

Solutions Manual to Applied Partial Differential Equations

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Dedicated to those who came before me and those who will come after.

The Author thanks DuChateau and Zachmann for such a good book. The internet for endless resources.

ABSTRACT. This work consists of solutions to the exercises from the volume “Applied Partial Differential Equations” by Paul DuChateau and David Zachmann.

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Preface

This document consists of computer-based solutions to problems in “Applied Partial Differential Equations” by Paul DuChateau and David Zachmann.

CHAPTER 1

Mathematical Modeling and Partial Differential Equations

1. Equation of Heat Conduction

1. Consider an infinitely long rod for which the parameters K , ϵ , σ , C are such that $\beta = 0.1$. Then equation (1.2.8*) becomes

$$(1.1) \quad u_n^{j+1} = 0.1u_{n+1}^j + 0.8u_n^j + 0.1u_{n-1}^j$$

Suppose

$$(1.2) \quad u_n^0 = \begin{cases} 1 & \text{for } n = 4, 5, 6 \\ 0 & \text{for all other } n \end{cases}$$

Then use (1.2.8*) and this initial condition to compose u_n^j for $n = -5, \dots, 5$ for $j = 1, \dots, 5$. For each value of j , for how many n is u_n^j different from zero?

2. Repeat Exercise 1 for the situation in which the rod is of finite length L with $10\epsilon = L$. Suppose

$$(1.3) \quad u_0^j = 1 \text{ and } u_{10}^j = -1 \text{ for all } j > 0$$

and

$$(1.4) \quad u_n^0 = 0 \text{ for all } n$$

Then use (1.2.8*) to compute u_n^j for $n = 1, \dots, 9$ and $j = 1, \dots, 5$.

```
import numpy as np
import matplotlib.pyplot as plt
```

```
N = 10 # Length of rod
T = 5 # Duration of simulation
```

```
def matrix_power(x, n):
    y = x.copy()
    if n > 1:
        for i in np.arange(n - 1):
            y = np.matmul(y, x)
    return y
```

```
coefficients = np.zeros((N, N))
for i in np.arange(N):
    if i == 0:
```

```

        coefficients[i, 0] = 1
    elif i == N - 1:
        coefficients[i, N - 1] = 1
    else:
        coefficients[i, i - 1] = 0.1
        coefficients[i, i] = 0.8
        coefficients[i, i + 1] = 0.1

initial_conditions = np.zeros((N, 1))
initial_conditions[0, 0] = 1
initial_conditions[N - 1, 0] = -1
y = [np.squeeze(np.transpose(np.matmul(matrix_power(coefficients, i),\
initial_conditions))) for i in np.arange(T)]

length_intervals = np.arange(N)
plt.figure()
for i in np.arange(T):
    plt.plot(length_intervals, y[i])
plt.show()
print("The number of non-zero elements is {0}.".format(\
np.count_nonzero(y[T - 1] != 0)))

```


CHAPTER 2

Finite Difference Methods for Parabolic Equations

1. Computational Methods

4. Use Algorithm 8.1 to approximate the solution of the initial-boundary-value problem

$$(1.1) \quad u_t - u_{xx} = -2e^{x-t}, 0 < x < 1, t > 0$$

$$(1.2) \quad u(x, 0) = e^x, 0 < x < 1$$

$$(1.3) \quad u(0, t) = e^{-t}, u(1, t) = e^{1-t}, t > 0$$

(a) Choose $k = 0.0025$ and $nmax = 9$ (so $h = 0.1$) and compare the numerical and exact solutions, $u(x, t) = e^{x-t}$, at time $t = 0.5$.

(b) Choose $k = 0.01$ and $nmax = 9$ and explain the numerical results.

```
import numpy as np

# Forward Difference Method - Dirichlet Initial-Boundary-Value Problem
def algorithm_8_1(diffusivity,
                 endpoint,
                 time_step,
                 number_of_time_steps,
                 number_of_nodes,
                 right_side,
                 initial_condition,
                 boundary_condition_left,
                 boundary_condition_right):
    # Define a grid
    increment = endpoint / (number_of_nodes + 1)
    coefficient_r = diffusivity * time_step / increment ** 2
    if coefficient_r > 0.5:
        print("WARNING: algorithm_8_1 is unstable")

    # Initialize numerical solution
    t = np.zeros((number_of_time_steps + 1,))
    #x = np.zeros((1, number_of_nodes + 2))
    x = np.zeros((number_of_nodes + 2,))
    x[0] = 0
    #V = np.zeros((1, number_of_nodes + 2))
    V = np.zeros((number_of_nodes + 2,))
    V[0] = (boundary_condition_left(0) + initial_condition(0)) / 2
    for n in np.arange(number_of_nodes):
        x[n + 1] = x[n] + increment
```

```

        V[n + 1] = initial_condition(x[n + 1])
    x[number_of_nodes + 1] = endpoint
    V[number_of_nodes + 1] = (boundary_condition_right(0) + \
        initial_condition(endpoint)) / 2

    # Begin time stepping
    #U = np.zeros((1, number_of_nodes + 2))
    U = np.zeros((number_of_nodes + 2,))
    for j in np.arange(number_of_time_steps):
        # Advance solution one time step
        for n in np.arange(number_of_nodes):
            U[n + 1] = coefficient_r * V[n]
            U[n + 1] += (1 - 2 * coefficient_r) * V[n + 1]
            U[n + 1] += coefficient_r * V[n + 2]
            U[n + 1] += time_step * right_side(x[n + 1], t[j])
        t[j + 1] = t[j] + time_step
        U[0] = boundary_condition_left(t[j + 1])
        U[number_of_nodes + 1] = boundary_condition_right(t[j + 1])
        # Output numerical solution
        # Prepare for next time step
        for n in np.arange(number_of_nodes + 2):
            V[n] = U[n]

    #x = x[1:-1]
    t = t[1:-1]

    return U, x, t

import math

# right_side
def S(x, t):
    return -2.0 * math.e ** (x - t)

# initial_condition
def f(x):
    return math.e ** x

# boundary_condition_left
def p(t):
    return math.e ** -t

# boundary_condition_right
def q(t):
    return math.e ** (1 - t)

# exact_answer
def u(x, t):
    return math.e ** (x - t)

a2 = 1 # diffusivity
L = 1 # endpoint
k = 0.0025 # time_step

```

```

nmax = 9 # number_of_nodes
end_time = 0.5
jmax = int(end_time / k) # number_of_time_steps

numerical_answer, x, t = algorithm_8_1(a2,
                                       L,
                                       k,
                                       jmax,
                                       nmax,
                                       S,
                                       f,
                                       p,
                                       q)
exact_answer = u(x, t[-1])
answer_error = (numerical_answer - exact_answer) / (exact_answer)
answer_error = answer_error * 100

from matplotlib import pyplot as plt

fig, ax1 = plt.subplots()

ax1.set_xlabel('Position')
ax1.set_ylabel('Temperature')
ax1.plot(x, numerical_answer, 'r', label='Numerical Solution')
ax1.plot(x, exact_answer, 'g', label='Exact Solution')

ax2 = ax1.twinx()
ax2.set_ylabel('Percent Error')
ax2.plot(x, answer_error, 'b', label='Percent Error')

ax1.legend()
ax2.legend()

plt.show()

```

5. Use Algorithm 8.3 or 8.4 to approximate the solution of the initial-boundary-value problem

$$(1.4) \quad u_t - u_{xx} = -2e^{x-t}, 0 < x < 1, t > 0$$

$$(1.5) \quad u(x, 0) = e^x, 0 < x < 1$$

$$(1.6) \quad u(0, t) = e^{-t}, u(1, t) = e^{1-t}, t > 0$$

(a) Choose $k = 0.0025$ and $nmax = 9$ (so $h = 0.1$) and compare the numerical and exact solutions, $u(x, t) = e^{x-t}$, at time $t = 0.5$.

(b) Choose $k = 0.01$ and $nmax = 9$ and compare the numerical and exact solutions at time $t = 0.5$.

(c) Choose $k = 0.01$ and $nmax = 99$ (so $h = 0.01$) and compare the numerical and exact solutions at time $t = 0.5$ at the positions $x = 0.1, 0.2, \dots, 0.9$.

```

import numpy as np

# Solution of a Tridiagonal Linear System

```

```

def algorithm_8_2(a, # subdiagonal
                 b, # diagonal
                 c, # superdiagonal
                 d, # right-hand side
                 number_of_nodes=None):
    if number_of_nodes is None:
        number_of_nodes = d.size
    # Forward substitute to eliminate subdiagonal
    for n in np.arange(number_of_nodes - 1):
        ratio = a[n + 2] / b[n + 1]
        b[n + 2] = b[n + 2] - ratio * c[n + 1]
        d[n + 2] = d[n + 2] - ratio * d[n + 1]
    # Back substitute and store in solution array in d
    d[number_of_nodes] = d[number_of_nodes] / b[number_of_nodes]
    for l in np.arange(number_of_nodes - 1):
        n = number_of_nodes - (l + 1)
        d[n] = (d[n] - c[n] * d[n + 1]) / b[n]
    return d

# Backward Difference Method - Dirichlet Initial-Boundary-Value Problem
def algorithm_8_3(diffusivity,
                 endpoint,
                 time_step,
                 number_of_time_steps,
                 number_of_nodes,
                 right_side,
                 initial_condition,
                 boundary_condition_left,
                 boundary_condition_right):
    # Define a grid
    increment = endpoint / (number_of_nodes + 1)
    coefficient_r = diffusivity * time_step / increment ** 2

    # Initialize numerical solution
    t = np.zeros((number_of_time_steps + 1,))
    x = np.zeros((number_of_nodes + 2,))
    x[0] = 0
    U = np.zeros((number_of_nodes + 2,))
    U[0] = (boundary_condition_left(0) + initial_condition(0)) / 2
    for n in np.arange(number_of_nodes):
        x[n + 1] = x[n] + increment
        U[n + 1] = initial_condition(x[n + 1])
    U[number_of_nodes + 1] = (boundary_condition_right(0) + \
        initial_condition(endpoint)) / 2
    x[number_of_nodes + 1] = endpoint

    term_a = np.zeros((number_of_nodes + 2,))
    term_b = np.zeros((number_of_nodes + 2,))
    term_c = np.zeros((number_of_nodes + 2,))
    term_d = np.zeros((number_of_nodes + 2,))
    # Begin time stepping
    for j in np.arange(number_of_time_steps):
        # Define tridiagonal system

```

```

    t[j + 1] = t[j] + time_step
    for n in np.arange(number_of_nodes):
        term_a[n + 1] = - coefficient_r
        term_b[n + 1] = 1 + 2 * coefficient_r
        term_c[n + 1] = - coefficient_r
        term_d[n + 1] = U[n + 1] + time_step * right_side(x[n + 1], t[j + 1])
    term_d[1] = term_d[1] + coefficient_r * boundary_condition_left(t[j + 1])
    term_d[number_of_nodes] = term_d[number_of_nodes] + \
        coefficient_r * boundary_condition_right(t[j + 1])
    # Advance solution one time step
    term_d = algorithm_8_2(term_a, term_b, term_c, term_d, number_of_nodes)
    for n in np.arange(number_of_nodes):
        U[n + 1] = term_d[n + 1]
    U[0] = boundary_condition_left(t[j + 1])
    # Output numerical solution
    U[0] = boundary_condition_left(t[j + 1])
    U[number_of_nodes + 1] = boundary_condition_right(t[j + 1])

    return U, x, t

import math

# right_side
def S(x, t):
    return -2.0 * math.e ** (x - t)

# initial_condition
def f(x):
    return math.e ** x

# boundary_condition_left
def p(t):
    return math.e ** -t

# boundary_condition_right
def q(t):
    return math.e ** (1 - t)

# exact_answer
def u(x, t):
    return math.e ** (x - t)

a2 = 1 # diffusivity
L = 1 # endpoint
k = 0.0025 # time_step
nmax = 9 # number_of_nodes
end_time = 0.5
jmax = int(end_time / k) # number_of_time_steps

numerical_answer, x, t = algorithm_8_3(a2,
                                     L,
                                     k,
                                     jmax,
```

```

        nmax,
        S,
        f,
        p,
        q)
exact_answer = u(x, t[-1])
answer_error = (numerical_answer - exact_answer) / (exact_answer)
answer_error = answer_error * 100

from matplotlib import pyplot as plt

fig, ax1 = plt.subplots()

ax1.set_xlabel('Position')
ax1.set_ylabel('Temperature')
ax1.plot(x, numerical_answer, 'r', label='Numerical Solution')
ax1.plot(x, exact_answer, 'g', label='Exact Solution')

ax2 = ax1.twinx()
ax2.set_ylabel('Percent Error')
ax2.plot(x, answer_error, 'b', label='Percent Error')

ax1.legend()
ax2.legend()

plt.show()

```

6. Use Algorithm 8.5 to approximate the solution of the initial-boundary-value problem

$$(1.7) \quad u_t - u_{xx} = -2e^{x-t}, 0 < x < 1, t > 0$$

$$(1.8) \quad u(x, 0) = e^x, 0 < x < 1$$

$$(1.9) \quad u(0, t) = e^{-t}, u(1, t) = e^{1-t}, t > 0$$

(a) Choose $k = 0.0025$ and $nmax = 9$ (so $h = 0.1$) and compare the numerical and exact solutions, $u(x, t) = e^{x-t}$, at time $t = 0.5$.

(b) Choose $k = 0.01$ and $nmax = 9$ and compare the numerical and exact solutions at time $t = 0.5$.

(c) Choose $k = 0.01$ and $nmax = 99$ (so $h = 0.01$) and compare the numerical and exact solutions at time $t = 0.5$.

```

import numpy as np

# Backward Difference Method - Neumann Initial-Boundary-Value Problem
def algorithm_8_5(diffusivity,
                 endpoint,
                 time_step,
                 number_of_time_steps,
                 number_of_nodes,
                 right_side,
                 initial_condition,
                 boundary_condition_left,

```

```

        boundary_condition_right):
# Define a grid
increment = endpoint / (number_of_nodes - 1)
coefficient_r = diffusivity * time_step / increment ** 2

# Initialize numerical solution
t = np.zeros((number_of_time_steps + 1,))
x = np.zeros((number_of_nodes + 1,))
x[0] = -increment
U = np.zeros((number_of_nodes + 1,))
for n in np.arange(number_of_nodes):
    x[n + 1] = x[n] + increment
    U[n + 1] = initial_condition(x[n + 1])

term_a = np.zeros((number_of_nodes + 1,))
term_b = np.zeros((number_of_nodes + 1,))
term_c = np.zeros((number_of_nodes + 1,))
term_d = np.zeros((number_of_nodes + 1,))
# Begin time stepping
for j in np.arange(number_of_time_steps):
    # Define tridiagonal system
    t[j + 1] = t[j] + time_step
    for n in np.arange(number_of_nodes):
        term_a[n + 1] = - coefficient_r
        term_b[n + 1] = 1 + 2 * coefficient_r
        term_c[n + 1] = - coefficient_r
        term_d[n + 1] = U[n + 1] + time_step * right_side(x[n + 1], t[j + 1])
    term_d[1] = term_d[1] - \
        2 * increment * coefficient_r * boundary_condition_left(t[j + 1])
    term_d[number_of_nodes] = term_d[number_of_nodes] + \
        2 * increment * coefficient_r * boundary_condition_right(t[j + 1])
    term_c[1] = -2 * coefficient_r
    term_a[number_of_nodes] = -2 * coefficient_r
    # Advance solution one time step
    term_d = algorithm_8_2(term_a, term_b, term_c, term_d, number_of_nodes)
    for n in np.arange(number_of_nodes):
        U[n + 1] = term_d[n + 1]
# Output numerical solution

#t = t[:-1]
x = x[1:]
U = U[1:]

return U, x, t

import math

# right_side
def S(x, t):
    return -2.0 * math.e ** (x - t)

# initial_condition
def f(x):

```

```

    return math.e ** x

# boundary_condition_left
def p(t):
    return math.e ** -t

# boundary_condition_right
def q(t):
    return math.e ** (1 - t)

# exact_answer
def u(x, t):
    return math.e ** (x - t)

a2 = 1 # diffusivity
L = 1 # endpoint
k = 0.0025 # time_step
nmax = 9 # number_of_nodes
end_time = 0.5
jmax = int(end_time / k) # number_of_time_steps

numerical_answer, x, t = algorithm_8_5(a2,
                                       L,
                                       k,
                                       jmax,
                                       nmax,
                                       S,
                                       f,
                                       p,
                                       q)
exact_answer = u(x, t[-1])
answer_error = (numerical_answer - exact_answer) / (exact_answer)
answer_error = answer_error * 100

from matplotlib import pyplot as plt

fig, ax1 = plt.subplots()

ax1.set_xlabel('Position')
ax1.set_ylabel('Temperature')
ax1.plot(x, numerical_answer, 'r', label='Numerical Solution')
ax1.plot(x, exact_answer, 'g', label='Exact Solution')

ax2 = ax1.twinx()
ax2.set_ylabel('Percent Error')
ax2.plot(x, answer_error, 'b', label='Percent Error')

ax1.legend()
ax2.legend()

plt.show()

```


2. Fourier's Method for Difference Equations

5. Given the initial-boundary-value problem

$$(2.1) \quad u_t - u_{xx} = 0, 0 < x < 1, t > 0$$

$$(2.2) \quad u(x, 0) = x^2, 0 < x < 1$$

$$(2.3) \quad u_x(0, t) = 0 = u_x(1, t), t > 0$$

and the grid $x_1 = 0$, $x_2 = 1/2$, and $x_3 = 1$, so $N = 3$,

- (a) Use the matrix \mathbf{F}_2 to formulate a forward-in-time difference system.
- (b) Use the matrix \mathbf{F}_2 to formulate a backward-in-time difference system.
- (c) Use the matrix \mathbf{F}_2 to formulate a Crank-Nicolson difference system.
- (d) Use Fourier's method to solve the system (a).
- (e) Use Fourier's method to solve the system (b).
- (f) Use Fourier's method to solve the system (c).

6. Given the initial-boundary-value problem

$$(2.4) \quad u_t - u_{xx} = 0, 0 < x < 1, t > 0$$

$$(2.5) \quad u(x, 0) = x^2, 0 < x < 1$$

$$(2.6) \quad u_x(0, t) = 0 = u_x(1, t), t > 0$$

and the grid $x_1 = 0$, $x_2 = 1/3$, and $x_3 = 2/3$, so $N = 3$,

- (a) Use the matrix \mathbf{F}_2 to formulate a forward-in-time difference system.
- (b) Use the matrix \mathbf{F}_2 to formulate a backward-in-time difference system.
- (c) Use the matrix \mathbf{F}_2 to formulate a Crank-Nicolson difference system.
- (d) Use Fourier's method to solve the system (a).
- (e) Use Fourier's method to solve the system (b).
- (f) Use Fourier's method to solve the system (c).

7. Given the initial-boundary-value problem

$$(2.7) \quad u_t - u_{xx} = 0, 0 < x < 1, t > 0$$

$$(2.8) \quad u(x, 0) = x, 0 < x < 1$$

$$(2.9) \quad u_x(0, t) = 0 = u_x(1, t), t > 0$$

and the grid $x_1 = 0$, $x_2 = 1/4$, $x_3 = 1/2$, $x_4 = 3/4$, and $x_5 = 1$, so $N = 5$,

- (a) Use the matrix \mathbf{F}_2 to formulate a forward-in-time difference system.
- (b) Use the matrix \mathbf{F}_2 to formulate a backward-in-time difference system.
- (c) Use the matrix \mathbf{F}_2 to formulate a Crank-Nicolson difference system.
- (d) Use Fourier's method to solve the system (a).
- (e) Use Fourier's method to solve the system (b).
- (f) Use Fourier's method to solve the system (c).

CHAPTER 3

Numerical Solutions of Hyperbolic Equations

1. Difference Methods for a Scalar Initial-Value Problem

1. Modify Algorithm 9.1 to implement

- (a) FTBS method
- (b) FTFS method
- (c) Lax-Friedrichs method
- (d) leapfrog method

2. Approximate the solution of the initial-value problem of Example 9.1.3 on the interval $0 \leq x \leq 1$ for $0 \leq t_j \leq 1.5$ with $h = 0.1$ and $k = 0.075$ using

- (a) FTBS method
- (b) Lax-Friedrichs method
- (c) leapfrog method

3. Repeat exercise 2 with $h = 0.1$ and $k = 0.1$.

4. Given the initial-value problem

$$(1.1) \quad u_x + u_t = 2xt + x^2, \quad -\infty < x < \infty, \quad t > 0$$

$$(1.2) \quad u(x, 0) = 0, \quad -\infty < x < \infty$$

$$(1.3)$$

whose exact solution is $u(x, t) = x^2t$. Find the local truncation error for this problem when using (a) FTBS method

(b) Lax-Wendroff method

Bibliography

- [1] DuChateau, P., Zachmann, D. *Applied Partial Differential Equations*, Dover Publications Inc, Mineola, New York, 1989