

Logică Propozițională Clasică – demonstrații algebrice și semantice

SEMINAR DE LOGICĂ MATEMATICĂ ȘI COMPUTAȚIONALĂ

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Universitatea din București, Facultatea de Matematică și Informatică, Semestrul I, 2022-2023

Exercițiu: Se se dă un algebră \mathcal{E} ,
 $(\forall \varphi, \psi, x \in \mathcal{E})$ (cu loc.)

(i) $\vdash (\varphi \wedge \psi) \rightarrow \psi$
(ii) $\vdash \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$
(iii) $\vdash (\varphi \rightarrow (\psi \rightarrow x)) \leftrightarrow (\psi \rightarrow (\varphi \rightarrow x))$

rezolvare:

(i) și (ii) sunt de fapt necesare
pt. construire algebră Lindenbaum –
Tarski, pe baza căreia se fac
demonstrații algebrice.

$(\vdash \varphi \in \mathcal{E})$ (not. cu $\varphi \rightarrow \perp$ close line
și în algebră Lindenbaum – Tarski
 \mathcal{E}/\sim).

(i) $x := (\varphi \wedge \psi) \rightarrow \psi \in \mathcal{E}/\sim$.

$\vdash x \quad \text{(demonstrare)} \quad x = 1,$

$\boxed{x = (\varphi \wedge \psi) \rightarrow \psi = 1, \text{ pt. } \varphi \wedge \psi \leq \psi \text{ în algebră Boole } \mathcal{E}/\sim}$

$\Rightarrow \vdash \alpha$
 (iii) $\beta := \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi)) \in E$
 Note: $x := \overline{\varphi}, y := \overline{\psi}$

$$\begin{aligned}
 \Rightarrow \overline{\beta} &= \overline{\varphi} \rightarrow (\overline{\psi} \rightarrow (\overline{\varphi} \wedge \overline{\psi})) = \text{(i)} \\
 &= x \rightarrow (y \rightarrow (x \wedge y)) = \text{(ii)} \\
 &= \overline{x} \vee \overline{y} \vee (x \wedge y) = \\
 &= (\overline{x} \vee \overline{y} \vee x) \wedge (\overline{x} \vee \overline{y} \vee y) = \\
 &= (1 \vee \overline{y}) \wedge (\overline{x} \vee 1) = \text{(a)} \quad 1 \wedge 1 = 1 \Rightarrow \\
 &\stackrel{\text{(end)}}{\Rightarrow} \vdash \beta
 \end{aligned}$$

(iii) $\gamma := (\varphi \rightarrow (\psi \rightarrow x)) \leftrightarrow (\psi \rightarrow (\varphi \rightarrow x)) \in E$
 Note: $x := \overline{\varphi}, y := \overline{\psi}, z := \overline{x}$

$$\begin{aligned}
 \widehat{\gamma} &= (\overline{\varphi} \rightarrow (\overline{\psi} \rightarrow \overline{x})) \leftrightarrow (\overline{\psi} \rightarrow (\overline{\varphi} \rightarrow \overline{x})) = \\
 &= (x \rightarrow (y \rightarrow z)) \leftrightarrow (y \rightarrow (x \rightarrow z)) = \\
 &= (\overline{x} \vee \overline{y} \vee z) \leftrightarrow (\overline{y} \vee \overline{x} \vee z) = \\
 &= (\overline{x} \vee \overline{y} \vee z) \leftrightarrow (\overline{x} \vee \overline{y} \vee z) = 1,
 \end{aligned}$$

yb. so e o echar dool, autre 2
 termeni egeli. $\stackrel{\text{(end)}}{\Rightarrow} \vdash \gamma$

Exercițiu: Se se dă, semnificația unor
reguli de deducție: $(\forall \Gamma_1, \Gamma_2, \Gamma_3 \subseteq E)$
 $(\forall \varphi, \psi, \delta \in E)$

$$\frac{\Gamma_1 \vdash \varphi \vee \psi \quad \Gamma_2 \cup \{\varphi\} \vdash \delta \quad \Gamma_3 \cup \{\psi\} \vdash \delta}{\Gamma_2 \cup \Gamma_3 \vdash \delta}$$

rezolvare:

$$\text{Pp. } \Leftrightarrow \left| \begin{array}{l} \Gamma_2 \vdash \varphi \vee \psi \xrightarrow{(\text{def})} \Gamma_1 \vdash \varphi \vee \psi, (\star) \\ \Gamma_2 \cup \{\varphi\} \vdash \delta \xrightarrow{(\text{def})} \Gamma_2 \cup \{\varphi\} \vdash \delta, (\star) \\ \Gamma_3 \cup \{\psi\} \vdash \delta \xrightarrow{(\text{def})} \Gamma_3 \cup \{\psi\} \vdash \delta, (\star) \end{array} \right.$$

Rezolvare: $\lambda: V \rightarrow \mathbb{Z}_2$

$$\Leftrightarrow \left| \begin{array}{l} \lambda \models \Gamma_1 \xrightarrow{(\star)} \widetilde{\lambda}(\varphi \vee \psi) = 1 \Leftrightarrow \widetilde{\lambda}(\varphi) \vee \widetilde{\lambda}(\psi) = 1 \\ \lambda \models \Gamma_2 \quad (2) \\ \lambda \models \Gamma_3 \quad (3) \end{array} \right.$$

$$\begin{aligned} &\text{Sc. } \widetilde{\lambda}(\varphi) = 1, \text{ st. (2)} \quad \widetilde{\lambda}(\varphi) = 1 \\ &\Rightarrow \lambda \models \Gamma_2 \cup \{\varphi\} \xrightarrow{(\star)} \widetilde{\lambda}(\delta) = 1 \\ &\text{Sc. } \widetilde{\lambda}(\psi) = 1, \text{ st. (3)} \quad \widetilde{\lambda}(\psi) = 1 \\ &\Rightarrow \lambda \models \Gamma_3 \cup \{\psi\} \xrightarrow{(\star)} \widetilde{\lambda}(\delta) = 1 \\ &\Rightarrow \text{In fiecare caz, } \widetilde{\lambda}(\delta) = 1 \Rightarrow \end{aligned}$$

$$\Rightarrow \Gamma_2 \cup \Gamma_3 \vdash \delta \xrightarrow{(\text{def})} \Gamma_2 \cup \Gamma_3 \vdash \delta$$

Exerc. Fixe $\alpha, \beta, \gamma \in E$, fixe
se dem. (a) in logica prop.
descriere.

$$\{ \alpha, \beta \rightarrow \gamma \} \vdash \neg \alpha \vee \neg \beta \vee \gamma.$$

(a) algebraic; (b) semantic.

RESOLVARE:

$$\text{Fixe } \Sigma = \{ \alpha, \beta \rightarrow \gamma \} \vdash \neg \alpha \vee \neg \beta \vee \gamma.$$

(a) Avem, pentru orice
 $q \in E$: $\Sigma \vdash q \Leftrightarrow q \vee \neg \Sigma = 1$
in algebra Boole $(\bigvee_{\alpha \in \Sigma} \alpha) \vee (\bigvee_{\alpha \in \neg \Sigma} \neg \alpha)$
 $\neg \Sigma \leq \Sigma \Rightarrow \neg \Sigma \vdash \neg \Sigma = 1$,

Notand cu $\rightarrow \Sigma$ implicatia
lui de la in cele doua

algebra Boole si folosind
faptul ca pentru orice
elemente x, y de la unii

algebra Boole B , avem:

$$x \rightarrow y = 1 \Leftrightarrow x \leq y \text{ obiectiv.}$$

$$\begin{aligned}
 & \gamma, \beta \rightarrow \gamma, \gamma \rightarrow \gamma \alpha \in \Sigma \Rightarrow \\
 & \vdash \alpha \Leftrightarrow \alpha / \gamma \Sigma = \gamma \Sigma \quad (*) \\
 & \vdash \beta \rightarrow \gamma \Leftrightarrow \gamma \Sigma = (\beta \rightarrow \gamma) / \gamma \Sigma \\
 & = \beta / \gamma \Sigma \rightarrow \gamma \Sigma \beta / \gamma \Sigma \Leftrightarrow \\
 & \Leftrightarrow \beta / \gamma \Sigma \leq \gamma \Sigma \beta / \gamma \Sigma \quad (**)
 \end{aligned}$$

$$\begin{aligned}
 & \vdash \gamma \rightarrow \gamma \alpha \Leftrightarrow \gamma \Sigma = \\
 & = (\gamma \rightarrow \gamma \alpha) / \gamma \Sigma = \gamma / \gamma \Sigma \rightarrow \Sigma \\
 & \rightarrow \Sigma \alpha / \gamma \Sigma \Leftrightarrow \gamma / \gamma \Sigma \leq \gamma \Sigma \\
 & \Leftrightarrow \alpha / \gamma \Sigma \leq \gamma / \gamma \Sigma \quad . \quad (***) \\
 & (*) \quad (***) \Rightarrow \gamma \Sigma = \gamma / \gamma \Sigma \leq \gamma \Sigma \\
 & = \gamma / \gamma \Sigma \cdot (1) \\
 & (***) \Rightarrow \gamma / \gamma \Sigma \leq \gamma (\beta / \gamma \Sigma) \cdot
 \end{aligned}$$

$$\Rightarrow \gamma_{\Sigma} = \left(\frac{\beta}{n_{\Sigma}}\right)^{\Sigma} = \frac{(\gamma_{\beta})}{n_{\Sigma}} \cdot (2)$$

$$(2), (2) \Rightarrow (\gamma_{\beta} \cdot \gamma_{\beta}) / n_{\Sigma} = \\ = (\gamma_{\beta}) / n_{\Sigma} \approx (\gamma_{\beta}) / n_{\Sigma} = \gamma_{\Sigma} \gamma_{\Sigma} = \\ = \gamma_{\Sigma} \Rightarrow \Sigma \vdash \gamma_{\beta} \cdot \gamma_{\beta}.$$



! ATENȚIE: Nu este
neobișnuit $E/n_{\Sigma} = E_{\Sigma}, \gamma_{\Sigma}$
și nu este ușor $\sigma_{\Sigma} \neq \gamma_{\Sigma}$
căci nu sună fără
efectiv un răsonament
reducere la absurd,

$$(b) \quad \Sigma \vdash \gamma_{\beta} \cdot \gamma_{\beta} \text{ (fct)}$$

$$\cancel{\text{fct}} \Rightarrow \Sigma \vdash \gamma_{\beta} \cdot \gamma_{\beta} \text{ (def)}$$

$$\cancel{\text{def}} \quad (\vdash \text{d}: V \rightarrow \mathcal{L}_2)$$

$$(\text{d} \vdash \Sigma \Rightarrow \text{d} \vdash \gamma_{\beta} \cdot \gamma_{\beta})$$

$$\text{Fie } \text{d}: V \rightarrow \mathcal{L}_2 \text{ e. i.}$$

$$d \models \Sigma, \text{def} (\forall \tau \in \Sigma) (d \models \tau) \Leftrightarrow$$

$$\text{def} (\forall \tau \in \Sigma) (\tilde{\nu}(\tau) = \top). \Leftrightarrow$$

$$\begin{cases} \tilde{\nu}(\alpha) = \top \\ \top = \tilde{\nu}(\beta \rightarrow \gamma) = \tilde{\nu}(\beta) \rightarrow \tilde{\nu}(\gamma) \\ \top = \tilde{\nu}(\gamma \rightarrow \beta) = \tilde{\nu}(\gamma) \rightarrow \tilde{\nu}(\beta) \end{cases}$$

$$\text{def} \quad \tilde{\nu}(\gamma) \rightarrow \top = \tilde{\nu}(\gamma) \rightarrow 0. \Rightarrow$$

$$\Rightarrow \tilde{\nu}(\gamma) \leq 0 \Leftrightarrow \tilde{\nu}(\gamma) = 0. \text{def}$$

$$\Rightarrow \tilde{\nu}(\beta) \leq 0 \Leftrightarrow \tilde{\nu}(\beta) = 0.$$

$$\begin{aligned} \Rightarrow \tilde{\nu}(\neg \beta \wedge \neg \gamma) &= \overline{\tilde{\nu}(\beta)} \wedge \overline{\tilde{\nu}(\gamma)} = \\ &= \top \wedge \top = \top \wedge \top = \top. \Rightarrow d \models \neg \beta \wedge \neg \gamma. \end{aligned}$$

$$\text{def} \quad \Sigma \models \neg \beta \wedge \neg \gamma \Leftrightarrow \Sigma \vdash \neg \beta \wedge \neg \gamma.$$

Intuit:

$$\tilde{\nu}: E \rightarrow L_2 = \{0, \top\}$$

in $0 \neq \top$, and we can proceed
to prove reduction to absurd:

$\frac{\text{P.P. prove absurd}}{\tilde{\nu}(\neg \beta \wedge \neg \gamma) \neq \top}$

$$\tilde{\nu}(\neg \beta \wedge \neg \gamma) \neq \top \Leftrightarrow \tilde{\nu}(\neg \beta \wedge \neg \gamma) = 0$$

$$\begin{aligned}
 & \Leftrightarrow \overline{\text{In}(\beta) \wedge \text{In}(\gamma)} = 0 \Leftrightarrow \\
 & \Leftrightarrow \overline{\text{In}(\beta)} = 0 = \overline{1} \text{ sau } \overline{\text{In}(\gamma)} = 0 = \overline{1} \\
 & \Leftrightarrow \overline{\text{In}(\beta)} = 1 \text{ sau } \overline{\text{In}(\gamma)} = 1 \\
 & \Leftrightarrow \overline{\text{In}(\gamma)} = 1. \\
 & \text{dintre } \circlearrowleft \text{ si } \circlearrowright, \text{ In}(\beta) = 1 \Rightarrow \\
 & \qquad \qquad \qquad \Rightarrow \overline{\text{In}(\gamma)} = 1. \\
 & \text{Dar } \overline{\text{In}(\gamma)} \leq \overline{\text{In}(\alpha)}, \text{ pentru} \\
 & \Leftarrow \overline{\text{In}(\gamma)} \rightarrow \overline{\text{In}(\alpha)} = 1. \\
 & \Rightarrow 1 \leq \overline{\text{In}(\alpha)} \Leftrightarrow \overline{\text{In}(\alpha)} = 1 \Leftrightarrow \\
 & \Leftrightarrow \overline{\text{In}(\alpha)} = \overline{1} = 0 \neq 1 \xrightarrow{\text{au } L_2} \text{ sau } \circlearrowleft \\
 & \Rightarrow \overline{\text{In}(\beta \wedge \gamma)} = 1. \xrightarrow{\substack{\text{(def)} \\ \text{tot}}} \sum \vdash \neg \beta \wedge \neg \gamma \Rightarrow \\
 & \qquad \qquad \qquad \xrightarrow{\substack{\text{tot} \\ \text{tot}}} \sum \vdash \neg \beta \wedge \neg \gamma.
 \end{aligned}$$

Exerc.: Fie $\alpha, \varphi, \psi, \chi \in E$, a.s.

$\alpha = (\varphi \rightarrow \neg(\neg \psi \rightarrow \chi)) \leftrightarrow ((\varphi \rightarrow \neg \psi) \wedge (\chi \rightarrow \neg \psi))$, demonstra

$\vdash \alpha$: $\begin{cases} (a) & \text{algebraic} \\ (b) & \text{semantic} \end{cases}$

$\varphi = (s)$

RESOLVARE: kabis s das s true

(a) Fie $x := \overline{\varphi}, y := \overline{\psi}$ și $z := \overline{x}$
în E/\sim'

$$\begin{aligned} \Rightarrow \overline{z} &= (\overline{\varphi} \rightarrow (\overline{\psi} \rightarrow \overline{x})) \Leftrightarrow \\ \Leftrightarrow ((\overline{\psi} \rightarrow \overline{\varphi}) \wedge (\overline{x} \rightarrow \overline{\varphi})) &= \\ = (x \rightarrow (\overline{\psi} \rightarrow z)) \Leftrightarrow (y \rightarrow \overline{x}) \wedge (z \rightarrow \overline{x}) &= \\ = (x \rightarrow (\overline{\psi} \vee z)) \Leftrightarrow ((\overline{\psi} \vee x) \wedge (\overline{z} \vee x)) &= \\ = (x \vee (\overline{\psi} \vee z)) \Leftrightarrow ((\overline{\psi} \wedge z) \vee x) &\quad (\text{de Morgan}) \\ = (x \vee (\overline{\psi} \wedge \overline{z})) \Leftrightarrow (x \vee (\overline{\psi} \wedge \overline{z})) &= 1 \end{aligned}$$

find \Rightarrow echivalentă booleană
într-o formă termenă foli,

$$\text{Deci } \overline{z} = 1 \Rightarrow \vdash z$$

(b)

Demonstrație $\vdash \vdash z$

Fie $h: V \rightarrow L_2$, arbitrară, fixată.

$$\begin{aligned} \widetilde{h}(z) &= (\widetilde{h}(\varphi) \rightarrow (\widetilde{h}(\psi) \rightarrow \widetilde{h}(x))) \Leftrightarrow \\ \Leftrightarrow ((\widetilde{h}(\psi) \rightarrow \widetilde{h}(\varphi)) \wedge (\widetilde{h}(x) \rightarrow \widetilde{h}(\varphi))) &= \dots \end{aligned}$$

Deoicică putem continua cu un

calcul boolean, ca le punctul (α),

seu putem procede astfel: notam

$$\beta := \varphi \rightarrow \neg(\neg\psi \rightarrow x) \in E \quad \text{si}$$

$$\gamma := (\psi \rightarrow \neg\varphi) \wedge (x \rightarrow \neg\varphi) \in E.$$

$$\tilde{\alpha}(\alpha) = \tilde{\alpha}(\beta) \leftrightarrow \tilde{\alpha}(\gamma),$$

Avem: $\tilde{\alpha}(\alpha) = 1 \Leftrightarrow \tilde{\alpha}(\beta) = \tilde{\alpha}(\gamma).$

Caz 1: $\tilde{\alpha}(\beta) = 0 \Leftrightarrow$

$$\Leftrightarrow \tilde{\alpha}(\varphi) \rightarrow \overline{(\tilde{\alpha}(\psi) \rightarrow \tilde{\alpha}(x))} = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \tilde{\alpha}(\varphi) = 1 \\ \tilde{\alpha}(\psi) = 0 \end{cases} \quad \text{bunăstătare} \quad \text{bunăstătare} \quad \text{bunăstătare}$$

$$\Leftrightarrow \overline{\tilde{\alpha}(\psi) \rightarrow \tilde{\alpha}(x)} = 0 \Leftrightarrow$$

$$\Leftrightarrow \overline{\tilde{\alpha}(\psi) \rightarrow \tilde{\alpha}(x)} = 1 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \tilde{\alpha}(\psi) = 0 \\ \tilde{\alpha}(x) = 1 \end{cases} \quad \text{bunăstătare} \quad \text{bunăstătare} \quad \text{bunăstătare}$$

Avem: $\tilde{\alpha}(\beta) = 0 \Leftrightarrow \begin{cases} \tilde{\alpha}(\varphi) = \tilde{\alpha}(\psi) = 1 \\ \tilde{\alpha}(x) = 1 \end{cases}$

$$\Leftrightarrow \tilde{\alpha}(\varphi) = \tilde{\alpha}(x) = 1$$

$$\begin{aligned}
 &\Leftrightarrow \left\{ \begin{array}{l} \tilde{\alpha}(\psi) = 1 \quad \text{si} \quad \overline{\tilde{\alpha}(\varphi)} = 0 \\ \text{sau} \\ \tilde{\alpha}(x) = 1 \quad \text{si} \quad \overline{\tilde{\alpha}(\varphi)} = 0 \end{array} \right. \\
 &\Leftrightarrow \left\{ \begin{array}{l} \tilde{\alpha}(\psi) \rightarrow \overline{\tilde{\alpha}(\varphi)} = 0 \\ \text{sau} \\ \tilde{\alpha}(x) \rightarrow \overline{\tilde{\alpha}(\varphi)} = 0 \end{array} \right. \Leftrightarrow \\
 &\Leftrightarrow (\tilde{\alpha}(\psi) \rightarrow \overline{\tilde{\alpha}(\varphi)}) \wedge (\tilde{\alpha}(x) \rightarrow \overline{\tilde{\alpha}(\varphi)}) = \\
 &= 0. \Leftrightarrow \tilde{\alpha}(y) = 0.
 \end{aligned}$$

$$\begin{aligned}
 &\text{Cas 2: } \tilde{\alpha}(\beta) = 1. \Leftrightarrow \tilde{\alpha}(y) = 1. \\
 &\Rightarrow \tilde{\alpha}(\beta) = \tilde{\alpha}(y) \Leftrightarrow \tilde{\alpha}(x) = 1. \\
 &\Rightarrow \models x. \Leftrightarrow \vdash x.
 \end{aligned}$$

ALTE METODE la (b):

- (5) Amintesc că într-o teorie Booleană B , pt. orice $a \in B$, avem:

$$\begin{aligned}
 a = a &\Leftrightarrow a \leftrightarrow a = 1; & (\textcircled{1}) \\
 0 \leq a &\Leftrightarrow 0 \rightarrow a = 1; & (\textcircled{*}) \\
 a \leq 1 &\Leftrightarrow a \rightarrow 1 = 1; & (\textcircled{**}) \\
 a \rightarrow 0 &= \overline{a} \vee 0 = \overline{a}; & (\textcircled{***}) \\
 1 \rightarrow a &= \overline{1} \vee a = 0 \vee a = a. & (\textcircled{****})
 \end{aligned}$$

Fie $\ln: V \rightarrow L_2 = \{0, 1\}$, arbitrară fixată.
 $\widetilde{\ln}(\alpha) = (\widetilde{\ln}(\varphi) \rightarrow (\widetilde{\ln}(\psi) \rightarrow \widetilde{\ln}(x))) \leftrightarrow$
 $\leftrightarrow ((\widetilde{\ln}(\psi) \rightarrow \widetilde{\ln}(\varphi)) \wedge (\widetilde{\ln}(x) \rightarrow \widetilde{\ln}(\varphi))).$

Caz 1: Dacă $\widetilde{\ln}(\varphi) = 0 \Rightarrow \widetilde{\ln}(\varphi) = 0 =$
 $= 1 \Rightarrow \widetilde{\ln}(\alpha) = (0 \rightarrow (\widetilde{\ln}(\psi) \rightarrow \widetilde{\ln}(x))) \leftrightarrow$
 $\leftrightarrow ((\widetilde{\ln}(\psi) \rightarrow 1) \wedge (\widetilde{\ln}(x) \rightarrow 1)) \quad (\textcircled{*}), (\textcircled{**})$
 $= 1 \leftrightarrow (1 \wedge 1) = 1 \leftrightarrow 1 \stackrel{\textcircled{1}}{=} 1.$

Caz 2: Dacă $\widetilde{\ln}(\varphi) = 1 \Rightarrow \widetilde{\ln}(\varphi) = 1 =$
 $= 0 \Rightarrow \widetilde{\ln}(\alpha) = (1 \rightarrow (\widetilde{\ln}(\psi) \rightarrow \widetilde{\ln}(x))) \leftrightarrow$
 $\leftrightarrow ((\widetilde{\ln}(\psi) \rightarrow 0) \wedge (\widetilde{\ln}(x) \rightarrow 0)) \quad (\textcircled{***}), (\textcircled{****})$

$$\begin{aligned}
 &= (\widetilde{\ln}(\psi) \rightarrow \widetilde{\ln}(x)) \leftrightarrow (\widetilde{\ln}(\psi) \wedge \widetilde{\ln}(x)) = \\
 &= \widetilde{\ln}(\psi) \vee \widetilde{\ln}(x) = \widetilde{\ln}(\psi) \vee \widetilde{\ln}(x) = \\
 &= \widetilde{\ln}(\psi) \wedge \widetilde{\ln}(x) \leftrightarrow \widetilde{\ln}(\psi) \wedge \widetilde{\ln}(x) \stackrel{\textcircled{2}}{=} 1,
 \end{aligned}$$

$$\Rightarrow (\forall \ln: V \rightarrow L_2)(\ln \models \alpha) \Leftrightarrow$$

$$\Leftrightarrow \vdash \alpha \Leftrightarrow \vdash \alpha.$$

Exerc. să se demonstreze
următoarele regulă de deducție:
daca $\Gamma \subseteq E$ și daca $\varphi, \psi \in E$,
 $\vdash \Gamma \cup \{\varphi \vee \psi\} \vdash \varphi \vee \psi$

REZONARE:

Fie $\Gamma \subseteq E$ și $\varphi, \psi \in E$,
 $\vdash \Gamma \cup \{\varphi \vee \psi\} \vdash \varphi \vee \psi$ (1)
 $\vdash \Gamma \cup \{\varphi \vee \psi\} \vdash \varphi \vee \psi$. (2)

$\vdash \Gamma \cup \{\varphi \vee \psi\} \vdash \varphi \vee \psi$

Fie $\tilde{h}: V \rightarrow L_2$, c.i.
 $h \vdash \Gamma \cup \{\varphi \vee \psi\} \Leftrightarrow \begin{cases} h \vdash \Gamma & (*) \\ \tilde{h}(\varphi \vee \psi) = 1 \Leftrightarrow \end{cases}$

$\Leftrightarrow \tilde{h}(\varphi) \vee \tilde{h}(\psi) = 1.$

$\tilde{h}(\varphi), \tilde{h}(\psi) \in L_2 = \{0, 1\} \Rightarrow$

$\Rightarrow \tilde{h}(\varphi) = 1 \text{ sau } \tilde{h}(\psi) = 1.$

Caz 1: dacă $\tilde{h}(\varphi) = 1 \Leftrightarrow$
 $\Leftrightarrow h \vdash \Gamma \cup \{\varphi\} \Leftrightarrow \tilde{h}(\varphi) = 1 \Leftarrow$

Caz 2: dacă $\tilde{h}(\psi) = 1 \Leftrightarrow$
 $\Leftrightarrow h \vdash \Gamma \cup \{\psi\} \stackrel{(2)}{\Leftrightarrow} \tilde{h}(\psi) = 1 \Leftarrow$

$\Rightarrow \Gamma \cup \{\varphi \vee \psi\} \vdash \varphi \vee \psi$.

$\Leftrightarrow \Gamma \cup \{\varphi \vee \psi\} \vdash \varphi \vee \psi$.

Exercitiu: Să se demonstreze următoarea regulă de deducție: pentru orice $\sum_1, \sum_2, \sum_3 \in \mathcal{P}(\Theta)$ și orice $\varphi, \psi, \chi \in E$:

$$\frac{\sum_1 \cup \{\varphi\} \vdash \psi, \sum_2 \cup \{\psi \wedge \chi\} \vdash \varphi, \sum_3 \cup \{\psi\} \vdash \chi}{\sum_1 \cup \sum_2 \cup \sum_3 \vdash \varphi \leftrightarrow (\psi \wedge \chi)}.$$

REZOLVARE: Fie $\sum_1, \sum_2, \sum_3 \in \mathcal{P}(\Theta)$
și $\varphi, \psi, \chi \in E$ a.s.

au. loc deducibile sintactice:

$$\left\{ \begin{array}{l} \sum_1 \cup \{\varphi\} \vdash \psi \Leftrightarrow \sum_1 \cup \{\varphi\} \models \psi; (1) \\ \sum_2 \cup \{\psi \wedge \chi\} \vdash \varphi \Leftrightarrow \sum_2 \cup \{\psi \wedge \chi\} \models \varphi; (2) \\ \sum_3 \cup \{\psi\} \vdash \chi \Leftrightarrow \sum_3 \cup \{\psi\} \models \chi. (3) \end{array} \right.$$

Fie $lh: V \rightarrow L_2$ a.s. $lh \models \sum_1 \cup \sum_2 \cup \sum_3$
 $\Leftrightarrow lh \models \sum_1, lh \models \sum_2, lh \models \sum_3$

Caz 1: dacă $lh(\varphi) = 1$ ($lh \models \sum_1$)

$$\Rightarrow lh \models \sum_1 \cup \{\varphi\} \stackrel{(1)}{\Leftrightarrow} lh(\psi) = 1 \stackrel{(lh \models \sum_3)}{\Leftrightarrow}$$

$$\Rightarrow lh \models \sum_3 \cup \{\psi\} \stackrel{(3)}{\Leftrightarrow} lh(\chi) = 1 \Rightarrow$$

$$\Rightarrow lh(\varphi \leftrightarrow (\psi \wedge \chi)) = lh(\varphi) \leftrightarrow (lh(\psi) \wedge lh(\chi)) = 1 \leftrightarrow (1 \wedge 1) = 1 \leftrightarrow 1 = 1.$$

Caz 2: dacă $lh(\varphi) = 0$:

$$\text{Prezumem prin absurd } \Rightarrow lh(\psi \wedge \chi) = 1 \Rightarrow \\ (lh \models \sum_3) \Rightarrow lh \models \sum_2 \cup \{\psi \wedge \chi\} \stackrel{(2)}{\Leftrightarrow} lh(\varphi) = 1 \Rightarrow$$

$$\Rightarrow lh(\psi \wedge \chi) = 0 \Rightarrow lh(\varphi \leftrightarrow (\psi \wedge \chi)) =$$

$$= lh(\varphi) \leftrightarrow lh(\psi \wedge \chi) = 0 \leftrightarrow 0 = 1,$$

$$\Rightarrow \sum_1 \cup \sum_2 \cup \sum_3 \vdash \varphi \leftrightarrow (\psi \wedge \chi) \stackrel{\text{fct}}{\Leftrightarrow} \sum_1 \cup \sum_2 \cup \sum_3 \vdash \varphi \leftrightarrow (\psi \wedge \chi).$$

Exerc. (tip examen (PT. 1 PUNCI)

din logice proposionale clasice).

Fie $\alpha \in T$ și $\beta \notin T$.

Dacă multimea $\{\alpha \rightarrow \beta\}$

$(\beta \vee \gamma) \rightarrow \neg \alpha\}$ e inconsistentă,

rezolvare:

Pă. primul obiect este acceptată
multimea e consistentă, (prop.) Aceasta
multime e satisfăcătoare, (def.)

$\Leftrightarrow (\exists h: V \rightarrow \mathcal{L}_2) (h \models \alpha \rightarrow \beta, h \models (\beta \vee \gamma) \rightarrow \neg \alpha)$.

$h \models \alpha \rightarrow \beta \quad \nexists \quad h \models (\beta \vee \gamma) \rightarrow \neg \alpha$

$$\begin{aligned} &\Leftrightarrow \begin{cases} 1 = \tilde{h}(\alpha \rightarrow \beta) = \tilde{h}(\alpha) \rightarrow \tilde{h}(\beta) \\ 2 = \tilde{h}((\beta \vee \gamma) \rightarrow \neg \alpha) = \\ = (\tilde{h}(\beta) \vee \tilde{h}(\gamma)) \rightarrow \overline{\tilde{h}(\alpha)} \end{cases} \end{aligned}$$

(proprietate
antrenă
o) $\tilde{h}(\alpha) \leq \tilde{h}(\beta) \quad (1)$

(definiție
Boole) $\tilde{h}(\beta) \vee \tilde{h}(\gamma) \leq \overline{\tilde{h}(\alpha)} \quad (2)$

Conform ipotesi, $\vdash \alpha, \beta \models \alpha$.

$$\Rightarrow \tilde{t}(\alpha) = 1 \xrightarrow{\text{def}} \tilde{t}(\beta) = 1$$

$$\tilde{t}(\beta) \vee \tilde{t}(\gamma) \leq 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow \tilde{t}(\beta) \vee \tilde{t}(\gamma) = 0.$$

$\Rightarrow 0 = 1 \vee \tilde{t}(\gamma) = 1$ in L_2 . \Rightarrow

\Rightarrow multime $\{\alpha \rightarrow \beta, (\beta \vee \gamma) \rightarrow \beta\}$ e inconsistent.

Exerc. (Ap examen (PT. 2 FUNCT) 2020-2022),
din logice propositionalo clasice
Sem. I) pt. Orice Σ, Δ
 $\in \mathcal{P}(E)$ si orice $\alpha, \beta, \gamma \in E$
era loc unde sunt regule
de deductie in logica
propositionalo clasice.

$$\frac{\sum \vdash (\alpha \vee \beta) \rightarrow \gamma, \Delta \vdash \neg \alpha \rightarrow \gamma}{\sum \Delta \vdash \alpha \leftrightarrow \gamma}$$

REZOLVARE:

Fie $\Sigma, \Delta \in \mathcal{P}(E)$ si

$\alpha, \beta, \gamma \in E$ satte erwt:

$$\left\{ \begin{array}{l} \sum \vdash (\alpha \vee \beta) \rightarrow \gamma \\ \text{if } \end{array} \right.$$

$$\left\{ \begin{array}{l} \Delta \vdash \neg \alpha \rightarrow \neg \gamma \\ \text{if } \\ \sum \vdash (\alpha \vee \beta) \rightarrow \gamma \\ \text{if } \\ \Delta \vdash \neg \alpha \rightarrow \neg \gamma. \end{array} \right. \quad (\#)$$

Die $\ln: V \rightarrow L_2$ s.a.

$$\vdash \sum \cup \Delta. \quad (\text{def}) \quad \vdash \sum \quad (*)$$

$$\vdash \Delta. \quad (**)$$

$$(\#), (*) \xrightarrow{\text{(def)}} \vdash (\alpha \vee \beta) \rightarrow \gamma \quad (\text{def})$$

$$\vdash \neg = \widetilde{\ln}((\alpha \vee \beta) \rightarrow \gamma) =$$

$$= (\widetilde{\ln}(\alpha) \vee \widetilde{\ln}(\beta)) \rightarrow \widetilde{\ln}(\gamma) \quad (\text{prop. vertreter of Rule})$$

$$\Leftrightarrow \widetilde{\ln}(\alpha) \vee \widetilde{\ln}(\beta) \leq \widetilde{\ln}(\gamma). \quad (\neg)$$

$$(\#), (***) \xrightarrow{\text{(def)}} \vdash \neg \alpha \rightarrow \neg \gamma \quad (\text{def}) \quad \vdash =$$

$$= \widetilde{\ln}(\neg \alpha \rightarrow \neg \gamma) = \overline{\widetilde{\ln}(\alpha) \rightarrow \widetilde{\ln}(\gamma)}.$$

$$\widetilde{\ln}(\alpha) \leq \widetilde{\ln}(\beta) \Leftrightarrow \widetilde{\ln}(\beta) \leq \widetilde{\ln}(\alpha)$$

$$(\neg) \Rightarrow \widetilde{\ln}(\alpha) \leq \widetilde{\ln}(\beta) \vee \widetilde{\ln}(\beta) \leq \widetilde{\ln}(\alpha) \Rightarrow$$

$$\Rightarrow \widetilde{\ln}(\alpha) \leq \widetilde{\ln}(\beta). \quad (\bullet)$$

$$(\bullet), (\bullet) \Rightarrow \widetilde{\ln}(\alpha) = \widetilde{\ln}(\beta) \quad (\text{prop. ordn. ej. Bsp.})$$

$$\Leftrightarrow \vdash = \widetilde{\ln}(\alpha) \leftrightarrow \widetilde{\ln}(\beta) = \widetilde{\ln}(\alpha \leftrightarrow \beta)$$

$$\xrightarrow{\text{(def)}} \sum \cup \Delta \vdash \alpha \leftrightarrow \beta \quad (\text{def})$$

$$\xrightarrow{\text{(def)}} \sum \cup \Delta \vdash \alpha \leftrightarrow \beta.$$

Exerc.: Seu, se, pende
que $\alpha, \beta \in E$ se log
an logic prop. besides, they
 $\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\gamma \rightarrow \beta)).$
RESOLVA:

MÉTODO I: ALGEBRICO:

álgebra Lindenbaum-Tarski

E/\sim se log.

$$\begin{aligned} & ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\gamma \rightarrow \beta))) / \sim = \\ & = (\alpha / \sim \rightarrow \beta / \sim) \rightarrow (\alpha / \sim \rightarrow (\gamma / \sim \rightarrow \beta / \sim)) \\ & = (\overline{\alpha / \sim} \vee \overline{\beta / \sim}) \rightarrow (\overline{\alpha / \sim} \vee \overline{\gamma / \sim} \vee \overline{\beta / \sim}) \end{aligned}$$

\Rightarrow adicione $\overline{\alpha / \sim} \vee \overline{\beta / \sim} \leq$

$$\leq \overline{\alpha / \sim} \vee \overline{\gamma / \sim} \vee \overline{\beta / \sim} \text{ exder,}$$

conform que temos,

$$\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\gamma \rightarrow \beta)).$$

MÉTODO A II-A: SEMANTICAS:

File $\ln: V \rightarrow L_2$, arbitraria,

Puteam efectua un calcul boolean ca mai sus sau putem proceda prin reducere la absurd, folosind faptul că
în valoarele booleene sunt
în L_2 : pp. prin absurd \Leftrightarrow

$$\tilde{h}((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\gamma \rightarrow \beta))) = 0 \Leftrightarrow$$

$$\Leftrightarrow \tilde{h}(\alpha \rightarrow \beta) \rightarrow \tilde{h}(\alpha \rightarrow (\gamma \rightarrow \beta)) = 0$$

$$\Leftrightarrow \tilde{h}(\alpha \rightarrow \beta) = 1 \quad \cancel{\text{xx}}$$

$$\tilde{h}(\alpha \rightarrow (\gamma \rightarrow \beta)) = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \tilde{h}(\alpha) \rightarrow \tilde{h}(\beta) = 1 & (*) \\ \exists i \end{cases}$$

$$\Leftrightarrow \tilde{h}(\alpha) \rightarrow (\tilde{h}(\gamma) \rightarrow \tilde{h}(\beta)) = 0 \Leftrightarrow$$

$$\begin{cases} \tilde{h}(\alpha) = 1 \\ \exists i \end{cases}$$

$$\Leftrightarrow \begin{cases} \exists i \\ \tilde{h}(\gamma) \rightarrow \tilde{h}(\beta) = 0 \end{cases} \Leftrightarrow \begin{cases} \tilde{h}(\gamma) = 1 \\ \exists i \\ \tilde{h}(\beta) = 0. \end{cases}$$

$$\Rightarrow \tilde{h}(\alpha) \rightarrow \tilde{h}(\beta) = 1 \rightarrow 0 = 0, \cancel{\text{xx}} \Rightarrow 0 = 1$$

$$\Rightarrow \tilde{h}((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\gamma \rightarrow \beta))) = 1 \quad \text{în } L_2, \cancel{\text{xx}}$$

$$\Rightarrow \models (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\gamma \rightarrow \beta)) \quad \cancel{\text{xx}}$$

$\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\beta \rightarrow \gamma))$.

Exerc.: Fixe $\alpha, \beta, \gamma \in E$ s.t.

$$\Sigma = \{\alpha \rightarrow (\beta \rightarrow \gamma), \beta \wedge \neg \gamma\}.$$

Sot se dem. \vdash em

logice prop. desired.

(1) data $\alpha, \beta, \gamma \in V$ s.t $\alpha \neq \beta \neq \gamma$, show Σ e consistent.

(2) data $(\exists \varphi \in E)(\alpha = \varphi \vee \neg \varphi)$
show Σ e inconsistent.

RESOLVARE:

Fixe $\ln: V \rightarrow L_2$, s.t.

$$\vdash \Sigma \Leftrightarrow \begin{cases} \perp = \overline{\ln(\alpha \rightarrow (\beta \rightarrow \gamma))} = \\ = \overline{\ln(\alpha)} \rightarrow (\overline{\ln(\beta)} \rightarrow \overline{\ln(\gamma)}) \\ \perp = \overline{\ln(\beta \wedge \neg \gamma)} = \overline{\ln(\beta)} \wedge \end{cases}$$

$$\Leftrightarrow \begin{cases} \overline{\ln(\alpha)} \vee \overline{\ln(\beta)} \vee \overline{\ln(\gamma)} = \perp \\ \perp \end{cases}$$

$$\Leftrightarrow \begin{cases} \overline{\ln(\alpha)} = \perp \quad \text{s.t. } \overline{\ln(\beta)} = \overline{\ln(\gamma)} = \perp \\ \perp \end{cases}$$

$$\Leftrightarrow \begin{cases} \overline{\ln(\beta)} = \perp, \overline{\ln(\gamma)} = 0 \quad \text{s.t. } \perp = \overline{\perp} = 0 \\ \overline{\ln(\alpha)} \vee \perp \vee 0 = \perp \Leftrightarrow \overline{\ln(\alpha)} = \overline{\perp} = 0 \end{cases}$$

$$\Leftrightarrow \tilde{h}(\alpha) = \tilde{h}(\gamma) = 0 \quad \text{si} \quad \tilde{h}(\beta) = 1.$$

(2) Dacă $\alpha, \beta, \gamma \in V$ și
 $\alpha \neq \beta \neq \gamma$, atunci \exists (dator
 și infinitate de interpretări)
 $h: V \rightarrow \mathbb{Z}_2$ cu
 $\begin{cases} h(\alpha) = h(\gamma) = 0 \\ h(\beta) = 1 \end{cases} \Leftrightarrow$

$$\begin{cases} \tilde{h}(\alpha) = \tilde{h}(\gamma) = 0 \\ \tilde{h}(\beta) = 1 \end{cases} \Leftrightarrow h \neq \sum \text{(prop)}$$

$\Rightarrow \sum \rightarrow$ nedistribuibile (prop)
 $\Leftrightarrow \sum \rightarrow$ consistentă.

(2) Dacă $(\exists \varphi \in E)(\alpha = \varphi \vee \neg \varphi)$,
 (PTES este TC)
 sau direct + $h: V \rightarrow \mathbb{Z}_2$, avem

$$\begin{aligned} \tilde{h}(\alpha) &= \tilde{h}(\varphi \vee \neg \varphi) = \tilde{h}(\varphi) \vee \tilde{h}(\neg \varphi) = \\ &= 1 \quad (\text{prop}) \quad (\text{prop}) \end{aligned}$$

$$\Leftrightarrow \sum \rightarrow$$
 nedistribuibile (prop) $\sum \rightarrow$ inconsistentă.

Exerc.: să se demonstreze pt.

dacă $\alpha, \beta, \gamma \in E$ și dacă

$\sum, \Delta \in \mathcal{P}(E)$, are loc, în

logic propositionals desired:

$$\frac{\sum \vdash \alpha \rightarrow (\beta \vee \gamma), \Delta \vdash \beta \rightarrow \gamma}{\sum \cup \Delta \vdash \alpha \rightarrow \gamma}.$$

RESOLVARE:

Fix $\alpha, \beta, \gamma \in E$ s.t. $\sum \Delta \in P(E)$, estrel amst.

$$\begin{cases} \sum \vdash \alpha \rightarrow (\beta \vee \gamma) \text{ (1)} \\ \Delta \vdash \beta \rightarrow \gamma \end{cases}$$

$$\begin{cases} \sum \vdash \alpha \rightarrow (\beta \vee \gamma) & (*) \\ \Delta \vdash \beta \rightarrow \gamma & (***) \end{cases}$$

Fix $\tilde{h}: V \rightarrow L_2$ s.t. α, β, γ

$$\vdash \sum \Delta, \quad \vdash \sum \quad \text{and} \quad \vdash \Delta. \quad \text{and} \quad \vdash \Delta.$$

$$\begin{aligned} (*) &\Leftrightarrow \vdash \tilde{h}(\alpha \rightarrow (\beta \vee \gamma)) = \\ &= \tilde{h}(\alpha) \rightarrow (\tilde{h}(\beta) \vee \tilde{h}(\gamma)) \Leftrightarrow \\ &\Leftrightarrow \tilde{h}(\alpha) \leq \tilde{h}(\beta) \vee \tilde{h}(\gamma). \quad (1) \\ (**) &\Leftrightarrow \vdash \tilde{h}(\beta \rightarrow \gamma) = \\ &= \tilde{h}(\beta) \rightarrow \tilde{h}(\gamma) \Leftrightarrow \tilde{h}(\beta) \leq \tilde{h}(\gamma) \Leftrightarrow \\ &\Leftrightarrow \tilde{h}(\beta) \vee \tilde{h}(\gamma) = \tilde{h}(\gamma). \quad (2) \end{aligned}$$

$$\begin{aligned} (1), (2) &\Rightarrow \tilde{h}(\alpha) \leq \tilde{h}(\gamma) \Leftrightarrow \\ &\Leftrightarrow \tilde{h}(\alpha) \rightarrow \tilde{h}(\gamma) = \vdash \Leftrightarrow \\ &\Leftrightarrow \tilde{h}(\alpha \rightarrow \gamma) = \vdash \quad \text{(s.t. } \tilde{h}: V \rightarrow L_2 \text{ arbitra)} \\ &\quad \text{and } \vdash \vdash \sum \Delta \quad \text{top} \\ &\Rightarrow \sum \cup \Delta \vdash \alpha \rightarrow \gamma \quad \text{top} \\ &\Leftrightarrow \sum \cup \Delta \vdash \alpha \rightarrow \gamma. \quad \text{top} \end{aligned}$$

Prop. *, $\Sigma \subseteq E$, atunci:

(1) $\Sigma \cup \{\varphi\} \rightarrow$ inconsistentă \Leftrightarrow

$$\Leftrightarrow \Sigma \vdash \neg \varphi$$

(2) $\Sigma \cup \{\neg \varphi\} \rightarrow$ inconsistentă \Leftrightarrow

$$\Leftrightarrow \Sigma \vdash \varphi$$

Prop. *, $\Sigma \subseteq E$, atunci $\Sigma \rightarrow$ consistentă \Leftrightarrow

Exerc. Considerăm următorul text:

(a) Dacă nu am chef și nu dispunem materiale predată, atunci nu vom avea la curs,

(b) Nu am dispus materiale predată, pentru că am chef,

(c) Nu am chef și nu stăd în se dau subiectele de examen,

(d) Dacă nu vom avea la curs, atunci nu se dau subiectele de examen,

(e) Nu dispunem materiale predată, să se erăt că acest text este inconsistent (sau nu în limbaj natural).

REZOLVARE: "contradicție"

Fie $P, Q, R, S \in V$, considerăm următoarele

unotarele valori:

p: "am. displice materie predatorii"

q: "am. chef"

r: "nu am la curs"

s: "nu se dau subiectele de examen"

Astfel, emisiunile core corespund frazelor din textul dat sunt:

(a) $\varphi = (\exists x \wedge p) \rightarrow \forall r \in E$

(b) $\psi = \exists x \rightarrow \forall p \in E$

(c) $\chi = s \rightarrow \forall r \in E$

(d) $\gamma = \forall r \rightarrow \forall s \in E$

(e) $p \vee \forall s \in E,$

Nat., $\Sigma = \{\varphi, \psi, \chi, \gamma, p\} \subseteq E.$

Amen de dem, este inconsistentă.
Metoda I: Aplicând Prop.** e suficient să arătăm că $\Sigma \vdash \neg p$ și $\vdash \neg p$.

Nu vom aplică, ci va ecuațiă metoda,
Metoda II: Aplicând Prop.***,

$\vdash p$, prin absurd $\vdash \Sigma$ este
consistentă. $\xrightarrow{\text{Prop.**}} (\vdash \perp; \vee \rightarrow \perp)$
 $\vdash \perp$ și $\vdash \neg p$. $(\vdash \perp \models \Sigma),$

Fie $\vdash \perp \models \Sigma$, c.e. $\vdash \perp \models \Sigma \models$

$$\begin{aligned}
& \Leftrightarrow \widetilde{\ln}(q) = \widetilde{\ln}(\psi) = \widetilde{\ln}(x) = \widetilde{\ln}(y) = \\
& = \widetilde{\ln}(p) = 1 \Leftrightarrow \text{An loc:} \\
& z = \widetilde{\ln}(q) = \widetilde{\ln}((\neg q \wedge p) \rightarrow \neg r) = \\
& = \frac{\widetilde{\ln}(q) \wedge \widetilde{\ln}(p) \rightarrow \widetilde{\ln}(r)}{=} \\
& = \frac{\widetilde{\ln}(q) \wedge \widetilde{\ln}(p)}{=} \vee \frac{\widetilde{\ln}(r)}{=} \\
& = \frac{\widetilde{\ln}(q) \vee \widetilde{\ln}(p) \vee \widetilde{\ln}(r)}{=} \quad (1) \\
& z = \widetilde{\ln}(\psi) = \widetilde{\ln}(q \rightarrow \neg p) = \\
& = \frac{\widetilde{\ln}(q) \rightarrow \widetilde{\ln}(p)}{=} = \\
& = \frac{\widetilde{\ln}(q)}{=} \vee \frac{\widetilde{\ln}(p)}{=} \quad (2) \\
& z = \widetilde{\ln}(x) = \widetilde{\ln}(z \rightarrow r) = \\
& = \frac{\widetilde{\ln}(z) \rightarrow \widetilde{\ln}(r)}{=} = \widetilde{\ln}(z) \vee \widetilde{\ln}(r) \quad (3) \\
& z = \widetilde{\ln}(y) = \widetilde{\ln}(\neg r \rightarrow z) = \\
& = \frac{\widetilde{\ln}(r) \rightarrow \widetilde{\ln}(z)}{=} = \widetilde{\ln}(r) \vee \widetilde{\ln}(z) \quad (4) \\
& z = \widetilde{\ln}(p) \quad (5)
\end{aligned}$$

$$\begin{aligned}
 (3), (4) \Rightarrow 1 &= 1 \wedge 1 = (\tilde{h}(r) \vee \\
 &\quad \sqrt{\tilde{h}(s)}) \wedge (\tilde{h}(r) \vee \tilde{h}(s)) = \\
 &= \tilde{h}(r) \vee (\tilde{h}(s) \wedge \tilde{h}(s)) = \tilde{h}(r) \\
 \text{---} \quad (2) \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\
 \Rightarrow 1 &= \tilde{h}(q) \vee \overline{\tilde{h}(p)} \vee \overline{1} = \\
 &= \tilde{h}(q) \vee \overline{\tilde{h}(p)} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\
 (2), (6) \Rightarrow 1 &= 1 \wedge 1 = (\tilde{h}(q) \vee \\
 &\quad \sqrt{\tilde{h}(p)}) \wedge (\tilde{h}(q) \vee \overline{\tilde{h}(p)}) = \\
 &= (\tilde{h}(q) \wedge \tilde{h}(q)) \vee \overline{\tilde{h}(p)} = \overline{\tilde{h}(p)} \\
 \Rightarrow \tilde{h}(p) &= \overline{\tilde{h}(p)} = \overline{1} = 0 \quad \text{---} \\
 \Rightarrow 0 &= 1 \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\
 \text{in } L_2 \Rightarrow \sum & \text{ este inconsistentă.}
 \end{aligned}$$

Am efectuat un calcul boolean puțin diferit față de cel din cealaltă rezolvare a acestui exercițiu dintr-un material de seminar anterior. Referitor la notării: puteam observa faptul că $\tilde{h}(v) = h(v)$ pentru fiecare $v \in \{p, q, r, s\}$.

Metoda I:

$$\Sigma \setminus \{p\} \models // \neg p. \quad \text{The } \ln: V \rightarrow L_2 \text{ s.t. } \ln \models \Sigma \setminus \{p\}. \Leftrightarrow$$

$$\begin{cases} 1 = \ln(\varphi) = (\overline{\ln(g)} \wedge \overline{\ln(p)}) \rightarrow \overline{\ln(r)} \\ 1 = \ln(\psi) = \overline{\ln(g)} \rightarrow \overline{\ln(p)} \quad (*) \\ 1 = \ln(x) = \overline{\ln(s)} \rightarrow \overline{\ln(r)} = \overline{\ln(s)} \\ 1 = \ln(y) = \overline{\ln(r)} \rightarrow \overline{\ln(s)} = \overline{\ln(r)} \end{cases}$$

$$= \overline{\ln(r)} \vee \overline{\ln(s)} = \overline{\ln(r)} \vee \overline{\ln(s)} = \overline{\ln(s)} \quad \checkmark$$

$$\Rightarrow 1 = 1 \wedge 1 = (\overline{\ln(s)} \vee \overline{\ln(r)}) \wedge$$

$$\wedge (\overline{\ln(s)} \vee \overline{\ln(r)}) = (\overline{\ln(s)} \wedge \overline{\ln(s)}) \vee$$

$$\vee \overline{\ln(r)} = 0 \vee \overline{\ln(r)} = \overline{\ln(r)}, \Rightarrow$$

$$\Rightarrow 1 = (\overline{\ln(g)} \wedge \overline{\ln(p)}) \rightarrow 1 =$$

$$= (\overline{\ln(g)} \wedge \overline{\ln(p)}) \rightarrow 0. \quad \begin{array}{c} 1 \rightarrow 0 = 0 \\ 0 \rightarrow 0 = 1 \end{array}$$

$$\Rightarrow 0 = \overline{\ln(g)} \wedge \overline{\ln(p)}. \quad (***)$$

$$\text{P.p. obs. s.t. } \overline{\ln(p)} = 1, \Rightarrow$$

$$\begin{cases} \text{dim } (***) \Rightarrow \overline{\ln(g)} = 0 \Leftrightarrow \overline{\ln(g)} = \\ = 0 \Leftrightarrow \overline{\ln(g)} = 1. \end{cases}$$

$$\begin{cases} \text{dim } (*) \Rightarrow 1 = \overline{\ln(g)} \rightarrow 1 = \\ = \overline{\ln(g)} \rightarrow 0. \quad \begin{array}{c} 0 \rightarrow 0 = 0 \\ 0 \rightarrow 0 = 1 \end{array} \end{cases}$$

$$\Rightarrow 1 = 0 \text{ in } L_2 \rightarrow X_0, \Rightarrow \overline{\ln(p)} = 0. \Rightarrow$$

$$\Rightarrow \overline{\ln(\neg p)} = \overline{\overline{\ln(p)}} = \overline{0} = 1 \Rightarrow \Sigma \setminus \{p\} \models \neg p$$

$$\Leftrightarrow \Sigma \setminus \{p\} \models \neg p. \quad \text{(*)} \quad \Sigma \rightarrow \text{inconsist.}$$

Exerci: să se demonstreze algebric

$$\text{ș: } (\forall \varphi, \psi, x \in E)$$

$$H(\varphi \rightarrow (\psi \rightarrow x)) \leftrightarrow (\psi \rightarrow (\varphi \rightarrow x))$$

rez.: $(\forall x \in E)(\vdash x \models x = 1$ în
 E/\sim = algebra Lindenbaum-Tarski

a logicii propositionale clasice)

Așa că E/\sim este o algebra Boole.

În E/\sim ,
Notă: $x := \widehat{\varphi}, y := \widehat{\psi}, z := \widehat{x}$

$$\begin{aligned} & ((\psi \rightarrow (\varphi \rightarrow x)) \leftrightarrow (\psi \rightarrow (\varphi \rightarrow x))) \\ & (\varphi \rightarrow (\psi \rightarrow x)) \leftrightarrow (\psi \rightarrow (\varphi \rightarrow x)) = \\ & \widehat{\varphi} \rightarrow (\widehat{\psi} \rightarrow \widehat{x}) \leftrightarrow (\widehat{\psi} \rightarrow (\widehat{\varphi} \rightarrow \widehat{x})) = \end{aligned}$$

transformare conectoare logice
în operații booleene ale lui E/\sim

$$= (x \rightarrow (y \rightarrow z)) \leftrightarrow (y \rightarrow (x \rightarrow z)) =$$

$$= (\overline{x} \vee \overline{y} \vee z) \leftrightarrow (\overline{y} \vee \overline{x} \vee z) = 1,$$

an E/\sim . $\xrightarrow{\text{(fund)}}$ $\vdash (\varphi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$.

Exerciștă să se demonstreze algebric
zi semantic $\vdash : \forall \varphi, \psi \in E$
 $\vdash (\neg \varphi \wedge \psi) \rightarrow (\neg \psi \vee \varphi)$.

Raz. Notă: $\alpha := \vdash (\neg \varphi \wedge \psi) \rightarrow (\neg \psi \vee \varphi)$,

Fie $\varphi, \psi, \chi \in E$
Algebric: Notă: $x := \overline{\varphi}, y = \overline{\psi}$ an

$$\begin{aligned} \tilde{x} &= (\overline{\varphi} \wedge \overline{\psi}) \rightarrow (\overline{\psi} \vee \overline{\varphi}) = \\ &= (\overline{x} \wedge \overline{y}) \rightarrow (\overline{y} \vee \overline{x}) = \\ &= (\overline{x} \wedge \overline{y}) \rightarrow (\overline{y} \vee \overline{x}) = 1, \text{ pts. } \text{d.} \end{aligned}$$

$x \wedge y \leq x \leq \overline{y} \vee x$, prin urmare

$x \wedge y \leq \overline{y} \vee x$, prin transformare,
deci an $\vdash \alpha = 1 \xrightarrow{\text{(fund)}}$ $\vdash \alpha$.

Fie $\mathcal{A} = \langle \text{dom}, \vdash \rangle$ de valori de adevar

$\mathcal{A} = \langle \text{dom}, \vdash \rangle$ variabili propozitionale
fixate.
de adevar date de \mathcal{A} valori de adevar
 $L_2 = \{0, 1\} = \text{alg. Boole standard}$

(vici) falsul aderant
alg. Boole e valori de adevar

In algebra Boole standard,
 L_2 , avem:

a	b	$a \rightarrow b$
0	0	1
0	1	1
1	0	0
1	1	1

Folosind acest lucru, calculam
 $\tilde{h}(\alpha) = (\tilde{h}(\varphi) \rightarrow \tilde{h}(\psi)) \rightarrow (\tilde{h}(\psi) \rightarrow \tilde{h}(\varphi))$
 not: $\tilde{T}_2(\varphi, \psi)$ not: $\tilde{T}_2(\psi, \varphi)$
 cu ajutorul acestui tabel semantic:

$\tilde{h}(\varphi)$	$\tilde{h}(\psi)$	$\tilde{T}_2(\varphi, \psi)$	$\tilde{T}_2(\psi, \varphi)$	$\tilde{h}(\alpha)$
0	0	0	1	1
0	1	0	0	1
1	0	0	1	1
1	1	1	1	1

$$\Rightarrow (\text{f. h: } V \rightarrow L_2 = \{0, 1\}) (\tilde{h}(\alpha) = 1) \Leftrightarrow \vdash \alpha, \vdash \alpha.$$

Exercițiu: Fie se dem., semantica următoare regula de deducție:

$$(\forall \Gamma \subseteq E) (\forall \varphi, \psi) \varphi, \psi \in E \models_{\Gamma} \varphi \vee \psi \Leftrightarrow \begin{cases} \Gamma \models \varphi \vee \psi \\ \Gamma \models \varphi \rightarrow \psi \end{cases}$$

rez.: Fie $\Gamma \subseteq E$ și $\varphi, \psi \in E$,

$$\text{a.d.} \left\{ \begin{array}{l} \Gamma \models \varphi \vee \psi \xrightarrow{\text{fact}} \Gamma \models \varphi \rightarrow \psi (1) \\ \Gamma \models \varphi \rightarrow \psi \xrightarrow{\text{fact}} \Gamma \models \varphi \vee \psi (2) \end{array} \right.$$

dem. că: $\Gamma \models \varphi \vee \psi \rightarrow \varphi$.

$$\text{Fie } h: V \rightarrow L_2, \text{ a.d. } h \models \Gamma \models \varphi \vee \psi \Leftrightarrow \begin{cases} h \models \Gamma \\ h(\varphi \vee \psi) = 1 \Leftrightarrow h(\varphi) \vee h(\psi) = 1 \end{cases}$$

in $\mathcal{L}_2 \Leftrightarrow \tilde{\alpha}(\varphi) = 1$ sau $\tilde{\alpha}(\psi) = 1$.

Caz 1: d.c. $\tilde{\alpha}(\varphi) = 1$, $\xrightarrow{*}$

$\xrightarrow{*} h \models \Gamma \cup \{\varphi\} \xrightarrow{*} \tilde{\alpha}(\varphi) = 1$.

Caz 2: d.c. $\tilde{\alpha}(\psi) = 1$, $\xrightarrow{*}$

$\xrightarrow{*} h \models \Gamma \cup \{\psi\} \xrightarrow{*} \tilde{\alpha}(\psi) = 1$.

$\Rightarrow (\forall h: V \rightarrow \mathcal{L}_2)(h \models \Gamma \cup \{\varphi \vee \psi\} \Rightarrow$

$\Rightarrow \tilde{\alpha}(\varphi \vee \psi) = 1) \xrightarrow{\text{def}} \Gamma \cup \{\varphi \vee \psi\} \vdash \varphi \xrightarrow{\text{def}}$

$\xrightarrow{\text{def}} \Gamma \cup \{\varphi \vee \psi\} \vdash \varphi$.

Exerc.: să se demonstreze cu regulile de deducție: $(\forall \Sigma_1, \Sigma_2, \Sigma_3 \subseteq E)(\forall \varphi, \psi, \chi \in E)$

$$\frac{(\Sigma_1 \cup \{\varphi\}) \vdash \psi, \Sigma_2 \cup \{\psi \wedge \chi\} \vdash \varphi, \Sigma_3 \cup \{\psi \wedge \chi\} \vdash \chi}{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash \varphi \leftrightarrow (\psi \wedge \chi)}$$

rez.: Fie $\Sigma_1, \Sigma_2, \Sigma_3 \subseteq E$

$\exists \varphi, \psi, \chi \in E$ a.s.a. au loc deducțile sintactice:

$$\begin{cases} \sum_1 \cup \{\varphi\} \vdash \psi & \text{if } \sum_1 \cup \{\varphi\} \models \psi, (1) \\ \sum_2 \cup \{\psi \wedge x\} \vdash \psi & \text{if } \sum_2 \cup \{\psi \wedge x\} \models \psi, (2) \\ \sum_3 \cup \{\psi\} \vdash x & \text{if } \sum_3 \cup \{\psi\} \models x, (3). \end{cases}$$

Dem. $\vdash = \sum_1 \cup \sum_2 \cup \sum_3 \vdash \varphi \leftrightarrow (\psi \wedge x)$

Fie $h: V \rightarrow \mathcal{L}_2$, a.d. $h \vdash \sum_1 \cup \sum_2 \cup \sum_3$.

$$\vdash \sum_3 \Rightarrow \begin{cases} h \vdash \sum_1 & (*) \\ h \vdash \sum_2 & (** \\ h \vdash \sum_3 & (***) \end{cases}$$

Case 1: $\text{def. } \tilde{h}(\varphi) = 1 \Rightarrow h \vdash \sum_1 \cup \{\varphi\}$

$\xrightarrow{(1)} \tilde{h}(\psi) = 1 \xrightarrow{(***)} h \vdash \sum_3 \cup \{\psi\}, (3)$

$\xrightarrow{(3)} \tilde{h}(x) = 1 \Rightarrow \tilde{h}(\varphi \leftrightarrow (\psi \wedge x)) =$

$$= \tilde{h}(\varphi) \leftrightarrow (\tilde{h}(\psi) \wedge \tilde{h}(x)) =$$

$$= 1 \leftrightarrow (1 \wedge 1) = 1 \leftrightarrow 1 = 1.$$

Case 2: $\text{def. } \tilde{h}(\varphi) = 0$:

P.p. prn absurd $\vdash \tilde{h}(\psi \wedge x) = 1$

$\xrightarrow{(***)} h \vdash \sum_2 \cup \{\psi \wedge x\} \xrightarrow{(2)} \tilde{h}(\varphi) = 1$

$\xrightarrow{\tilde{h}(\varphi) = 0} 0 = 1 \text{ an } \mathcal{L}_2 \rightarrow \mathcal{L}_0 \Rightarrow$

$\Rightarrow \tilde{h}(\psi \wedge x) = 0 \Rightarrow \tilde{h}(\varphi \leftrightarrow (\psi \wedge x)) =$

$$= \tilde{h}(\varphi) \leftrightarrow \tilde{h}(\psi \wedge x) = 0 \leftrightarrow 0 = 1.$$

$$\begin{aligned}
 &\Rightarrow (\vdash h: V \rightarrow L_3) (h \vdash \Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \\
 &\Rightarrow \tilde{\Delta}(\varphi \leftrightarrow (\psi \wedge x)) = 1 \quad (\text{def}) \\
 &\stackrel{(\text{def})}{\Rightarrow} \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash \varphi \leftrightarrow (\psi \wedge x) \\
 &\stackrel{(\text{fct})}{\Rightarrow} \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash \varphi \leftrightarrow (\psi \wedge x).
 \end{aligned}$$

Exerc: Se se dem. semantică urm.
 reg. de ded.: $(\vdash \Sigma_1, \Sigma_2, \Sigma_3 \subseteq E)$
 $(\vdash \varphi, \psi, x \in E)$

$$\begin{aligned}
 &\Sigma_1 \cup \Sigma_2 \vdash (\varphi \vee \psi) \leftrightarrow (x \rightarrow \gamma \psi) \quad \frac{\Sigma_1 \cup \Sigma_3 \vdash \varphi \wedge \Sigma_2 \vdash \psi}{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash x \rightarrow \psi} \\
 &\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash x \rightarrow \psi
 \end{aligned}$$

rez.; Fix $\Sigma_1, \Sigma_2, \Sigma_3 \subseteq E$ și

$\varphi, \psi, x \in E$, a.d.:

$$\begin{cases} \Sigma_1 \cup \Sigma_2 \vdash (\varphi \vee \psi) \leftrightarrow (x \rightarrow \gamma \psi) \\ \Sigma_1 \cup \Sigma_3 \cup \{\varphi \leftrightarrow x\} \vdash \psi \end{cases}$$

$$\begin{cases} \Sigma_1 \cup \Sigma_2 \vdash (\varphi \vee \psi) \leftrightarrow (x \rightarrow \gamma \psi) \quad (1) \\ \Sigma_1 \cup \Sigma_3 \cup \{\varphi \leftrightarrow x\} \vdash \psi \quad (2). \end{cases}$$

dem. că: $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash x \rightarrow \psi$.

Fix $h: V \rightarrow L_2$, a.d. $h \vdash \Sigma_1 \cup \Sigma_2$

$$\begin{cases} h \vdash \Sigma_1 \cup \Sigma_2 \quad (*) \\ h \vdash \Sigma_1 \cup \Sigma_3 \quad (***) \end{cases}$$

Caz 1: a.d. $\tilde{\Delta}(\varphi) = 1 \Rightarrow$

$$\Rightarrow \widetilde{h}(x \rightarrow \psi) = \widetilde{h}(x) \rightarrow \widetilde{h}(\psi) = \\ = \widetilde{h}(x) \rightarrow 1 = 1.$$

Case 2: s.t. $\widetilde{h}(\psi) = 0$, at:

p. from absurd $\Rightarrow \widetilde{h}(\varphi \leftrightarrow x) =$
 $= 1$. $\xrightarrow{\text{(*)}} h \models \sum_1 \cup \sum_2 \cup \{\varphi \leftrightarrow x\} \Rightarrow$
 $\xrightarrow{\text{(*)}} \widetilde{h}(\varphi) = 1 \Rightarrow 0 = 1$ in $L \vdash x \Rightarrow$
 $\Rightarrow \widetilde{h}(\varphi \leftrightarrow x) = 0 \Leftrightarrow \widetilde{h}(\varphi) \leftrightarrow \widetilde{h}(x) =$
 $\neq 1 \Leftrightarrow \widetilde{h}(\varphi) \neq \widetilde{h}(x)$. (I)

$\xrightarrow{\text{(*)}} (1) \Rightarrow \widetilde{h}((\varphi \vee \psi) \leftrightarrow (x \rightarrow \gamma_\psi))$
 $= 1 \Leftrightarrow (\widetilde{h}(\varphi) \vee \widetilde{h}(\psi)) \leftrightarrow (\widetilde{h}(x) \rightarrow$
 $\rightarrow \widetilde{h}(\psi)) = 1 \Leftrightarrow \widetilde{h}(\varphi) \vee \widetilde{h}(\psi) =$
 $= \widetilde{h}(x) \rightarrow \widetilde{h}(\psi) \quad \xleftarrow{\widetilde{h}(\psi) = 0}$
 $\Leftrightarrow \widetilde{h}(\varphi) \vee 0 = \widetilde{h}(x) \rightarrow 0 \Leftrightarrow$
 $\Leftrightarrow \widetilde{h}(\varphi) = \widetilde{h}(x) \rightarrow 1 \Leftrightarrow \widetilde{h}(\varphi) = 1$
 $\xrightarrow{\text{(*)}} \widetilde{h}(x) = 0 \Rightarrow \widetilde{h}(x \rightarrow \psi) =$
 $= \widetilde{h}(x) \rightarrow \widetilde{h}(\psi) = 0 \rightarrow 0 = 1.$

$$\Rightarrow \sum_1 \cup \sum_2 \cup \sum_3 \models x \rightarrow \psi. \quad \xrightarrow{\text{frt}}$$

$$\xrightarrow{\text{frt}} \sum_1 \cup \sum_2 \cup \sum_3 \vdash x \rightarrow \psi.$$

ALTĂ REZOLVARE:

$$\begin{aligned}
 & \text{Fie } \Sigma_1, \Sigma_2, \Sigma_3 \subseteq E \text{ și} \\
 & \varphi, \psi, x \in E, \text{ c.d.:} \\
 & \left\{ \begin{array}{l} \Sigma_1 \cup \Sigma_2 \vdash (\varphi \vee \psi) \leftrightarrow (x \rightarrow \neg \psi) \\ \Sigma_1 \cup \Sigma_3 \vdash \varphi \leftrightarrow x \exists \vdash \psi. \end{array} \right. \\
 & \Leftrightarrow \left\{ \begin{array}{l} \Sigma_1 \cup \Sigma_2 \vdash (\varphi \vee \psi) \leftrightarrow (x \rightarrow \neg \psi) \quad (*) \\ \Sigma_1 \cup \Sigma_3 \vdash \varphi \leftrightarrow x \exists \vdash \psi \quad (***) \end{array} \right. \\
 & \text{Fie } h: V \rightarrow L_2 = \{\varnothing, \exists\} \text{ c.d.,} \\
 & h \vdash \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \Rightarrow \left\{ \begin{array}{l} h \vdash \Sigma_1 \cup \Sigma_3 \\ h \vdash \Sigma_1 \cup \Sigma_2 \end{array} \right. \\
 & \Leftrightarrow \begin{aligned} \exists &= \tilde{h}((\varphi \vee \psi) \leftrightarrow (x \rightarrow \neg \psi)) = \\ &= (\tilde{h}(\varphi) \vee \tilde{h}(\psi)) \leftrightarrow (\tilde{h}(x) \rightarrow \overline{\tilde{h}(\psi)}) = \\ &= (\tilde{h}(\varphi) \vee \tilde{h}(\psi)) \leftrightarrow (\overline{\tilde{h}(x)} \vee \overline{\tilde{h}(\psi)}). \end{aligned} \\
 & \text{Deci } \exists \text{ dacă } \tilde{h}(\varphi \leftrightarrow \psi) = \exists, \\
 & \text{atunci } h \vdash \Sigma_1 \cup \Sigma_3 \vdash \varphi \leftrightarrow x \exists. \quad \text{(**)}
 \end{aligned}$$

$$\xrightarrow{(*)} \tilde{h}(\psi) = 1 \Rightarrow \tilde{h}(x \rightarrow \psi) = \\ = \tilde{h}(x) \rightarrow \tilde{h}(\psi) = \tilde{h}(x) \rightarrow 1 = 1$$

Caso 2: Seja $\tilde{h}(\varphi \leftrightarrow x) = 0$ (1)

$$\text{d}. \quad \tilde{h}(\varphi) \leftrightarrow \tilde{h}(x) = 0 \neq 1 \Rightarrow$$

$$\Rightarrow \tilde{h}(\varphi) \neq \tilde{h}(x).$$

$$\tilde{h}(\varphi), \tilde{h}(x) \in L_2 = \{0, 1\},$$

$$\Rightarrow \tilde{h}(\varphi) = \overline{\tilde{h}(x)} \xrightarrow{(*)} 1 =$$

$$= (\overline{\tilde{h}(x)} \vee \overline{\tilde{h}(\psi)}) \leftrightarrow (\overline{\tilde{h}(x)} \vee$$

$$\overline{\tilde{h}(\psi)}). \Rightarrow \overline{\tilde{h}(x)} \vee \overline{\tilde{h}(\psi)} = \\ = \overline{\tilde{h}(x)} \vee \overline{\tilde{h}(\psi)}.$$

$$\tilde{h}(\psi) \in L_2 = \{0, 1\}; \overline{0} = 1; \overline{1} = 0,$$

$$\Rightarrow \overline{\tilde{h}(x)} \vee 0 = \overline{\tilde{h}(x)} \vee 1 \Leftrightarrow$$

$$\Leftrightarrow \tilde{w}(x) = 1 \Leftrightarrow \tilde{w}(x) = 0, \Rightarrow$$

$$\Rightarrow \tilde{w}(x \rightarrow \psi) = \tilde{w}(x) \rightarrow \tilde{w}(\psi) = \\ = 0 \rightarrow \tilde{w}(\psi) = 1. \quad (2)$$

$$(1), (2) \Rightarrow \sum_1 \cup \sum_2 \cup \sum_3 \vdash x \rightarrow \psi. \Leftrightarrow \\ (\text{fct}) \quad \sum_1 \cup \sum_2 \cup \sum_3 \vdash x \rightarrow \psi.$$

Exerc.: $\alpha \beta \gamma \delta, \Sigma \vdash x \rightarrow \psi$.

$$\Sigma \vdash x \rightarrow (\psi \rightarrow \beta), \alpha \rightarrow (\beta \rightarrow \delta),$$

$$\varepsilon \rightarrow \alpha, \varepsilon \rightarrow \beta, \varepsilon \rightarrow \gamma, \varepsilon \rightarrow \delta,$$

Dem. se Σ é inconsistente.

RESOLVA:

P.p. form absurd $\rightarrow \Sigma$ e inconsistente. $\Leftrightarrow \Sigma$ e satisfacibile, \Leftrightarrow

$\Leftrightarrow (\exists h: V \rightarrow L_2) (ht \models \Sigma)$, (cf. extensi defini)
pentru dege unu

Fie $h: V \rightarrow L_2$ s.t. $ht \models \Sigma \Leftrightarrow$

$$\left\{ \begin{array}{l} z = \tilde{h}(\alpha \rightarrow (\beta \rightarrow \gamma)) = \tilde{h}(\alpha) \rightarrow \\ \quad \rightarrow (\tilde{h}(\beta) \rightarrow \tilde{h}(\gamma)) \\ z = \tilde{h}(\alpha \rightarrow (\beta \rightarrow \delta)) = \tilde{h}(\alpha) \rightarrow \\ \quad \rightarrow (\tilde{h}(\beta) \rightarrow \tilde{h}(\delta)) \\ z = \tilde{h}(\varepsilon \rightarrow \alpha) = \tilde{h}(\varepsilon) \rightarrow \tilde{h}(\alpha) \\ z = \tilde{h}(\varepsilon \rightarrow \beta) = \tilde{h}(\varepsilon) \rightarrow \tilde{h}(\beta) \\ \tilde{h}(\varepsilon) = z \\ z = \tilde{h}(\neg \delta) = \overline{\tilde{h}(\delta)} \Leftrightarrow \tilde{h}(\delta) = 0. \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} \tilde{h}(\alpha) \geq \tilde{h}(\varepsilon) = z \Rightarrow \tilde{h}(\alpha) = \tilde{h}(\beta) = z \\ \tilde{h}(\beta) \geq \tilde{h}(\varepsilon) = z \Rightarrow \tilde{h}(\beta) = 0 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} z = z \rightarrow (z \rightarrow \tilde{h}(\beta)) = \overline{z} \vee \overline{z} \vee \tilde{h}(\beta) = \\ = 0 \vee \tilde{h}(\beta) = \tilde{h}(\beta). \\ z = z \rightarrow (\tilde{h}(\beta) \rightarrow 0) \end{array} \right. \Rightarrow z = z \rightarrow (z \rightarrow 0) = z \rightarrow 0 = 0 \Rightarrow$$

$$\Rightarrow 0 = z \text{ in } L_2. \text{ X.} \Rightarrow \Sigma \text{ e inconsistente.}$$