Homework 2 - November 14, 2025

Deadline and delivery: 08:00 am, November 21, 2025, by e-mail (fe.olariu@gmail.com)

- **The homework can be solved by teams of at most four students. Each student must declare itself as leader for at least one problem.**
- **The solutions will be drafted in LaTeX or in a word processor from an Office suite like Word, Writer etc.**
- **LaTeX (correctly) written solutions will receive a bonus of 2 points.**
- **Don't write again the texts of the problems! Each problem needs at most 1 page for its solution.**
- **ANY COPIED SOLUTION FOR A PROBLEM FROM BELOW WILL BE PENALISED WITH 2 POINTS.**
- **SEMINAR INSTRUCTORS HAVE THE RIGHT TO VERIFY THE ORIGINALITY AND THE AUTHENTICITY OF THE SOLUTIONS.**
- **The solutions will be delivered by e-mail until 08:00 am, November 21, 2025 at the following address:** fe.olariu@gmail.com
- **The e-mail must contain the source (a .doc, .odt or .tex file) and the .pdf file, both with the following name format:**

  Name1_group_Name2_group_Homework1.tex (or .doc, .odt)

  Name1_group_Name2_group_Homework1.pdf

- **Examples:**

  IonescuPVasile_A8_VasilescuTIon_X1_Homework1.tex (or .doc, .odt, .pdf)

  IonescuPVasile_B6_VasilescuTIon_E5_Homework1.tex (or .doc, .odt, .pdf)

1. Let $G = (V, E)$ be a graph; a **m-matching** in G is a matching M such that there is no matching M' in G with $M \subsetneq M'$. Let M and N be two m-matchings of G.

(a) Define a function $\varphi : M \setminus N \to N \setminus M$ in the following way: for $e = uv \in M \setminus N$, there exists an edge $f \in N \setminus M$ such that f is incident with u or v, take $\varphi(e) = f$. Prove that φ is well-defined: both $M \setminus N$ and $N \setminus M$ are non-empty and an edge f like above really exists.

(b) Show that $|M \setminus N| \leqslant 2|\varphi(M \setminus N)| \leqslant 2|N \setminus M|$.

(c) Prove that $|M| \leqslant 2|N|$.

(2 + 2 + 1 = 5 points)

2. Let $G = (V, E)$ be a connected graph $c : E \to \mathbb{R}$ a cost function on its edges, and $T = (V, E')$ a minimum cost spanning tree of G with respect to c. For any two vertices $x, y \in V$ we denote by $P_T(x, y)$ the xy-path in T.

(a) Let $uv \in E \setminus E'$ and $\alpha(uv) = \max\{c(xy) \,:\, xy \in E(P_T(u,v))\}$. Prove that decreasing the cost of uv by no more than $(c(uv) - \alpha(uv))$, T remains a minimum cost spanning tree of G with respect to the new cost function.

(b) Let $uv \in E'$ and $\beta(uv) = \min\{c(xy) \,:\, xy \in E \setminus E' \text{ s. t. } uv \in E(P_T(x,y))\}$. Prove that increasing the cost of uv by no more than $(\beta(uv) - c(uv))$, T remains minimum cost spanning tree of G with respect to the new cost function.

(3 + 3 = 6 points)

3. Let $G = (V, E)$ be a connected graph with n vertices and m edges. Let $c : E \to \{c_1, c_2\}$ be a cost function on its edges ($c_1 < c_2$). Let $G_i = (V, E_i)$ a spanning graph of G with $E_i = \{uv \in E : c(uv) = c_i\}$, for $i \in \{1, 2\}$. Denote by p_i the number of connected components of G_i.

(a) Prove that the minimum cost of a spanning tree in G is $[c_1(n - p_1) + c_2(p_1 - 1)]$.

(b) Devise an efficient algorithm for finding a minimum cost spanning tree in G.

(1 + 1 = 2 points)

4. Let $G = (V, E)$ be a graph with $|G| \geqslant 3$.

(a) Let G be 2-connected and $u, v \in V$. Build the graph G_1 by adding a new vertex x to G and linking x to u and v. Prove that G_1 is also 2-connected.

(b) Show that G is 2-connected if and only if for any three distinct vertices u, v, and w there exists a path from u to v containing w.

(1 + 2 = 3 points)