

Logică Propozițională Clasică – demonstrații sintactice

SEMINAR DE LOGICĂ MATEMATICĂ ȘI COMPUTAȚIONALĂ

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Def.: $n \in \mathbb{N}^*$; $\varphi_1, \dots, \varphi_n \in E$.
Se definește $\bigwedge_{i=1}^n \varphi_i = \begin{cases} \varphi_1, & \text{dacă } n=1 \\ (\bigwedge_{i=1}^{n-1} \varphi_i) \wedge \varphi_n, & \text{dacă } n \geq 2. \end{cases}$

Exerc.: $n \in \mathbb{N}^*$; $\varphi_1, \dots, \varphi_n, \varphi \in E$.
Să se dem. că: $\vdash (\bigwedge_{i=1}^n \varphi_i) \rightarrow \varphi \Leftrightarrow \vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow (\dots \rightarrow (\varphi_n \rightarrow \varphi) \dots))$.
REZOLVARE:

Inducție matematică, după $n \in \mathbb{N}^*$.
 $n=1$: $\vdash \varphi_1 \rightarrow \varphi \Leftrightarrow \vdash \varphi_1 \rightarrow \varphi$.
 $n=2$: $\vdash (\varphi_1 \wedge \varphi_2) \rightarrow \varphi \Leftrightarrow \vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi)$.
 $\vdash \varphi_2 \rightarrow (\varphi_2 \rightarrow \varphi) \Leftrightarrow \vdash \varphi_2 \rightarrow \varphi$.
 $\Leftrightarrow \{\varphi_2, \varphi_2\} \vdash \varphi \quad (I)$

de otra parte: $\vdash (\gamma_1 \wedge \gamma_2) \rightarrow \varphi$

$$\Rightarrow [\gamma_1 \wedge \gamma_2] \vdash \varphi, \quad (\text{II})$$

Sea $[\gamma_1 \wedge \gamma_2] \vdash \varphi$ entonces:

$$\vdash (\gamma_1 \wedge \gamma_2) \rightarrow \varphi, \Rightarrow [\gamma_1, \gamma_2] \vdash (\gamma_1 \wedge \gamma_2) \rightarrow \varphi$$

$$\left. \begin{array}{l} [\gamma_1, \gamma_2] \vdash \gamma_1 \\ [\gamma_1, \gamma_2] \vdash \gamma_2 \end{array} \right\} \xrightarrow{\text{conj.}} [\gamma_1, \gamma_2] \vdash \gamma_1 \wedge \gamma_2$$

$$\xrightarrow{(MP)} [\gamma_1, \gamma_2] \vdash \varphi, \quad (\text{III})$$

Sea $[\gamma_1, \gamma_2] \vdash \varphi$, entonces:

$$\begin{aligned} &\vdash \gamma_1 \rightarrow (\gamma_2 \rightarrow \varphi) \quad (\text{cf. (I)}) \Rightarrow \\ &\Rightarrow [\gamma_1 \wedge \gamma_2] \vdash \gamma_2 \rightarrow (\gamma_2 \rightarrow \varphi), \\ &\text{por } \vdash (\gamma_1 \wedge \gamma_2) \rightarrow \gamma_2 \xrightarrow{(\text{II})} [\gamma_1 \wedge \gamma_2] \vdash \gamma_2 \end{aligned}$$

$$\xrightarrow{(MP)} [\gamma_1 \wedge \gamma_2] \vdash \gamma_2 \rightarrow \varphi, \quad \xrightarrow{(\text{II})} [\gamma_1 \wedge \gamma_2] \vdash \gamma_2$$

$$\xrightarrow{(MP)} [\gamma_1 \wedge \gamma_2] \vdash \varphi, \quad (\text{IV})$$

$$\begin{aligned} &(\text{I}), (\text{II}), (\text{III}), (\text{IV}) \Rightarrow \vdash (\gamma_1 \wedge \gamma_2) \rightarrow \varphi \\ &\Leftrightarrow \vdash \gamma_1 \rightarrow (\gamma_2 \rightarrow \varphi) \end{aligned}$$

$$\frac{n \geq 2 \rightarrow n+1}{}$$

Prin ip. de ind, $\vdash \left(\bigwedge_{i=1}^n \delta_i \right) \rightarrow \varphi \Leftrightarrow$
 $\Leftrightarrow \vdash \delta_1 \rightarrow (\delta_2 \rightarrow (\dots \rightarrow (\delta_n \rightarrow \varphi) \dots))$

$\vdash \left(\bigwedge_{i=1}^{n+1} \delta_i \right) \rightarrow \varphi \stackrel{(\text{def})}{\Leftrightarrow}$
 $\Leftrightarrow \vdash \left(\left(\bigwedge_{i=1}^n \delta_i \right) \wedge \delta_{n+1} \right) \rightarrow \varphi \stackrel{(\text{casul } n=2)}{\Leftrightarrow}$

$\Leftrightarrow \vdash \left(\bigwedge_{i=1}^n \delta_i \right) \rightarrow (\delta_{n+1} \rightarrow \varphi) \stackrel{(\text{ip. de ind})}{\Leftrightarrow}$

$\Leftrightarrow \vdash \delta_1 \rightarrow (\delta_2 \rightarrow (\dots \rightarrow (\delta_n \rightarrow (\delta_{n+1} \rightarrow \varphi) \dots)))$

Exerc.: $\Sigma \subseteq E; \varphi \in E$.

At. $\Sigma \vdash \varphi \Leftrightarrow (\exists n \in \mathbb{N}^*) (\exists \delta_2, \dots, \delta_n \in \Sigma) \left(\vdash \left(\bigwedge_{i=1}^n \delta_i \right) \rightarrow \varphi \right)$

RESOLVARE:

$\Sigma \vdash \varphi \stackrel{(\text{prop.})}{\Leftrightarrow} (\exists \Gamma \subseteq \Sigma) (|\Gamma| < \omega \wedge \Gamma \vdash \varphi)$

*de. nipl e 0, i.e. $\Gamma = \emptyset$
 et. $\text{luam (npluam)} \Gamma = \{\delta\}, n=1$*

de. $\Gamma \neq \emptyset$ find: $n=0$.

de. $\Gamma = \{\delta_2, \dots, \delta_n\}, \Gamma = \{\delta_2, \dots, \delta_n\}$

$$\text{Axioma: } \Sigma \vdash \varphi \Leftrightarrow (\exists n \in \mathbb{N}^*)$$

$$(\exists \gamma_1, \dots, \gamma_n \in \Sigma) (\{\gamma_1, \dots, \gamma_n\} \vdash \varphi)$$

$$\text{donc: } \{\gamma_1, \dots, \gamma_n\} \vdash \varphi \Leftrightarrow$$

$$\Leftrightarrow \{\gamma_1, \dots, \gamma_{n-1}\} \vdash \gamma_n \rightarrow \varphi$$

$$\Leftrightarrow \{\gamma_1, \dots, \gamma_{n-2}\} \vdash \gamma_{n-1} \rightarrow (\gamma_n \rightarrow \varphi)$$

$$\Leftrightarrow \dots \Leftrightarrow \vdash \gamma_1 \rightarrow (\gamma_2 \rightarrow (\dots \rightarrow (\gamma_n \rightarrow \varphi) \dots))$$

(exercice)

$$\vdash \left(\bigwedge_{i=1}^n \gamma_i \right) \rightarrow \varphi$$

$$\text{Pour montrer: } \Sigma \vdash \varphi \Leftrightarrow$$

$$\Leftrightarrow (\exists n \in \mathbb{N}^*) (\exists \gamma_1, \dots, \gamma_n \in \Sigma) \left(\vdash \gamma_1 \rightarrow (\gamma_2 \rightarrow (\dots \rightarrow (\gamma_n \rightarrow \varphi) \dots)) \right)$$

$$\Leftrightarrow (\exists n \in \mathbb{N}^*) (\exists \gamma_1, \dots, \gamma_n \in \Sigma) \left(\vdash \left(\bigwedge_{i=1}^n \gamma_i \right) \rightarrow \varphi \right)$$

OBS: Si on résolvait de nos jours
résultat: $(\forall n \in \mathbb{N}^*) (\forall \gamma_1, \dots, \gamma_n \in \Sigma) \Leftrightarrow$
donc: $\{\gamma_1, \dots, \gamma_n\} \vdash \varphi \Leftrightarrow \vdash \left(\bigwedge_{i=1}^n \gamma_i \right) \rightarrow \varphi$
 $\Leftrightarrow \vdash \gamma_1 \rightarrow (\gamma_2 \rightarrow (\dots \rightarrow (\gamma_n \rightarrow \varphi) \dots))$
 $\Leftrightarrow \left(\bigwedge_{i=1}^n \gamma_i \right) \vdash \varphi$