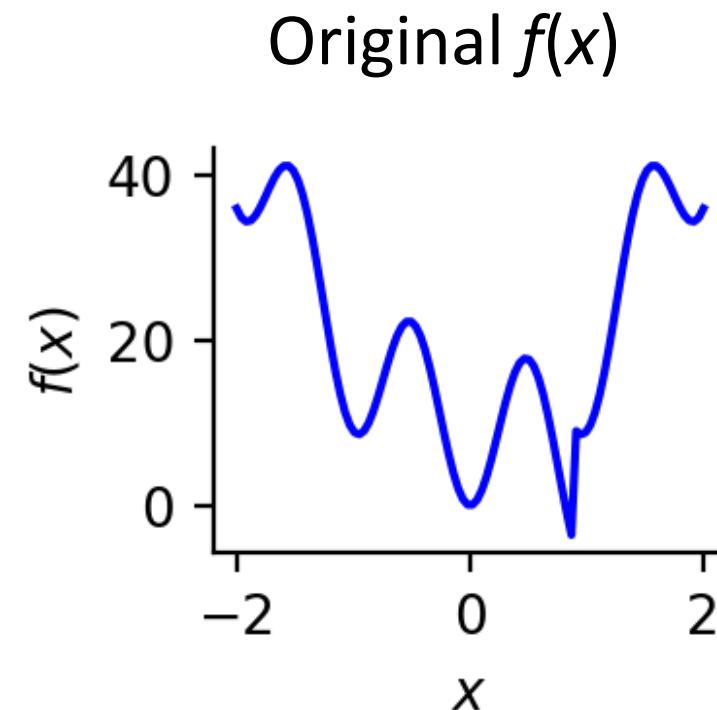


Sampling-based optimization



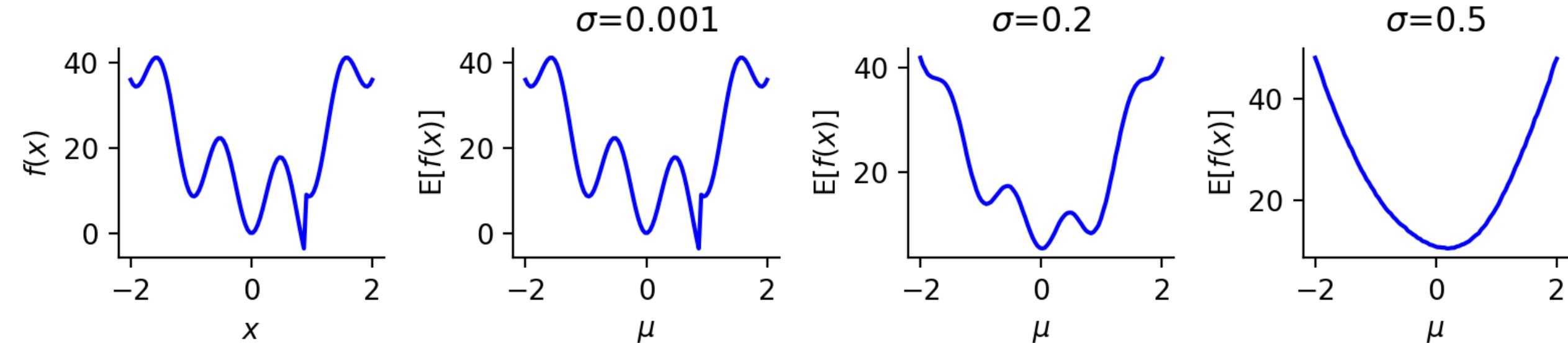
Key intuition: A multimodal function can be smoothed to make it convex

- The most common and successful multimodal optimization methods apply some form of Gaussian blurring to the objective function to avoid gradient-based optimization from getting stuck



Key intuition: A multimodal function can be smoothed to make it convex

- Mathematically, optimize $\mathbb{E}_{x \sim p(x)}[f(x)] \approx \frac{1}{N} \sum_{i=1}^N f(x)$ instead of $f(x)$
- $p(x)$ is a normal distribution with some mean μ and standard deviation σ . The sum is over a batch of N random samples.
- The expectation plotted over μ corresponds to Gaussian-blurred $f(x)$



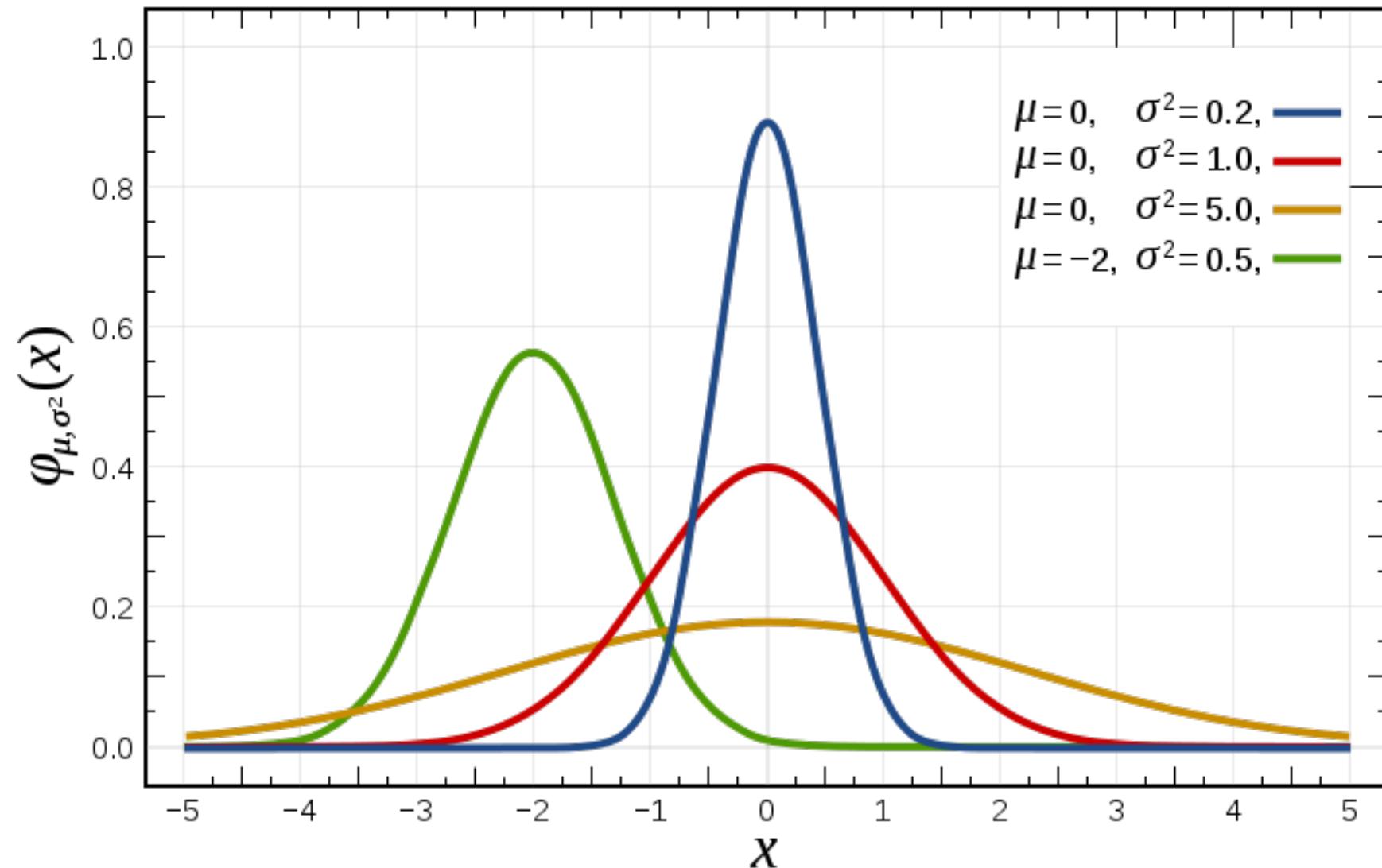
Key intuition: A multimodal function can be smoothed to make it convex

- The expectation is a bound for the true minimum:

$$\min_x f(x) \leq \mathbb{E}_{x \sim p(x)}[f(x)]$$

- The bound becomes tighter when σ approaches zero.
- Implication: To find optimal x , optimize μ, σ . This will drive σ to 0, at which point the ***optimal μ equals the optimal x .***

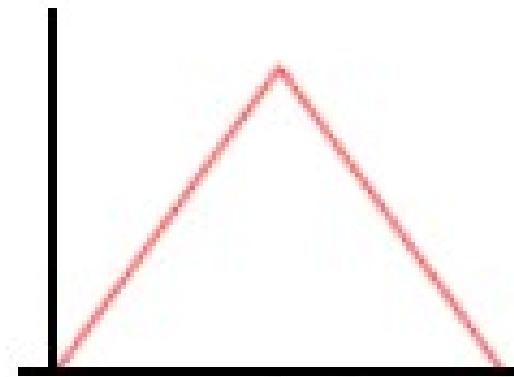
Reminder: normal (Gaussian) distribution



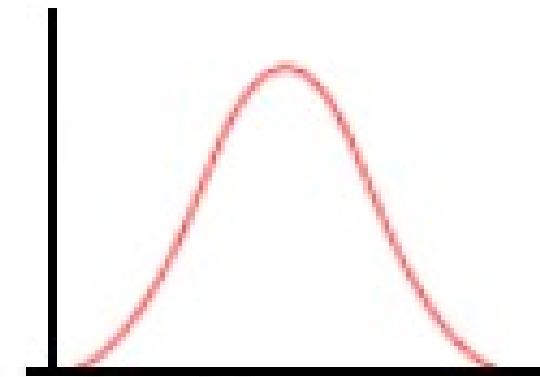
Preliminaries: normal (Gaussian) distribution



**uniform
distribution
1 die**

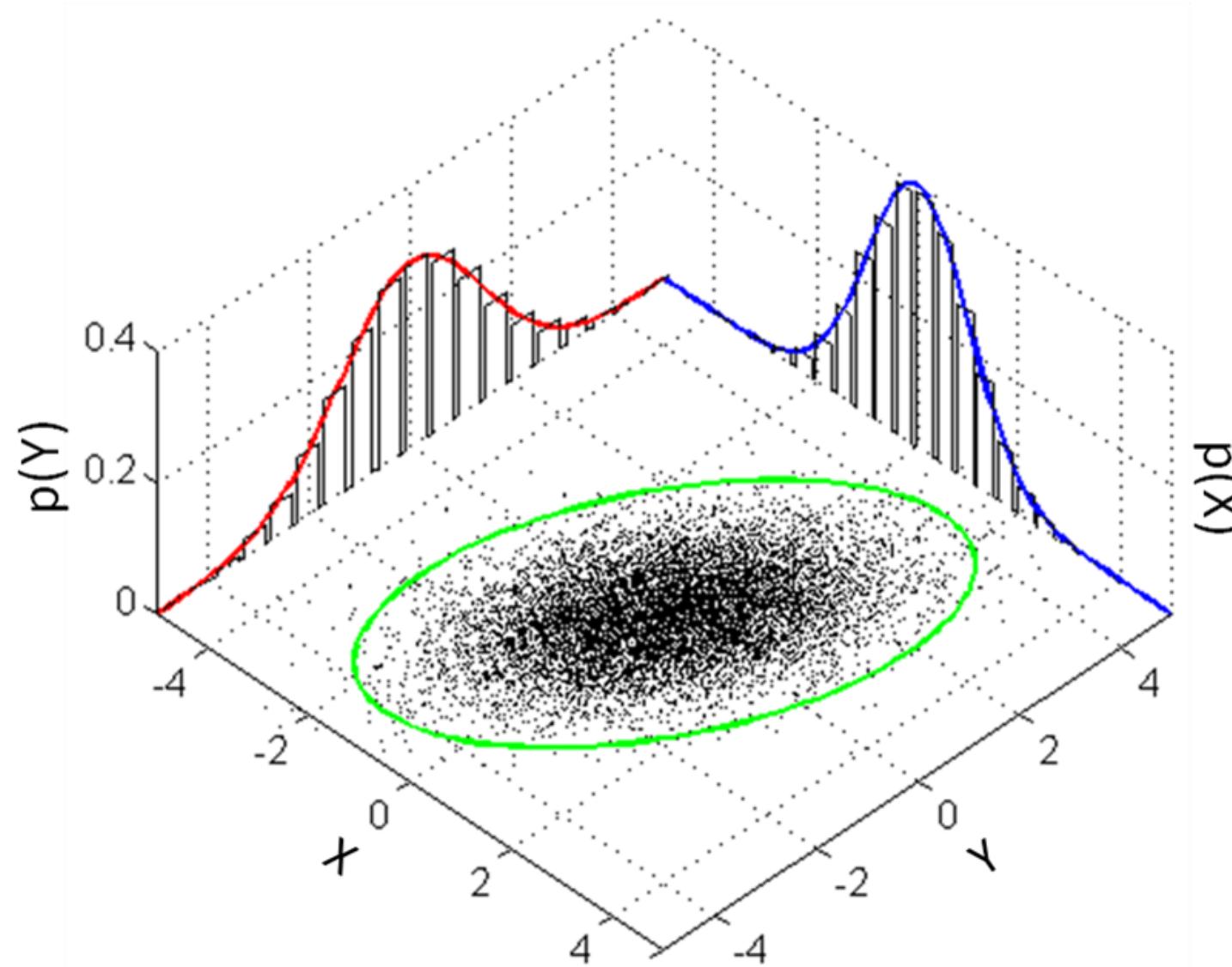


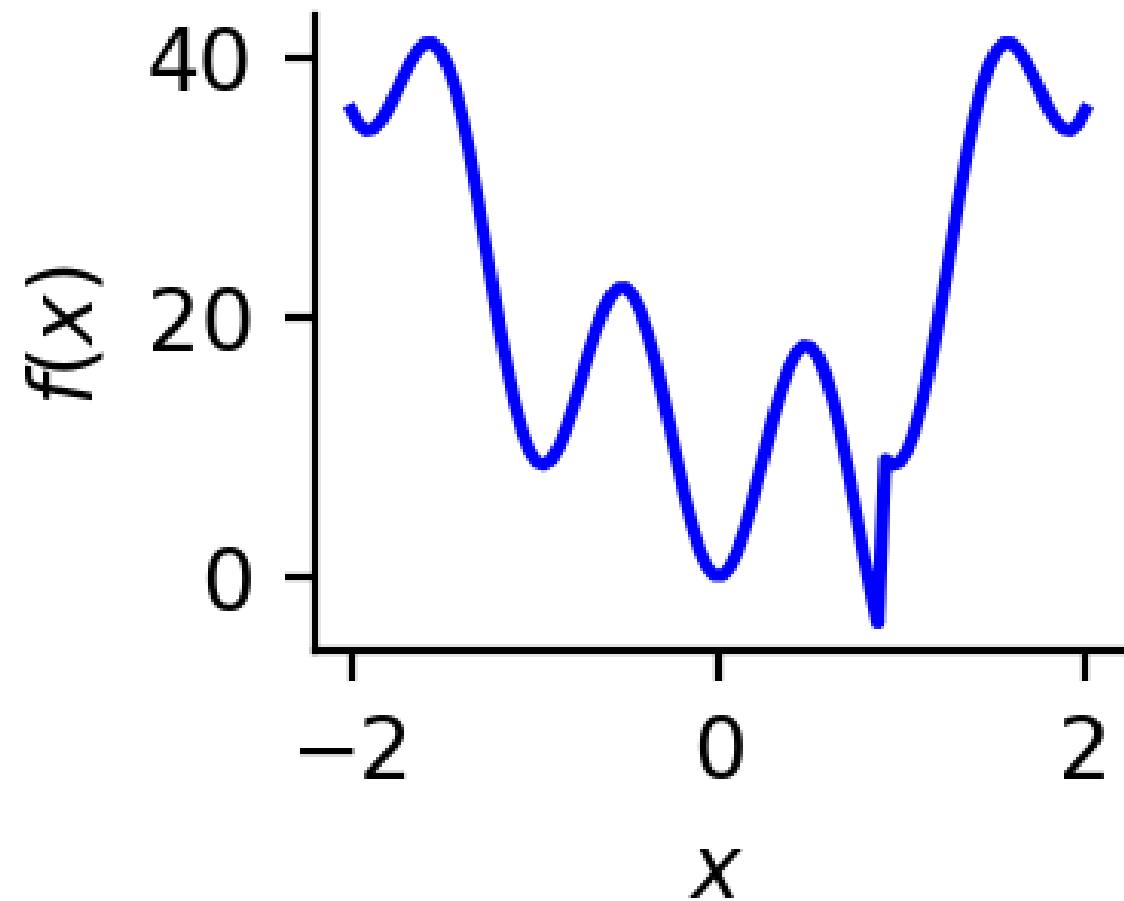
**uniform sum
distribution
2 dice**

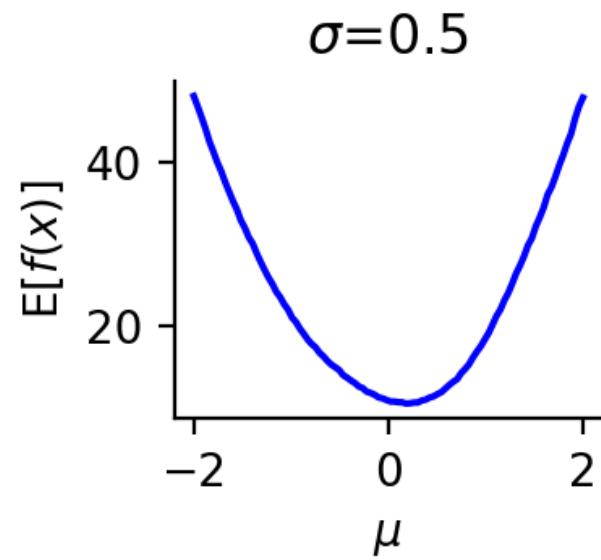
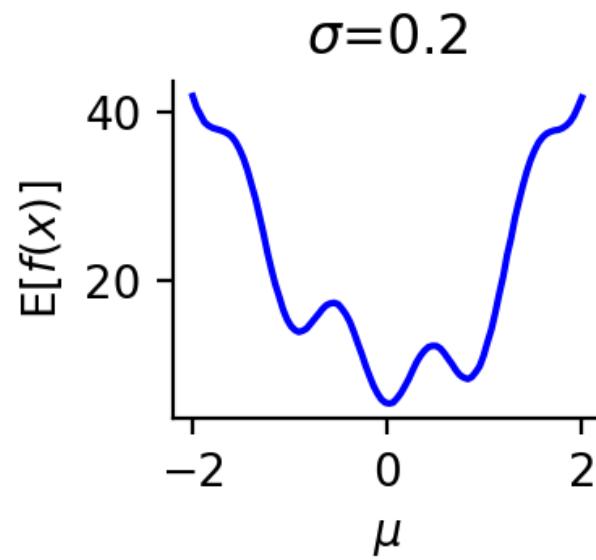
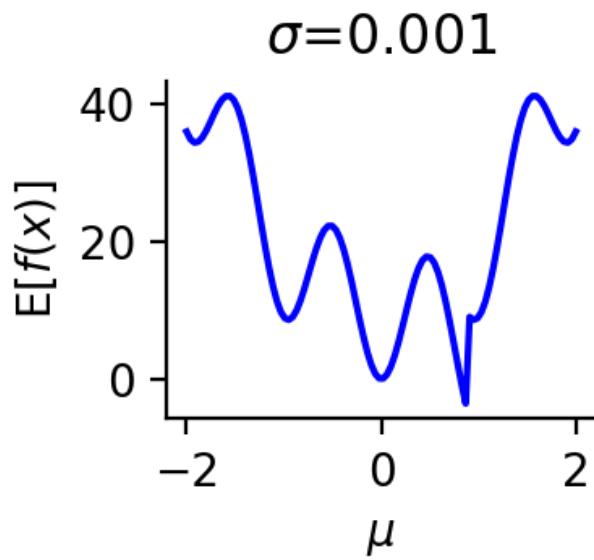
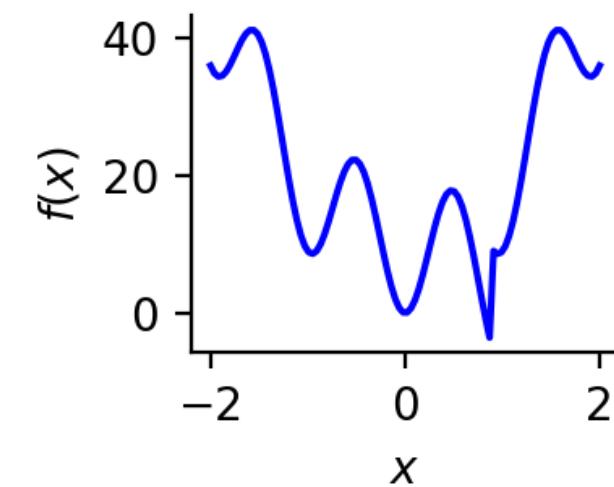


**uniform sum
distribution
3 dice**

Preliminaries: normal (Gaussian) distribution

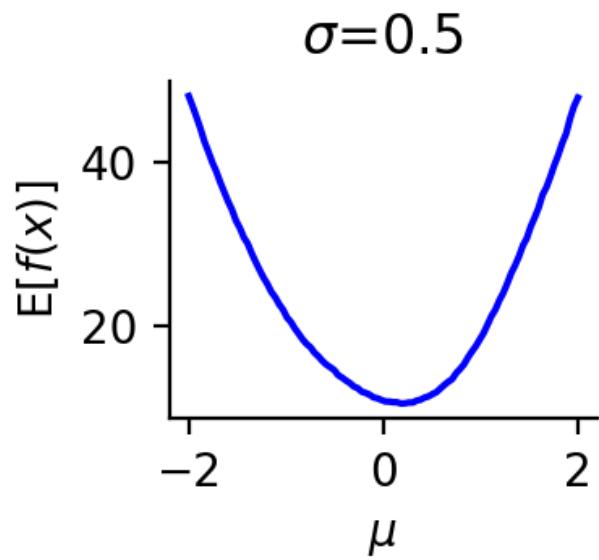
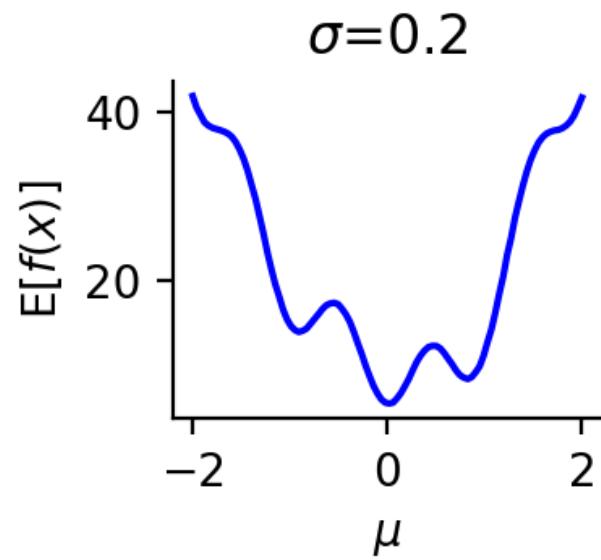
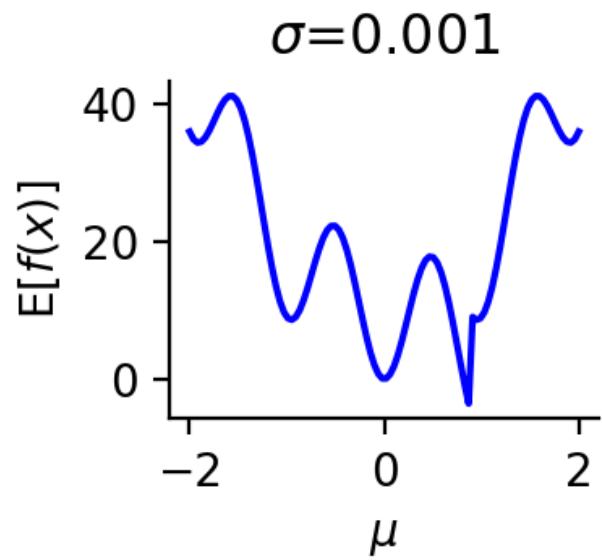
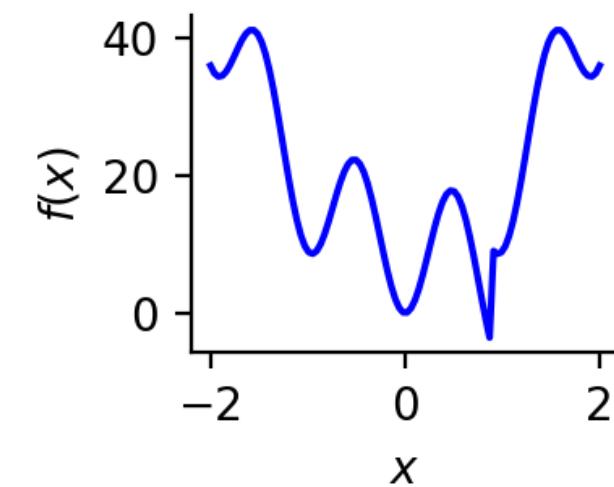




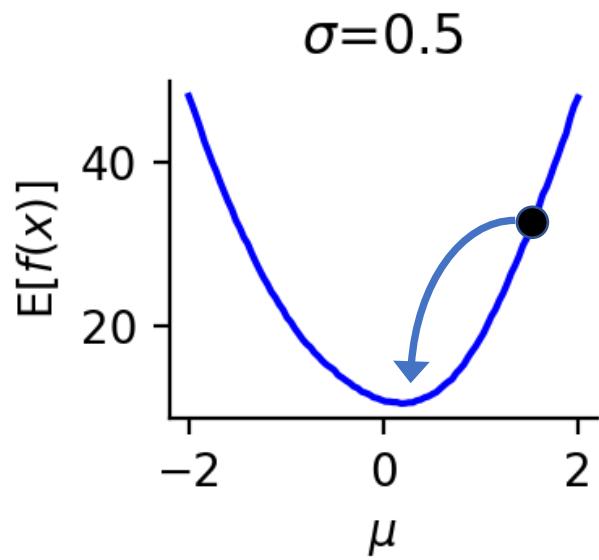
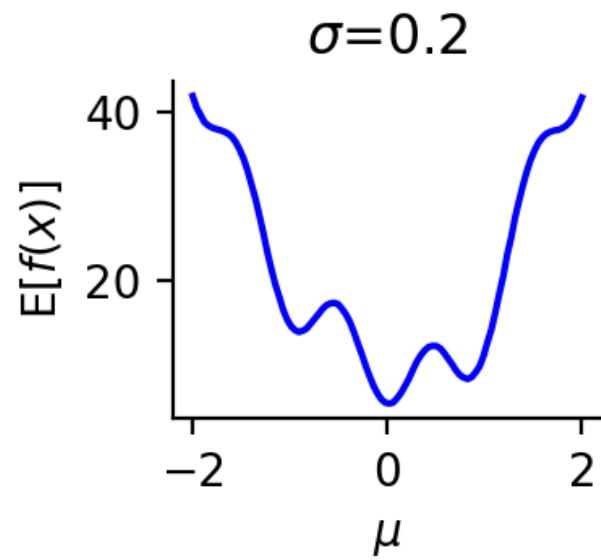
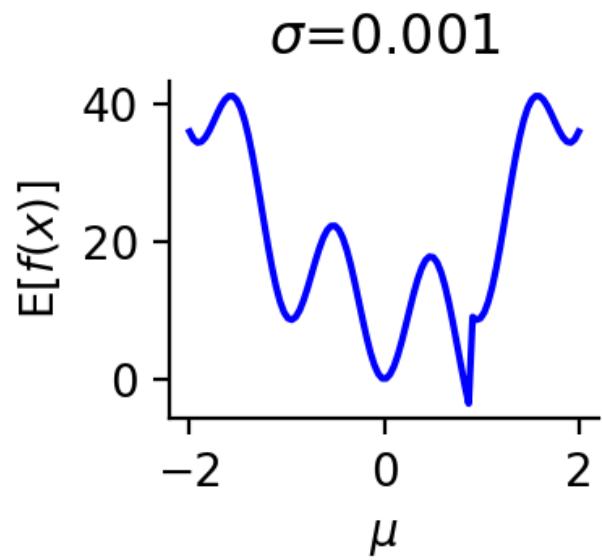
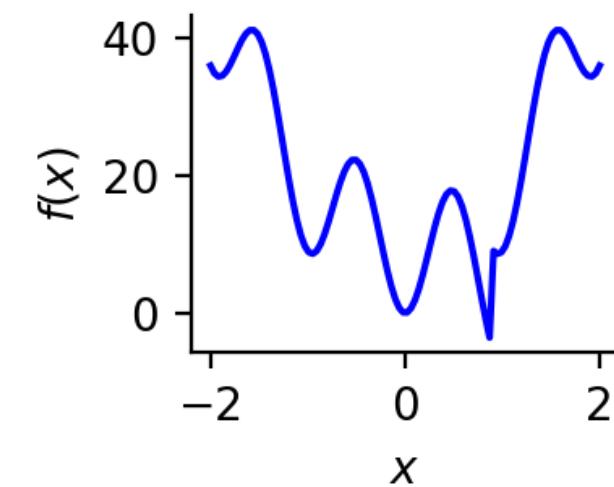


Gaussian exploration $p(x) = \text{Gaussian blurring of } f(x)$. Known to not introduce new extrema!

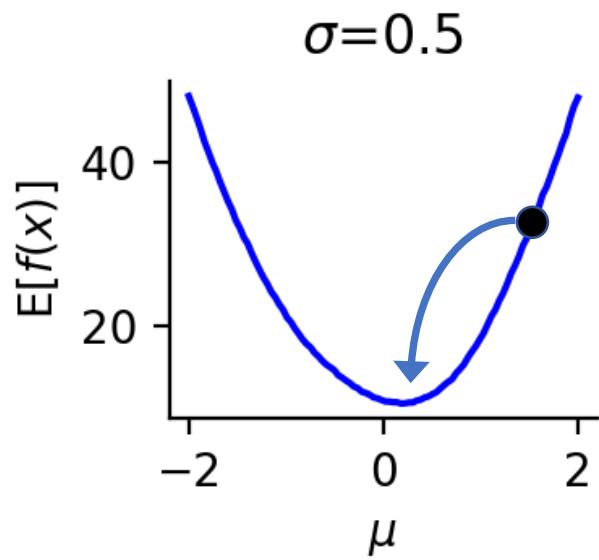
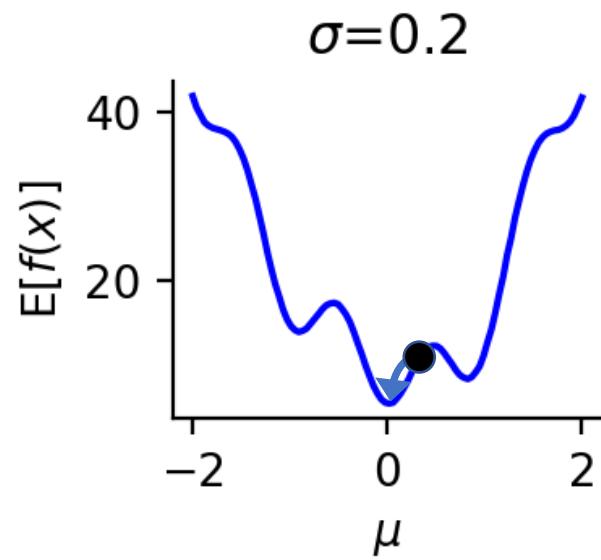
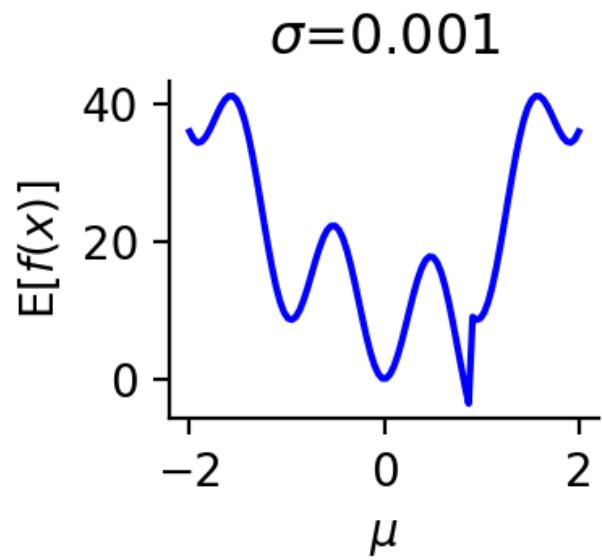
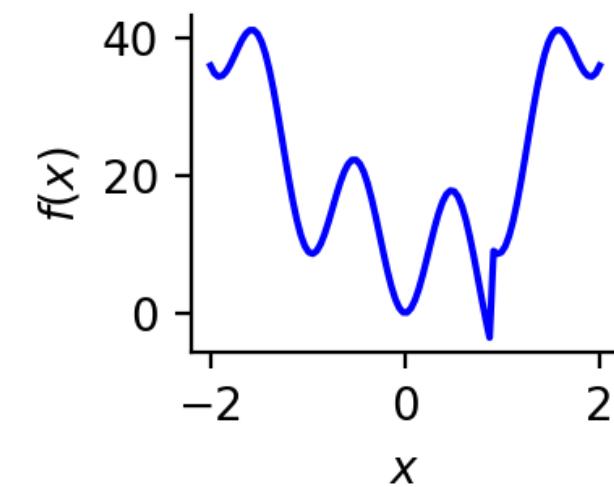
Key idea: can optimize μ with decreasing σ



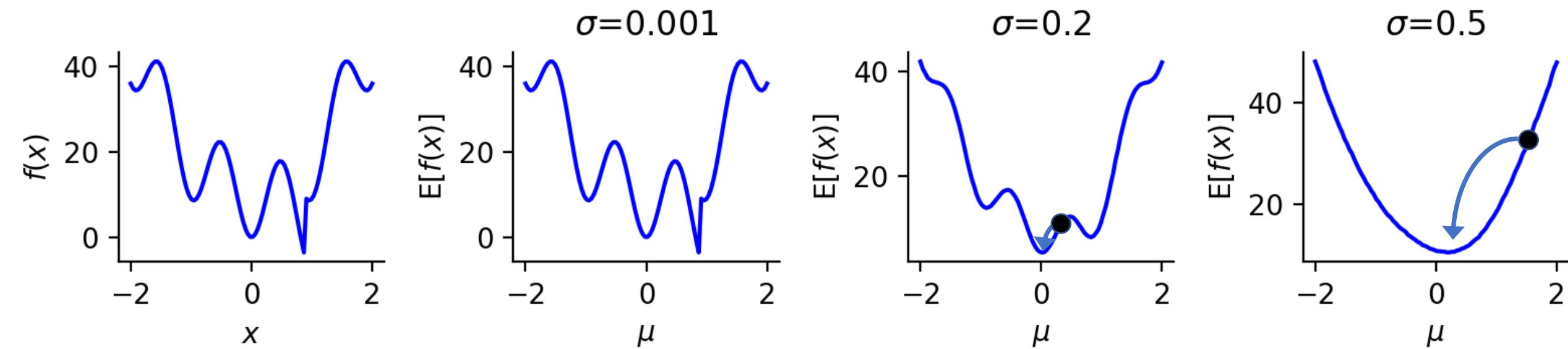
Key idea: can optimize μ with decreasing σ



Key idea: can optimize μ with decreasing σ

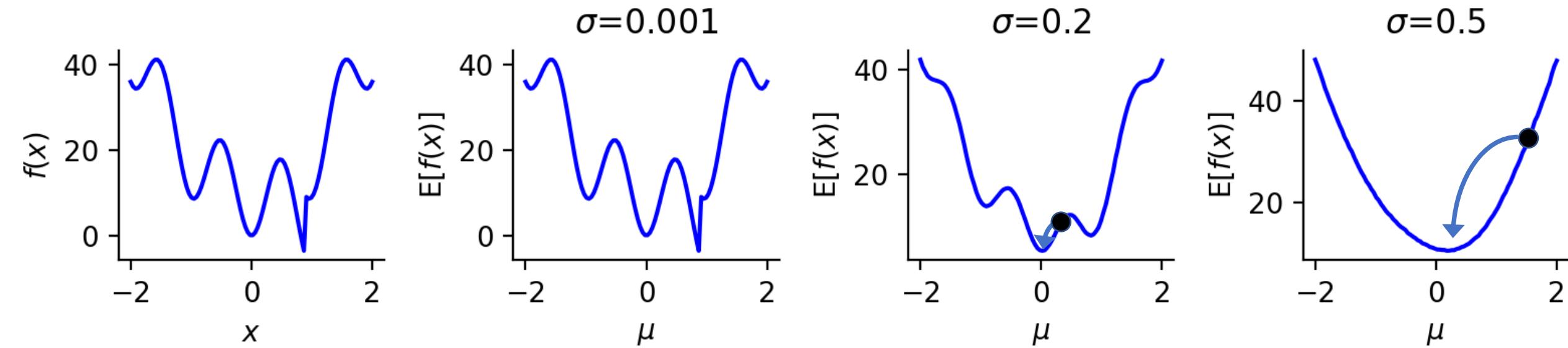


Key idea: can optimize μ with decreasing σ



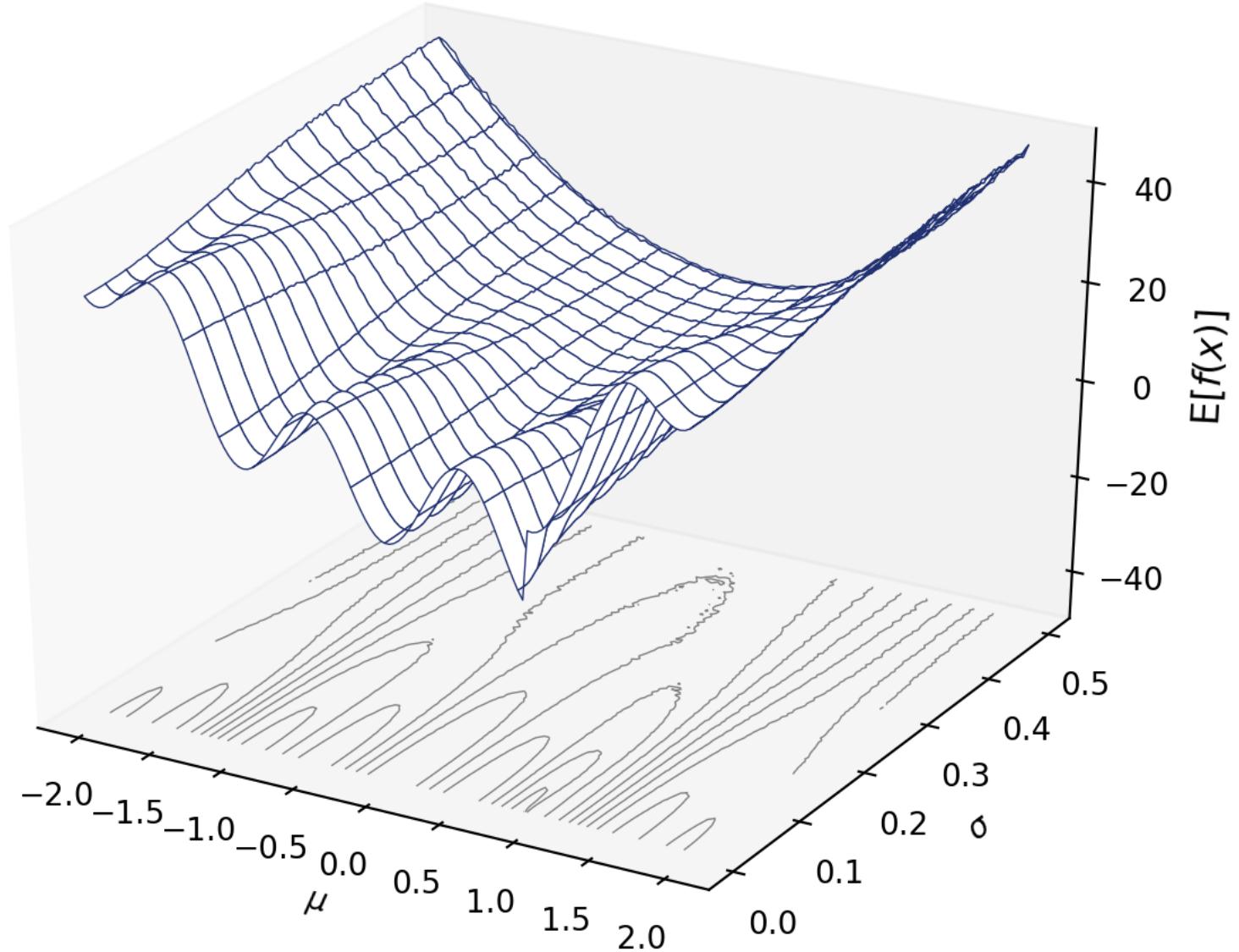
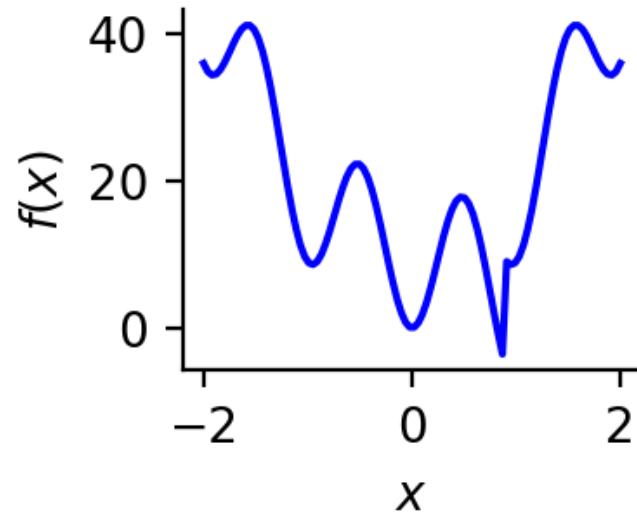
Will usually find a global optimum or a reasonably good local one.

Key idea: can optimize μ with decreasing σ



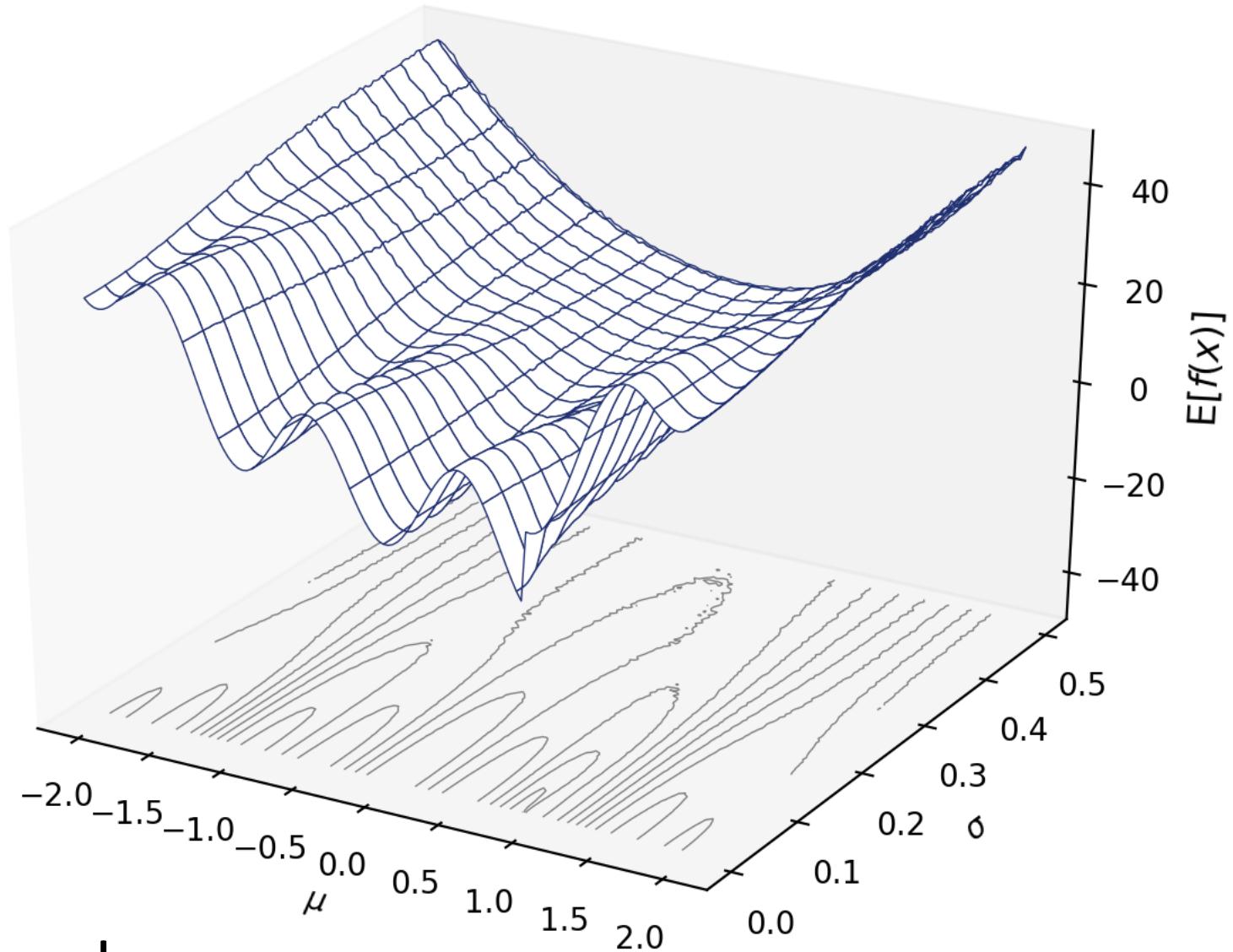
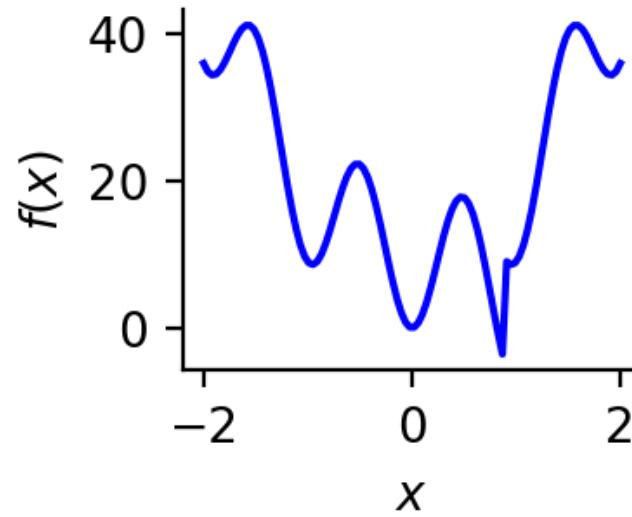
How to decide the annealing schedule for σ ?

Optimize both μ, σ , start with large σ



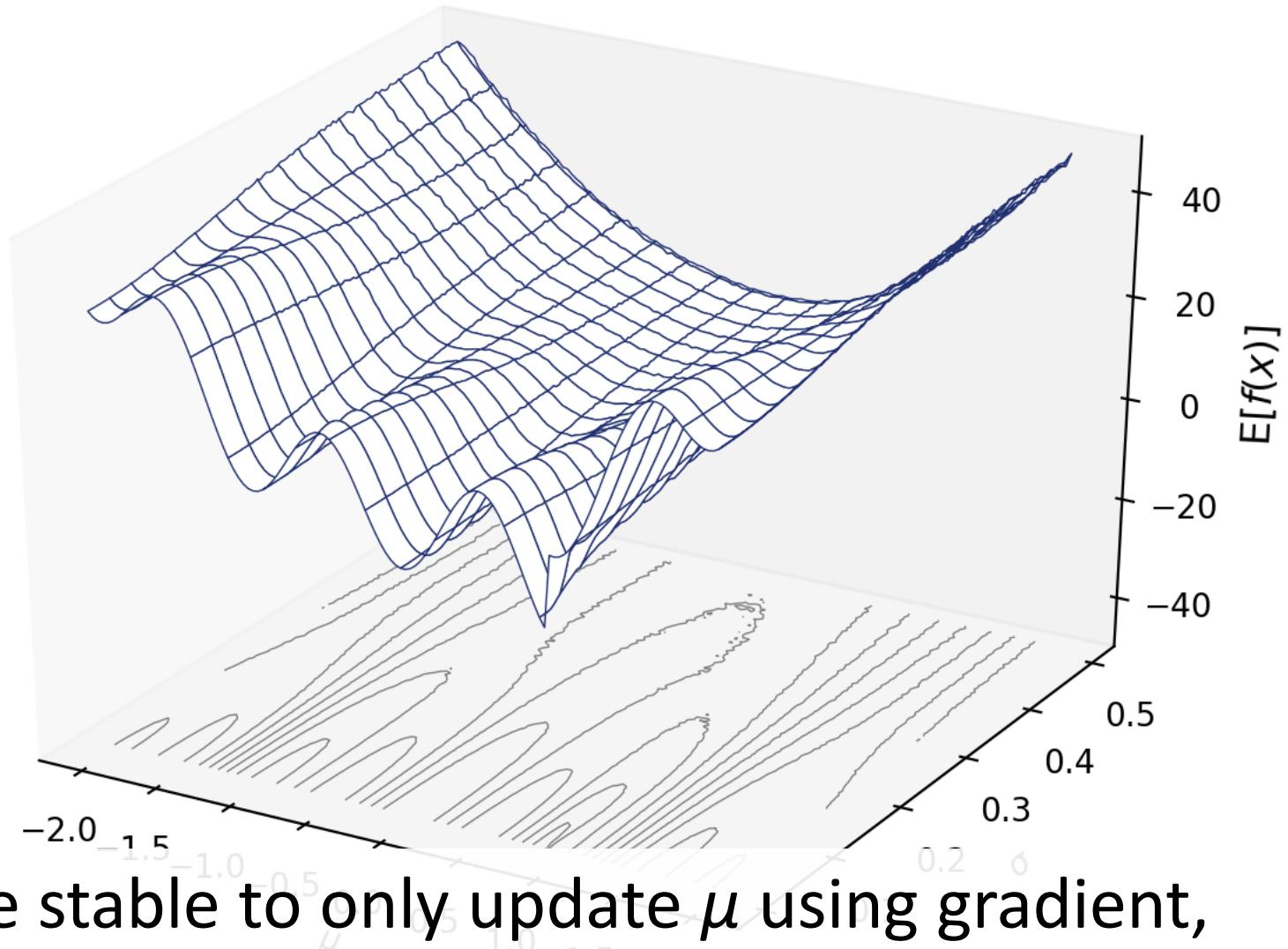
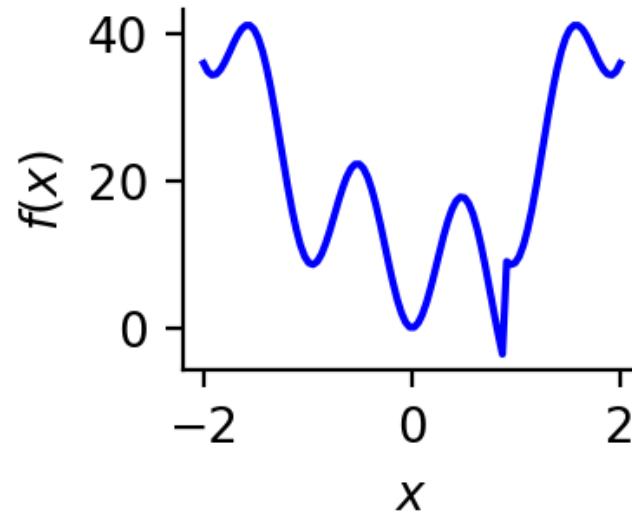
Just follow the gradient!

Optimize both μ, σ , start with large σ

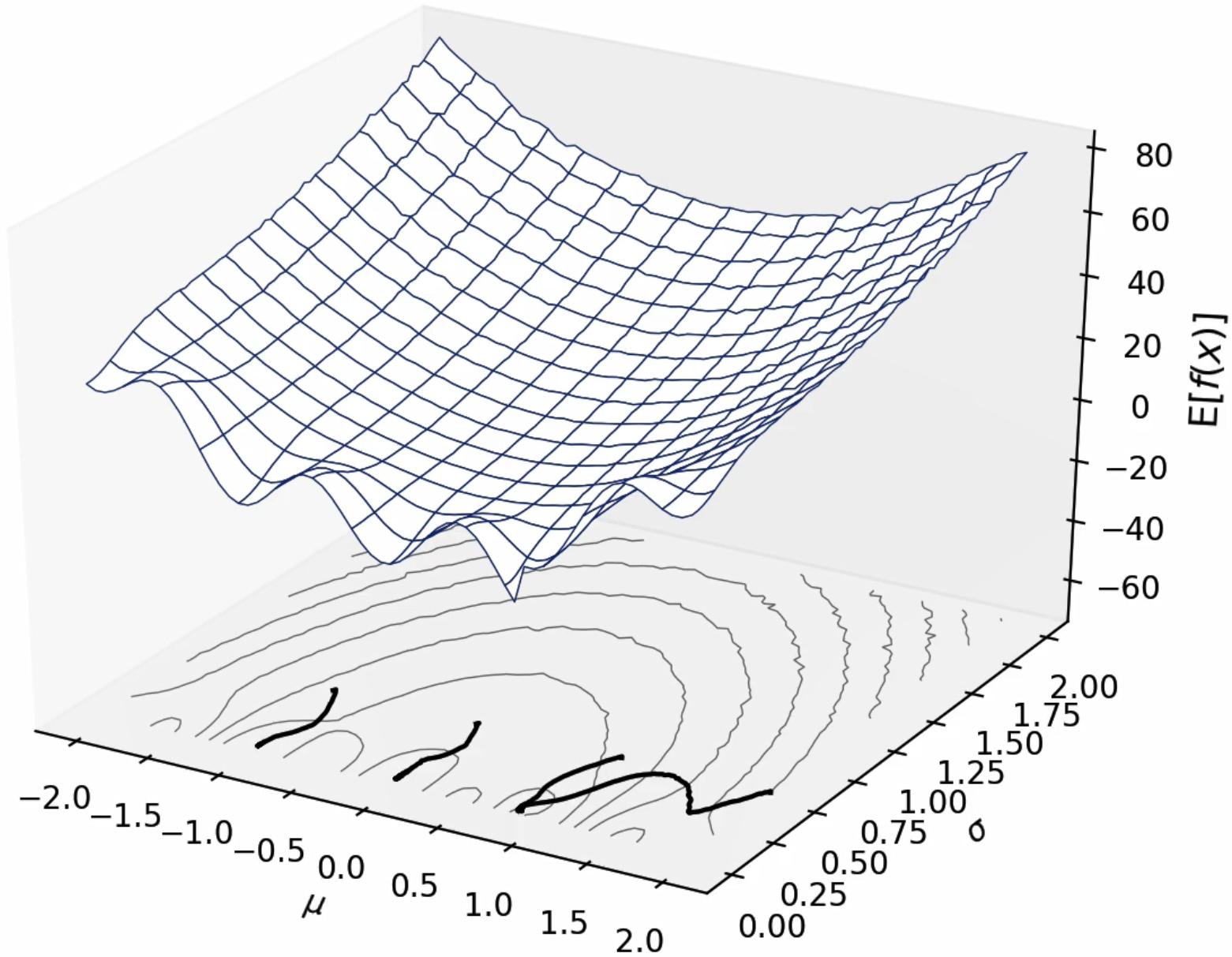
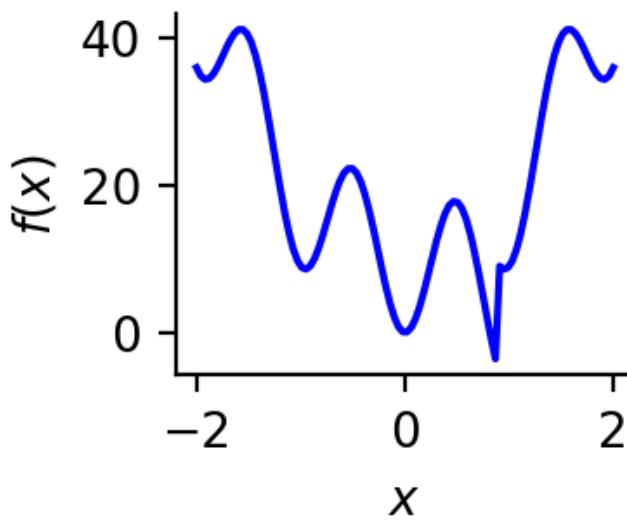


Gradient estimated from samples

Optimize both μ, σ , start with large σ



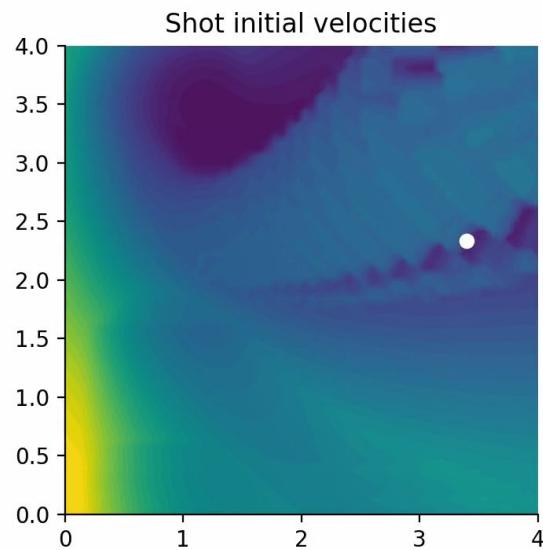
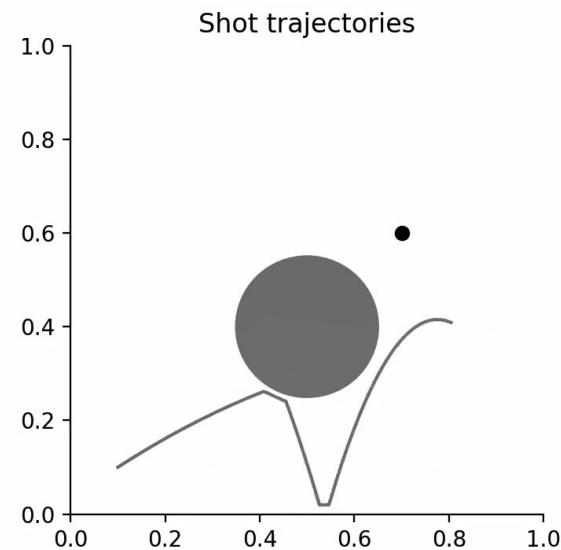
In practice, it may be more stable to only update μ using gradient, and linearly or exponentially drop σ towards zero.



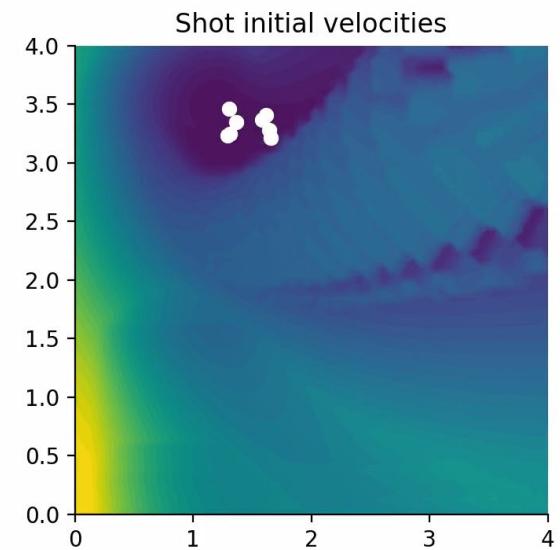
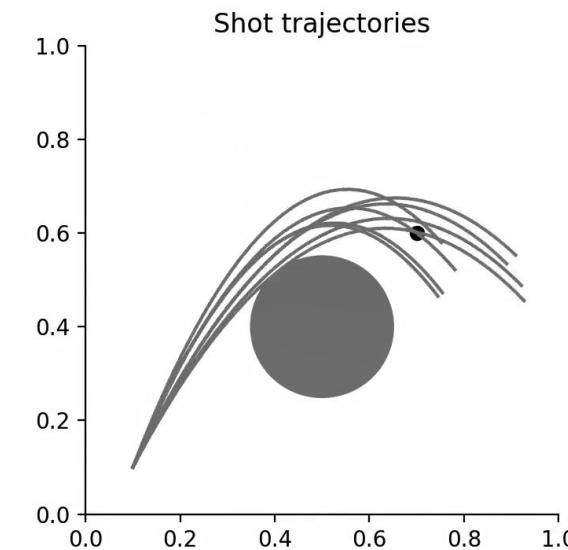
Video: Optimizing with different initial μ, σ . Large sigma: all runs converge to same optimum, independent of initial μ .

Didactic example: Ballistic shot with collisions

Adam (optimize $f(x)$)



Adam (optimize $E[f(x)]$)



Algorithm

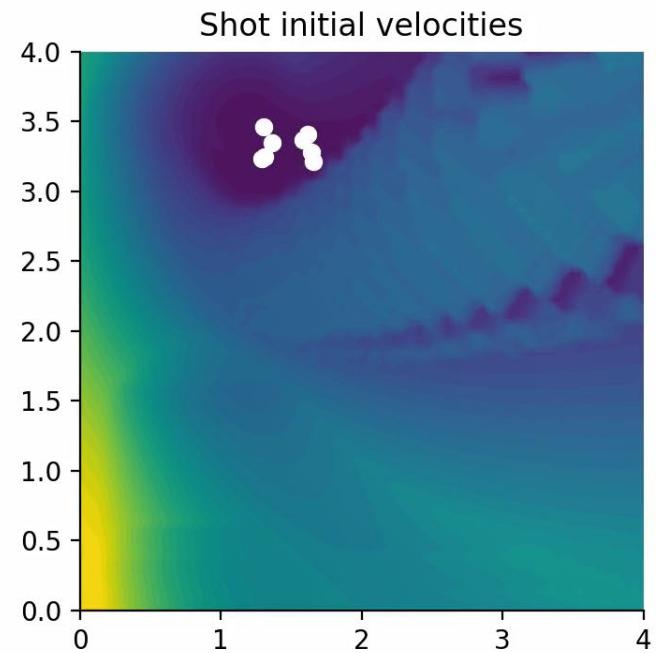
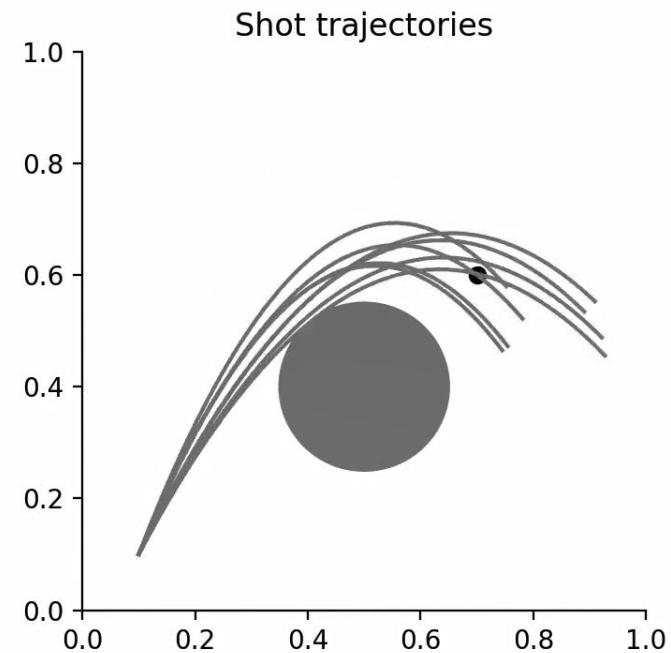
One iteration (repeat until results are good enough):

1. Sample multiple \mathbf{x} from the exploration distribution $p(\mathbf{x})$
2. Evaluate $f(\mathbf{x})$ for all sampled \mathbf{x}
3. Follow the gradient: Nudge the parameters of the exploration distribution (e.g., μ , σ) so that it becomes more probable to sample again the \mathbf{x} with high $f(\mathbf{x})$

Algorithm in code

```
shotVels=optimizer.ask()  
fvals,trajectories=simulateBatch(shotVels)  
loss=optimizer.tell(fvals)
```

Sample the $\mathbf{x} \sim p(\mathbf{x})$
Evaluate the $f(\mathbf{x})$
Use gradient to update $p(\mathbf{x})$



x_1

x_2

How to estimate the gradient from samples?



Gradient estimator for non-differentiable $f(\mathbf{x})$

Score function estimator, Variational Optimization, REINFORCE (pretty much all RL):

$$\nabla_{\theta} \mathbb{E}_{\mathbf{x} \sim p(\mathbf{x})} [f(\mathbf{x})] \approx \frac{1}{N} \sum_{i=1}^N f(\mathbf{x}_i) \nabla_{\theta} \log p_{\theta}(\mathbf{x}_i)$$

Here, θ denotes the parameters of $p(\mathbf{x})$, e.g., $\theta = [\mu, \sigma]$

Gradient estimator for non-differentiable $f(\mathbf{x})$

Score function estimator, Variational Optimization, REINFORCE (pretty much all RL):

$$\nabla_{\theta} \mathbb{E}_{\mathbf{x} \sim p(\mathbf{x})} [f(\mathbf{x})] \approx \boxed{\frac{1}{N} \sum_{i=1}^N f(\mathbf{x}_i) \nabla_{\theta} \log p_{\theta}(\mathbf{x}_i)}$$

Mean over N samples,
 \mathbf{x}_i sampled from $p_{\theta}(\mathbf{x})$

Gradient estimator for non-differentiable $f(\mathbf{x})$

Score function estimator, Variational Optimization, REINFORCE (pretty much all RL):

$$\nabla_{\theta} \mathbb{E}_{\mathbf{x} \sim p(\mathbf{x})} [f(\mathbf{x})] \approx \frac{1}{N} \sum_{i=1}^N f(\mathbf{x}_i) \boxed{\nabla_{\theta} \log p_{\theta}(\mathbf{x}_i)}$$

Which way to change θ
to increase the
probability of sampled \mathbf{x}_i

Gradient estimator for non-differentiable $f(\mathbf{x})$

Score function estimator, Variational Optimization, REINFORCE (pretty much all RL):

$$\nabla_{\theta} \mathbb{E}_{\mathbf{x} \sim p(\mathbf{x})} [f(\mathbf{x})] \approx \frac{1}{N} \sum_{i=1}^N f(\mathbf{x}_i) \nabla_{\theta} \log p_{\theta}(\mathbf{x}_i)$$

Increase the probability of positive $f(\mathbf{x})$, decrease the probability of negative $f(\mathbf{x})$

Gradient estimator for non-differentiable $f(\mathbf{x})$

Score function estimator, Variational Optimization, REINFORCE (pretty much all RL):

$$\nabla_{\theta} \mathbb{E}_{\mathbf{x} \sim p(\mathbf{x})} [f(\mathbf{x})] \approx \frac{1}{N} \sum_{i=1}^N f(\mathbf{x}_i) \nabla_{\theta} \log p_{\theta}(\mathbf{x}_i)$$

Increase the probability of positive $f(\mathbf{x})$, decrease the probability of negative $f(\mathbf{x})$

See slide notes for links to more details, e.g., the derivation of this identity through the so-called log derivative trick.



Gradient estimator for differentiable $f(\mathbf{x})$

Reparameterization trick:

$$\nabla_{\theta} \mathbb{E}_{\mathbf{x} \sim p_{\theta}(\mathbf{x})} [f(\mathbf{x})] = \nabla_{\theta} \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z})} [f(\mathbf{x}(\mathbf{z}, \theta))]$$

Gradient estimator for differentiable $f(\mathbf{x})$

Reparameterization trick:

$$\nabla_{\theta} \mathbb{E}_{\mathbf{x} \sim p_{\theta}(\mathbf{x})}[f(\mathbf{x})] = \nabla_{\theta} \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z})}[f(\mathbf{x}(\mathbf{z}, \theta))] \approx \frac{1}{N} \sum_{i=1}^N \nabla_{\theta} f(\mathbf{x}_i(\mathbf{z}_i, \theta))$$

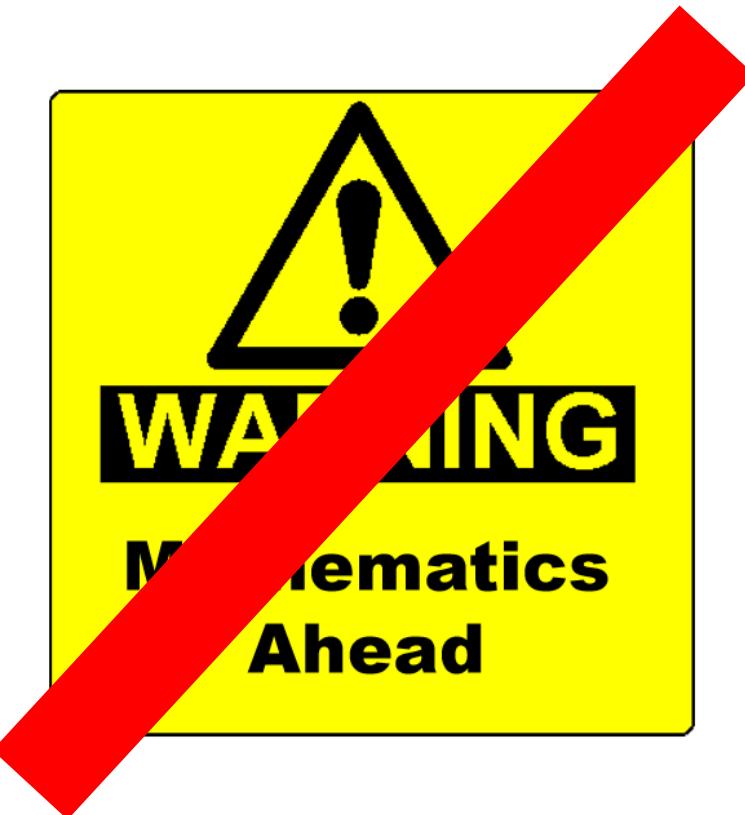
Compute gradient of $f(\mathbf{x})$ at all sampled \mathbf{x}_i and average over the samples

Gradient estimator for differentiable $f(\mathbf{x})$

Reparameterization trick:

$$\nabla_{\theta} \mathbb{E}_{\mathbf{x} \sim p_{\theta}(\mathbf{x})}[f(\mathbf{x})] = \nabla_{\theta} \mathbb{E}_{\mathbf{z} \sim p(\mathbf{z})}[f(\mathbf{x}(\mathbf{z}, \theta))] \approx \frac{1}{N} \sum_{i=1}^N \nabla_{\theta} f(\mathbf{x}_i(\mathbf{z}_i, \theta))$$

$$\theta = [\mu, \sigma], \quad \mathbf{z} \sim N(\mathbf{z}; 0, \mathbf{I}), \quad \mathbf{x} = \mu + \sigma \mathbf{z}$$

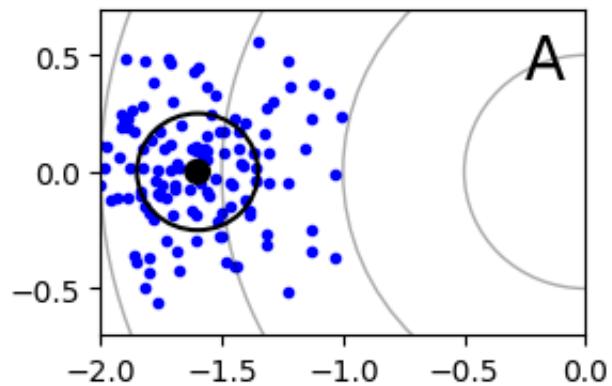


Improving optimization efficiency

- Taking a single gradient step for each batch of samples can be expensive
- Cross-Entropy Method (CEM): Fit the exploration distribution to the best samples
 - This is equal to 1) setting the $f(\mathbf{x})$ values to 1 for the best samples and 0 for others, and 2) taking multiple gradient steps with the same \mathbf{x} and $f(\mathbf{x})$ to maximize the probability of sampling the best \mathbf{x} in the next iteration
- Covariance Matrix Adaptation Evolution Strategy (CMA-ES): Similar to CEM, but implementing a form of momentum.

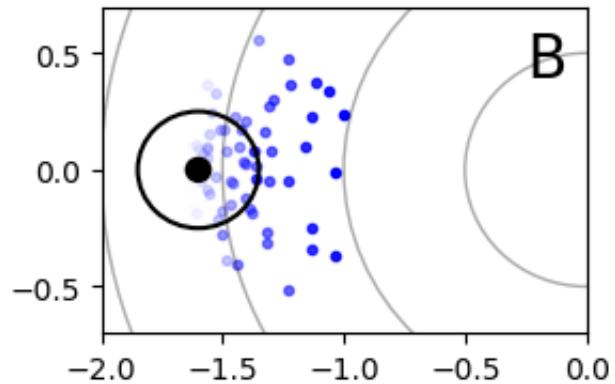
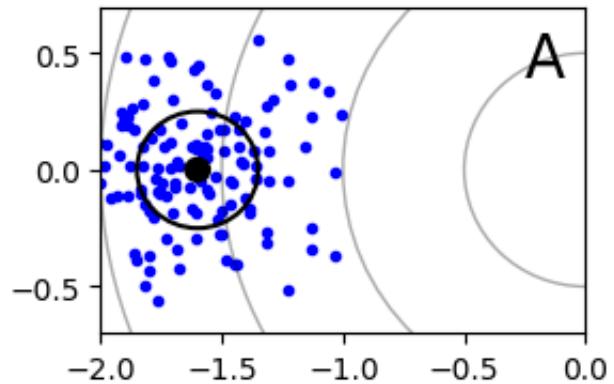
CMA-ES and CEM

Sample some \mathbf{x}



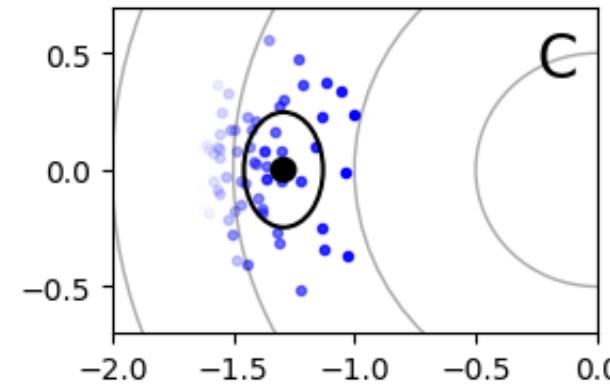
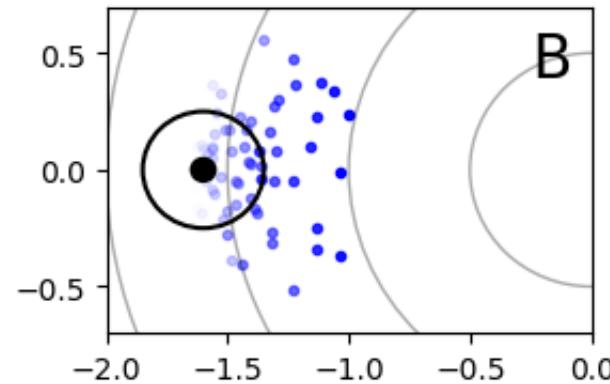
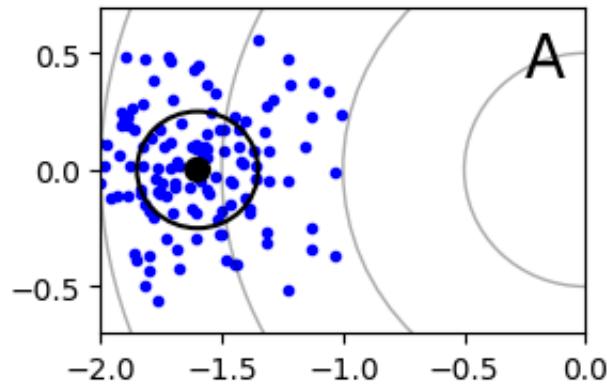
CMA-ES and CEM

Only keep the best x



CMA-ES and CEM

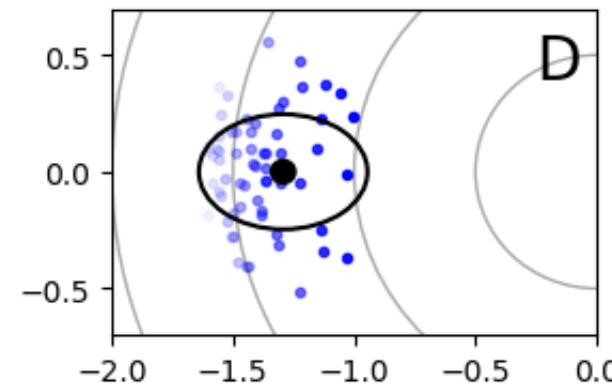
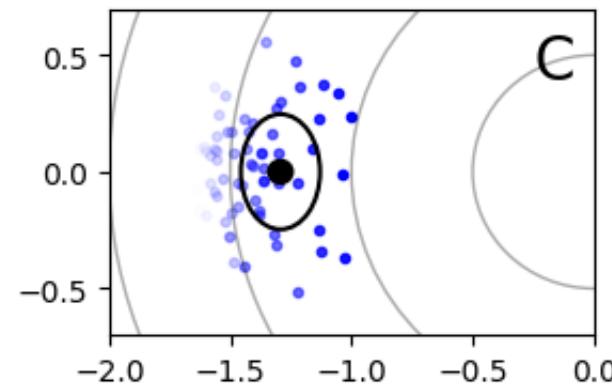
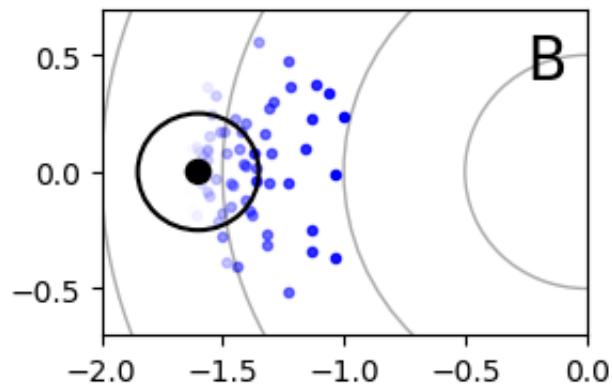
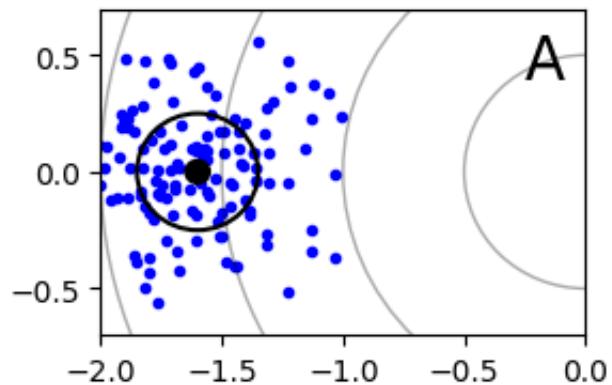
CEM: fit $p(\mathbf{x})$
to the best \mathbf{x}



CMA-ES and CEM

CEM: fit $p(\mathbf{x})$
to the best \mathbf{x}

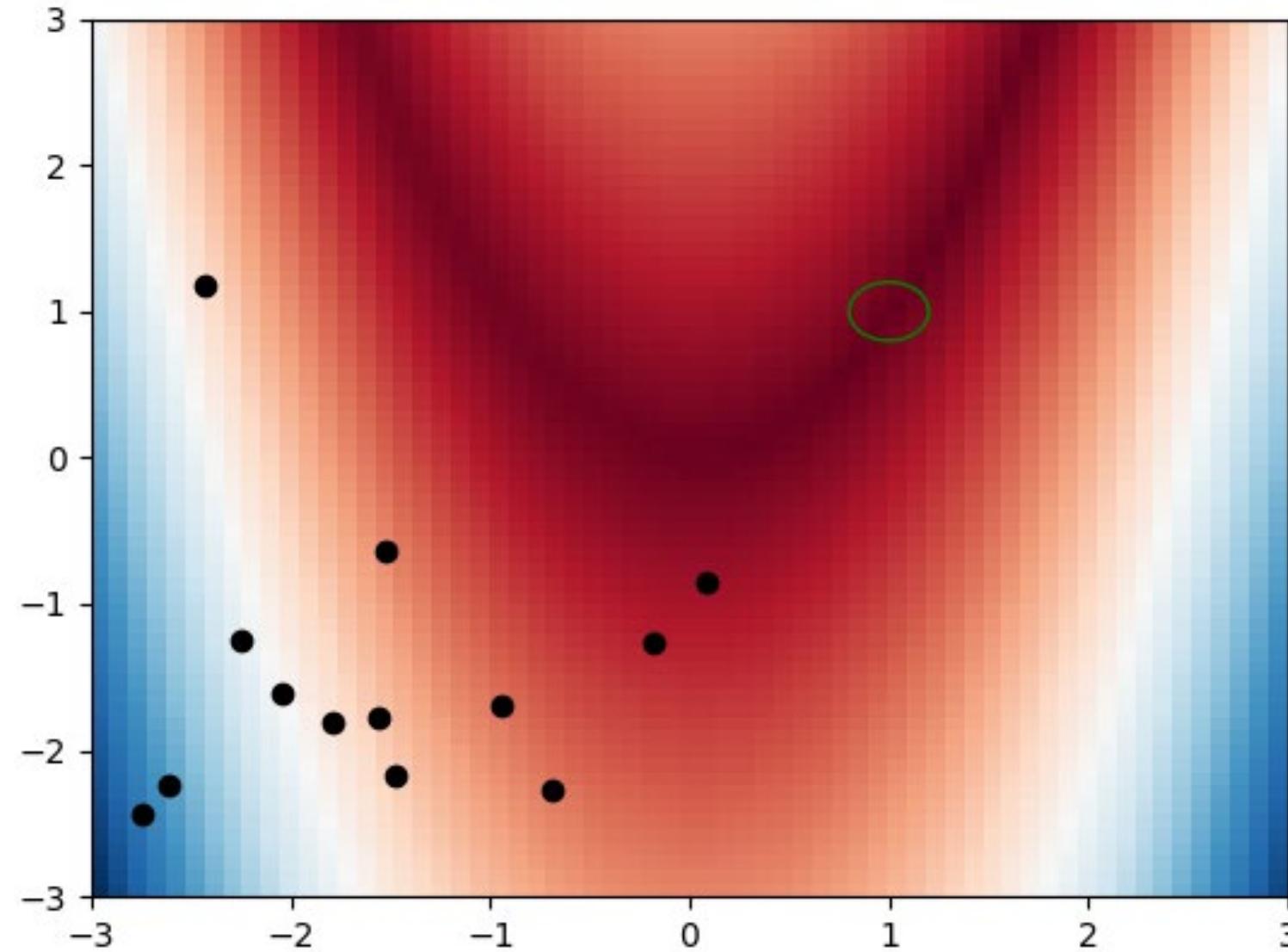
CMA-ES: $p(\mathbf{x})$
elongated in the
progress direction

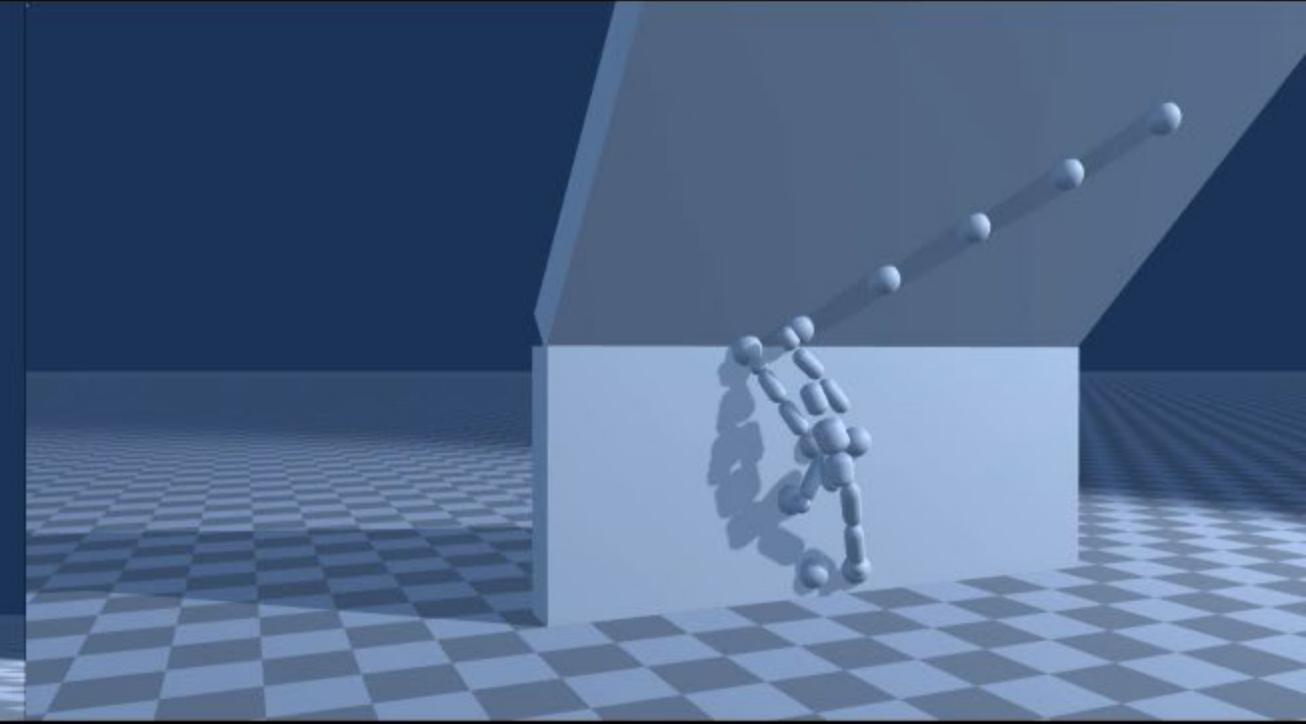


TODO: add an animation comparing single gradient step per samples, CEM, and CMA-ES



CMA-ES on Rosenbrock's function







The CMA Evolution Strategy: A Tutorial

Nikolaus Hansen

Inria

Research centre Saclay–Île-de-France

Université Paris-Saclay, LRI

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A Tutorial on the Cross-Entropy Method

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Received January 2003; Revised November 2003; Accepted January 2004

Abstract. The cross-entropy (CE) method is a new generic approach to combinatorial and multi-extremal optimization and rare event simulation. The purpose of this tutorial is to give a gentle introduction to the CE method. We present the CE methodology, the basic algorithm and its modifications, and discuss applications in combinatorial optimization and machine learning.

<https://people.smp.uq.edu.au/DirkKroese/ps/aortut.pdf>



The Differentiable Cross-Entropy Method

Brandon Amos¹ Denis Yarats^{1,2}

Abstract

We study the cross-entropy method (CEM) for the non-convex optimization of a continuous and parameterized objective function and introduce a differentiable variant that enables us to differentiate the output of CEM with respect to the objective function’s parameters. In the machine learning setting this brings CEM inside of the end-to-end learning pipeline where this has otherwise been impossible. We show applications in a synthetic energy-based structured prediction task and in non-convex continuous control. In the control setting we show how to embed optimal action sequences into a lower-dimensional space. DCEM enables us to fine-tune CEM-based controllers with policy optimization.

End-to-end learning in these settings can be done by defining a loss function \mathcal{L} on top of \hat{x} and taking gradient steps $\nabla_{\theta}\mathcal{L}$. If f_{θ} were *convex* this gradient is easy to analyze and compute when it exists and is unique (Gould et al., 2016; Johnson et al., 2016; Amos et al., 2017; Amos & Kolter, 2017). Analyzing and computing a “derivative” through the *non-convex* arg min in eq. (1) is not as easy and is challenging in theory and practice. The derivative may not exist or may be *uninformative* in theory, it might not be unique, and even if it does, the numerical solver being used to compute the solution may not find a global or even local optimum of f . One promising direction to sidestep these issues is to approximate the arg min operation with an explicit optimization procedure that is interpreted as another compute graph and *unrolled* through, *i.e.* seen as a sequence of differentiable computations. This is most commonly done with gradient descent as in Domke (2012); Metz et al. (2016);



Limited-Memory Matrix Adaptation for Large Scale Black-box Optimization

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Abstract

The Covariance Matrix Adaptation Evolution Strategy (CMA-ES) is a popular method to deal with nonconvex and/or stochastic optimization problems when the gradient information is not available. Being based on the CMA-ES, the recently proposed Matrix Adaptation Evolution Strategy (MA-ES) provides a rather surprising result that the covariance matrix and all associated operations (e.g., potentially unstable eigendecomposition) can be replaced in the CMA-ES by a updated transformation matrix without any loss of performance. In order to further simplify MA-ES and reduce its $\mathcal{O}(n^2)$ time and storage complexity to $\mathcal{O}(n \log(n))$, we present the Limited-Memory Matrix Adaptation Evolution Strategy (LM-MA-ES) for efficient zeroth order large-scale optimization. The algorithm demonstrates state-of-the-art performance on a set of established large-scale benchmarks. We explore the algorithm on the problem of generating adversarial inputs for a (non-smooth) random forest classifier, demonstrating a surprising vulnerability of the classifier.

LM-MA-ES (Loschilov et al. 2017)

- Limited memory variant which scales to millions of variables, $O(N \log(N))$
- LM-MA-ES works even in generating adversarial images, something that had previously only been done with gradient-based optimization
- Unity implementation provided in the course repository

Optimize

Reward Shaping

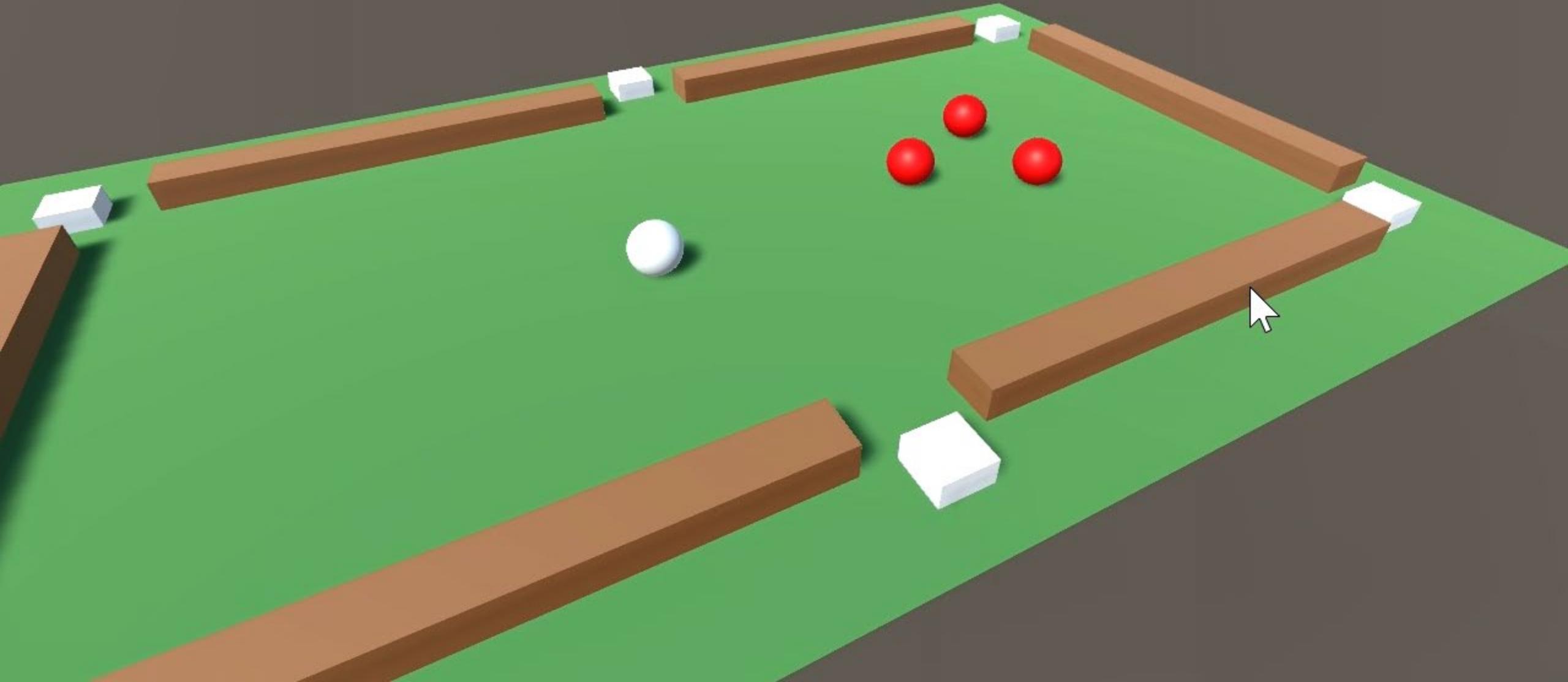
Predicted score: 0

Max Iter:

50

Population size:

16

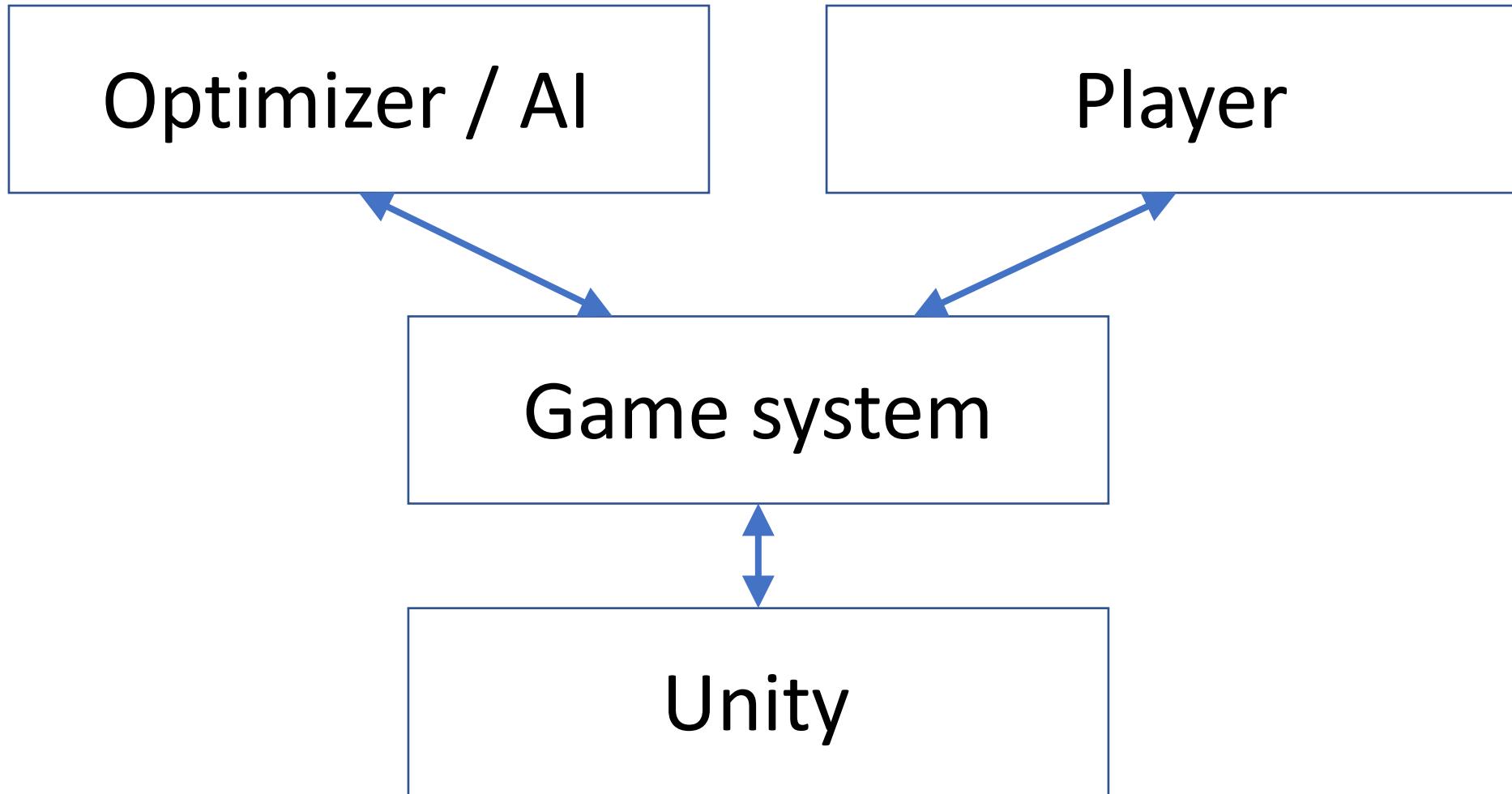


(LM-)MAES Unity interface

```
public interface IMAES
{
    int recommendedPopulationSize(int dim);
    void init(int dim, int populationSize, double[] initialMean, float init
    void generateSamples(OptimizationSample[] samples, int nInitialGuesses
    double update(OptimizationSample[] samples);
    double[] getBest();
    double getBestObjectiveFuncValue();
    void optimize(Func<double[], int, double> objectiveFunc, int maxIter);
}
```

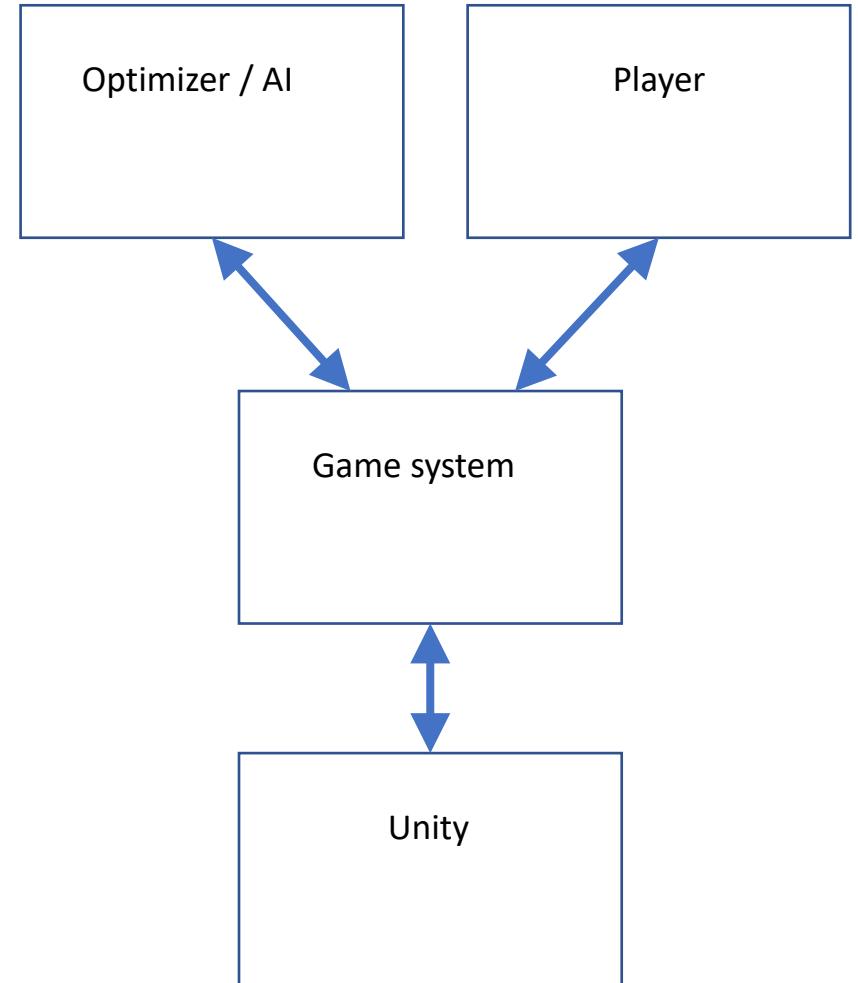
Plain C# implementation, no math or machine learning libraries needed.

Design pattern: AI and Player as game system controllers



Game system

- Implement actions
- Simulate actions speculatively (for evaluating $f(\mathbf{x})$ without affecting system state)
- Save & restore state (needed for the above)
- The speculative simulation may also be useful for the player!



Optimize

Reward Shaping

Predicted score: 0

Max Iter:

50

Population size:

16





IntelligentPool code walkthrough...

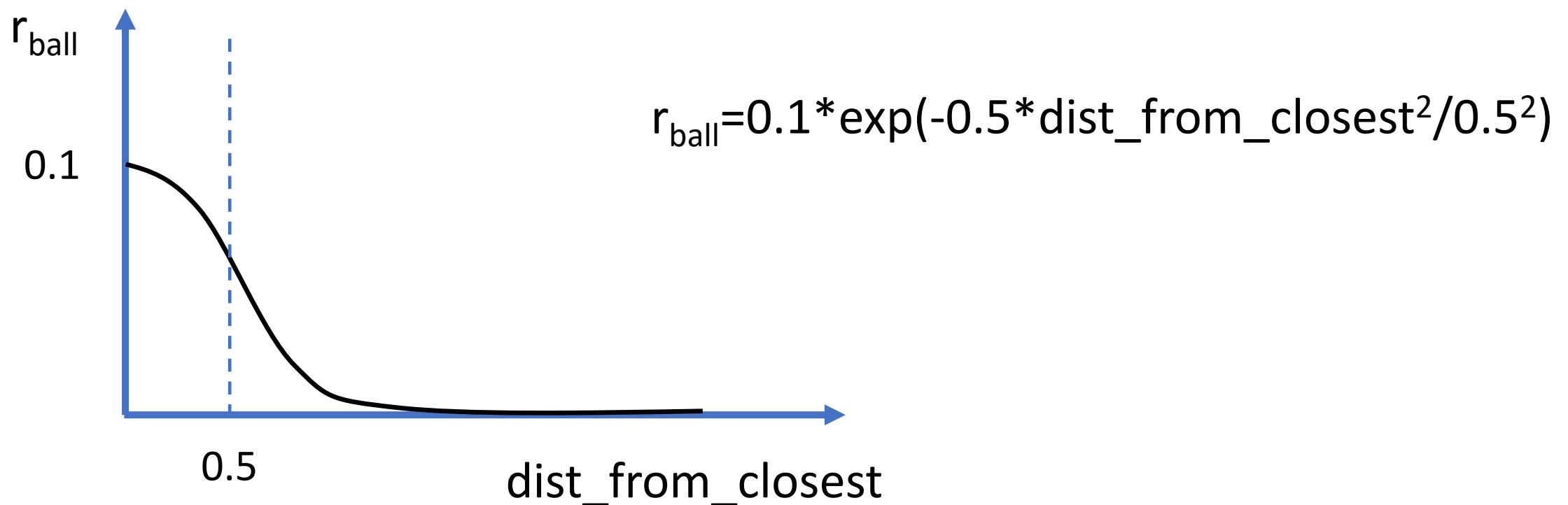


Key things to remember in sampling-based optimization

- Larger sample batches (CMA-ES population size) => less gradient noise.
Harder problems require more samples
- Reward design: avoid flat $f(\mathbf{x})$ with no gradients, e.g., simply giving a +1 score for each pocketed billiards ball
- Reward shaping, proxy objective: give a small bonus for balls that are close to pockets at the end of the shot

Reward shaping

- Simply giving a +1 score for each pocketed ball => mostly flat $f(\mathbf{x})$
- Reward shaping, proxy objective: give a small bonus for balls that are close to pockets at the end of the shot



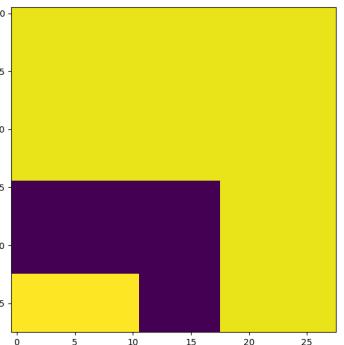
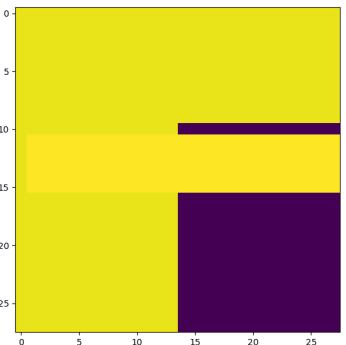
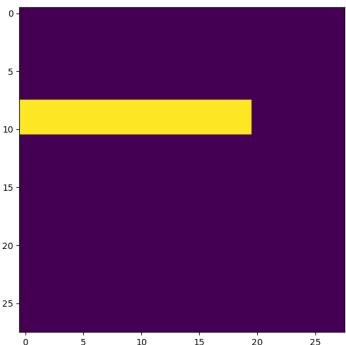
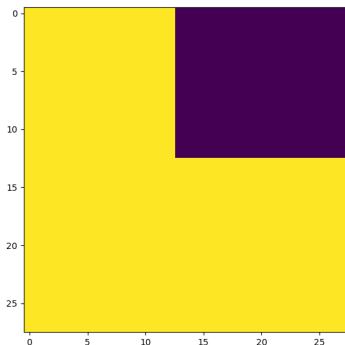
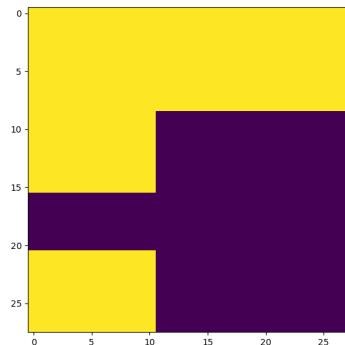
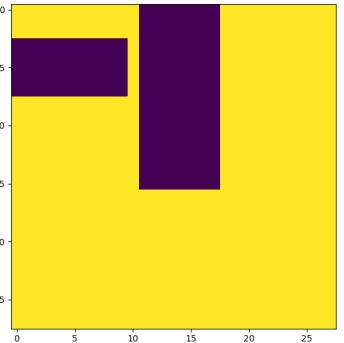
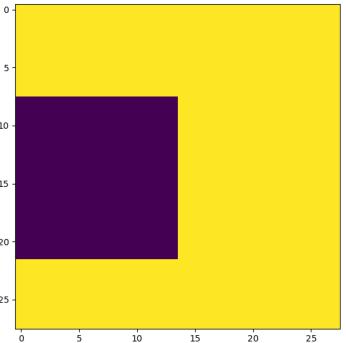
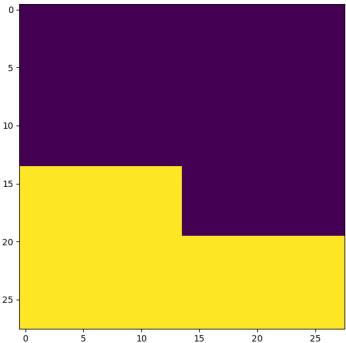
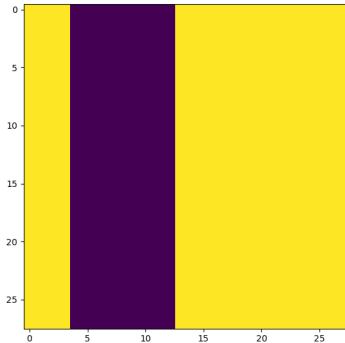
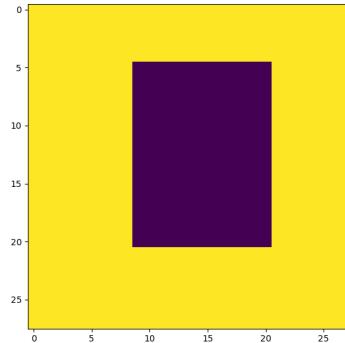
Parallel computing

- Sampling-based methods are usually trivially parallelizable
- Threading overhead negligible if $f(\mathbf{x})$ evaluation is costly
- However, physics simulators typically not thread-safe
- Unity: physics simulation automatically parallelized if groups of objects not interacting (not used in the billiards demo) => create copies of the scene that are placed well apart

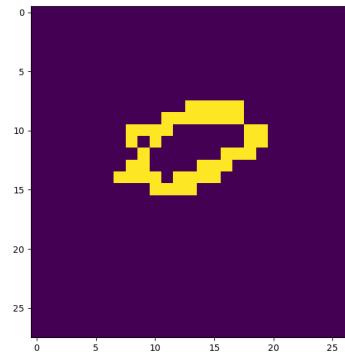
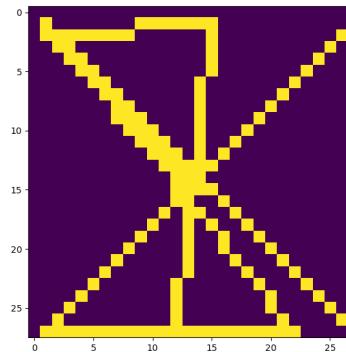
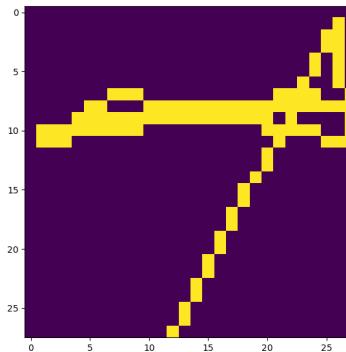
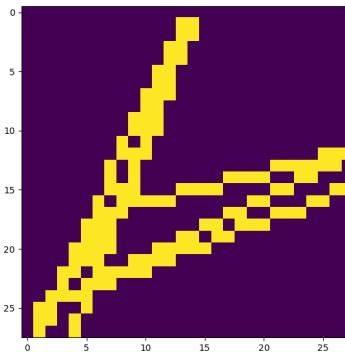
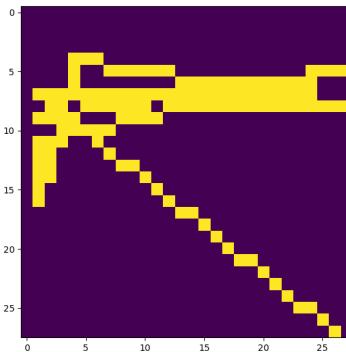
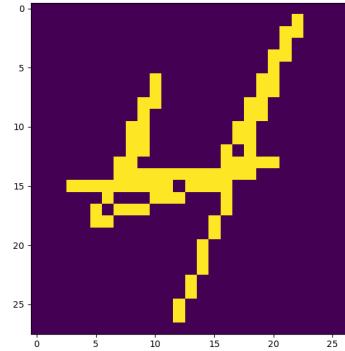
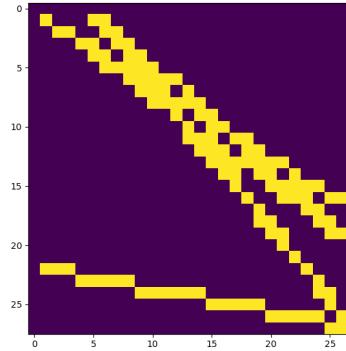
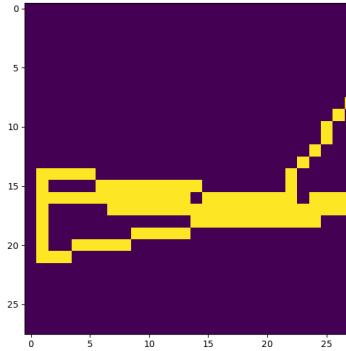
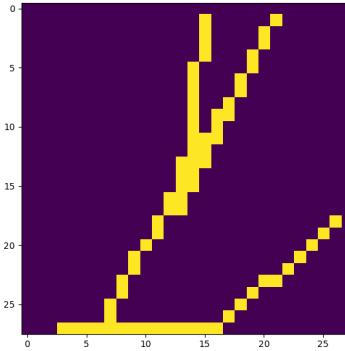
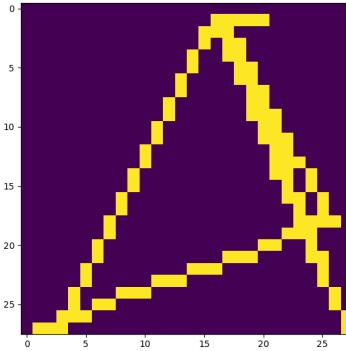
Exercise: CMA-ES for Abstract Adversarial Images

- \mathbf{x} contains the parameters of a painting composed of primitives like rectangles (parameters: corner coordinates, color)
- $f(\mathbf{x})$ is the target class probability of a convolutional image classifier network, similar to what we trained in the day 1 exercise.
- Results tell us what the network finds relevant, sometimes aesthetically interesting. Could one produce high-quality abstract art with this method? Shadow puppetry with posed animation characters?
- Colab notebooks in the course repository: `CMA-ES_Art.ipynb`, `AdversarialMNIST.ipynb`

MNIST using 3 rectangles



MNIST using 7 lines



CIFAR-10 dataset

airplane



automobile



bird



cat



deer



dog



frog



horse



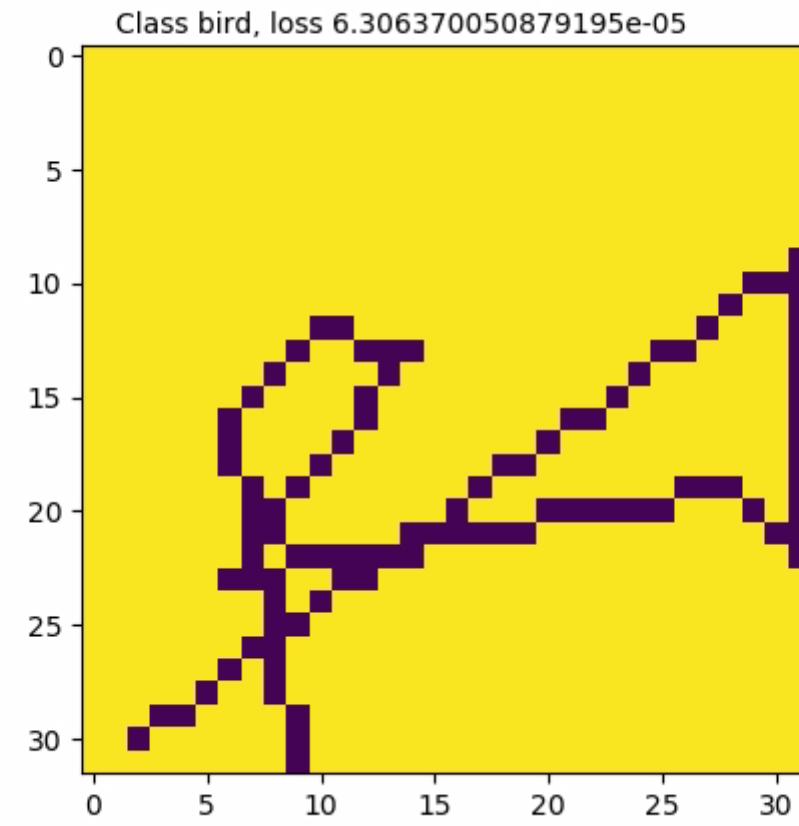
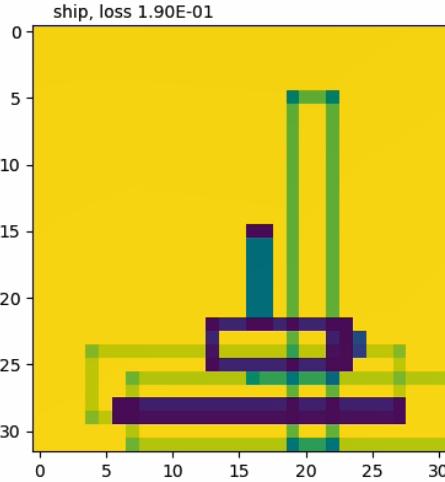
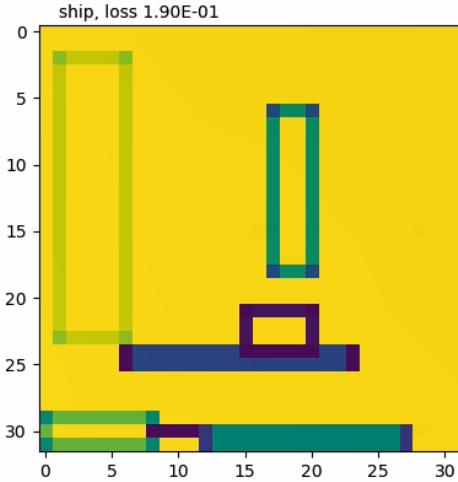
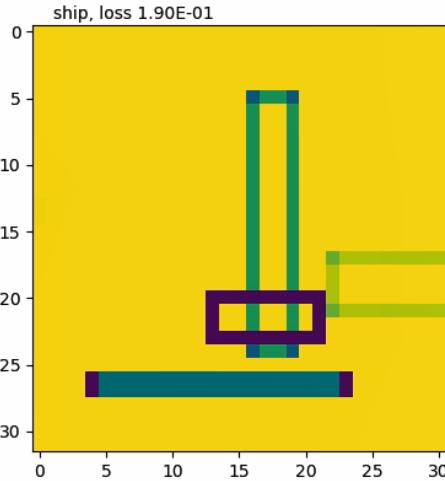
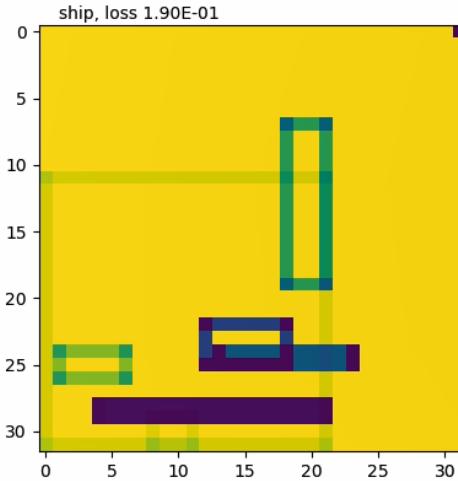
ship



truck



Optimizing CIFAR-10 images



Same idea with a bigger neural network and more sophisticated drawing primitives



Perception Engines: cello, cabbage, hammerhead shark, iron, tick, starfish, binoculars, measuring cup, blow dryer, and jack-o'-lantern

Discrete/combinatorial optimization

Discrete/combinatorial optimization

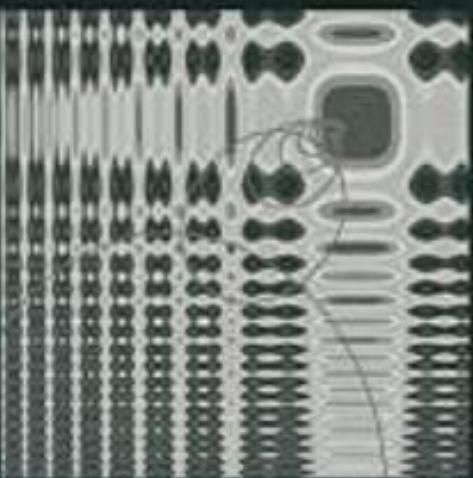
- Q: How to compute gradient of $f(x)$ with respect to discrete variables, e.g., which button to press when controlling a game character?
- A: You can't, as per the definition of gradient.
- Fortunately, optimizing $E[f(x)]$ works just as well if x is sampled from a discrete distribution that defines the probabilities of each button
- For multiple variables (combinatorial optimization), one can often assume independence as $p(x,y)=p(x)p(y)$, sample x,y from their own discrete distributions, and optimize $E[f(x,y)]$ using the REINFORCE gradient estimator
- A popular alternative: CEM

The Cross-Entropy Method

A Unified Approach to Combinatorial Optimization, Monte-Carlo Simulation, and Machine Learning

Reuven Y. Rubinstein

Dirk P. Kroese



*Information
Science
& Statistics*

Combinatorial Optimization of Graphical User Interface Designs

This article surveys combinatorial optimization as a flexible and powerful tool for computational generation and adaptation of graphical user interfaces (GUIs).

By ANTTI OULASVIRTA^{1b}, NIRAJ RAMESH DAYAMA, MORTEZA SHIRIPOUR, MAXIMILIAN JOHN,
AND ANDREAS KARRENBAUER^{1d}

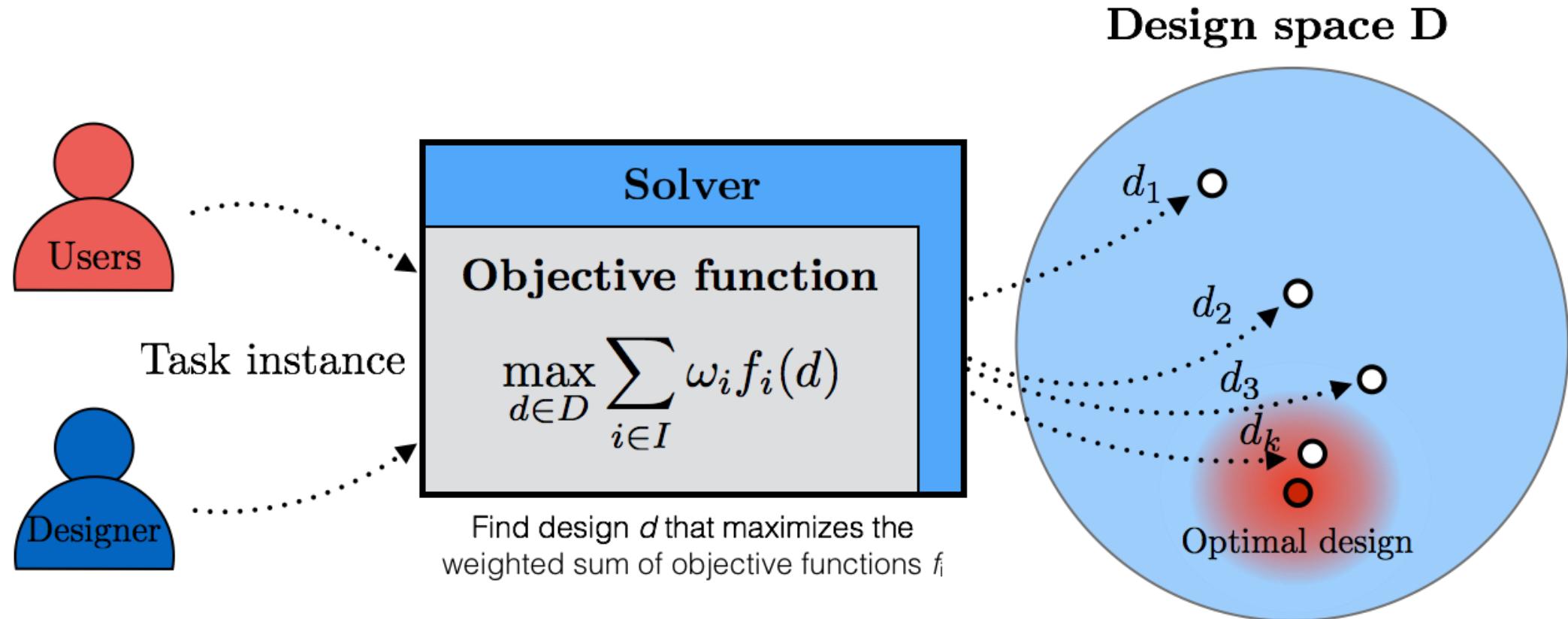
ABSTRACT | The graphical user interface (GUI) has become the prime means for interacting with computing systems. It leverages human perceptual and motor capabilities for elementary tasks such as command exploration and invocation, information search, and multitasking. For designing a GUI, numerous interconnected decisions must be made such that the outcome strikes a balance between human factors and technical objectives. Normally, design choices are specified manually and coded within the software by professional designers and developers. This article surveys combinatorial optimization as a flexible and powerful tool for computational generation and adaptation of GUIs. As recently as 15 years ago, applications were limited to keyboards and widget layouts. The obstacle has been the mathematical definition of design tasks, on the one hand, and the lack of objective functions that capture essential aspects of human behavior, on the other. This article presents definitions of layout design problems as integer programming tasks, a coherent formalism that permits identification of problem types, analysis of their complexity, and exploitation

of known algorithmic solutions. It then surveys advances in formulating evaluative functions for common design-goal foci such as user performance and experience. The convergence of these two advances has expanded the range of solvable problems. Approaches to practical deployment are outlined with a wide spectrum of applications. This article concludes by discussing the position of this application area within optimization and human-computer interaction research and outlines challenges for future work.

KEYWORDS | Combinatorial optimization; computational design; graphical user interfaces (GUIs); human-computer interaction (HCI); integer programming; interactive optimization; meta-heuristic optimization.

I. INTRODUCTION

This article surveys combinatorial optimization approaches for graphical user interface (GUI) design. GUIs have become the prime user interface type for interacting with computing systems. They leverage our perceptual and



Oulasvirta's awesome Jupyter notebook on combinatorial optimization for UI design:
<https://urly.fi/1W9O>



No Free Lunch Theorems for Optimization

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William G. Macready
Santa Fe Institute
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Santa Fe, NM, 87501

December 31, 1996

Abstract

A framework is developed to explore the connection between effective optimization algorithms and the problems they are solving. A number of “no free lunch” (NFL) theorems are presented that establish that for any algorithm, any elevated performance over one class of problems is exactly paid for in performance over another class. These theorems result in a geometric interpretation of what it means for an algorithm to be well suited to an optimization problem. Applications of the NFL theorems to information theoretic aspects of optimization and benchmark measures of performance are also presented. Other issues addressed are time-varying optimization problems and *a priori* “head-to-head” minimax distinctions between optimization algorithms, distinctions that can obtain despite the NFL theorems’ enforcing of a type of uniformity over all algorithms.

Heuristic for choosing a good enough method:

if differentiable $f(\mathbf{x})$ with no local optima:

 Use Adam

elif $f(\mathbf{x})$ is continuous and differentiable:

 Try Adam. If local optima are a problem (results are highly dependent on initialization), try Adam with reparameterized gradient from samples.

 If both fail, try CMA-ES or LM-MA-ES (see below).

elif \mathbf{x} is real-valued:

 if \mathbf{x} is very high-dimensional (hundreds or thousands of variables) :

 Use LM-MA-ES

 else:

 Use CMA-ES

else:

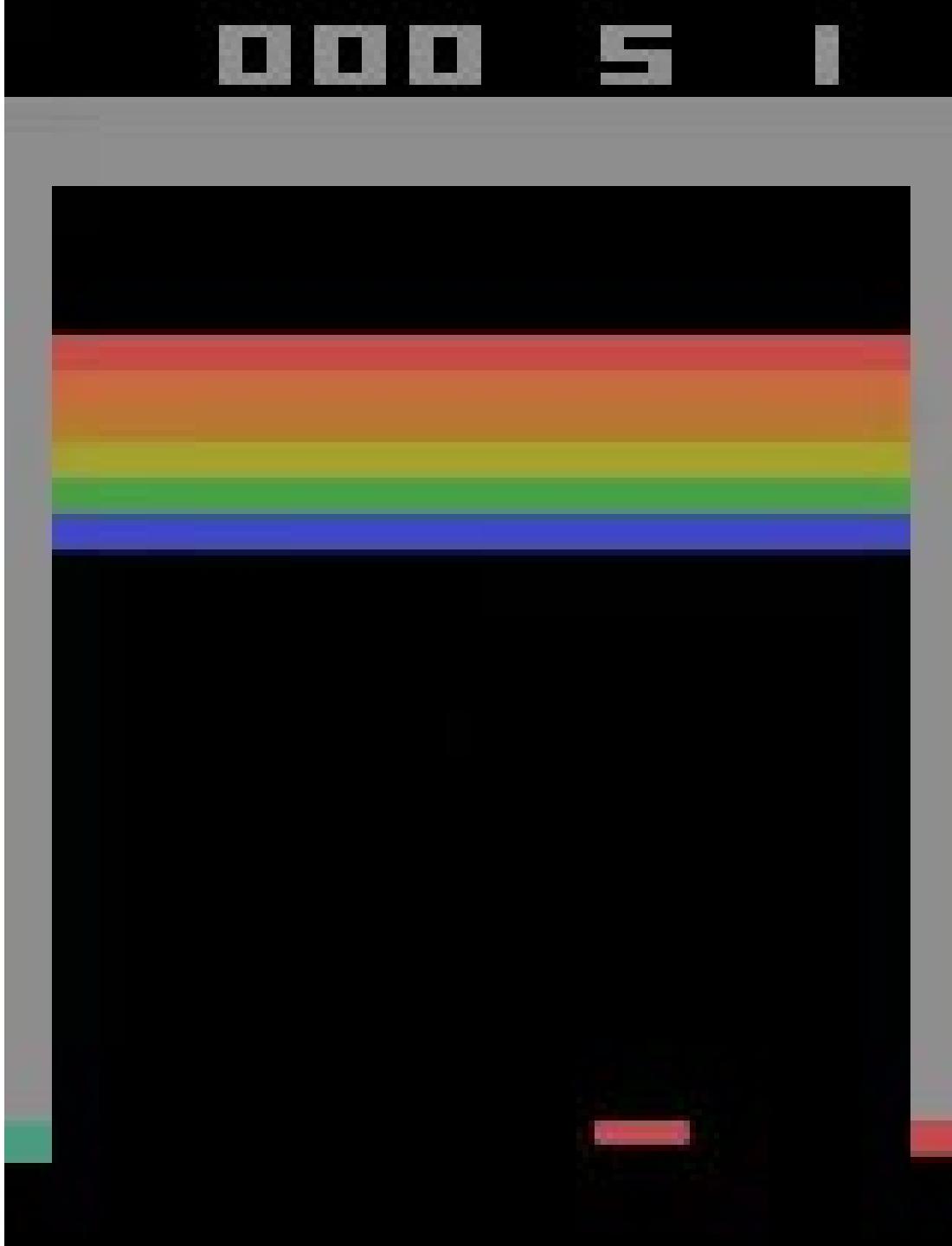
 We have a discrete/combinatorial problem => use CEM or Adam with REINFORCE-based gradient estimation from samples



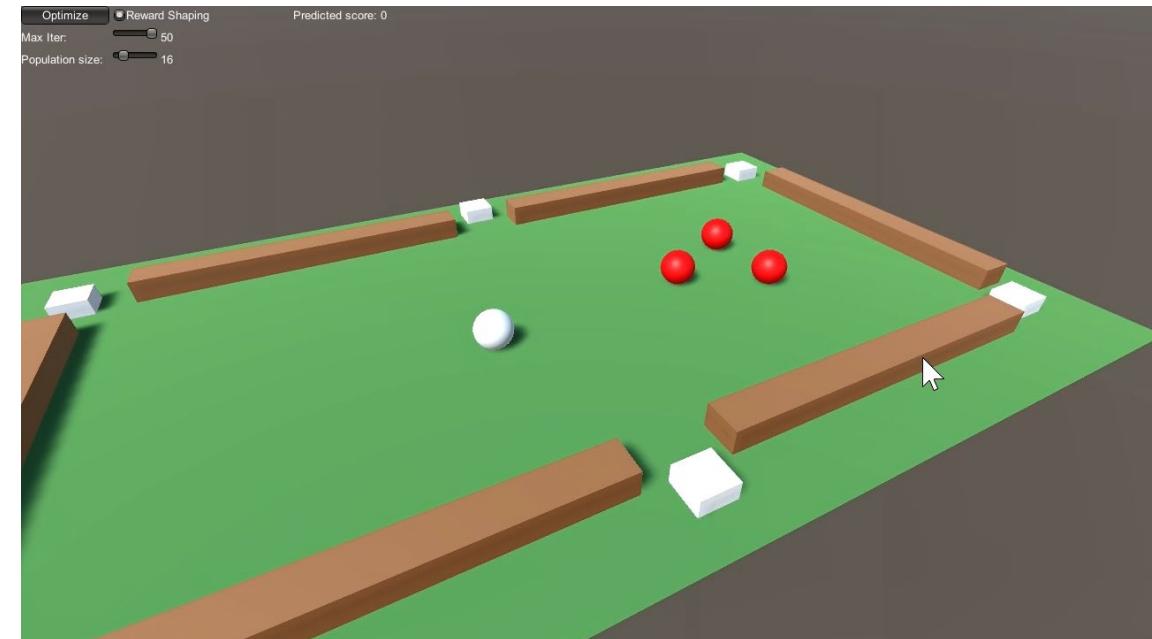
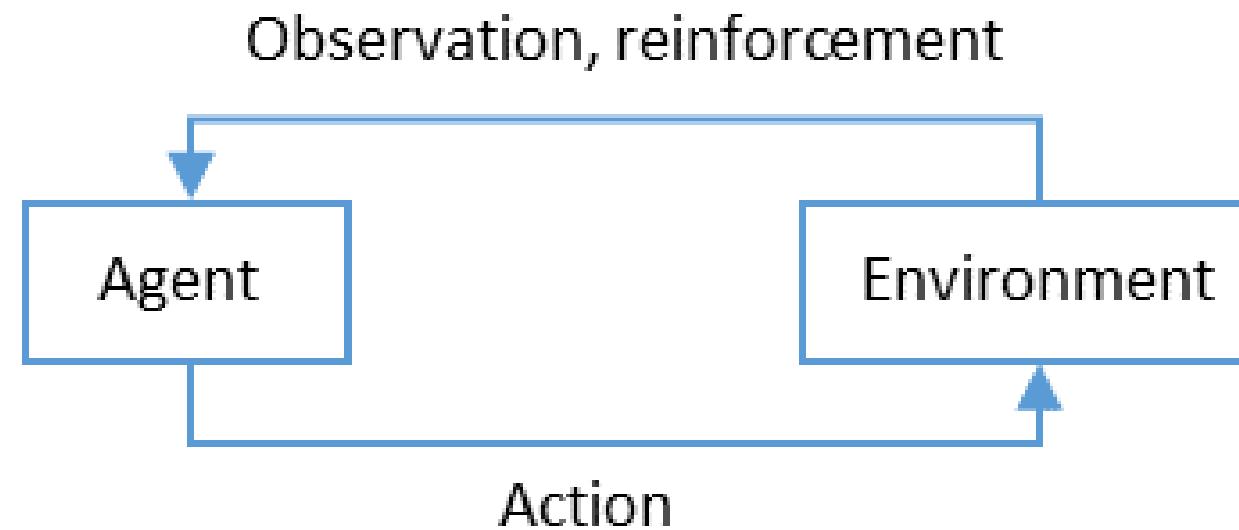
(Deep) Reinforcement learning

Mnih et al. 2015:

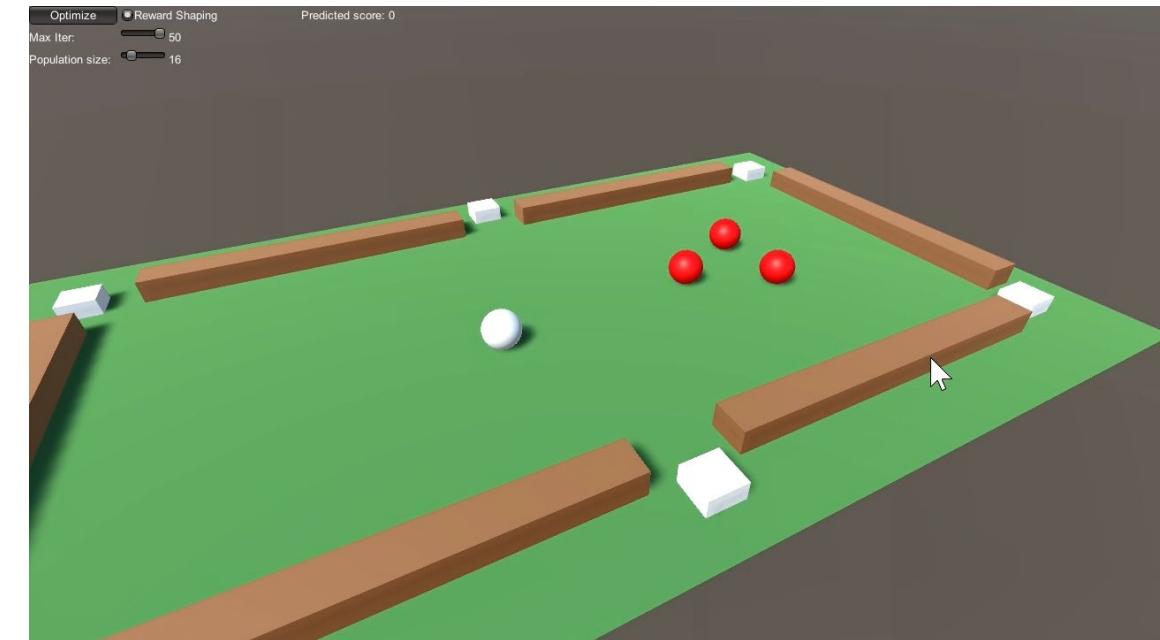
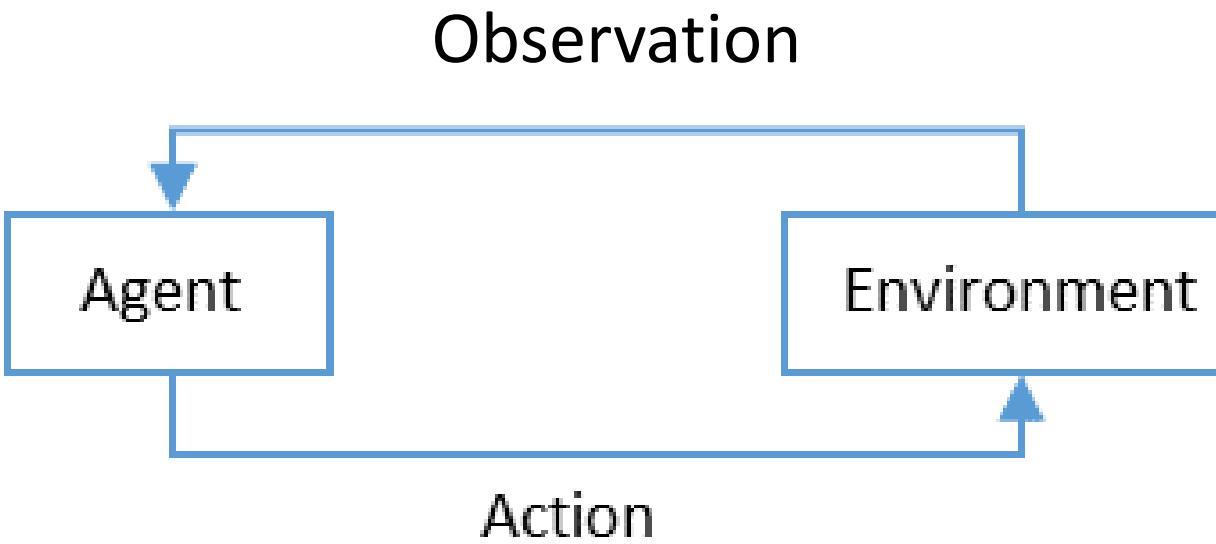
Human-level control
of Atari games using
Deep Reinforcement
Learning (DRL)



Real life: Countless variations of the same problems

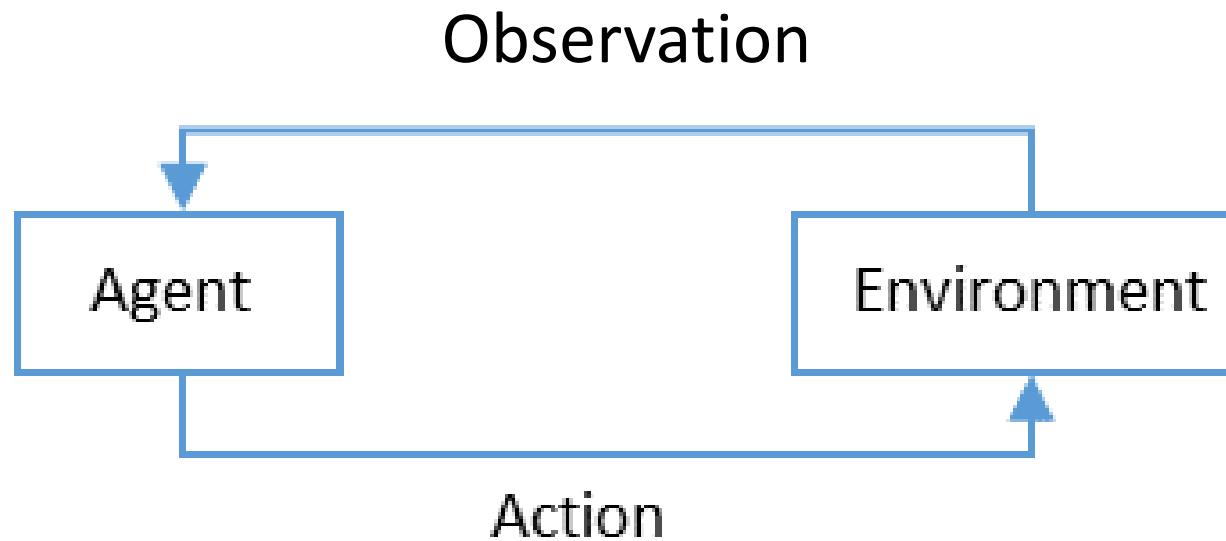


Real life: Countless variations of the same problems



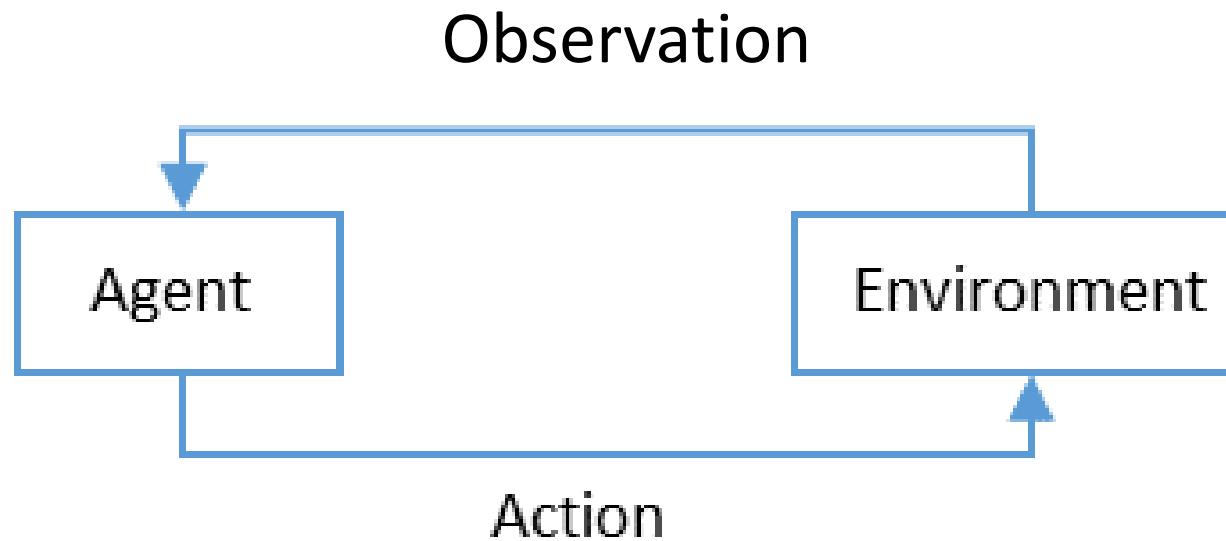
Observation: configuration of balls, i.e., problem definition for optimizing action

Real life: Countless variations of the same problems



DRL goal: Instead of optimizing a single action a , optimize the parameters θ of a policy function $a=\pi_\theta(o)$ that gives the best action for any observation o .

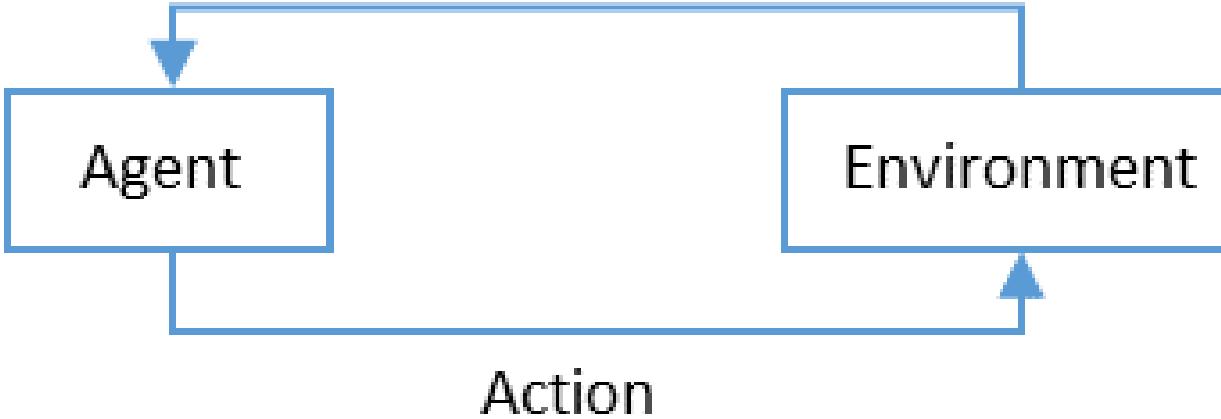
Real life: Countless variations of the same problems



Essentially: Explore/sample different actions in different situations, memorize the best actions, and interpolate/extrapolate to be able to act in new situations.

Real life: Countless variations of the same problems

Observation and reward

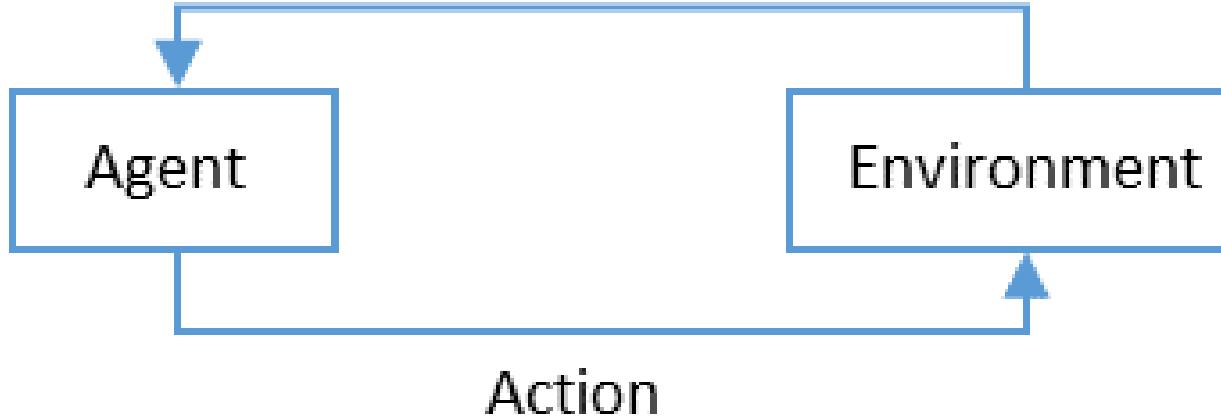


Optimization objective: Maximize the expected sum of (future) rewards:

$$\mathbb{E} \left[\sum_t \gamma^t r(\mathbf{a}_t, \mathbf{o}_t) \right]$$

Real life: Countless variations of the same problems

Observation and reward



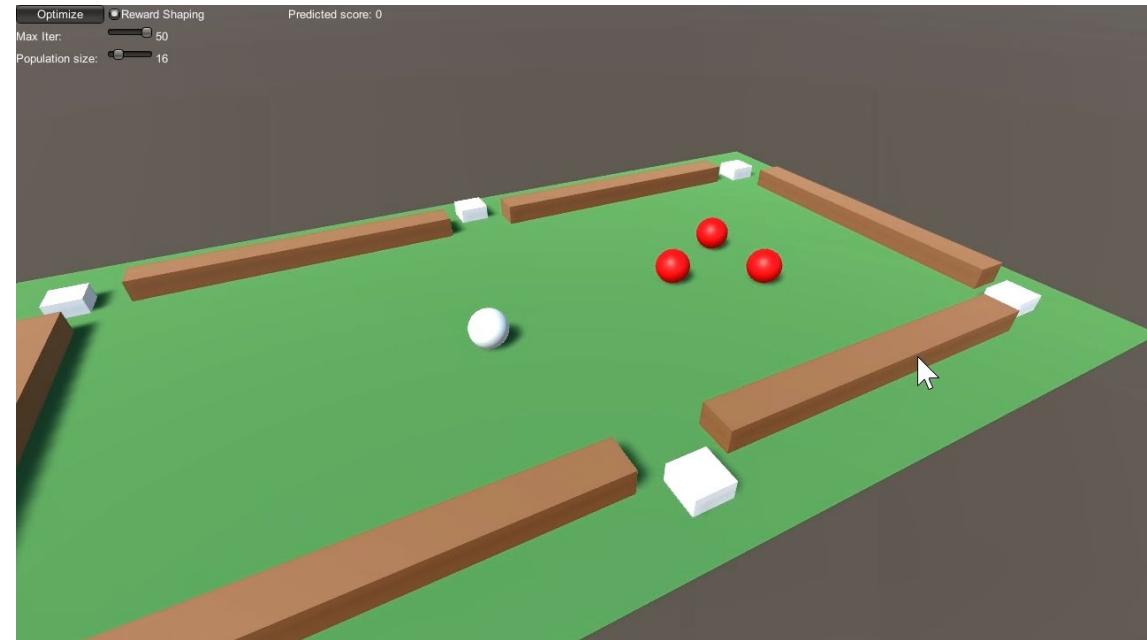
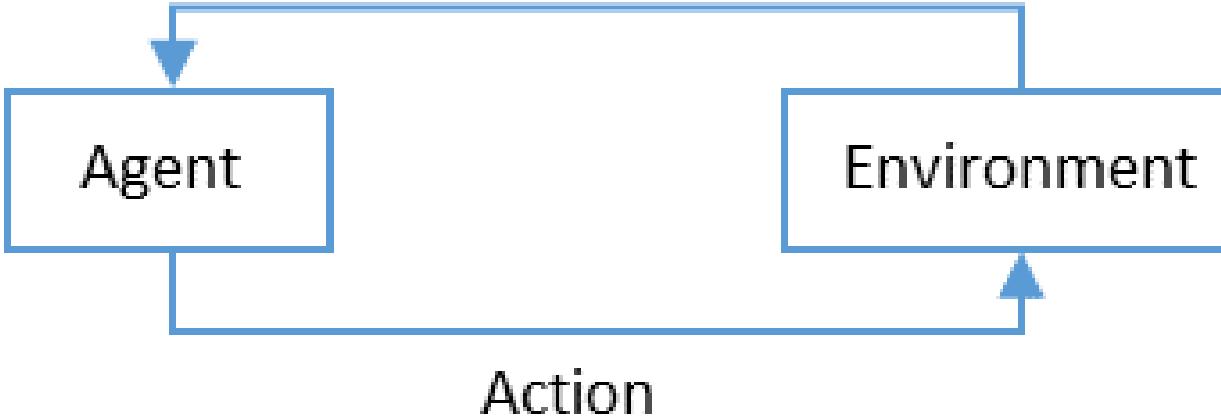
Optimization objective: Maximize the expected sum of (future) rewards:

$$\mathbb{E} \left[\sum_t \gamma^t r(\mathbf{a}_t, \mathbf{o}_t) \right]$$

Expectation: We're interested in the average behavior over all actions and observations

Real life: Countless variations of the same problems

Observation and reward



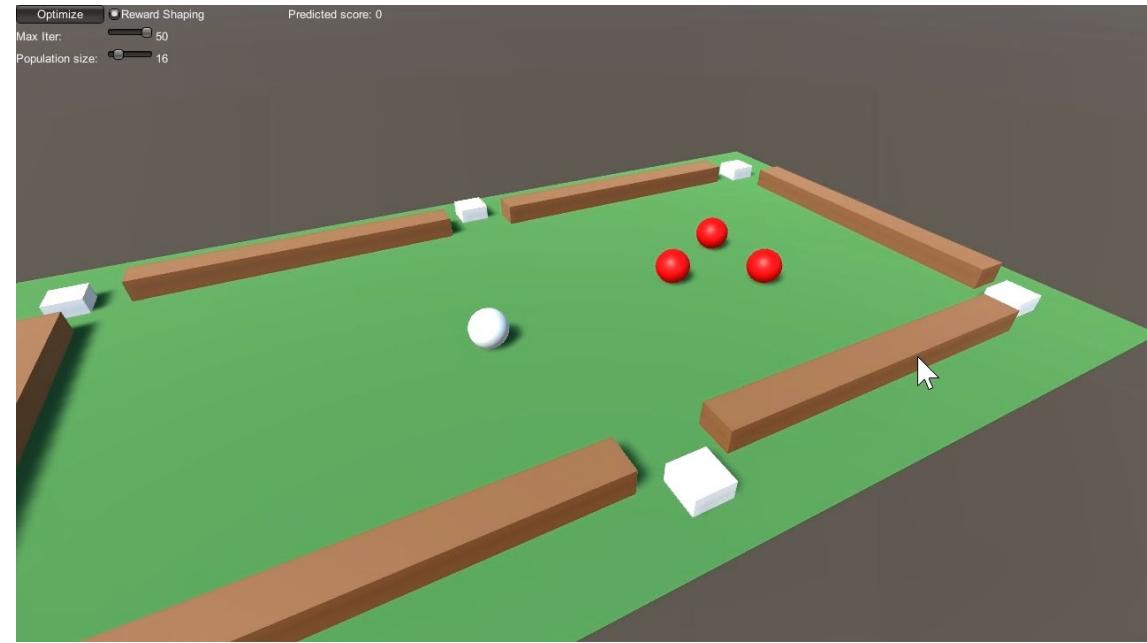
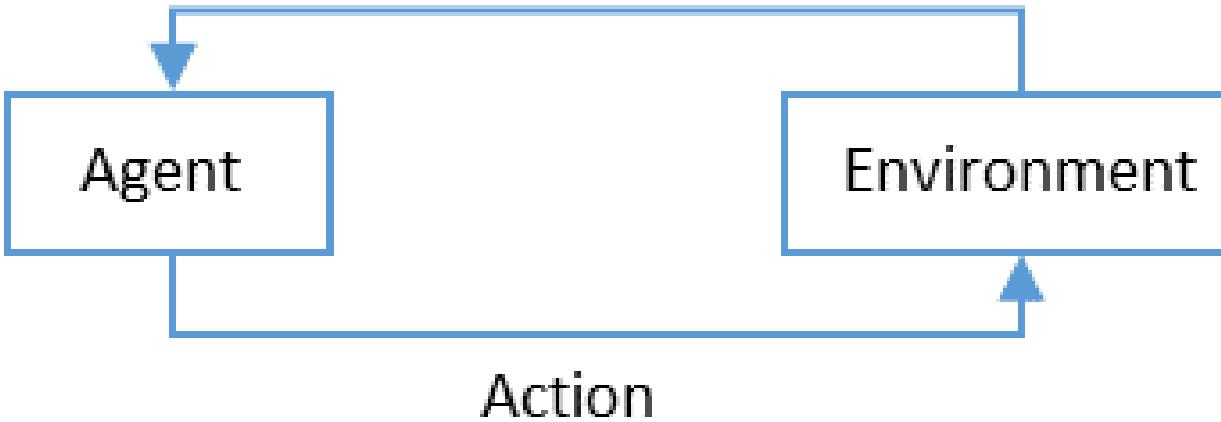
Optimization objective: Maximize the expected sum of (future) rewards:

$$\mathbb{E} \left[\sum_t \gamma^t r(\mathbf{a}_t, \mathbf{o}_t) \right]$$

Time. 0 means “current” time and other values are in the future

Real life: Countless variations of the same problems

Observation and reward



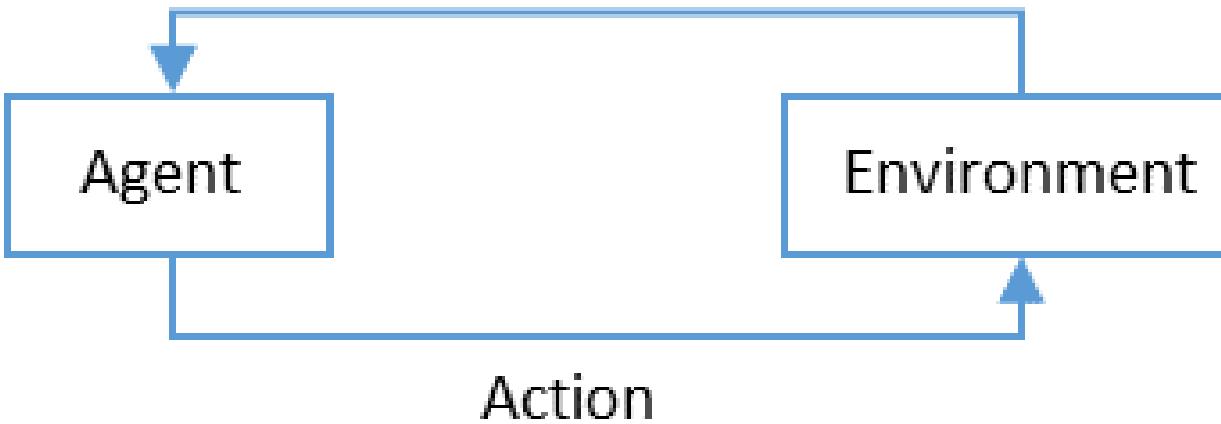
Optimization objective: Maximize the expected sum of (future) rewards:

$$\mathbb{E} \left[\sum_t \gamma^t r(\mathbf{a}_t, \mathbf{o}_t) \right]$$

Reward for action \mathbf{a}_t and observation \mathbf{o}_t at time t

Real life: Countless variations of the same problems

Observation and reward

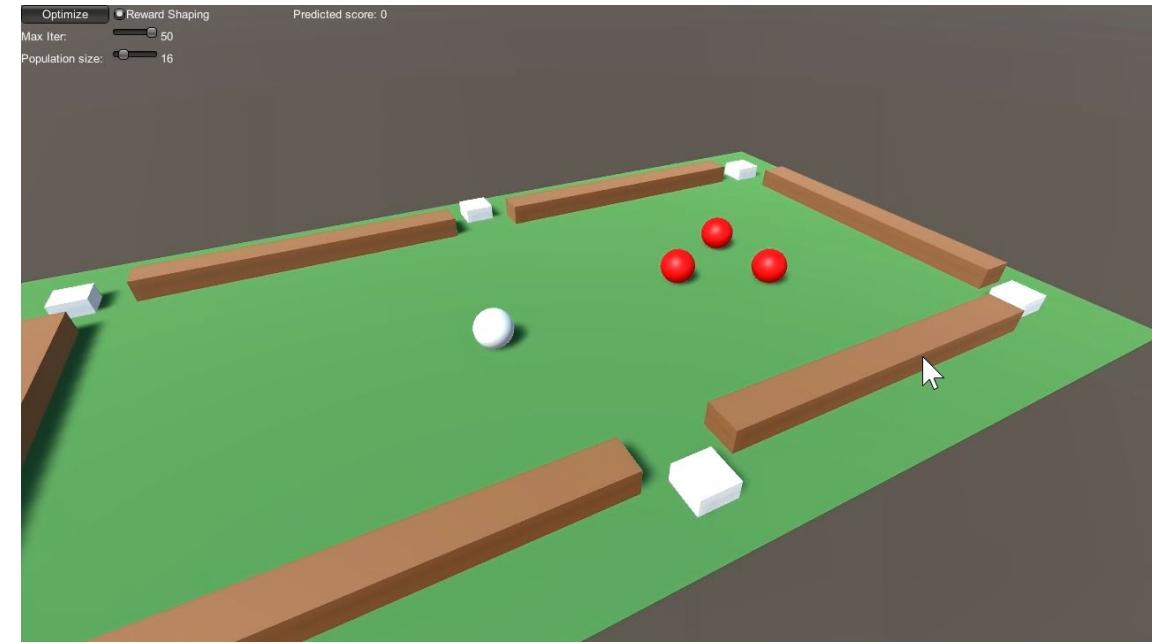
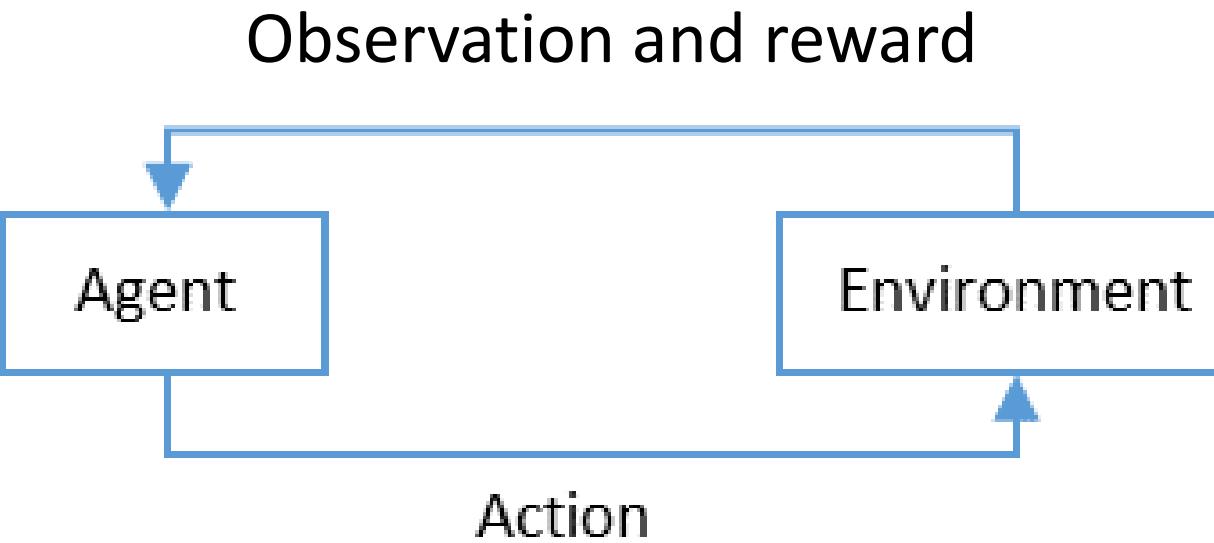


Optimization objective: Maximize the expected sum of (future) rewards:

$$\mathbb{E} \left[\sum_t \gamma^t r(\mathbf{a}_t, \mathbf{o}_t) \right]$$

Sum over time: Consider not just a single action but future actions too

Real life: Countless variations of the same problems

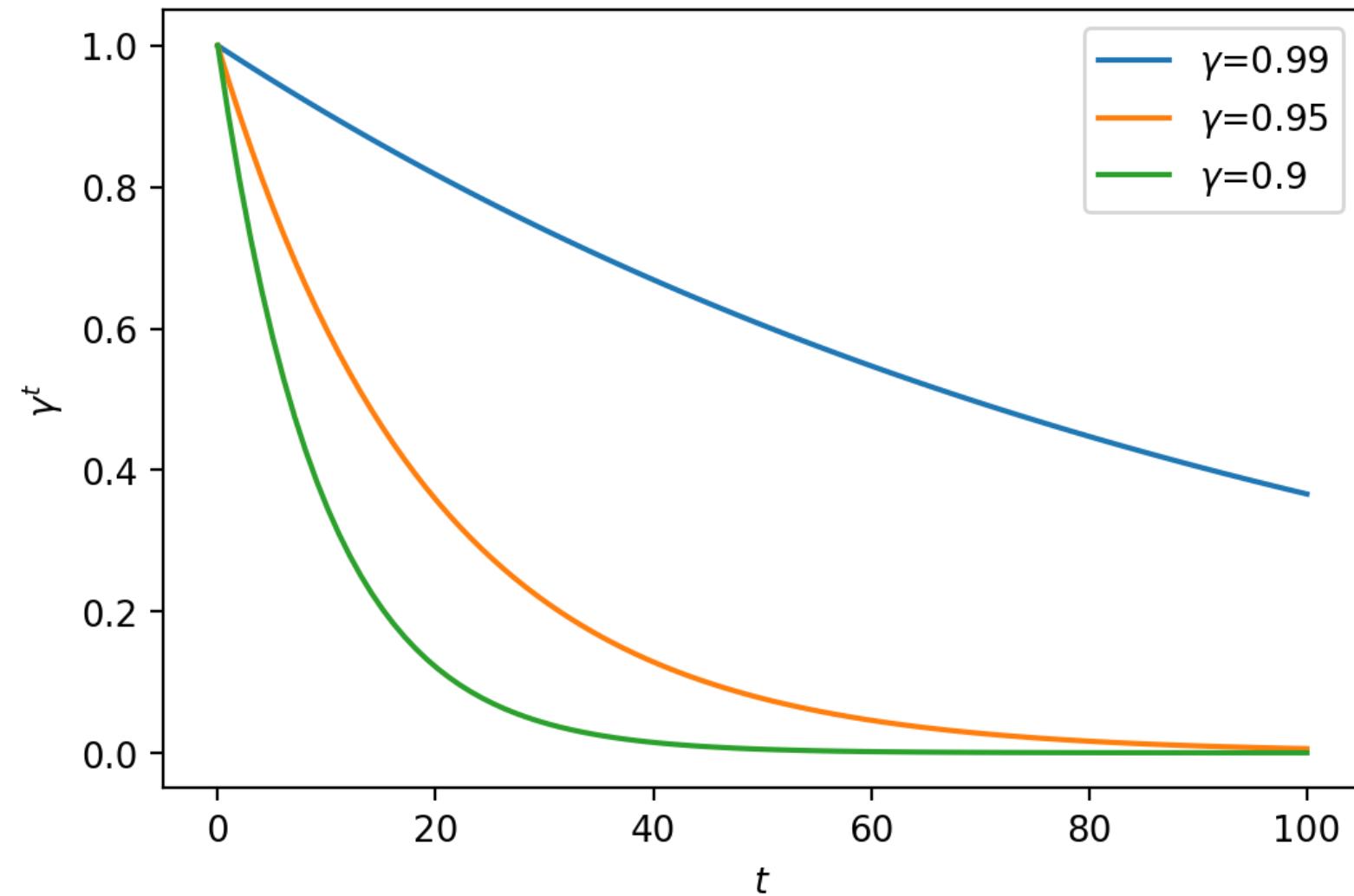


Optimization objective: Maximize the expected sum of (future) rewards:

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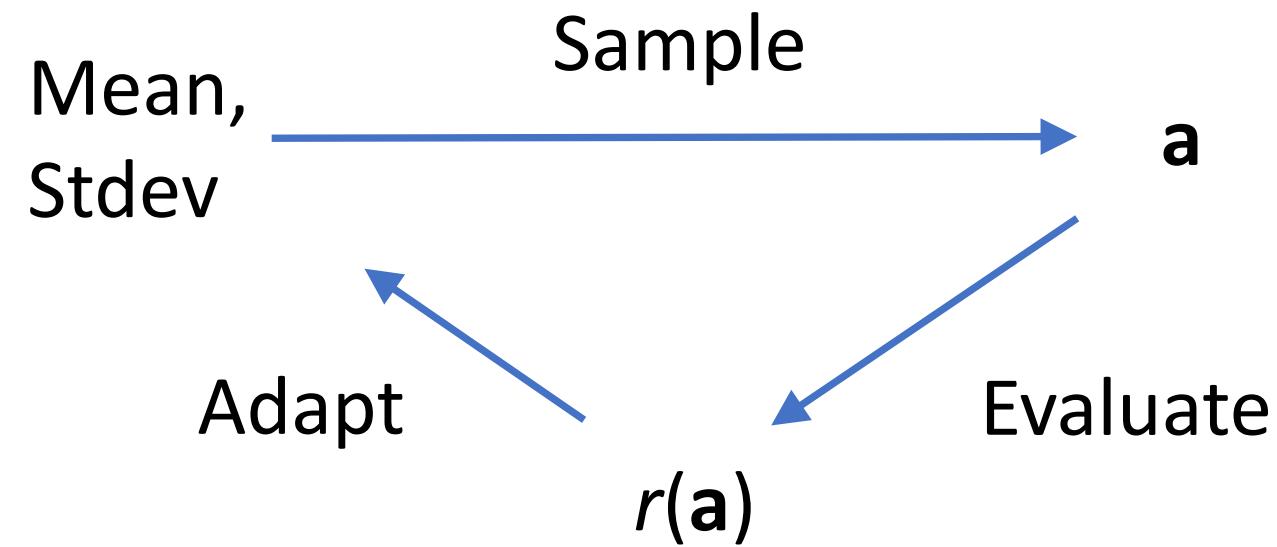
Discount factor

Small discount factor: Immediate rewards matter more



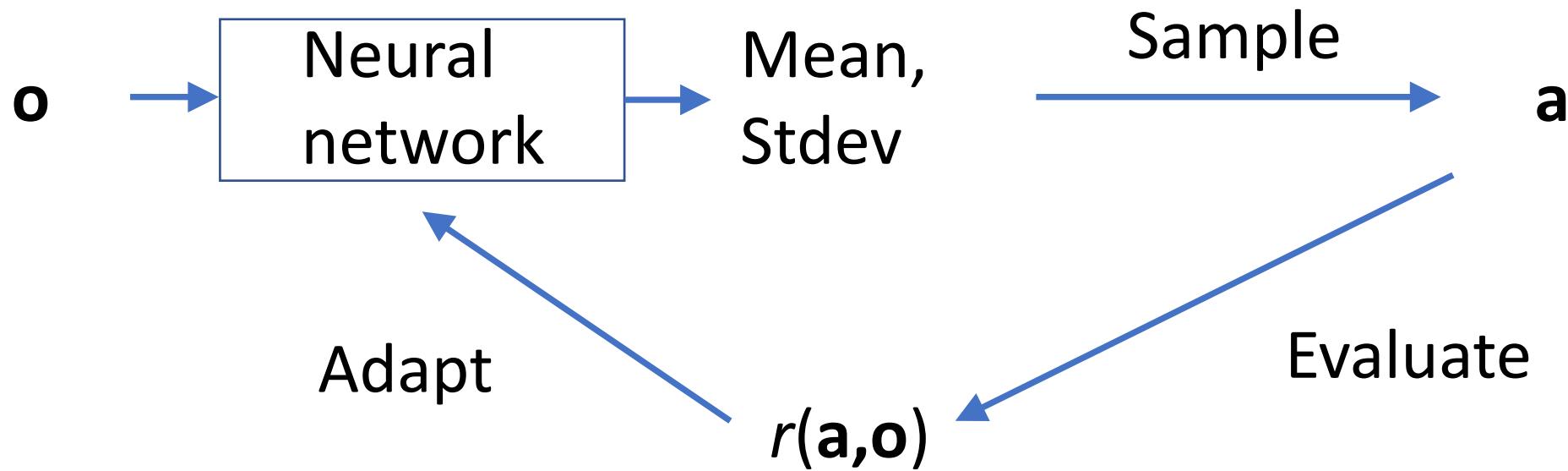


Optimizing actions independent of state (e.g., CMA-ES)



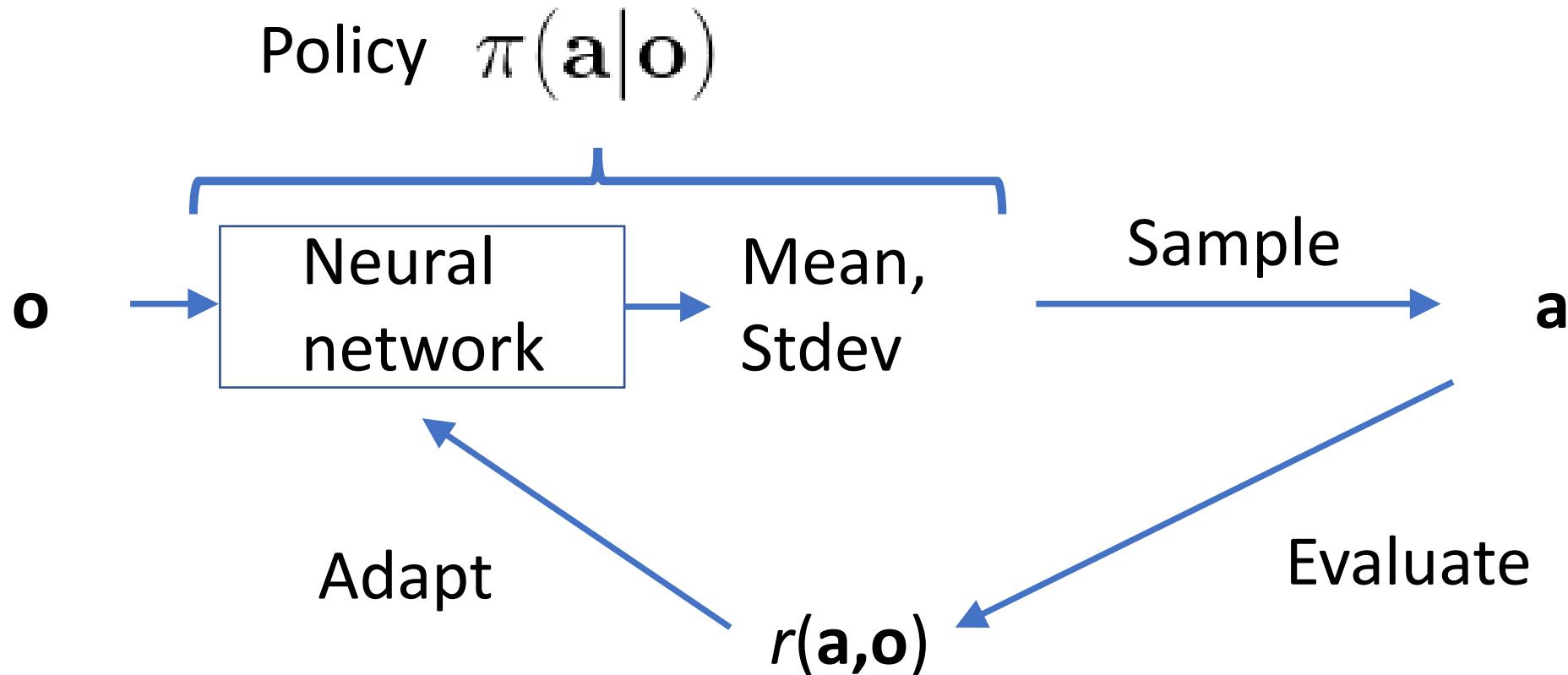


DRL: The action distribution depends on the observation (e.g., configuration of billiards balls)



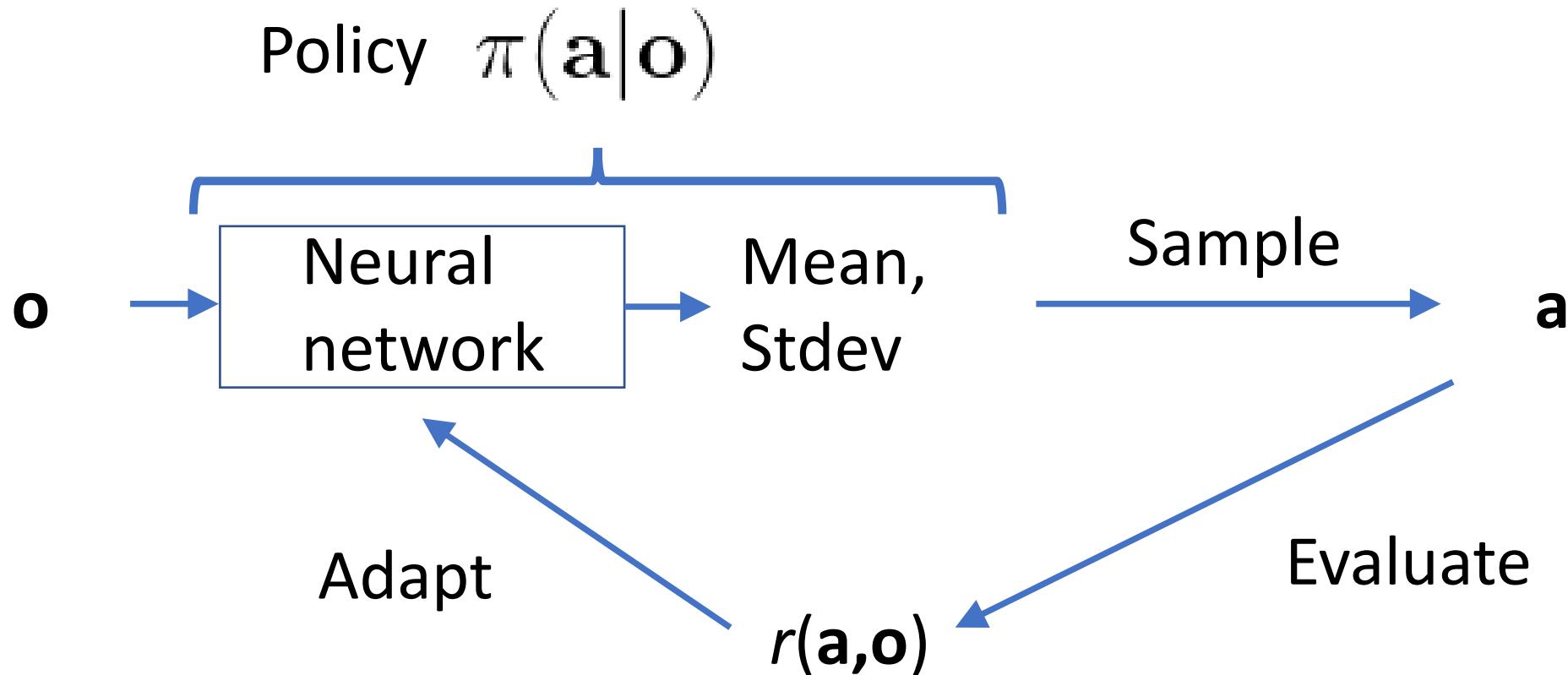


DRL: The action distribution depends on the observation (e.g., configuration of billiards balls)





DRL: The action distribution depends on the observation (e.g., configuration of billiards balls)





JULY 20, 2017

Proximal Policy Optimization

We're releasing a new class of reinforcement learning algorithms, **Proximal Policy Optimization (PPO)**, which perform comparably or better than state-of-the-art approaches while being much simpler to implement and tune. PPO has become the default reinforcement learning algorithm at OpenAI because of its ease of use and good performance.

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PPO and other episodic Reinforcement Learning methods as pseudocode

Until iteration simulation budget exhausted:

 Sample initial state **s**, observed as **o**

 Until terminal state encountered or time limit:

 Sample action **a** according to policy

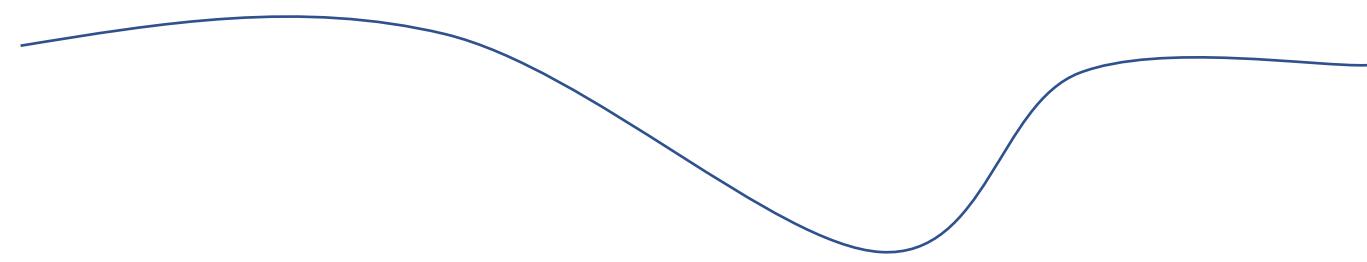
 Execute action (simulate world model)

 Observe new **o'** and reward **r**

*One
episode*

 Update the policy based on the collected experience tuples **[o,a,o',r]**

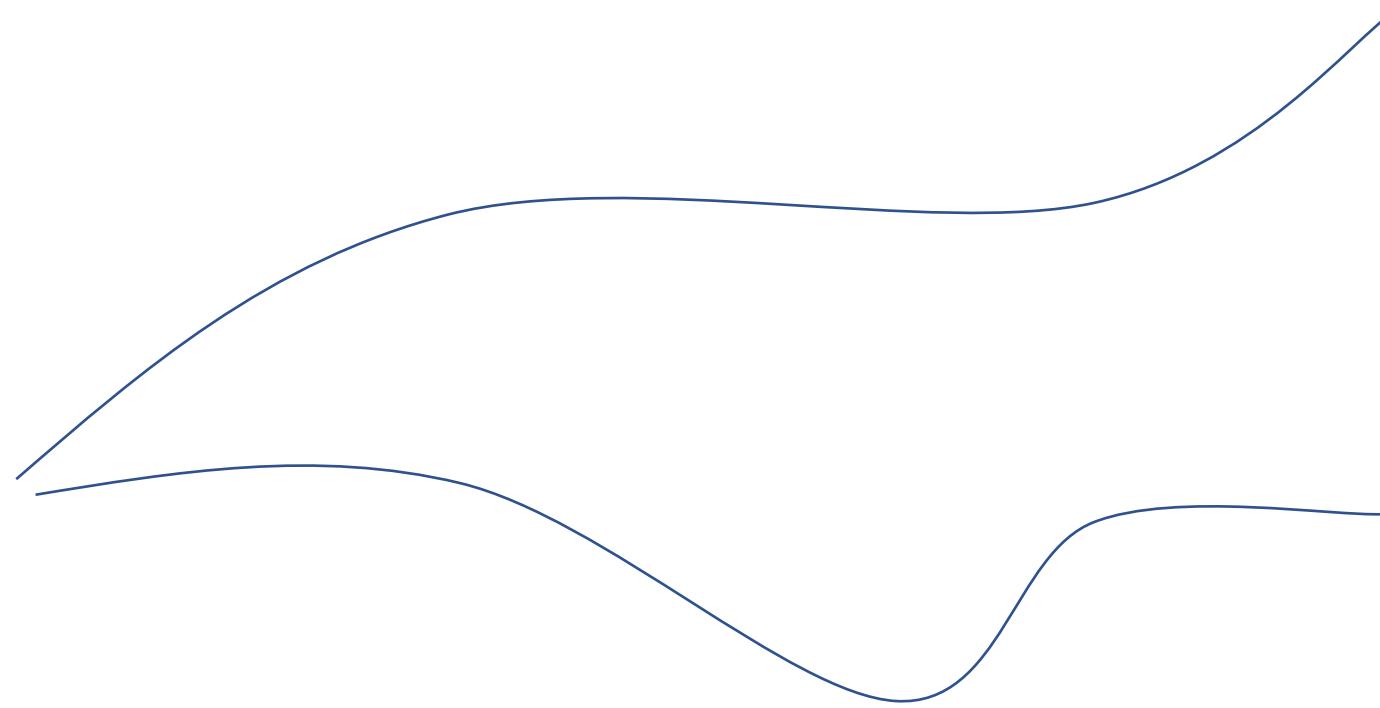
Game start



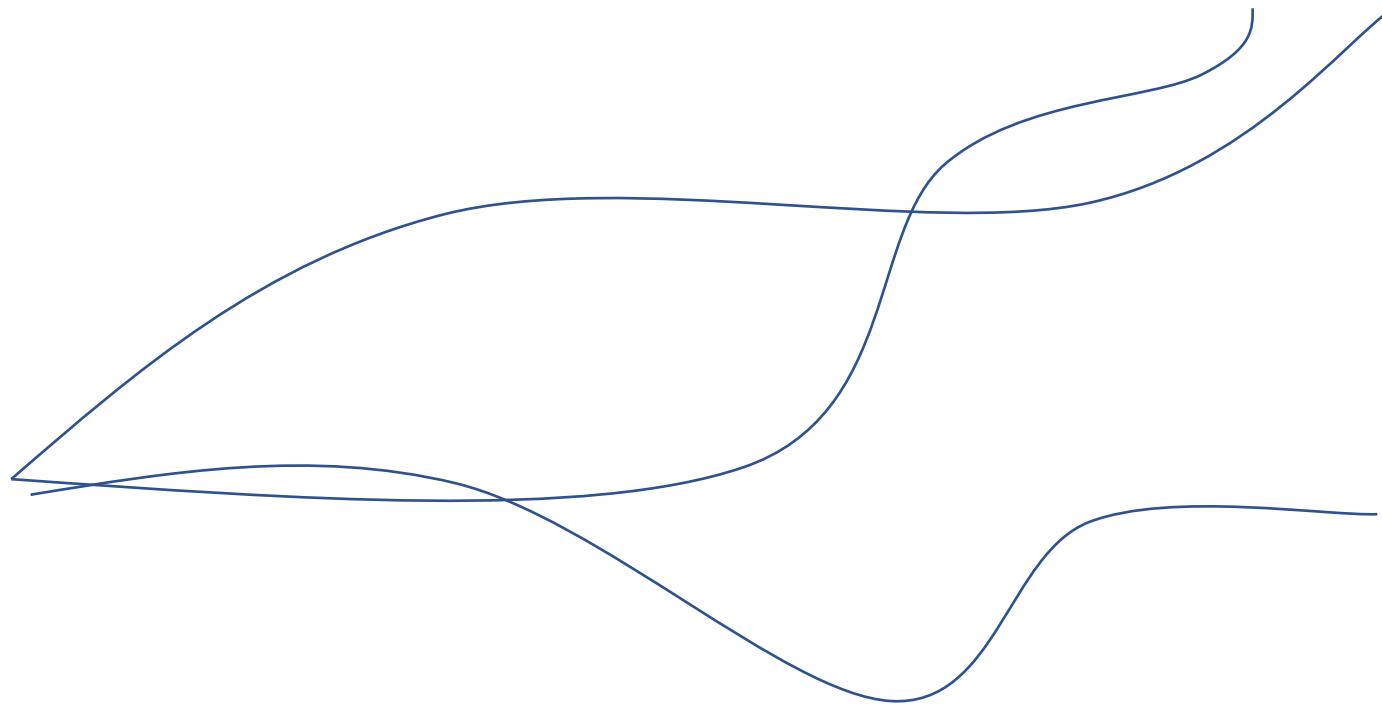
Game end



Game start

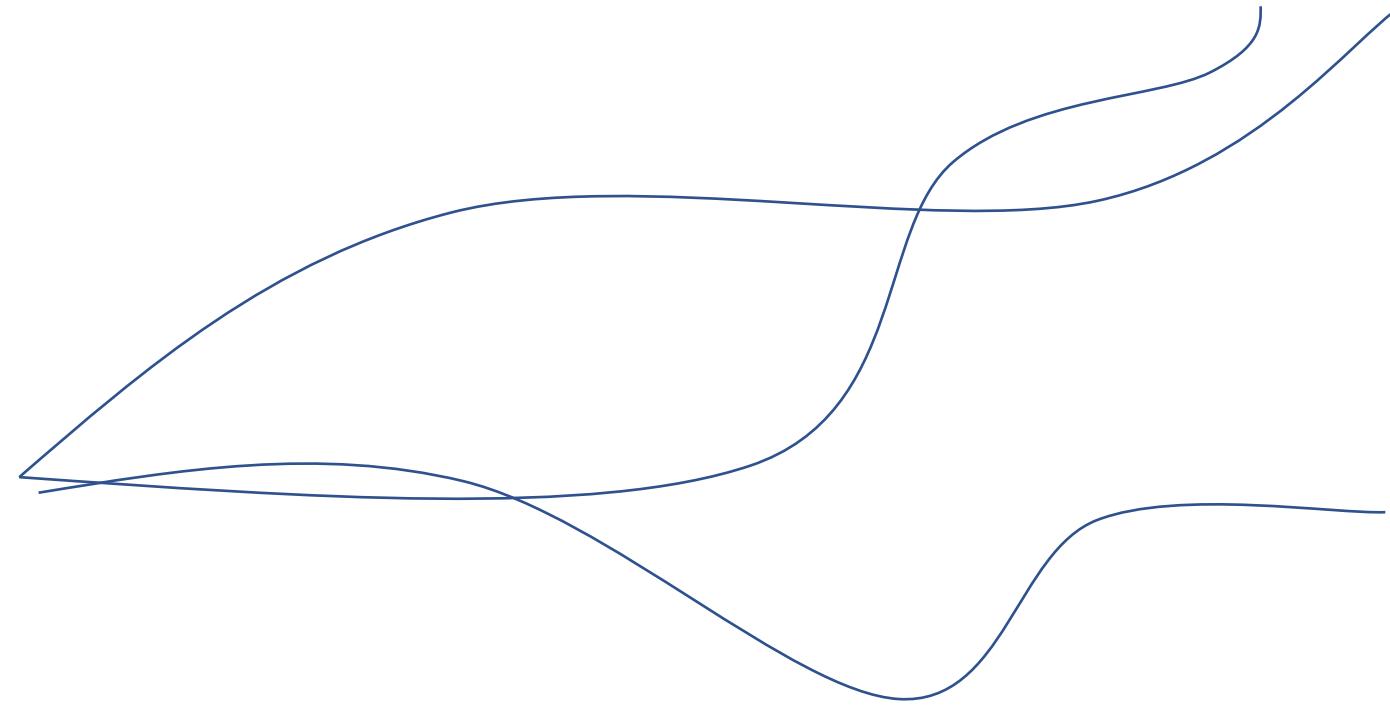


Game start

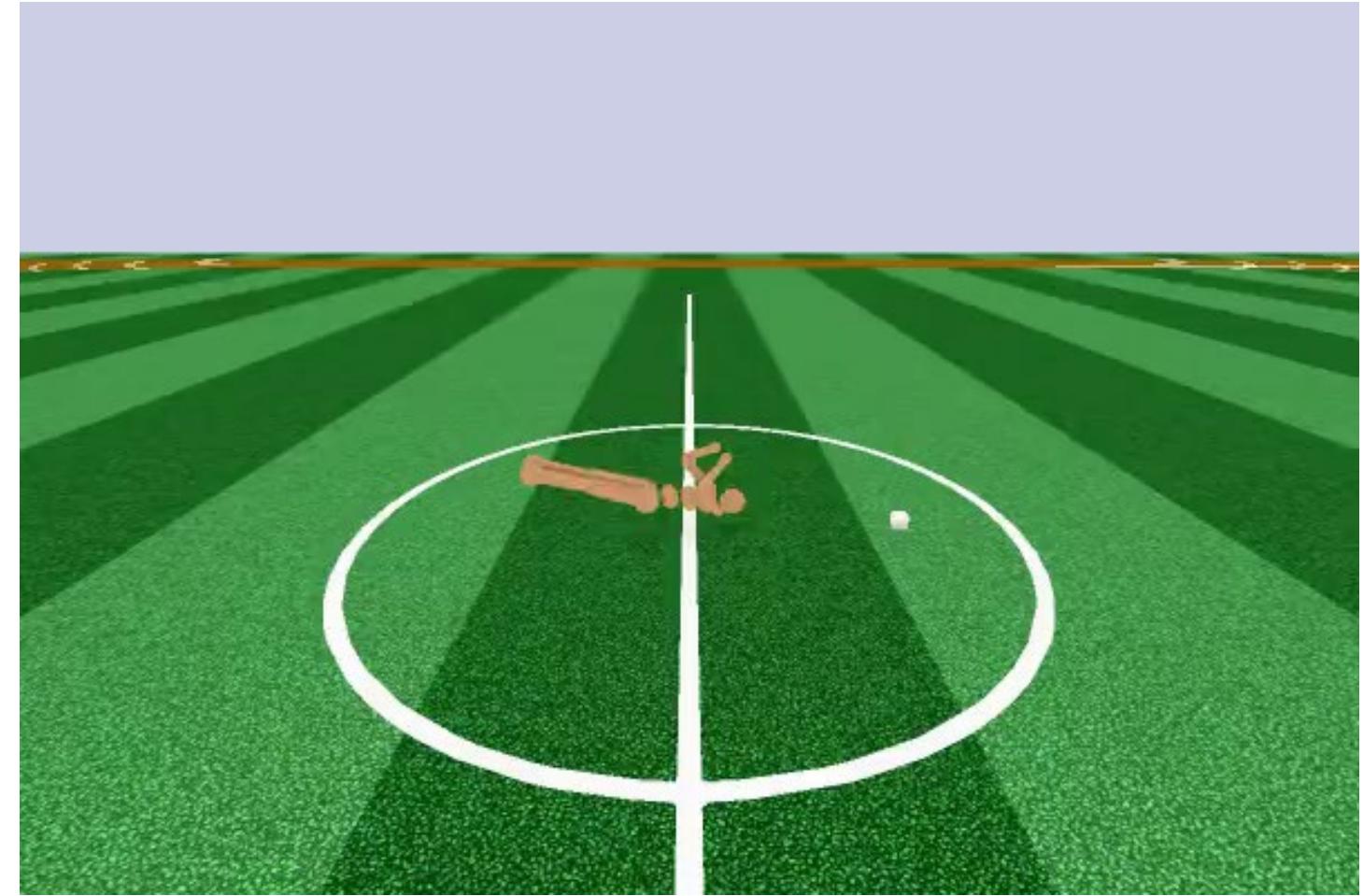
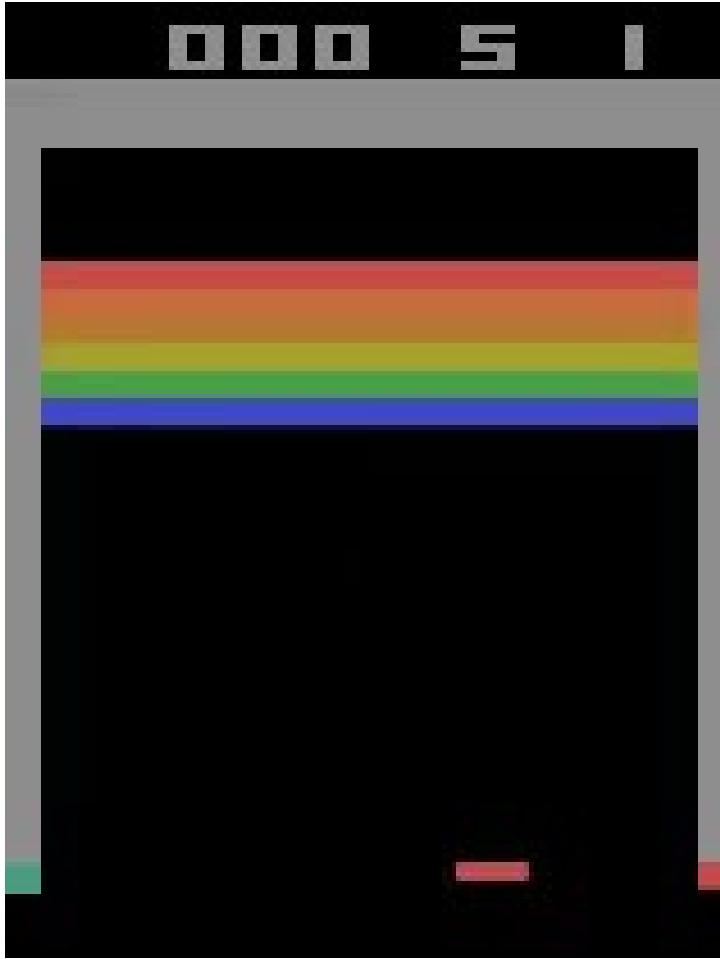




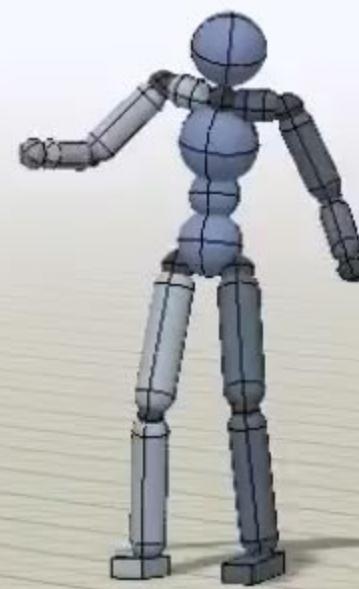
Game start



When enough episodes collected, use all the $[o, a, o', r]$ to compute the gradient



DeepMimic: Example-Guided Deep Reinforcement Learning of Physics-Based Character Skills

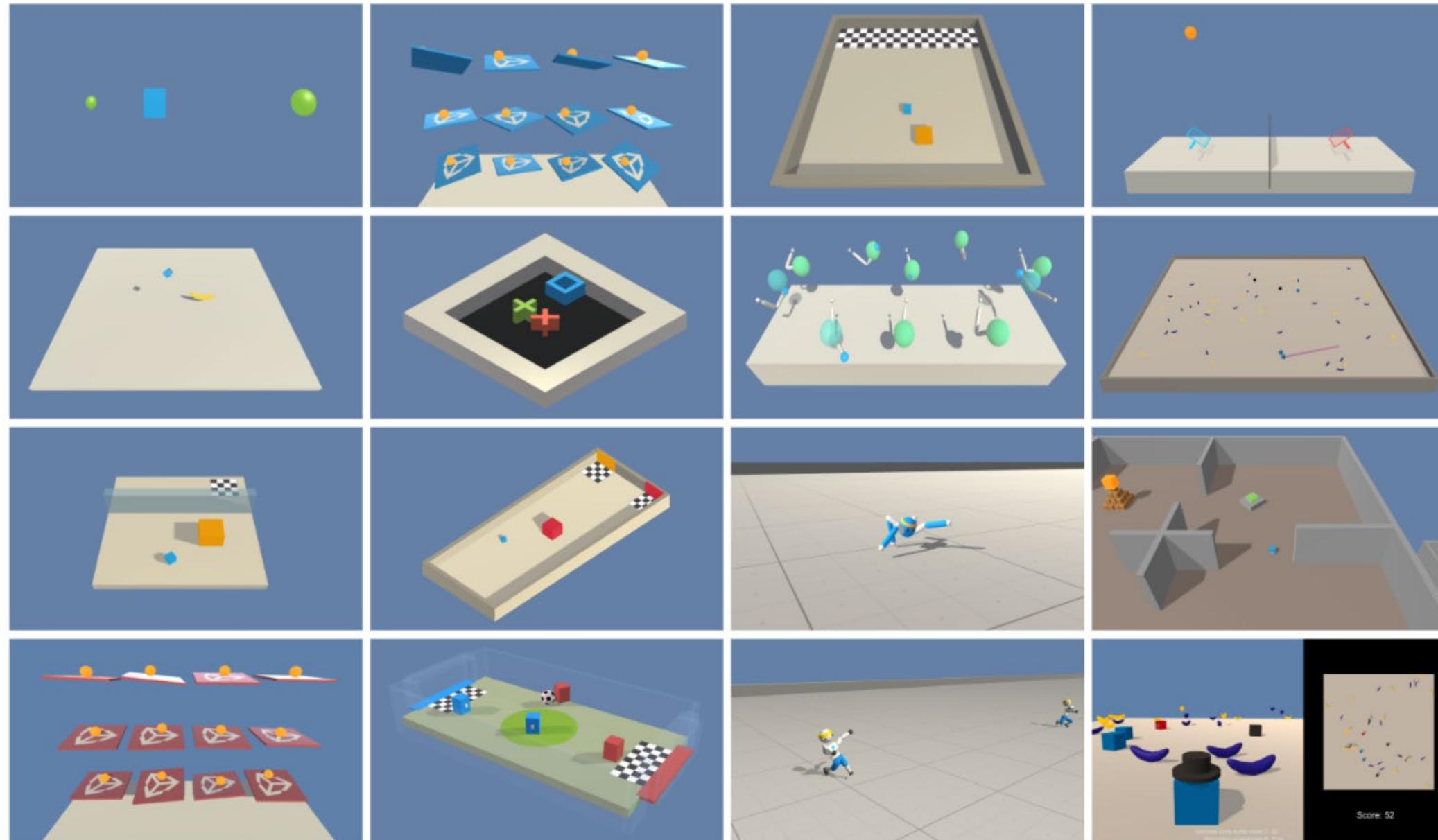


Xue Bin Peng¹, Pieter Abbeel¹, Sergey Levine¹, Michiel van de Panne²

¹ University of California
Berkeley

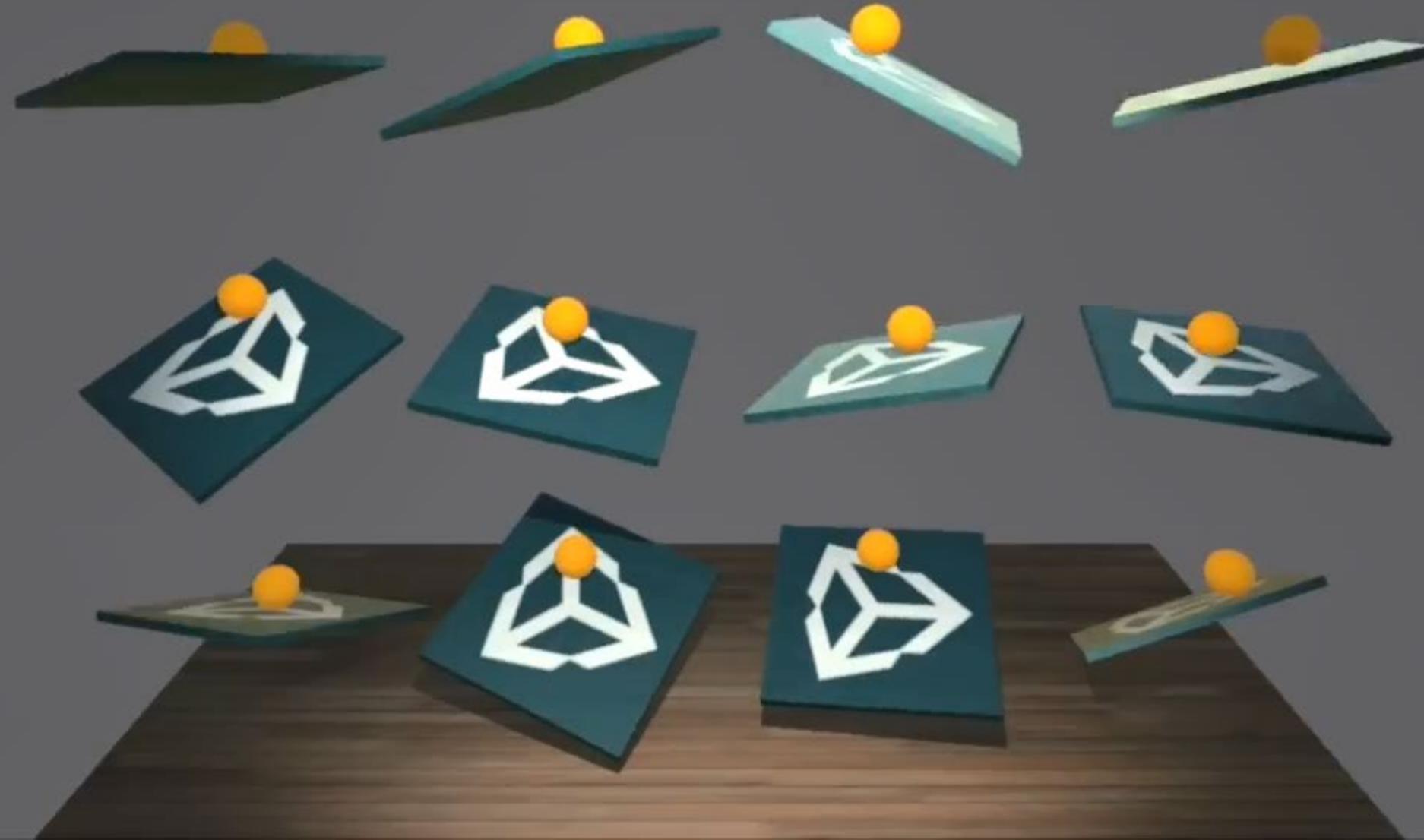


² University of British Columbia
The logo of the University of British Columbia, consisting of three wavy lines above the letters "UBC" and a stylized sunburst below it.



Unity ML-Agents Toolkit (Beta)

Unity Machine Learning Agents, PPO





<https://openai.com/blog/emergent-tool-use/>

SEPTEMBER 17, 2019 • 9 MINUTE READ

Emergent Tool Use from Multi-Agent Interaction

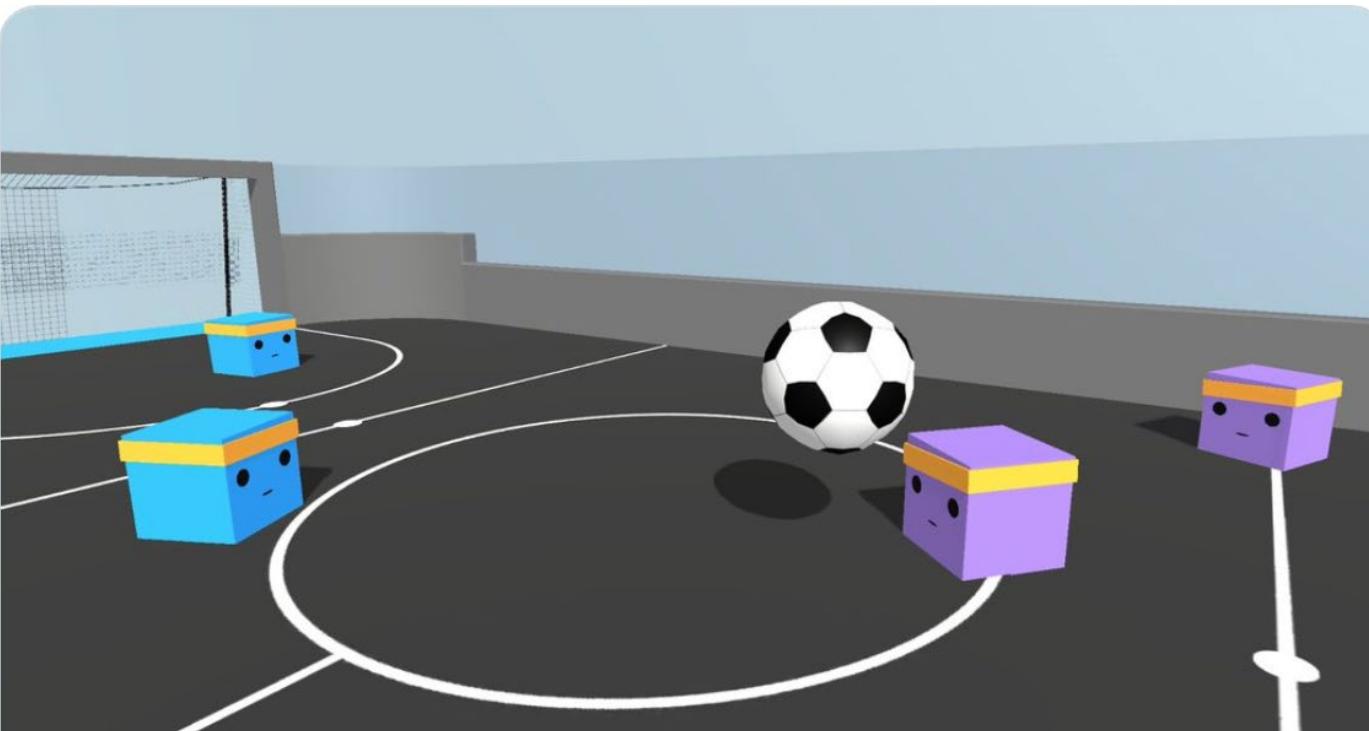
We've observed agents discovering progressively more complex tool use while playing a simple game of hide-and-seek. Through training in our new simulated hide-and-seek environment, agents build a series of six distinct strategies and counterstrategies, some of which we did not know our environment supported. The self-supervised emergent complexity in this



Unity ✅ @unity3d · Feb 28

Score! New with ML-Agents v0.14: self-play and the ability to train competitive agents in adversarial games! #mlagents

Play around with it in our soccer demo environment. ⚽



Training intelligent adversaries using self-play with ML-Agents - Unity T...

In the latest release of the ML-Agents Toolkit (v0.14), we have added a self-play feature that provides the capability to train competitive agents...

🔗 [blogs.unity3d.com](https://blogs.unity3d.com/2020/02/28/training-intelligent-adversaries-using-self-play-with-ml-agents/)

<https://blogs.unity3d.com/2020/02/28/training-intelligent-adversaries-using-self-play-with-ml-agents/>

Some methods to try

- PPO (Schulman et al. 2017). Works for both continuous actions (e.g., robot joint torques) and discrete actions (e.g., gamepad button presses)
- Soft Actor Critic (Haarnoja et al. 2018): Only for continuous actions.

Both PPO and SAC are available in Unity ML Agents.

SAC can be more efficient, but may require more fine-tuning of parameters.

In Python: OpenAI Gym

```
import gym
env = gym.make("CartPole-v1")
observation = env.reset()
for _ in range(1000):
    env.render()
    action = env.action_space.sample() # your agent here (this takes random actions)
    observation, reward, done, info = env.step(action)

    if done:
        observation = env.reset()
env.close()
```

Tonic: A Deep Reinforcement Learning Library for Fast Prototyping and Benchmarking

Fabio Pardo
Imperial College London
`f.pardo@imperial.ac.uk`

Abstract

Deep reinforcement learning has been one of the fastest growing fields of machine learning over the past years and numerous libraries have been open sourced to support research. However, most codebases have a steep learning curve or limited flexibility that do not satisfy a need for fast prototyping in fundamental research. This paper introduces Tonic, a Python library allowing researchers to quickly implement new ideas and measure their importance by providing: 1) a collection of configurable modules such as exploration strategies, replays, neural networks, and updaters 2) a collection of baseline agents: A2C, TRPO, PPO, MPO, DDPG, D4PG, TD3 and SAC built with these modules 3) support for the two most popular deep learning frameworks: TensorFlow 2 and PyTorch 4) support for the three most popular sets of continuous-control environments: OpenAI Gym, DeepMind Control Suite and PyBullet 5) a large-scale benchmark of the baseline agents on 70 continuous-control tasks 6) scripts to experiment in a reproducible way, plot results, and play with trained agents.

<https://github.com/fabiopardo/tonic>

Markov Decision Process (MDP)

- A term very often encountered if you read RL papers
- A formal way to define a behavior optimization problem
- Defined by: The states s , actions a , a reward function $r(s,a)$, and the discount factor γ
- (Deep) Reinforcement Learning is a tool that one can apply once a problem has been formulated as an MDP => one only needs to define/implement the s,a,r,γ
- If the full state is not observed, the formulation is a Partially Observable Markov Decision Process (PO-MDP), where s is replaced by o .
- In PO-MDP, the agent typically needs to observe a sequence of multiple timeteps to be able to infer the unobserved state and act accordingly.



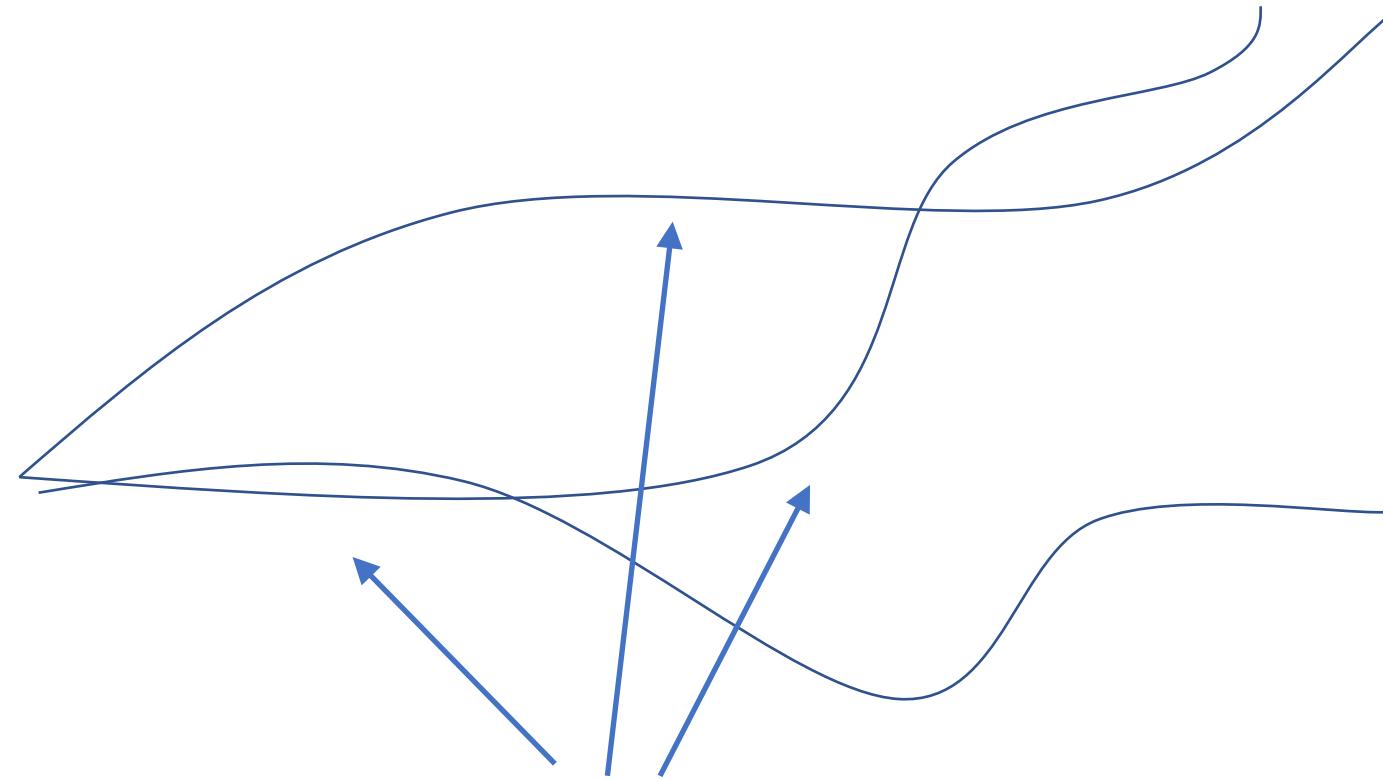
Game start

Backpropagate episode returns R

$$R(a,s) = r(a,s) + \gamma R(a,s')$$



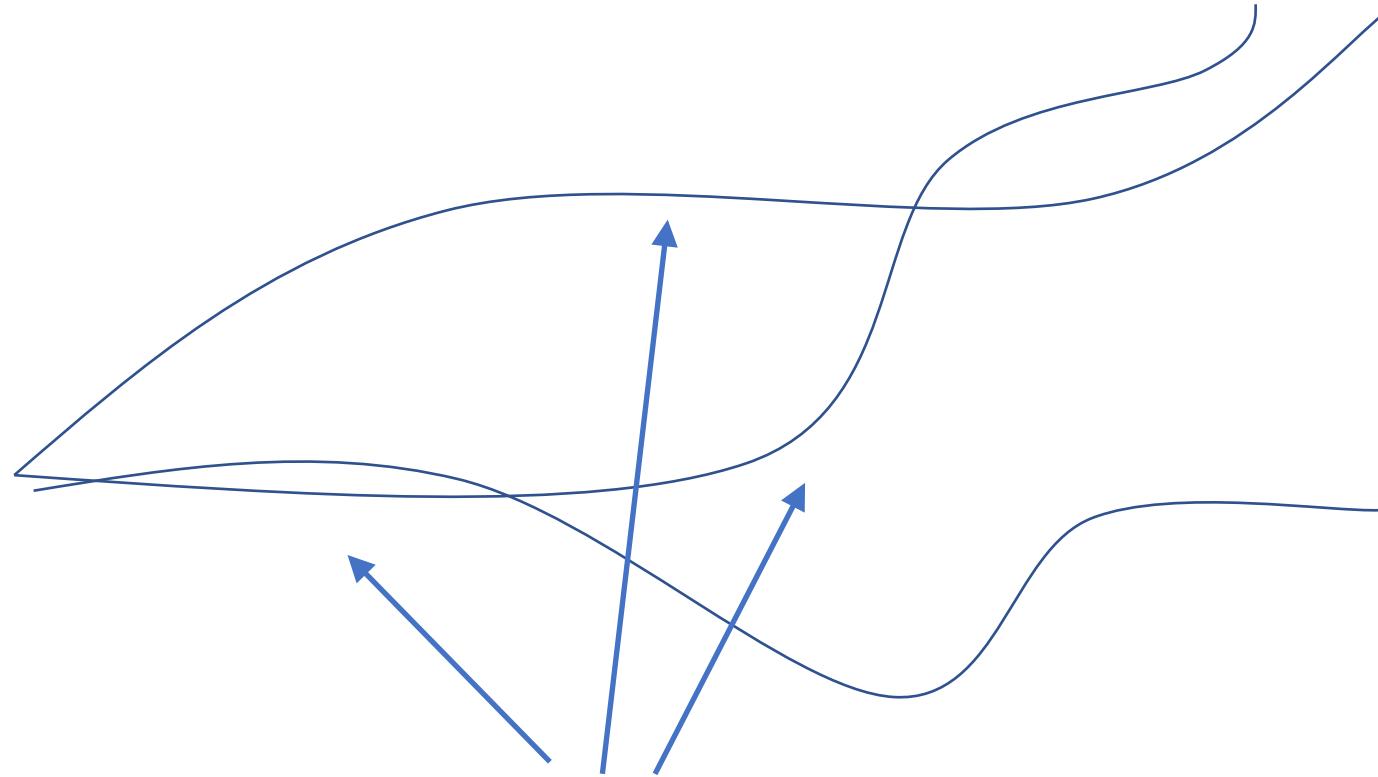
Game start



Train a value function $V(s)=E[R(s)]$ predictor network with the returns for each s in the episode trajectories.



Game start



Compute advantages as $A(a,s)=Q(a,s)-V(s)$,
 $Q(a,s)=r(a,s)+\gamma V(s')$ (Bellman recursion)

Training loss

$$\mathcal{L} = -\frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \log \pi_\theta(\mathbf{a}_i | \mathbf{s}_i)$$

Training loss

$$\mathcal{L} = -\frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \log \pi_\theta(\mathbf{a}_i | \mathbf{s}_i)$$

Training loss

Average over all samples in minibatch

$$\mathcal{L} = -\frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \log \pi_\theta(\mathbf{a}_i | \mathbf{s}_i)$$

Training loss

Minimizing the loss = maximizing the average

$$\mathcal{L} = \boxed{-\frac{1}{M}} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \log \pi_\theta(\mathbf{a}_i | \mathbf{s}_i)$$

Training loss

Probability of action \mathbf{a}_i
in state \mathbf{s}_i

$$\mathcal{L} = -\frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \log \boxed{\pi_\theta(\mathbf{a}_i | \mathbf{s}_i)}$$

Training loss

Parameters of the
policy neural network, i.e.,
the optimized variables

$$\mathcal{L} = -\frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \log \tau[\theta](\mathbf{a}_i | \mathbf{s}_i)$$

Training loss

The advantage values, treated as constants
(computed based on the iteration's experience data)

$$\mathcal{L} = -\frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \log \pi_\theta(\mathbf{a}_i | \mathbf{s}_i)$$

Training loss

Maximize this = maximize the probability
of actions with positive advantages

$$\mathcal{L} = -\frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \log \pi_\theta(\mathbf{a}_i | \mathbf{s}_i)$$

Gaussian policy with identity covariance matrix

$$\mathcal{L} = \frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \|\mathbf{a}_i - \mu_\theta(\mathbf{s})\|^2$$

Training with advantage-weighted samples

The sampling mean output
by the policy network.

$$\mathcal{L} = \frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \|\mathbf{a}_i - \boxed{\mu_\theta(\mathbf{s})}\|^2$$

Training with advantage-weighted samples

Neural network parameters
that we optimize

$$\mathcal{L} = \frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \|\mathbf{a}_i - \mu[\theta](\mathbf{s})\|^2$$

Training with advantage-weighted samples

Squared distance between
the policy and actions

$$\mathcal{L} = \frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \boxed{\|\mathbf{a}_i - \mu_\theta(\mathbf{s})\|^2}$$

Training with advantage-weighted samples

Advantages as weights:
minimize the distance to
positive-advantage actions

$$\mathcal{L} = \frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \|\mathbf{a}_i - \mu_\theta(\mathbf{s})\|^2$$

Updating both mean and variance

$$\mathcal{L} = \frac{1}{M} \sum_{i=1}^M A^\pi(\mathbf{s}_i, \mathbf{a}_i) \sum_j \left[\frac{(a_{i,j} - \mu_{j;\theta}(\mathbf{s}_i))^2}{c_{j;\theta}(\mathbf{s}_i)} + 0.5 \log c_{j;\theta}(\mathbf{s}_i) \right]$$



How to use DRL

- Implement a game or simulation, wrapped inside the OpenAI Gym Env interface in Python, or Unity ML's agent interface in C#
- Define the MDP or PO-MDP
 - Define how simulation state is mapped to a fixed length observation vector
 - Define how the episode initial state is sampled (in OpenAI gym, this happens when the DRL algorithm calls `env.reset()`)
 - Define what causes the episode to end (typically, timeout or failure/termination)
 - Define a mapping from a fixed length action vector into simulation actions, e.g., physics simulation torques or simulated gamepad button presses
 - Define the reward function
- Optional: define a curriculum, i.e., start training with some easier problem, then modify it as training progresses (pedagogy can matter even more than when training humans...)
- Test training with some DRL algorithm

Typical problems: Normalization

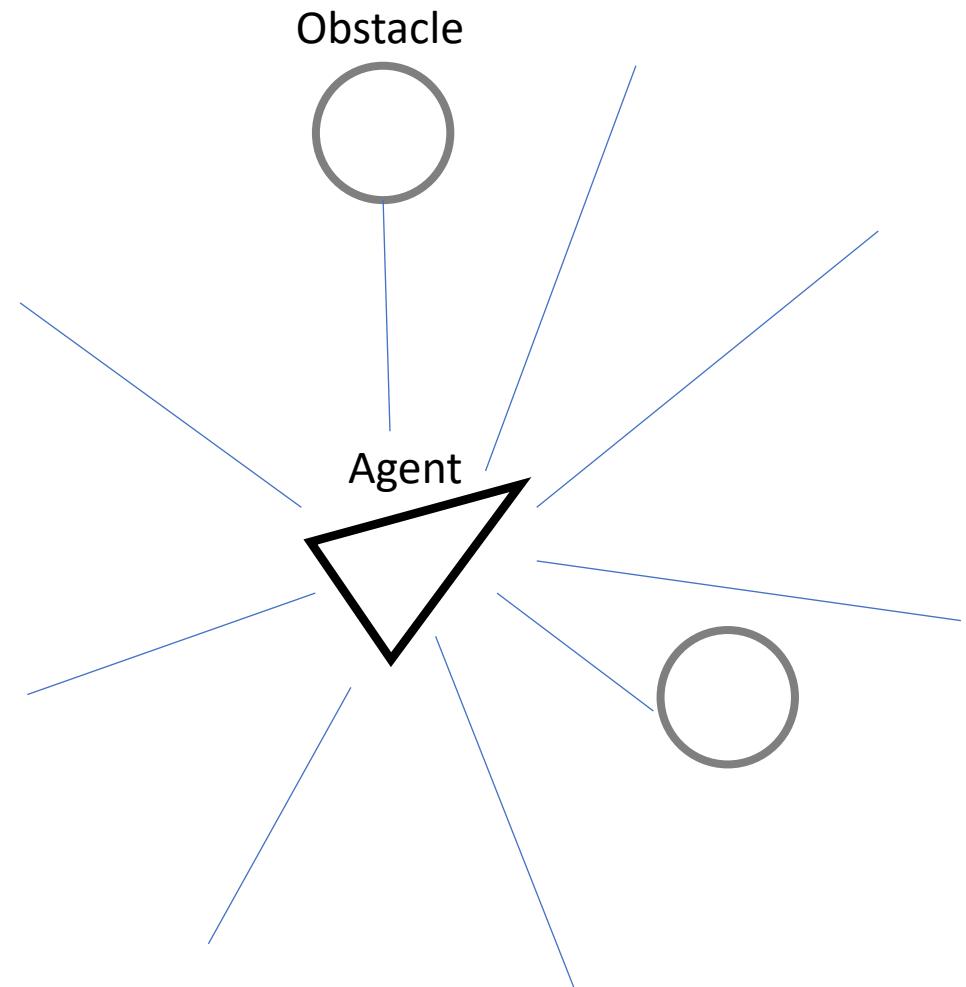
- Make sure that the reward function is in range 0...1 or close to it
- Make sure the observations are in range -1...1 or have mean 0 and standard deviation 1
- Reason: Neural network optimization is highly sensitive to initialization. The DRL networks are typically initialized assuming this kind of normalized rewards and observations, and the algorithms may fail otherwise.

Typical problems: Incomplete observations

- The next state/observation and the reward should be predictable (disregarding any sensor or dynamics noise) based on only the current observation. Otherwise, the problem is not an MDP and learning will be slow or impossible
- For instance, if training a spaceship AI, imagine yourself in the cockpit with no windows and all you can see is the observation vector. Is it enough for you to act optimally?

Typical problems: Incomplete observations

- Example: Raycast sensing in a game can be more efficient than training the agent simply based on the rendered game pixels
- Important: The observation vector should contain the hit distances for each ray, and possibly the velocities of the hit objects
- Simply observing which rays hit an obstacle is not enough for deciding how to avoid them



Typical problems: Early termination and negative rewards

- Episode termination when the agent fails (e.g., walking agent falls down) can reduce gradient noise and speed up training, compared to running every episode until a timeout
- However, if the agent can receive negative rewards, it may optimize by failing as fast as possible to avoid the negative rewards
- Solution: Make sure your rewards are positive. Print out the minimum reward encountered during training. Add a constant “alive bonus” to all rewards if the minimum is negative.

Typical problems: Zero or noisy gradient

- DRL can only learn if the random exploration manages to discover good actions (i.e., high rewards). Otherwise, there's no gradient to follow.
- Debug: Watch what happens at the beginning of the training. Are any of the random actions meaningful?
- Solution 1: Increase the amount of collected experience per algorithm iteration (more likely that at least some actions are good)
- Solution 2: Make sure your reward function is informative and not just zero most of the time. If possible, reduce gamma to reduce the noise in summing together the future rewards
- Solution 3: Design your actions so that they are more likely to produce rewards. For example, the policy could output navigation waypoints driven towards using some other controller, instead of the policy outputting raw steering actions
- Solution 4: Use a curiosity bonus that encourages exploration even with zero rewards (implemented in Unity ML, easy to toggle)

Raivo



YOU'RE THE TRAINER

YOUR ROBOT IS THE FIGHTER



THE WORLD IS YOUR ARENA

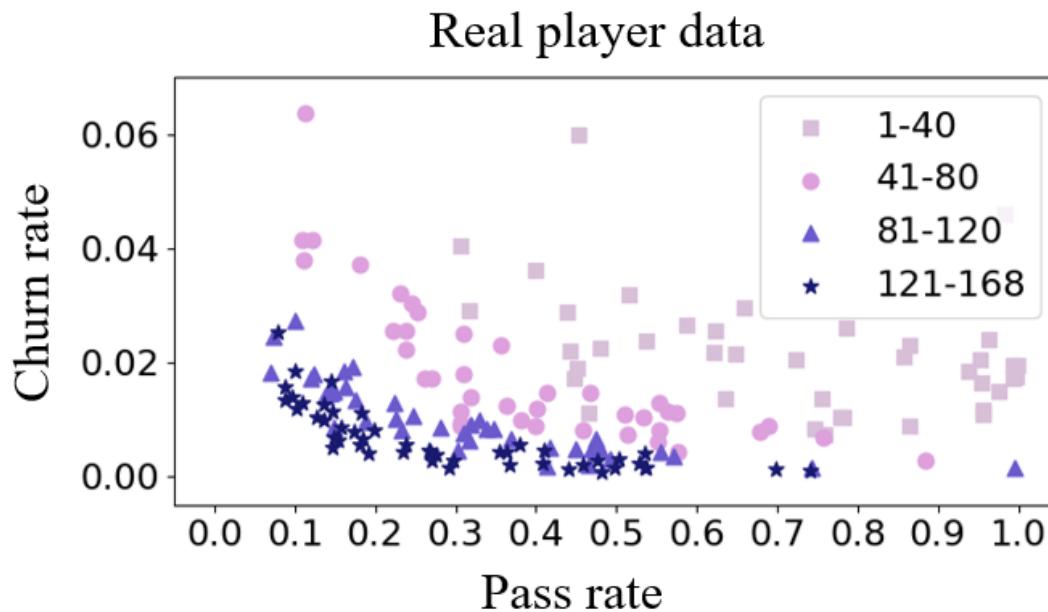


WELCOME TO RAIVO
MACHINE LEARNING GAMIFIED

<https://www.utoposgames.com/raivo>

Predicting player behavior and experience using DRL agents

- An Aalto-Rovio collaboration:
https://twitter.com/perttu_h/status/1302536834002628608?s=20
- We use DRL agents to predict pass rate (i.e., level difficulty, a crucial part of game experience) and churn rate (behavioral metric, how many players quit at each level)





Forward search methods

Selected mode: Getting up



Online Motion Synthesis using Sequential Monte Carlo



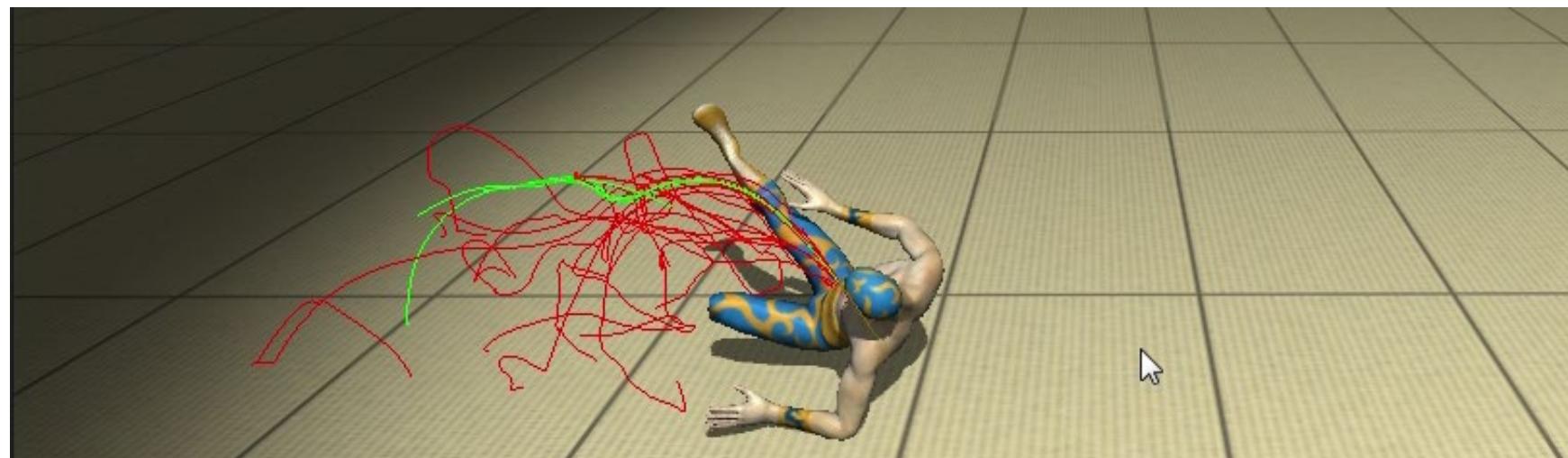
SIGGRAPH 2014

Live demo



Controlling an agent through forward search

1. Simulate multiple action strategies up to a planning horizon or termination. Run simulations in parallel threads if possible
2. Step the “master simulation” forward using the best strategy
3. Repeat from the resulting state, possibly informing the forward simulations based on previous results. Planning horizon is shifted one time step forward (rolling horizon, receding horizon)



Deep RL vs. Forward Search

Deep RL:

- + Once trained, very computationally efficient
- Training can be very slow and may not converge

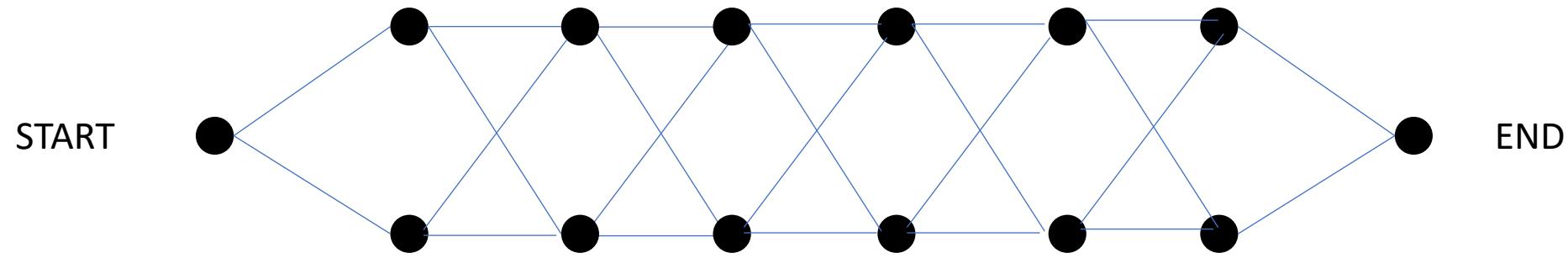
Search methods:

- + Get good results fast, without training neural networks
- Requires forward simulation (many times faster than real-time simulation)
- Requires capability to save and load simulation state

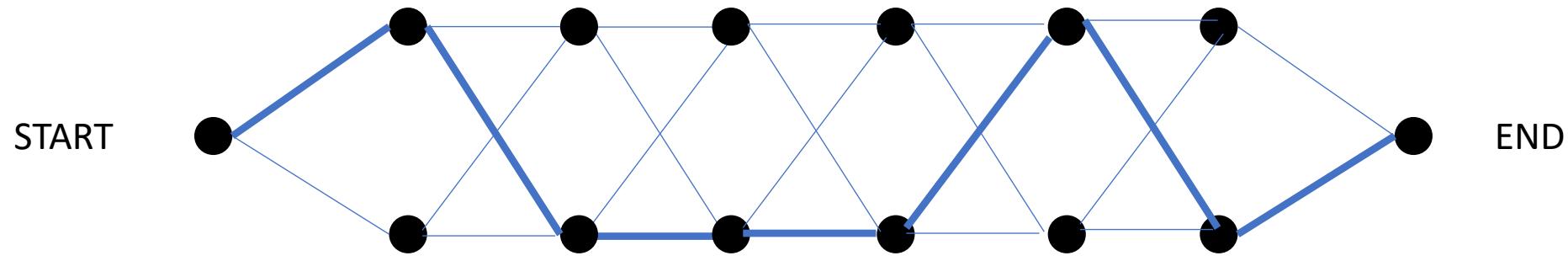
Combining search & neural networks can allow adjusting the tradeoffs.

Why is searching for action sequences hard?

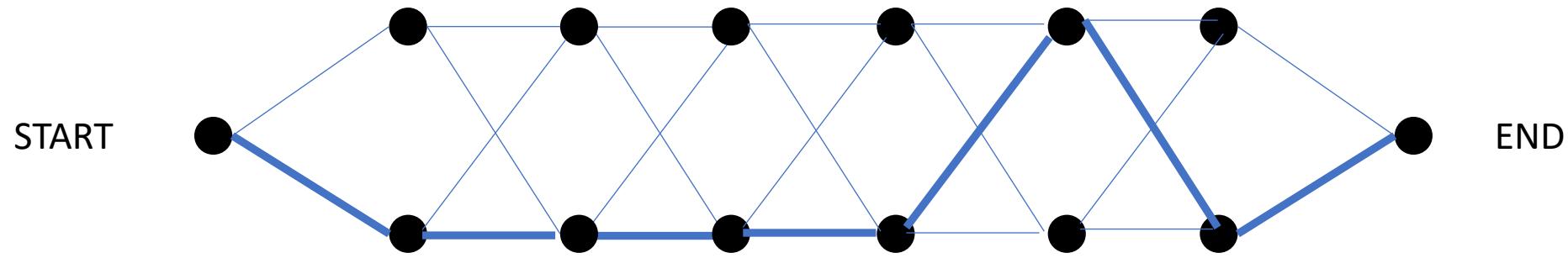
Curse of dimensionality



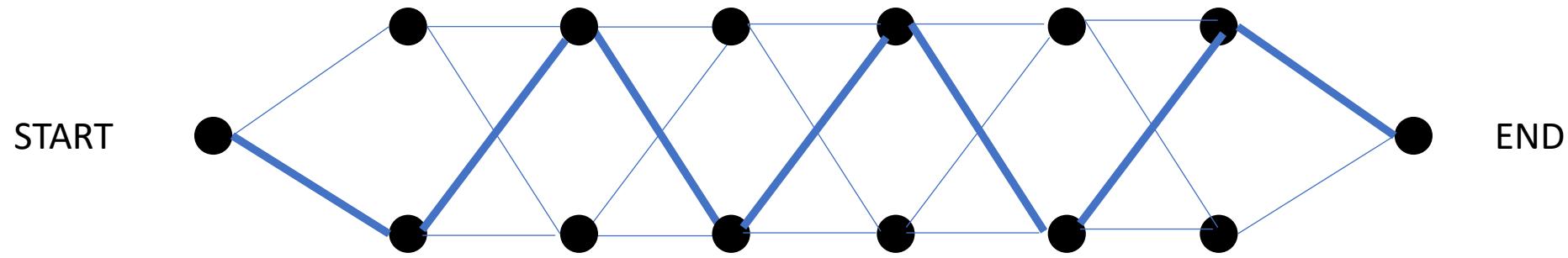
Curse of dimensionality



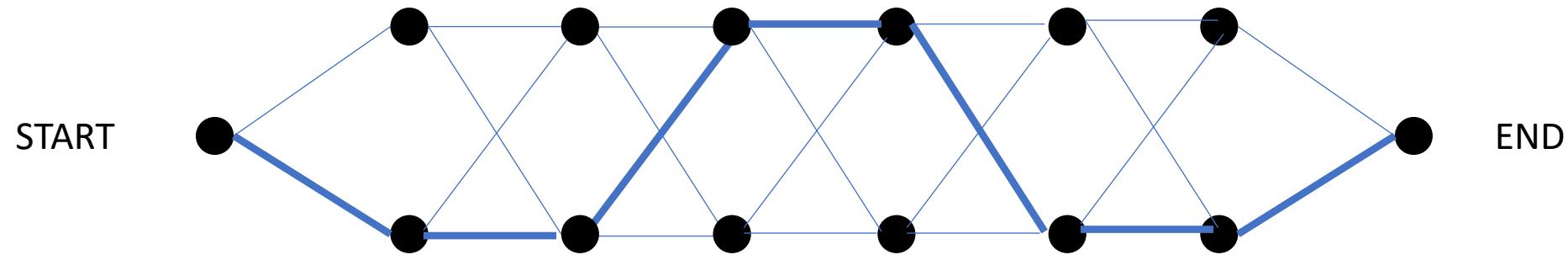
Curse of dimensionality



Curse of dimensionality



Curse of dimensionality



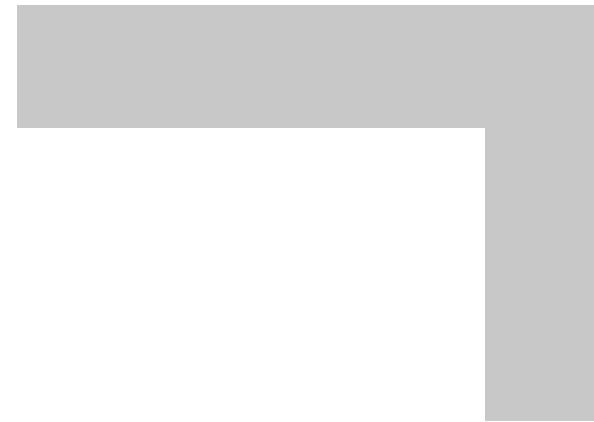


Dynamic programming: Dijkstra's method





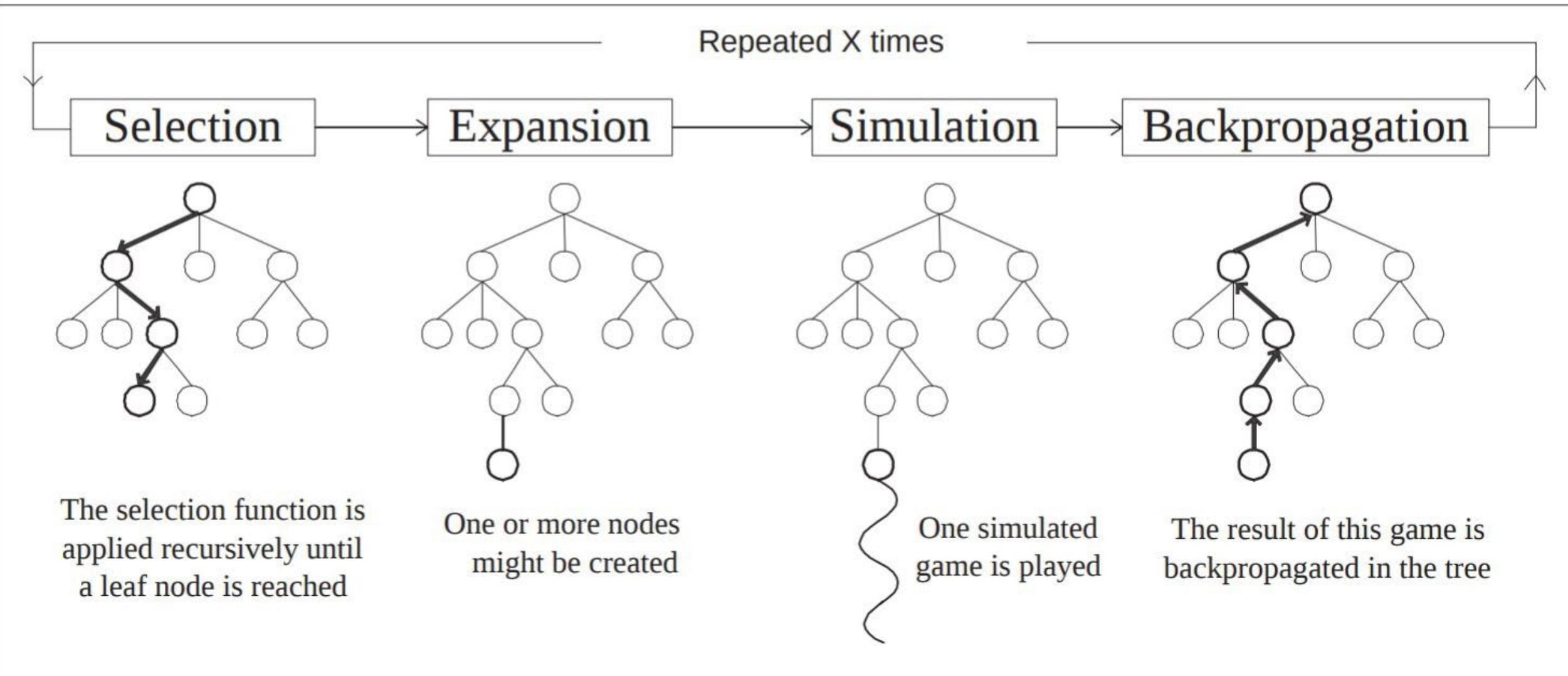
From Dijkstra to A*



Why is it still hard?

- In 2d or 3d, this is trivial
- Unity's NavMesh implements pathfinding, no need to code yourself
- However, the search process becomes very slow with dozens or hundreds of possible actions
- A* heuristics hard to design for problems beyond pathfinding

Monte Carlo Tree Search to the rescue



Node selection: Maximize the Upper Confidence Bound

$$\bar{X}_j + 2C_p \sqrt{\frac{2 \ln n}{n_j}}$$

Node selection: Maximize the Upper Confidence Bound

$$\boxed{\bar{X}_j} + 2C_p \sqrt{\frac{2 \ln n}{n_j}}$$

Utility of node j , i.e., mean of subtree rewards

Node selection: Maximize the Upper Confidence Bound

$$\bar{X}_j + 2C_p \sqrt{\frac{2 \ln n}{n_j}}$$

How many times this node has been selected

Node selection: Maximize the Upper Confidence Bound

$$\bar{X}_j + 2C_p \sqrt{\frac{2 \ln n}{n_j}}$$

How many times the parent node has been visited

Node selection: Maximize the Upper Confidence Bound

$$\bar{X}_j + 2C_p \sqrt{\frac{2 \ln n}{n_j}}$$

Tuning parameter, adjusts the balance between *exploration* and *exploitation*

Node selection: Maximize the Upper Confidence Bound

$$\boxed{\bar{X}_j} + 2C_p \sqrt{\frac{2 \ln n}{n_j}}$$

Exploitation:
maximize utility
(e.g., average number of wins)

Node selection: Maximize the Upper Confidence Bound

$$\bar{X}_j + \boxed{2C_p \sqrt{\frac{2 \ln n}{n_j}}}$$

Exploration:
High for nodes with a small number of visits n_j

[~ Mario AI Benchmark ~ 0.1.9]



DIFFICULTY: 99

SEED: 123456

TYPE: Overground (0)

LENGTH: 2 OF 320

HEIGHT: 2 OF 15

OBSTACLES: 0 OF 0

Agent: MCTS Agent

PRESSED KEYS:

ALL KILLS: 9

by Fire

by Shell

by Stomp

TIME:

399

FPS:

42

R>>



Intermediate reward: -1

A general reinforcement learning algorithm that masters chess, shogi, and Go through self-play

David Silver^{1,2,*†}, Thomas Hubert^{1,*}, Julian Schrittwieser^{1,*}, Ioannis Antonoglou¹, Matthew Lai¹, Arthur Guez¹, Marc Lancto...

* See all authors and affiliations

Science 07 Dec 2018;
Vol. 362, Issue 6419, pp. 1140-1144
DOI: 10.1126/science.aar6404

Article

Figures & Data

Info & Metrics

eLetters

PDF

One program to rule them all

Computers can beat humans at increasingly complex games, including chess and Go. However, these programs are typically constructed for a particular game, exploiting its properties, such as the symmetries of the board on which it is played. Silver et al. developed a program called AlphaZero, which taught itself to play Go, chess, and shogi (a Japanese version of chess) (see the Editorial, and the Perspective by Campbell). AlphaZero managed to beat state-of-the-art programs specializing in these three games. The ability of AlphaZero to adapt to various game rules is a notable step toward achieving a general game-playing system.

Science, this issue p. 1140; see also pp. 1087 and 1118

Science

Vol 362, Issue 6419
07 December 2018

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A Survey of Monte Carlo Tree Search Methods

Cameron Browne, *Member, IEEE*, Edward Powley, *Member, IEEE*, Daniel Whitehouse, *Member, IEEE*,
Simon Lucas, *Senior Member, IEEE*, Peter I. Cowling, *Member, IEEE*, Philipp Rohlfshagen,
Stephen Tavener, Diego Perez, Spyridon Samothrakis and Simon Colton

Abstract—Monte Carlo Tree Search (MCTS) is a recently proposed search method that combines the precision of tree search with the generality of random sampling. It has received considerable interest due to its spectacular success in the difficult problem of computer Go, but has also proved beneficial in a range of other domains. This paper is a survey of the literature to date, intended to provide a snapshot of the state of the art after the first five years of MCTS research. We outline the core algorithm's derivation, impart some structure on the many variations and enhancements that have been proposed, and summarise the results from the key game and non-game domains to which MCTS methods have been applied. A number of open research questions indicate that the field is ripe for future work.

Index Terms—Monte Carlo Tree Search (MCTS), Upper Confidence Bounds (UCB), Upper Confidence Bounds for Trees (UCT), Bandit-based methods, Artificial Intelligence (AI), Game search, Computer Go.



1 INTRODUCTION

MONTE Carlo Tree Search (MCTS) is a method for finding optimal decisions in a given domain by taking random samples in the decision space and building a search tree according to the results. It has already had a profound impact on Artificial Intelligence (AI) approaches for domains that can be represented as trees of sequential decisions, particularly games and planning problems.

In the five years since MCTS was first described, it has become the focus of much AI research. Spurred on by some prolific achievements in the challenging task of computer Go, researchers are now in the pro-

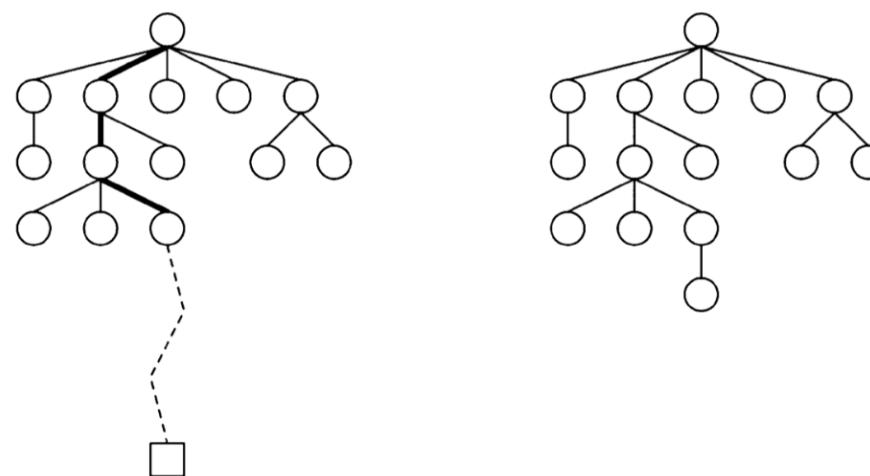


Fig. 1. The basic MCTS process [17].

Monte Carlo Tree Search resources

The best visualizations I've found:

<https://int8.io/monte-carlo-tree-search-beginners-guide/>

Game AI Book section on MCTS (page 45): <http://gameaibook.org/book.pdf>

Colab notebook: MCTS for OpenAI Gym environments

https://colab.research.google.com/github/yandexdataschool/Practical_RL/blob/spring19/week10_planning/seminar_MCTS.ipynb

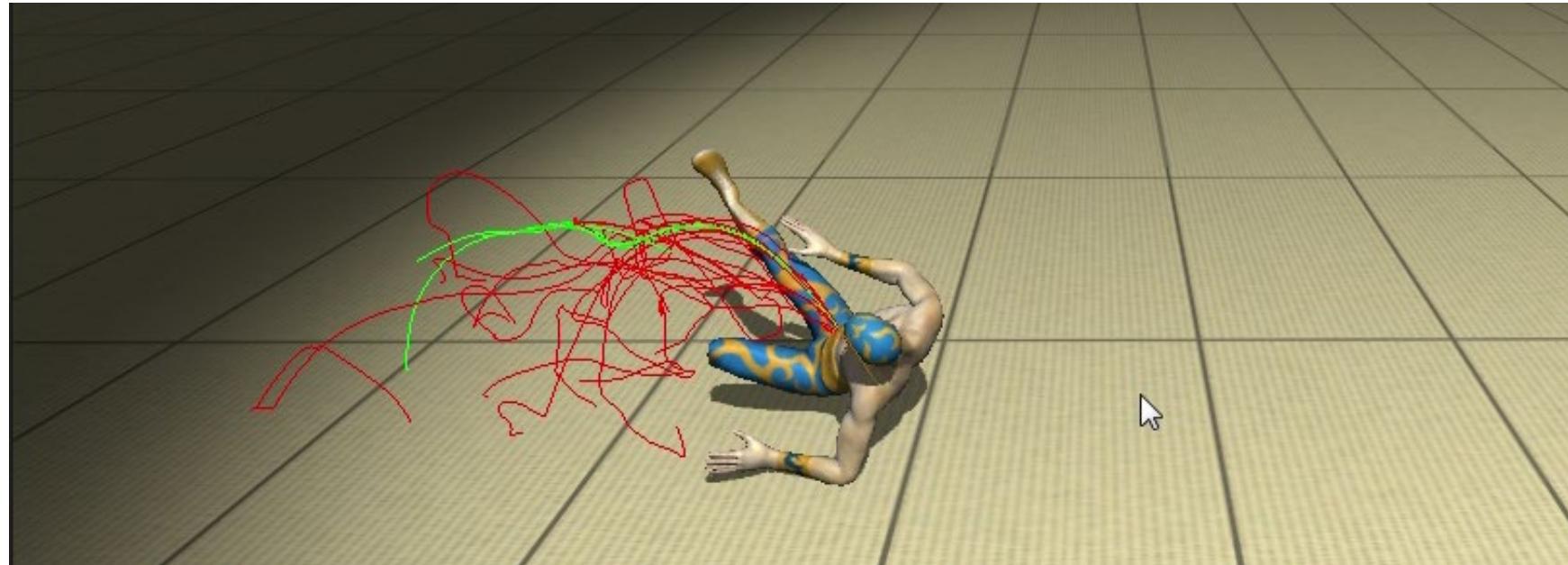
MCTS as a Deep Reinforcement Learning alternative

- Both find solutions to a Markov Decision Process defined by states, actions, and rewards
- MCTS benefit: usually gets results faster, but requires the ability to save and restore simulation state
 - IMPORTANT: You can quickly iterate on the MDP design, e.g., what kinds of rewards to use, without always waiting for a day for RL training
- MCTS drawback: requires more CPU than simply querying a policy network for an action based on the current observation.
- Recommended use: If you can save & restore state and have discrete actions, start with MCTS. Switch to RL when you're sure that your reward function produces good results.

What about continuous actions?

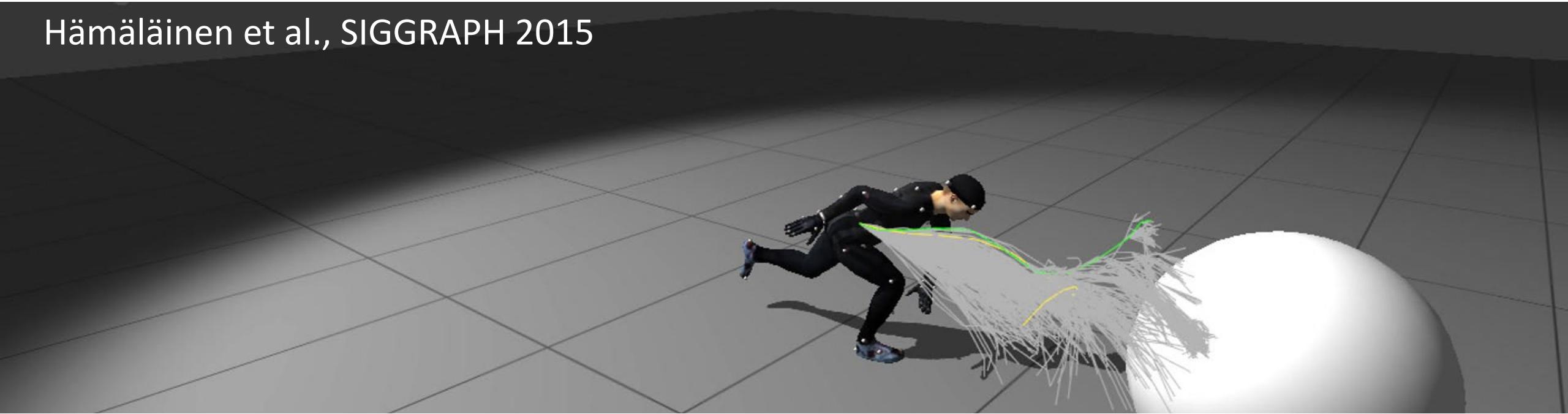
Our SIGGRAPH 2014 work: no tree search

- Simply run a number of rollouts from the current state
- Search space reduced by encoding action sequences as pose splines
- Take action following the best rollout
- The distribution of rollout actions is adapted



Control Particle Belief Propagation (C-PBP)

Hämäläinen et al., SIGGRAPH 2015



$$\mathcal{P}(\mathbf{z}) \propto \left(\prod_k \psi_k(\mathbf{z}_k) \right) \left(\prod_{k=1}^K \Psi_{\text{fwd}}(\mathbf{z}_{k-1}, \mathbf{z}_k) \right) \left(\prod_{k=0}^{K-1} \Psi_{\text{bwd}}(\mathbf{z}_{k+1}, \mathbf{z}_k) \right)$$

FDI-MCTS (Rajamäki & Hämäläinen, 2017)

- Simple MCTS variant for fixed-horizon continuous control.
Simplification of C-PBP, no cumbersome math.
- Combines deep neural networks and tree search
- Benefit: Very fast convergence, humanoid walking in less than a minute on a single computer
- Drawback: Requires forward simulation and the resulting policy not always as robust as, e.g., with PPO

Simulation time 00:00: 0 (frame 0), total computing time 0.08 s

Total training simulation steps: 2304

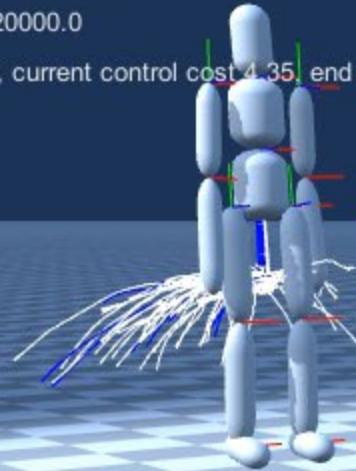
Number of trajectories: 64

Controller update: 84 ms,

Planning horizon: 1.2 seconds

Pruning threshold 20000.0

state cost 1085.57, current control cost 4.35, end state cost 334.36



FDI-MCTS algorithm overview

1. Simulate a number of trajectories forward
 - Trajectories utilize different types of information
 - One trajectory repeats the best trajectory of the previous frame
 - Some trajectories: neural network policy + noise
 - Some trajectories: find nearest memorized states, use their stored actions + noise
 - Some trajectories: random actions
2. During the forward simulation, terminate worst trajectories, and reassign their simulation resources by forking the best trajectories
3. After forward simulation, step the master simulation forward with the first action of the best trajectory.
4. Memorize the state and action taken
5. In a background thread, constantly retrain the neural network policy with the memorized states and actions. Also update the nearest neighbor search data structure (a decision forest)

Recent continuous control MCTS

- Moerland, Thomas M., et al. "AOC: Alpha zero in continuous action space." *arXiv preprint arXiv:1805.09613* (2018).

Predictive Physics Simulation in Game Mechanics

ACM CHI PLAY 2017

Perttu Hämäläinen (Aalto University)

Xiaoxiao Ma (Aalto University)

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Optimizing action sequences: Summary

Deep RL:

- + Produces a policy network that can compute actions very efficiently
- Training can be very slow

Forward search methods like MCTS:

- + Find good actions fast, without training or optimizing neural networks
- Requires forward simulation (many times faster than real-time simulation)
- Requires capability to save and load game or simulation state
- Seems to be hard to implement in a clean, generic, and easy-to-use manner in Unity. Challenge to students with Unity experience: Can you figure out a way?

Recap: Optimization theory

- Optimization: Hill climbing/descent on the optimization landscape defined by $f(\mathbf{x})$
- Gradient –based optimization: Follow the gradient (direction of steepest ascent/descend on the landscape), with momentum to avoid getting stuck at the bottom of valleys or top of ridges (e.g., Adam). Tensorflow and Pytorch allow computing gradients of any computation written using them.
- Sampling –based optimization: The $f(\mathbf{x})$ doesn't need to be directly differentiable to estimate gradients based on samples. A Gaussian sampling distribution = Gaussian-blurred $f(\mathbf{x})$ that can smooth away local optima.
Practical methods: CMA-ES, PPO, SAC.
- Discrete optimization: Sampling-based optimization still works.
- Action sequences: DRL, MCTS. What's needed: formulate your problem as a (PO-)MDP, i.e., define the actions, observations, rewards, and initial state distribution

Recap: Optimization applications

- Machine learning: Optimize the weights and biases of a neural network to minimize some error metric
- Image synthesis: Optimize a drawing to fool a neural network thinking it's some object
- Animation: Optimize a simulated humanoid's muscle activations to make it achieve a goal like walking or running
- Game AI, artificial life: Optimize the actions of agents to make them achieve goals like performing billiards trick shot, winning a game of Go, or the hiders avoiding the seekers in a game of hide-and-seek.
- Game content: Optimize procedural generation parameters based on some model of player behavior or experience
- Sometimes, the optimization or AI training can be made a game, challenging the player to design a learning environment

Appendix: Extra material

Neuroevolution

- Apply an evolutionary optimization method like CMA-ES directly to policy network parameters θ (neural network weights & biases)
- Some algorithms like NEAT optimize both network parameters and network architecture
- Optimization objective $f(\theta) = \text{sum of rewards over episode}$

One iteration:

1. Sample parameters $\theta_1, \dots, \theta_N$ for N neural networks
2. Run 1 or more episode with each network, compute $f(\theta)$
3. Update the sampling distribution



Geijtenbeek 2013 (CMA-ES neuroevolution)

Flexible Muscle-Based Locomotion for Bipedal Creatures

SIGGRAPH ASIA 2013

**Thomas Geijtenbeek
Michiel van de Panne
Frank van der Stappen**

Papers on neuroevolution

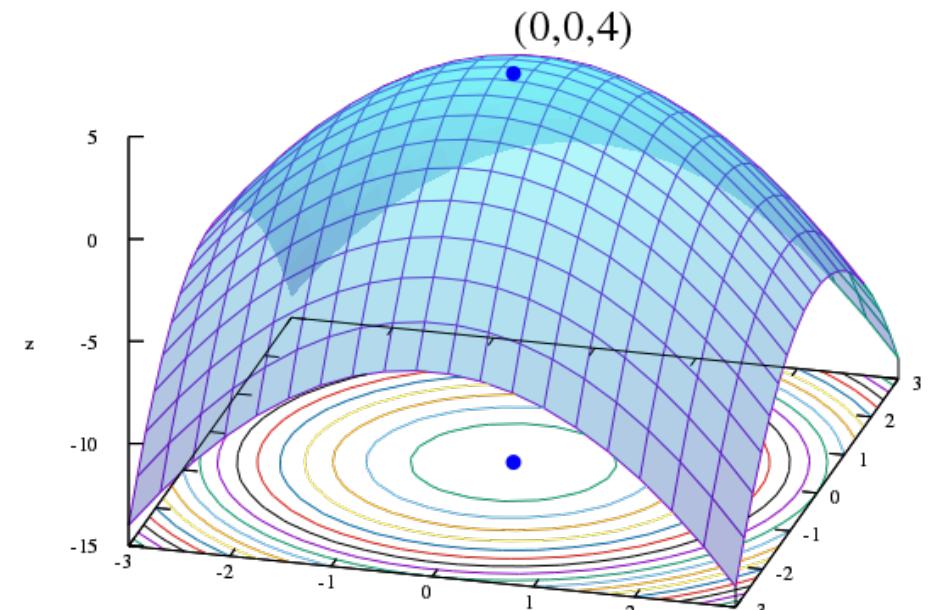
Such, Felipe Petroski, et al. "Deep Neuroevolution: Genetic Algorithms Are a Competitive Alternative for Training Deep Neural Networks for Reinforcement Learning." *arXiv preprint arXiv:1712.06567* (2017)

Salimans, Tim, et al. "Evolution strategies as a scalable alternative to reinforcement learning." *arXiv preprint arXiv:1703.03864* (2017).



Curvature

- Gradient descend: form a 1st order Taylor expansion, i.e., linear approximation of $f(\mathbf{x})$, use that to determine search direction
- Newton's method: a 2nd order model using 2nd derivatives
- If $f(\mathbf{x})$ is quadratic, model fits perfectly and directly gives the optimum
- An analogue of Newton's method for action sequences: Differential Dynamic Programming (DDP)





Curvature

- Newton's method requires the Hessian matrix of second derivatives, requires N^2 function evaluations and memory
- BFGS, L-BFGS: approximate Hessian indirectly
- If $f(\mathbf{x})$ is sum of squares: Gauss-Newton, Levenberg-Marquardt.
- An analogue of Gauss-Newton for action sequences: Iterative Linear Quadratic Gaussian (ILQG, e.g., Tassa et al. 2012)

