We will denote with $E_G(V_1, V_2)$ the cut generated by a bipartition (V_1, V_2) of $V(G)$.

(a)

o Let $\phi \in [0, c(uv) - \alpha(uv)]$.

Let the new cost function be $c' : E \to \mathbb{R}$ where $c'(xy) = \begin{cases} c(xy), & \text{if } xy \in E \setminus \{uv\} \\ c(xy) - \phi, & \text{if } xy = uv \end{cases}$

$\phi \leq c(uv) - \alpha(uv) \Rightarrow c'(uv) \geq c(uv) - (c(uv) - \alpha(uv)) \Rightarrow c'(uv) \geq \alpha(uv)$.

Proof by contradiction:

Assume there exists a spanning tree $T' \subseteq G$ such that $c'(T') < c'(T)$.

If $uv \notin E(T')$, then we have $c'(T') = c(T') \Rightarrow c(T') < c(T)$

This contradicts T being a minimum cost spanning tree of G with respect to c, therefore we know that:

$\forall\ T''$ such that T'' is an MST in G with respect to c', we have $uv \in E(T'')$ (i).

Since T' is a tree, we know uv is a bridge in T', and $T' - uv$ has exactly two connected components.

Let V_1 and V_2 be the two connected components of $T' - uv$. It is obvious that (V_1, V_2) is a partition of $V(T') = V(G)$.

Without loss of generality, assume that $u \in V_1$ and $v \in V_2$.

Consider $P_T(u, v)$, the unique path in T from u to v.

We know that $P_T(u, v)$ must contain an edge $xy \in E_G(V_1, V_2)$, since u and v are in different components of $T' - uv$.

We also know that $\alpha(uv) \leq c'(uv)$, which means that all edges on $P_T(u, v)$ have cost less than or equal to $c'(uv)$.

So we know that there exists an edge $xy \in E_G(V_1, V_2)$ such that $c'(xy) \leq c'(uv)$.

Then we can take $T'' = (V(T'), E(T') \setminus \{uv\} \cup \{xy\})$, which is a spanning tree of G.

But then we have that: $c'(T'') = c'(T') - c'(uv) + c'(xy) \leq c'(T') \Rightarrow T''$ is an MST of G with respect to c', but, since $uv \notin E(T'')$, this contradicts (i).

We have reached a contradiction, therefore our assumption was false, meaning there cannot exist a spanning tree of G that has a cost strictly less than T with respect to $c' \Rightarrow T$ is an MST of G with respect to c'.

(b)

Let $\phi \in [0, \beta(uv) - c(uv)]$.

Let the new cost function be $c' : E \to \mathbb{R}$ where $c'(xy) = \begin{cases} c(xy), & \text{if } xy \in E \setminus \{uv\} \\ c(xy) + \phi, & \text{if } xy = uv \end{cases}$

$\phi \leq \beta(uv) - c(uv) \Rightarrow c'(uv) \leq c(uv) + (\beta(uv) - c(uv)) \Rightarrow c'(uv) \leq \beta(uv)$.

Proof by contradiction:

Assume there exists a spanning tree $T' \subseteq G$ such that $c'(T') < c'(T)$.

Notice that $c'(T) = c(T) + \phi$.

If $uv \in E(T')$, then we have $c'(T') = c(T') + \phi, c'(T') < c'(T) \Rightarrow c(T') + \phi < c(T) + \phi \Rightarrow c(T') < c(T)$.

This contradicts T being an MST of G with respect to c, therefore we know that:

$\forall\ T''$ such that T'' is an MST in G with respect to c', we have $uv \notin E(T'')$ (ii).

Since T is a tree, we know uv is a bridge in T, and $T - uv$ has exactly two connected components.

Let V_1 and V_2 be the two connected components of $T - uv$. It is obvious that (V_1, V_2) is a partition of $V(T) = V(G)$.

Without loss of generality, assume that $u \in V_1$ and $v \in V_2$.

Consider $P_{T'}(u, v)$, the unique path in T' from u to v.

We know that $P_{T'}(u, v)$ must contain an edge $xy \in E_G(V_1, V_2)$, since u and v are in different components of $T - uv$.

We also know that $c'(uv) \leq \beta(uv)$, which means that all edges on $P_{T'}(u, v)$ have cost greater than or equal to $c'(uv)$.

So we know that there exists an edge $xy \in E_G(V_1, V_2)$ such that $c'(xy) \geq c'(uv)$, and $xy \in E(T')$.

Then we can take $T'' = (V(T'), E(T') \setminus \{xy\} \cup \{uv\})$, which is a spanning tree of G.

But then we have that: $c'(T'') = c'(T') - c'(xy) + c'(uv) \leq c'(T') \Rightarrow T''$ is an MST of G with respect to c', but, since $uv \in E(T'')$, this contradicts (ii).

We have reached a contradiction, therefore our assumption was false, meaning there cannot exist a spanning tree of G that has a cost strictly less than T with respect to $c' \Rightarrow T$ is an MST of G with respect to c'.