Filter Notes

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1 Transmission Lines

1.1 Group Velocity

In a source-free, linear, isotropic, homogeneous region, Maxwell's curl equations in phasor form are

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}, \tag{1}$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E}. \tag{2}$$

These yield the wave equations which are known as the Helmholtz equations

$$\nabla^2\vec{E} + \omega^2\mu\epsilon\vec{E} = 0, \tag{3}$$

$$\nabla^2\vec{H} + \omega^2\mu\epsilon\vec{H} = 0. \tag{4}$$

The constant $k = \omega\sqrt{\mu\epsilon}$ is known as the *propagation constant*, phase constant, or wave number. It is given in units of inverse length. We could obtain a plane-wave solution of the Helmholtz equation in an arbitrary direction. But for generality, let us assume an arbitrary direction r which could represent a radial component of a spherical solution to the Helmholtz equation. The velocity of a wave is called the phase velocity because it is the velocity at which a fixed phase point on the wave travels. It is given by

$$v_p = \frac{dr}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}. \tag{5}$$

In a lossy medium with conductivity σ Maxwell's curl equations are

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}, \tag{6}$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E} + \sigma\vec{E}. \tag{7}$$

The resulting Helmholtz wave equations are

$$\nabla^2\vec{E} + \omega^2\mu\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)\vec{E} = 0, \tag{8}$$

$$\nabla^2\vec{H} + +\omega^2\mu\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)\vec{H} = 0. \tag{9}$$

If we define the *complex propagation constant* as

$$\gamma = \alpha + j\beta = j\omega\sqrt{(\mu\epsilon)\left(1 - j\frac{\sigma}{\omega\epsilon}\right)} \tag{10}$$

with attenuation constant α and phase constant β. In this case, we have a phase velocity given by

$$v_p = \frac{dr}{dt} = \frac{\omega}{\beta}. \tag{11}$$

The telegrapher's equations yield the phasor form of wave propagation of voltage and current on a generalized transmission line in the direction $\mathbf{r}$

$$D^2_{\hat{\mathbf{r}}}\tilde{V}(\mathbf{r}) - \gamma^2(\omega)\tilde{V}(\mathbf{r}) = 0, \tag{12}$$

$$D^2_{\hat{\mathbf{r}}}\tilde{I}(\mathbf{r}) - \gamma^2(\omega)\tilde{I}(\mathbf{r}) = 0. \tag{13}$$

On a lossless line, the propagation constant and phase constant are

$$\gamma = j\omega\sqrt{LC}, \tag{14}$$

$$\beta = \omega\sqrt{LC}. \tag{15}$$

The phase velocity on a lossless line is

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}. \tag{16}$$

On a lossy line, the propagation constant and phase velocity are

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}, \tag{17}$$

$$v_p = \frac{\omega}{\beta(\omega)}. \tag{18}$$

The group velocity can be interpreted physically as the velocity at which a narrowband signal propagates. Consider a system whose transfer function is given by $H(\omega) = Ae^{-j\beta r}$ and whose input signal $x(t)$ is a baseband function $u(t)$ modulated by the tone $\mathrm{Re}\{e^{j\omega_0 t}\}$ so that $X(\omega) = U(\omega - \omega_0)$. In the time domain, the output signal is given by

$$y(t) = \frac{1}{2\pi}\mathrm{Re}\left\{\int_{-\infty}^{\infty} AU(\omega - \omega_0)e^{j(\omega t - \beta r)}\, d\omega\right\}. \tag{19}$$

If $U(\omega)$ is narrowband—$\omega_m \ll \omega_0$ where ω_m is the maximum frequency of the baseband function $u(t)$—then the propagation constant β can be linearized using a Taylor series expansion about ω_0,

$$\begin{aligned}\beta(\omega) &\approx \beta(\omega_0) + \left.\frac{d\beta}{d\omega}\right|_{\omega=\omega_0}(\omega - \omega_0)\\ &= \beta_0 + \beta_0{}'(\omega - \omega_0).\end{aligned} \tag{20}$$

This means that the output signal is

$$y(t) \approx Au(t - \beta_0{}'r)\cos(\omega_0 t - \beta_0 r) \tag{21}$$

and the group velocity is

$$v_g \approx \frac{1}{\beta_0{}'} = \left(\frac{d\beta}{d\omega}\right)^{-1}\Bigg|_{\omega=\omega_0}. \tag{22}$$

1.2 Group Delay

Group delay and phase delay describe the differential delay times of a signals frequency components as they propagate through an LTI system. Consider a system whose transfer function is given by $H(\omega) = |H(\omega)|e^{j\phi(\omega)} = Ae^{j\phi(\omega)}$ and whose input signal $x(t)$ is a baseband function $u(t)$ modulated by the tone $\mathrm{Re}\{e^{j\omega_0 t}\}$ so that $X(\omega) = U(\omega - \omega_0)$. In the time domain, the output signal is given by

$$y(t) = \frac{1}{2\pi}\mathrm{Re}\left\{\int_{-\infty}^{\infty} AU(\omega - \omega_0)e^{j(\omega t - \phi(\omega))}\,d\omega\right\}. \tag{23}$$

If $U(\omega)$ is narrowband—$\omega_m \ll \omega_0$ where ω_m is the maximum frequency of the baseband function $u(t)$—then the phase response ϕ can be linearized using a (first-order) Taylor series expansion about ω_0,

$$\begin{aligned} \phi(\omega) &\approx \phi(\omega_0) + \frac{d\phi}{d\omega}\bigg|_{\omega=\omega_0}(\omega - \omega_0) \\ &= \phi_0 + \phi_0{}'(\omega - \omega_0). \end{aligned} \tag{24}$$

We then make a change of variables $\xi = \omega - \omega_0$ to obtain

$$\begin{aligned} y(t) &\approx \frac{1}{2\pi}\mathrm{Re}\left\{\int_{-\infty}^{\infty} AU(\omega - \omega_0)e^{j(\omega t - \phi_0 - \phi_0{}'(\omega-\omega_0))}\,d\omega\right\} \\ &= \frac{A}{2\pi}\mathrm{Re}\left\{e^{j(\omega_0 t - \phi_0)}\int_{-\infty}^{\infty} AU(\xi)e^{j(t-\phi_0{}'\xi)}\,d\xi\right\} \\ &= A\mathrm{Re}\left\{u(t - \phi_0{}')e^{j(\omega_0 t - \phi_0)}\right\} \\ &= Au(t - \phi_0{}')\cos(\omega_0 t - \phi_0). \end{aligned} \tag{25}$$

Given that the group delay is defined as

$$\tau_g \equiv -\frac{d\phi}{d\omega}\bigg|_{\omega=\omega_0} \tag{26}$$

our output signal is

$$y(t) \approx Au(t - \tau_g)\cos(\omega_0 t - \phi_0). \tag{27}$$

The phase delay is defined as

$$\tau_\phi \equiv -\frac{\phi}{\omega}. \tag{28}$$

1.3 Conclusion

If we have a medium of length l with propagation constant $\gamma = \alpha + j\beta$ then the system phase is $\phi = -\beta l$ so that

$$\begin{aligned} \tau_g(\omega) &= -\frac{d\phi}{d\omega} \\ &= l\frac{d\beta}{d\omega} \\ &= \frac{l}{v_g}. \end{aligned} \tag{29}$$

2 Transmission Matrix

The transition matrix or $ABCD$ matrix allows cascading of network connections, and is thus a popular choice for representing networks in network analysis. The $ABCD$ matrix for a two-port network is defined as follows:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}. \tag{30}$$

In general, we may have the transition matrix between ports i and $i+1$:

$$\begin{bmatrix} V_i \\ I_i \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} V_{i+1} \\ I_{i+1} \end{bmatrix}. \tag{31}$$

And we may cascade the transition matrices between ports i, $i+1$ and $i+2$:

$$\begin{bmatrix} V_i \\ I_i \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} A_{i+1} & B_{i+1} \\ C_{i+1} & D_{i+1} \end{bmatrix} \begin{bmatrix} V_{i+2} \\ I_{i+2} \end{bmatrix}. \tag{32}$$

3 Impedance Matching

As long as the load impedance has a positive real part, a matching network can be found.

3.1 *L* Network

The L-section can be used as a matching network. It comes in two configurations: (1) a series reactive element X followed by a shunt reactive element B or (2) a shunt reactive element B followed by a series reactive element X. The reactive elements may be either inductors or capacitors. Configuration 1 is used the normalized load impedance is inside the $1 + jx$ circle on the Smith chart and configuration 2 should be used if the normalized load impedance is outside of this circle. The element values of the matching network may be solved either algebraically or with the use of a Smith chart.

3.2 Stub Tuning

A single stub either in series or in shunt (configuration) may be used to tune. A double-stub tuner circuit may also be used.