Filter Notes

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1 Introduction

The purpose of this article is to provide a theoretical treatment of non-linear electromagnetics which can be used in custom electromagnetic simulation software. (Incidentally, this software can be used in filter design – which was the original use case of this treatment.)

2 Transmission Lines

2.1 Group Velocity

In a source-free, linear, isotropic, homogeneous region, Maxwell's curl equations in phasor form are

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}, \tag{1}$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E}. \tag{2}$$

These yield the wave equations which are known as the Helmholtz equations

$$\nabla^2\vec{E} + \omega^2\mu\epsilon\vec{E} = 0, \tag{3}$$

$$\nabla^2\vec{H} + \omega^2\mu\epsilon\vec{H} = 0. \tag{4}$$

The constant $k = \omega\sqrt{\mu\epsilon}$ is known as the *propagation constant*, phase constant, or wave number. It is given in units of inverse length. We could obtain a plane-wave solution of the Helmholtz equation in an arbitrary direction. But for generality, let us assume an arbitrary direction r which could represent a radial component of a spherical solution to the Helmholtz equation. The velocity of a wave is called the phase velocity because it is the velocity at which a fixed phase point on the wave travels. It is given by

$$v_p = \frac{dr}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}. \tag{5}$$

In a lossy medium with conductivity σ Maxwell's curl equations are

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}, \tag{6}$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E} + \sigma\vec{E}. \tag{7}$$

The resulting Helmholtz wave equations are

$$\nabla^2 \vec{E} + \omega^2 \mu\epsilon \left(1 - j\frac{\sigma}{\omega\epsilon}\right) \vec{E} = 0, \tag{8}$$

$$\nabla^2 \vec{H} + +\omega^2 \mu\epsilon \left(1 - j\frac{\sigma}{\omega\epsilon}\right) \vec{H} = 0. \tag{9}$$

If we define the *complex propagation constant* as

$$\gamma = \alpha + j\beta = j\omega\sqrt{(\mu\epsilon)\left(1 - j\frac{\sigma}{\omega\epsilon}\right)} \tag{10}$$

with attenuation constant α and phase constant β. In this case, we have a phase velocity given by

$$v_p = \frac{dr}{dt} = \frac{\omega}{\beta}. \tag{11}$$

The telegrapher's equations yield the phasor form of wave propagation of voltage and current on a generalized transmission line in the direction $\mathbf{r}$

$$D^2_{\hat{\mathbf{r}}}\tilde{V}(\mathbf{r}) - \gamma^2(\omega)\tilde{V}(\mathbf{r}) = 0, \tag{12}$$

$$D^2_{\hat{\mathbf{r}}}\tilde{I}(\mathbf{r}) - \gamma^2(\omega)\tilde{I}(\mathbf{r}) = 0. \tag{13}$$

On a lossless line, the propagation constant and phase constant are

$$\gamma = j\omega\sqrt{LC}, \tag{14}$$

$$\beta = \omega\sqrt{LC}. \tag{15}$$

The phase velocity on a lossless line is

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}. \tag{16}$$

On a lossy line, the propagation constant and phase velocity are

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}, \tag{17}$$

$$v_p = \frac{\omega}{\beta(\omega)}. \tag{18}$$

The group velocity can be interpreted physically as the velocity at which a narrowband signal propagates. Consider a system whose transfer function is given by $H(\omega) = Ae^{-j\beta r}$ and whose input signal $x(t)$ is a baseband function $u(t)$ modulated by the tone $\mathrm{Re}\{e^{j\omega_0 t}\}$ so that $X(\omega) = U(\omega - \omega_0)$. In the time domain, the output signal is given by

$$y(t) = \frac{1}{2\pi}\mathrm{Re}\left\{\int_{-\infty}^{\infty} AU(\omega - \omega_0)e^{j(\omega t - \beta r)}\, d\omega\right\}. \tag{19}$$

If $U(\omega)$ is narrowband—$\omega_m \ll \omega_0$ where ω_m is the maximum frequency of the baseband function $u(t)$—then the propagation constant β can be linearized using a Taylor series expansion about ω_0,

$$\begin{aligned}\beta(\omega) &\approx \beta(\omega_0) + \left.\frac{d\beta}{d\omega}\right|_{\omega=\omega_0} (\omega - \omega_0) \\ &= \beta_0 + {\beta_0}'(\omega - \omega_0).\end{aligned} \tag{20}$$

This means that the output signal is

$$y(t) \approx Au(t - {\beta_0}'r)\cos(\omega_0 t - \beta_0 r) \tag{21}$$

and the group velocity is

$$v_g \approx \frac{1}{{\beta_0}'} = \left.\left(\frac{d\beta}{d\omega}\right)^{-1}\right|_{\omega=\omega_0}. \tag{22}$$

2.2 Group Delay

Group delay and phase delay describe the differential delay times of a signals frequency components as they propagate through an LTI system. Consider a system whose transfer function is given by $H(\omega) = |H(\omega)|e^{j\phi(\omega)} = Ae^{j\phi(\omega)}$ and whose input signal $x(t)$ is a baseband function $u(t)$ modulated by the tone $\mathrm{Re}\{e^{j\omega_0 t}\}$ so that $X(\omega) = U(\omega - \omega_0)$. In the time domain, the output signal is given by

$$y(t) = \frac{1}{2\pi}\mathrm{Re}\left\{\int_{-\infty}^{\infty} AU(\omega - \omega_0)e^{j(\omega t - \phi(\omega))}\, d\omega\right\}. \tag{23}$$

If $U(\omega)$ is narrowband—$\omega_m \ll \omega_0$ where ω_m is the maximum frequency of the baseband function $u(t)$—then the phase response ϕ can be linearized using a (first-order) Taylor series expansion about ω_0,

$$\begin{aligned}\phi(\omega) &\approx \phi(\omega_0) + \left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_0} (\omega - \omega_0) \\ &= \phi_0 + {\phi_0}'(\omega - \omega_0).\end{aligned} \tag{24}$$

We then make a change of variables $\xi = \omega - \omega_0$ to obtain

$$\begin{aligned}y(t) &\approx \frac{1}{2\pi}\mathrm{Re}\left\{\int_{-\infty}^{\infty} AU(\omega - \omega_0)e^{j(\omega t - \phi_0 - {\phi_0}'(\omega - \omega_0))}\, d\omega\right\} \\ &= \frac{A}{2\pi}\mathrm{Re}\left\{e^{j(\omega_0 t - \phi_0)}\int_{-\infty}^{\infty} AU(\xi)e^{j(t - {\phi_0}'\xi)}\, d\xi\right\} \\ &= A\mathrm{Re}\left\{u(t - {\phi_0}')e^{j(\omega_0 t - \phi_0)}\right\} \\ &= Au(t - {\phi_0}')\cos(\omega_0 t - \phi_0).\end{aligned} \tag{25}$$

Given that the group delay is defined as

$$\tau_g \equiv -\left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_0} \tag{26}$$

our output signal is

$$y(t) \approx Au(t-\tau_g)\cos(\omega_0 t - \phi_0). \tag{27}$$

The phase delay is defined as

$$\tau_\phi \equiv -\frac{\phi}{\omega}. \tag{28}$$

2.3 Conclusion

If we have a medium of length l with propagation constant $\gamma = \alpha + j\beta$ then the system phase is $\phi = -\beta l$ so that

$$\begin{aligned} \tau_g(\omega) &= -\frac{d\phi}{d\omega} \\ &= l\frac{d\beta}{d\omega} \\ &= \frac{l}{v_g}. \end{aligned} \tag{29}$$

3 Media

In a dielectric material, instead of the electric field $\vec{E}$ we use the electric flux density $\vec{D}$ – $\vec{D}$ may also be called the total displacement flux – and instead of the magnetic field $\vec{H}$ we use the magnetic flux density $\vec{B}$. (Really, we use both the field and the flux density.) The flux densities are related to the fields by

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}, \tag{30}$$

$$\vec{B} = \mu_0\vec{H} + \mu_0\vec{M}, \tag{31}$$

where $\vec{P}$ is the polarization density and $\vec{M}$ is the magnetization density. $\vec{P}$ may also be called the electric polarization or polarization and $\vec{M}$ may be called the magnetic polarization or magnetization. Let us call the electric polarization simply polarization and let us call the magnetic polarization simply magnetization. The electric polarization represents the (volumetric) density of permanent or induced electric dipole moments in a dielectric medium. The magnetization represents the permanent or induced dipole moments in a dieletric medium. The magnetization represents the permanent or induced dipole moments in a magnetic material.

In a linear, non-dispersive, homogeneous, isotropic dielectric medium, the polarization is

$$\vec{P} = \epsilon_0\chi\vec{E} \tag{32}$$

where χ or χ_e is the electric susceptibility. This gives us

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0(1+\chi)\vec{E} = \epsilon\vec{E}, \tag{33}$$

$$\epsilon = \epsilon' - j\epsilon'' = \epsilon_0(1+\chi) \tag{34}$$

where ϵ is known as the electric permittivity of the medium. If the medium is lossless then $\epsilon'' = 0$. Similarly, the magnetization is

$$\vec{M} = \chi \vec{H} \tag{35}$$

where χ or χ_m is the (volume) magnetic susceptibility – note that χ_m and χ_e are different event though both are written here with χ. We can now write the magnetic flux as

$$\vec{B} = \mu_0 \vec{H} \chi \vec{H} = \mu_0 (1 + \chi) \vec{H} = \mu \vec{H}, \tag{36}$$

$$\mu = \mu_0 (1 + \chi), \tag{37}$$

where μ is the magnetic permeability.

4 Nonlinear Optics

(This document will only treat parametric interactions.)

In a non-linear dielectric medium, the relationship between the polarization and electric field may be expanded in a Taylor series about $E = 0$:

$$P = \sum_{k=1}^{N} \frac{1}{k!} a_i E^i \tag{38}$$

We can stop at order 3 ($N = 3$) and use the following notation

$$1 \cdot a_1 = \epsilon_0 \chi \tag{39}$$

$$\frac{1}{2} a_2 = 2d = \epsilon_0 \chi^{(2)} \tag{40}$$

$$\frac{1}{6} a_3 = 4\chi^{(3)} \text{ or } \frac{1}{6} a_3 = \epsilon_0 \chi^{(3)} \tag{41}$$

5 Transmission Matrix

The transition matrix or $ABCD$ matrix allows cascading of network connections, and is thus a popular choice for representing networks in network analysis. The $ABCD$ matrix for a two-port network is defined as follows:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}. \tag{42}$$

In general, we may have the transition matrix between ports i and $i + 1$:

$$\begin{bmatrix} V_i \\ I_i \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} V_{i+1} \\ I_{i+1} \end{bmatrix}. \tag{43}$$

And we may cascade the transition matrices between ports i, $i + 1$ and $i + 2$:

$$\begin{bmatrix} V_i \\ I_i \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} A_{i+1} & B_{i+1} \\ C_{i+1} & D_{i+1} \end{bmatrix} \begin{bmatrix} V_{i+2} \\ I_{i+2} \end{bmatrix}. \tag{44}$$

6 Impedance Matching

As long as the load impedance has a positive real part, a matching network can be found.

6.1 *L* Network

The L-section can be used as a matching network. It comes in two configurations: (1) a series reactive element X followed by a shunt reactive element B or (2) a shunt reactive element B followed by a series reactive element X. The reactive elements may be either inductors or capacitors. Configuration 1 is used the normalized load impedance is inside the $1 + jx$ circle on the Smith chart and configuration 2 should be used if the normalized load impedance is outside of this circle. The element values of the matching network may be solved either algebraically or with the use of a Smith chart.

6.2 Stub Tuning

A single stub either in series or in shunt (configuration) may be used to tune. A double-stub tuner circuit may also be used.