Funcţii Caracteristice

SEMINAR DE LOGICĂ MATEMATICĂ ŞI COMPUTAŢIONALĂ

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Exerc.: Să se dem. asociativitatea lui Δ, folosind funcții caracteristice.

REZOLVARE:

Fie A, B, C mulțimi.

$A \Delta (B \Delta C) \overset{?}{=} (A \Delta B) \Delta C$.

Fie $T = A \cup B \cup C \cup \{0\}$. $\Rightarrow T \neq \emptyset$ și $A \subseteq T$, $B \subseteq T$, $C \subseteq T$. $\Rightarrow A \Delta (B \Delta C) \subseteq T$ și $(A \Delta B) \Delta C \subseteq T$.

Pt. orice $X \subseteq T$, fie χ_X funcția caracteristică a lui X raportat la T.

$$\chi_{A\Delta(B\Delta C)} = \chi_A + \chi_{B\Delta C} - 2\cdot\chi_A\cdot\chi_{B\Delta C} = \chi_A + \chi_B + \chi_C - 2\cdot\chi_B\cdot\chi_C - 2\cdot\chi_A\cdot(\chi_B + \chi_C - 2\chi_B\chi_C) = \chi_A + \chi_B + \chi_C - 2\chi_A\chi_B - 2\chi_A\chi_C - 2\chi_B\chi_C + 4\chi_A\chi_B\chi_C. \quad (*)$$

Altfel: știm că Δ e comutativă, așadar: $(A\Delta B)\Delta C = C\Delta(A\Delta B)$, deci:

$$\chi_{(A\Delta B)\Delta C} = \chi_{C\Delta(A\Delta B)} \overset{(*),\ \text{tripletul } (A,B,C) \to (C,A,B)}{=\!=\!=} \chi_C + \chi_A + \chi_B - 2\chi_C\chi_A - 2\chi_C\chi_B - 2\chi_A\chi_B + 4\chi_C\chi_A\chi_B \overset{(*)}{=} \chi_{A\Delta(B\Delta C)}$$

$$\chi_{(A\Delta B)\Delta C} = \chi_{A\Delta B} + \chi_C - 2\chi_{A\Delta B}\chi_C = \chi_A + \chi_B - 2\chi_A\chi_B + \chi_C - 2(\chi_A + \chi_B - 2\chi_A\chi_B)\cdot\chi_C = \chi_A + \chi_B + \chi_C - 2\chi_A\chi_B - 2\chi_A\chi_C - 2\chi_B\chi_C + 4\chi_A\chi_B\chi_C. \quad (**)$$

$(*), (**) \Rightarrow \chi_{A\Delta(B\Delta C)} = \chi_{(A\Delta B)\Delta C} \Leftrightarrow A\Delta(B\Delta C) = (A\Delta B)\Delta C$.

Exerc. (legile de distrib. generalizate pt. $\cup$ şi $\cap$ de mulţimi):

$A \to$ mulţime,

$I, J \to$ mulţimi, $I \neq \emptyset$, $J \neq \emptyset$,

$(A_i)_{i \in I}$, $(B_j)_{j \in J} \to$ familii de mulţimi.

Dem. că:

(1) $A \cup \left(\bigcap_{j \in J} B_j\right) = \bigcap_{j \in J} (A \cup B_j)$;

(2) $A \cap \left(\bigcup_{j \in J} B_j\right) = \bigcup_{j \in J} (A \cap B_j)$;

(3) $\left(\bigcap_{i \in I} A_i\right) \cup \left(\bigcap_{j \in J} B_j\right) = \bigcap_{i \in I} \bigcap_{j \in J} (A_i \cup B_j)$ $\left(= \bigcap_{j \in J} \bigcap_{i \in I} (A_i \cup B_j)\right)$;

(4) $\left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{j \in J} B_j\right) = \bigcup_{i \in I} \bigcup_{j \in J} (A_i \cap B_j)$ $\left(= \bigcup_{j \in J} \bigcup_{i \in I} (A_i \cap B_j)\right)$.

REZ: Fie $T := A \cup \bigcup_{i \in I} A_i \cup \bigcup_{j \in J} B_j \cup \{0\} \neq \emptyset$, şi $(\forall M \subseteq T)(\chi_M :=$

:= fct. caract. a lui M raportat la T).

(1) Notăm: $f := \chi_{A \cup \left(\bigcap_{j \in J} B_j\right)} : T \to \{0,1\}$ și $g := \chi_{\bigcap_{j \in J}(A \cup B_j)} : T \to \{0,1\}$. ~~$f \overset{?}{=} g$.~~

Fie $x \in T$.

$f(x) = \max\{\chi_A(x), \min\{\chi_{B_j}(x) \mid j \in J\}\}$

$g(x) = \min\{\max\{\chi_A(x), \chi_{B_j}(x)\} \mid j \in J\}$.

Caz 1: $\min\{\chi_{B_j}(x) \mid j \in J\} = 0 \Leftrightarrow \left(\exists\ j_0 \in J\right) \left(\chi_{B_{j_0}}(x) = 0\right) \Rightarrow f(x) = \max\{\chi_A(x), 0\} = \chi_A(x)$

$g(x) \overset{?}{=} \chi_A(x)$.

$g(x) = \min\{\max\{\chi_A(x), \chi_{B_j}(x)\} \mid j \in$
$\in J\} \overset{(j_0 \in J)}{\leq} \max\{\chi_A(x), \chi_{B_{j_0}}(x)\} =$
$= \max\{\chi_A(x), 0\} = \chi_A(x)$

$g(x) = \min\{\underbrace{\max\{\chi_A(x), \chi_{B_j}(x)\}}_{\geq \chi_A(x)} \mid$
$j \in J\} \geq \min\{\chi_A(x) \mid j \in J\} = \chi_A(x)$

$\Rightarrow g(x) = \chi_A(x) = f(x)$.

Case 2: $\min\{\chi_{B_j}(x) \mid j \in J\} = 1$
$\Leftrightarrow (\forall j \in J)(\chi_{B_j}(x) = 1)$.

$\Downarrow$

$f(x) =$
$= \max\{\chi_A(x), 1\} = 1$.

$g(x) = \min\{\max\{\chi_A(x), 1\} \mid j \in$
$\in J\} = \min\{1 \mid j \in J\} = 1 = f(x)$.

$\Rightarrow (\forall x \in T)(f(x) = g(x)) \Leftrightarrow f = g$

$$\Leftrightarrow \chi_{A \cup \left(\bigcap\limits_{j \in J} B_j\right)} = \chi_{\bigcap\limits_{j \in J} (A \cup B_j)} \Leftrightarrow$$

$$\Leftrightarrow A \cup \left(\bigcap_{j \in J} B_j\right) = \bigcap_{j \in J} \left(A \cup B_j\right).$$

(2) Analog.

(3) $\Leftarrow$ (1), astfel"

$$\left(\bigcap_{i \in I} A_i\right) \cup \left(\bigcap_{j \in J} B_j\right) \stackrel{(1)}{=} \bigcap_{j \in J} \left(\left(\bigcap_{i \in I} A_i\right) \cup B_j\right) \stackrel{(1)}{=} \bigcap_{j \in J} \bigcap_{i \in I} \left(A_i \cup B_j\right).$$

$$\stackrel{(1)}{=}$$

$$\bigcap_{i \in I} \left(A_i \cup \left(\bigcap_{j \in J} B_j\right)\right) \stackrel{(1)}{=} \bigcap_{i \in I} \bigcap_{j \in J} \left(A_i \cup B_j\right).$$

(4) $\Leftarrow$ (2), analog.

Exerc. (legile lui De Morgan generalizate):

$T, I \in Set$; $T \neq \emptyset$; $(A_i)_{i \in I} \subseteq \mathcal{P}(T)$; $(\forall A \in \mathcal{P}(T))(\overline{A} := T \setminus A)$.

Dem. că:

- $\overline{\bigcup_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i}$;
- $\overline{\bigcap_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$.

REZOLVARE: Avem: $(\overline{A_i})_{i \in I} \subseteq \mathcal{P}(T)$.

Cazul 1: $I = \emptyset$:

$$\overline{\bigcup_{i \in \emptyset} A_i} = \overline{\emptyset} = T = \bigcap_{i \in \emptyset} \overline{A_i};$$

$$\overline{\bigcap_{i \in \emptyset} A_i} = \overline{T} = \emptyset = \bigcup_{i \in \emptyset} \overline{A_i}.$$

Cazul 2: Pp. $I \neq \emptyset$.

METODA I: Fie $a \in T$.

$$a \in \overline{\bigcup_{i \in I} A_i} \Leftrightarrow a \notin \bigcup_{i \in I} A_i \Leftrightarrow$$

$$\Leftrightarrow (\nexists\, i \in I)(a \in A_i) \Leftrightarrow (\forall\, i \in I)(a \notin A_i)$$

$$\Leftrightarrow (\forall\, i \in I)(a \in \overline{A_i}) \Leftrightarrow a \in \bigcap_{i \in I} \overline{A_i}.$$

Cum $\overline{\bigcup_{i \in I} A_i} \subseteq T \supseteq \bigcap_{i \in I} \overline{A_i}$, rezultă că $\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i}$. (*)

Pentru a doua lege a lui De Morgan, putem proceda analog sau o putem folosi pe prima, plus faptul că $(\forall A\in\mathcal{P}(T))(\overline{\overline{A}}=A)$:

$$\bigcup_{i\in I} A_i = \bigcup_{i\in I} \overline{\overline{A_i}} = \overline{\overline{\bigcup_{i\in I} \overline{A_i}}} \overset{(*)}{=} \overline{\bigcap_{i\in I} \overline{\overline{A_i}}} = \overline{\bigcap_{i\in I} A_i}.$$

METODA II: $(\forall A\in\mathcal{P}(T))(\chi_A :=$ fct. caracteristică a lui A raportată la T);

$1 :=$ fct. constantă 1 pe T:

$1 : T \to \{0,1\}, (\forall x\in T)(1(x)=1)$.

Amintesc că, $\forall A\in\mathcal{P}(T)$:

$$\chi_{\overline{A}} = \chi_{T\setminus A} = 1 \setminus \chi_A.$$

Așadar:

$$\chi_{\overline{\bigcup_{i\in I} A_i}} = 1 - \chi_{\bigcup_{i\in I} A_i} \overset{\text{(v. cursul)}}{=}$$

$$= 1 - \max\{\chi_{A_i} \mid i\in I\} \overset{\text{(v. mai jos)}}{=}$$

$$= \min\{1 - \chi_{A_i} \mid i\in I\} =$$

$$= \min\{\chi_{\overline{A_i}} \mid i\in I\} \overset{\text{(v. cursul)}}{=} \chi_{\bigcap_{i\in I} \overline{A_i}}\,,$$

prin urmare $\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i}$,

deoarece, pt. orice $x \in T$, avem:

$\left(\mathbb{1} - \max\{\chi_{A_i} \mid i \in I\}\right)(x) =$
$= \mathbb{1}(x) - \max\{\chi_{A_i}(x) \mid i \in I\} =$
$= 1 - \max\{\chi_{A_i}(x) \mid i \in I\} \overset{\text{(vezi mai jos)}}{=}$
$= \min\{1 - \chi_{A_i}(x) \mid i \in I\} =$
$= \min\{\mathbb{1}(x) - \chi_{A_i}(x) \mid i \in I\} =$
$= \min\{(\mathbb{1} - \chi_{A_i})(x) \mid i \in I\} =$
$= \left(\min\{\mathbb{1} - \chi_{A_i} \mid i \in I\}\right)(x)$,

întrucât, dacă $i_0 \in I$ a.î. $\chi_{A_{i_0}}(x) = \max\{\chi_{A_i}(x) \mid i \in I\}$, atunci:

pt. fiecare $i \in I$, $\chi_{A_i}(x) \leq \chi_{A_{i_0}}(x)$,

deci $1 - \chi_{A_i}(x) \geq 1 - \chi_{A_{i_0}}(x)$,

așadar $1 - \max\{\chi_{A_i}(x) \mid i \in I\} = 1 - \chi_{A_{i_0}}(x) = \min\{1 - \chi_{A_i}(x) \mid i \in I\}$.

Procedăm la fel pt. a doua lege a lui De Morgan, cu toate că am arătat mai sus că rezultă din prima și autodualitatea complementării:

$$\chi_{\overline{\bigcap_{i\in I} A_i}} = 1 - \chi_{\bigcap_{i\in I} A_i} \overset{\text{(v. cursul)}}{=}$$

$$= 1 - \min\{\chi_{A_i} \mid i \in I\} \overset{\text{(analog dem. de mai sus)}}{=}$$

$$= \max\{1 - \chi_{A_i} \mid i \in I\} =$$

$$= \max\{\chi_{\overline{A_i}} \mid i \in I\} \overset{\text{(v. cursul)}}{=} \chi_{\bigcup_{i\in I} \overline{A_i}}$$

prin urmare $\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$.