Chapter 2

LINEAR ANISOTROPIC ELASTIC MATERIALS

The relations between stresses and strains in an anisotropic elastic material are presented in this chapter. A linear anisotropic elastic material can have as many as 21 elastic constants. This number is reduced when the material possesses a certain material symmetry. The number of elastic constants is also reduced, in most cases, when a two-dimensional deformation is considered. An important condition on elastic constants is that the strain energy must be positive. This condition implies that the 6×6 matrices of elastic constants presented herein must be positive definite.

2.1 Elastic Stiffnesses

Referring to a fixed rectangular coordinate system x_1, x_2, x_3, let σ_{ij} and ε_{ij} be the stress and strain, respectively, in an anisotropic elastic material. The stress-strain law can be written as

$$\sigma_{ij} = C_{ijks}\varepsilon_{ks} \tag{2.1-1}$$

in which C_{ijks} are the *elastic stiffnesses* which are components of a fourth rank tensor. They satisfy the full symmetry conditions

$$C_{ijks} = C_{jiks}, \quad C_{ijks} = C_{ijsk}, \quad C_{ijks} = C_{ksij}. \tag{2.1-2}$$

Before we present justifications for the three conditions in (2.1-2), we show that $(2.1\text{-}2)_1$ and $(2.1\text{-}2)_3$ imply $(2.1\text{-}2)_2$. Using $(2.1\text{-}2)_3$, $(2.1\text{-}2)_1$, and $(2.1\text{-}2)_3$ in that order we have

$$C_{ijks} = C_{ksij} = C_{skij} = C_{ijsk}$$

which proves $(2.1\text{-}2)_2$. Therefore the three conditions in (2.1-2) are written as

$$C_{ijks} = C_{jiks} = C_{ksij}. \tag{2.1-3}$$

One can also show that $(2.1\text{-}2)_2$ and $(2.1\text{-}2)_3$ imply $(2.1\text{-}2)_1$.

The first equation of (2.1-2) follows directly from the symmetry of the stress tensor $\sigma_{ij}=\sigma_{ji}$. The second equation of (2.1-2) does *not* follow directly from the symmetry of the strain tensor $\varepsilon_{ij}=\varepsilon_{ji}$. However, if the C_{ijks} in (2.1-1) do not satisfy $(2.1\text{-}2)_2$, we rewrite (2.1-1) as

$$\sigma_{ij} = \tfrac{1}{2}C_{ijks}\varepsilon_{ks} + \tfrac{1}{2}C_{ijks}\varepsilon_{ks} = \tfrac{1}{2}C_{ijks}\varepsilon_{ks} + \tfrac{1}{2}C_{ijsk}\varepsilon_{sk}$$

or, since $\varepsilon_{sk}=\varepsilon_{ks}$,

$$\sigma_{ij} = \tfrac{1}{2}(C_{ijks} + C_{ijsk})\varepsilon_{ks}. \tag{2.1-4}$$

The coefficients of ε_{ks} are symmetric with the subscripts ks. We can therefore redefine the coefficients of ε_{ks} in (2.1-4) as the new C_{ijks} which satisfy $(2.1\text{-}2)_2$.

The third equation follows from the consideration of strain energy. The strain energy W per unit volume of the material is

$$W = \int_0^{\varepsilon_{pq}} \sigma_{ij} d\varepsilon_{ij} = \int_0^{\varepsilon_{pq}} C_{ijks}\varepsilon_{ks} d\varepsilon_{ij}. \tag{2.1-5}$$

We demand that the integral be independent of the path ε_{ij} takes from 0 to ε_{pq}. If not, say path 1 yields a larger integral than path 2, one can consider loading the material from 0 to ε_{pq} through path 1, and unloading from ε_{pq} to 0 through the reverse of path 2. The energy gained is the difference between the W's for path 1 and path 2. If we repeat the process we can extract unlimited amount of energy from the material, which is physically impossible for a real material. Therefore the integral in (2.1-5) must be independent of the path taken by ε_{ij}, and W depends on the final strain ε_{pq} only. This implies that the integrand must be the total differential dW, i.e.,

$$C_{ijks}\varepsilon_{ks} d\varepsilon_{ij} = dW = \frac{\partial W}{\partial \varepsilon_{ij}} d\varepsilon_{ij}. \tag{2.1-6}$$

Since $d\varepsilon_{ij}$ is arbitrary we must have

$$\sigma_{ij} = C_{ijks}\varepsilon_{ks} = \frac{\partial W}{\partial \varepsilon_{ij}} \tag{2.1-7}$$

in which the first equality follows from (2.1-1). Differentiation of $(2.1\text{-}7)_2$ with ε_{ks} leads to

$$C_{ijks} = \frac{\partial^2 W}{\partial \varepsilon_{ks} \partial \varepsilon_{ij}}.$$

The double differentiations on the right are interchangeable. Therefore

$$C_{ijks} = C_{ksij}$$

is the condition for the integral in (2.1-5) to be independent of the loading path. This proves $(2.1\text{-}2)_3$.

With $(2.1\text{-}2)_3$, $(2.1\text{-}6)_1$ is written as

$$dW = C_{ijks}\varepsilon_{ks}d\varepsilon_{ij} = \tfrac{1}{2}d(C_{ijks}\varepsilon_{ij}\varepsilon_{ks}).$$

Hence

$$W = \tfrac{1}{2}C_{ijks}\varepsilon_{ij}\varepsilon_{ks} = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}. \tag{2.1-8}$$

For the strain energy to be positive we must have

$$C_{ijks}\varepsilon_{ij}\varepsilon_{ks} > 0 \tag{2.1-9}$$

for any real, nonzero, symmetric tensor ε_{ij}.

The displacements u_i (i=1,2,3) are related to the strains by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.1-10}$$

where the comma denotes differentiation. Let

$$\omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}) \tag{2.1-11}$$

which is skew-symmetric. We then have

$$u_{i,j} = \varepsilon_{ij} + \omega_{ij}. \tag{2.1-12}$$

It is readily shown that

$$C_{ijks}\omega_{ij} = 0, \quad C_{ijks}\omega_{ks} = 0. \tag{2.1-13}$$

The first equation follows from

$$C_{ijks}\omega_{ij} = C_{jiks}\omega_{ji} = -C_{jiks}\omega_{ij} = -C_{ijks}\omega_{ij}.$$

Similarly, one can prove the second equation. With (2.1-12) and (2.1-13) we have

$$C_{ijks}u_{i,j}u_{k,s} = C_{ijks}\varepsilon_{ij}\varepsilon_{ks}.$$

The condition (2.1-9) for the strain energy to be positive can therefore be replaced by the less restrictive condition

$$C_{ijks}\gamma_{ij}\gamma_{ks} > 0 \tag{2.1-14}$$

in which the γ_{ij} need not be symmetric.

2.2 Elastic Compliances

The inverse of (2.1-1) is written as

$$\varepsilon_{ij} = S_{ijks}\sigma_{ks} \tag{2.2-1}$$

where S_{ijks} are the *elastic compliances* which are components of a fourth rank tensor. They also possess the full symmetry

$$S_{ijks} = S_{jiks}, \quad S_{ijks} = S_{ijsk}, \quad S_{ijks} = S_{ksij}. \tag{2.2-2}$$

or, as in (2.1-3),

$$S_{ijks} = S_{jiks} = S_{ksij}. \tag{2.2-3}$$

The justifications of the first and second equations in (2.2-2) are similar to their counterparts in (2.1-2). The justification of $(2.2\text{-}2)_3$ also follows from the energy consideration. Integration by parts of $(2.1\text{-}5)_1$ leads to

$$\begin{aligned} W &= \sigma_{pq}\varepsilon_{pq} - \int_0^{\sigma_{pq}} \varepsilon_{ij} d\sigma_{ij} \\ &= \sigma_{pq}\varepsilon_{pq} - \int_0^{\sigma_{pq}} S_{ijks}\sigma_{ks} d\sigma_{ij}. \end{aligned}$$

If W depends on the final strain ε_{pq}, it depends on the final stress σ_{pq}. The last integral which represents the *complementary energy* must be independent of the path σ_{ij} takes from 0 to the final stress σ_{pq}. Following a similar argument for C_{ijks}, we deduce that $(2.2\text{-}2)_3$ must hold for the integral to be path independent.

For the strain energy to be positive, substitution of (2.2-1) into $(2.1\text{-}8)_2$ yields

$$S_{ijks}\sigma_{ij}\sigma_{ks} > 0. \tag{2.2-4}$$

The counterpart of (2.1-14) is

$$S_{ijks}\gamma_{ij}\gamma_{ks} > 0. \tag{2.2-5}$$

The σ_{ij} in (2.2-4) is symmetric while the γ_{ij} in (2.2-5) need not by symmetric.

2.3 Contracted Notations

Introducing the contracted notation (Voigt, 1910; Lekhnitskii, 1950; Jones, 1975; Christensen, 1979; see also Cowin et al., 1992)

$$\begin{aligned} &\sigma_{11} = \sigma_1, \quad \sigma_{22} = \sigma_2, \quad \sigma_{33} = \sigma_3, \\ &\sigma_{23} = \sigma_4, \quad \sigma_{31} = \sigma_5, \quad \sigma_{12} = \sigma_6, \end{aligned} \tag{2.3-1}$$

$$\begin{aligned} &\varepsilon_{11} = \varepsilon_1, \quad \varepsilon_{22} = \varepsilon_2, \quad \varepsilon_{33} = \varepsilon_3, \\ &2\varepsilon_{23} = \varepsilon_4, \quad 2\varepsilon_{31} = \varepsilon_5, \quad 2\varepsilon_{12} = \varepsilon_6, \end{aligned} \tag{2.3-2}$$

the stress-strain law (2.1-1) and (2.1-2) can be written as

$$\sigma_\alpha = C_{\alpha\beta}\varepsilon_\beta, \qquad C_{\alpha\beta} = C_{\beta\alpha}, \tag{2.3-3a}$$

or, in matrix notation,

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}, \qquad \mathbf{C} = \mathbf{C}^T. \tag{2.3-3b}$$

In the above $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are 6×1 column matrices and $\mathbf{C}$ is the 6×6 symmetric matrix given by

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}. \tag{2.3-4}$$

Only the upper triangle of the matrix is shown above since $\mathbf{C}$ is symmetric. The transformation between C_{ijks} and $C_{\alpha\beta}$ is accomplished by replacing the subscripts ij (or ks) by α (or β) using the following rules:

$$\begin{array}{ccc} ij \text{ (or } ks) & \leftrightarrow & \alpha \text{ (or } \beta) \\ 11 & \leftrightarrow & 1 \\ 22 & \leftrightarrow & 2 \\ 33 & \leftrightarrow & 3 \\ 23 \text{ or } 32 & \leftrightarrow & 4 \\ 31 \text{ or } 13 & \leftrightarrow & 5 \\ 12 \text{ or } 21 & \leftrightarrow & 6 \end{array} \tag{2.3-5a}$$

We may write the transformation in (2.3-5a) as

$$\alpha = \begin{cases} i, & \text{if } i = j, \\ 9 - i - j, & \text{if } i \neq j, \end{cases}$$
$$\beta = \begin{cases} k, & \text{if } k = s, \\ 9 - k - s, & \text{if } k \neq s. \end{cases} \tag{2.3-5b}$$

The presence of the factor 2 in $(2.3\text{-}2)_{4,5,6}$ but not in $(2.3\text{-}1)_{4,5,6}$ is necessary for the $\mathbf{C}$ to be symmetric. Had we omitted the factor 2 in $(2.3\text{-}2)_{4,5,6}$ the 6×6 matrix $\mathbf{C}$ would not be symmetric.

With (2.3-1) and (2.3-2), the stress-strain law (2.2-1) in contracted notation is

$$\varepsilon_\alpha = s_{\alpha\beta}\sigma_\beta, \qquad s_{\alpha\beta} = s_{\beta\alpha}, \tag{2.3-6a}$$

or

$$\boldsymbol{\varepsilon} = \mathbf{s}\boldsymbol{\sigma}, \qquad \mathbf{s} = \mathbf{s}^T, \tag{2.3-6b}$$

where

$$\mathbf{s} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ & & s_{33} & s_{34} & s_{35} & s_{36} \\ & & & s_{44} & s_{45} & s_{46} \\ & & & & s_{55} & s_{56} \\ & & & & & s_{66} \end{bmatrix}. \tag{2.3-7}$$

Again, only the upper triangle is shown for the matrix **s**. We employ the lower case **s** for the 6×6 matrix because the upper case **S** is reserved for something else (Section 5.4). The transformation between S_{ijks} and $s_{\alpha\beta}$ is similar to that between C_{ijks} and $C_{\alpha\beta}$ except the following:

$$\begin{aligned} S_{ijks} &= s_{\alpha\beta}, \text{ if both } \alpha,\beta \leq 3, \\ 2S_{ijks} &= s_{\alpha\beta}, \text{ if either } \alpha \text{ or } \beta \leq 3, \\ 4S_{ijks} &= s_{\alpha\beta}, \text{ if both } \alpha,\beta > 3. \end{aligned} \tag{2.3-8}$$

The strain energy W is, from $(2.1\text{-}8)_2$, $(2.3\text{-}3b)_1$, and $(2.3\text{-}6b)_1$,

$$W = \tfrac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\varepsilon} = \tfrac{1}{2}\boldsymbol{\varepsilon}^T\mathbf{C}\boldsymbol{\varepsilon} = \tfrac{1}{2}\boldsymbol{\sigma}^T\mathbf{s}\boldsymbol{\sigma}. \tag{2.3-9}$$

For the W to be positive we must have

$$\boldsymbol{\varepsilon}^T\mathbf{C}\boldsymbol{\varepsilon} > 0 \tag{2.3-10}$$

or

$$\boldsymbol{\sigma}^T\mathbf{s}\boldsymbol{\sigma} > 0. \tag{2.3-11}$$

Equations (2.3-10) and (2.3-11) indicate that the matrices **C** and **s** are both positive definite. Substitution of $(2.3\text{-}6b)_1$ into $(2.3\text{-}3b)_1$ yields

$$\mathbf{Cs} = \mathbf{I} = \mathbf{sC}. \tag{2.3-12}$$

The second equality follows from the first equality which says that the **C** and **s** are the inverses of each other, and hence their products commute.

2.4 Reduced Elastic Compliances

For the most part of this book we will be concerned with two-dimensional deformations of anisotropic elastic bodies for which $\varepsilon_3 = 0$. When $\varepsilon_3 = 0$, the stress-strain law $(2.3\text{-}3a)_1$ becomes

$$\sigma_\alpha = \sum_{\beta\neq 3} C_{\alpha\beta}\varepsilon_\beta, \qquad \alpha = 1,2,\ldots,6. \tag{2.4-1}$$

Ignoring the equation for σ_3 we may write (2.4-1) as

$$\boldsymbol{\sigma}^{\mathrm{o}} = \mathbf{C}^{\mathrm{o}}\boldsymbol{\varepsilon}^{\mathrm{o}}, \qquad \mathbf{C}^{\mathrm{o}} = (\mathbf{C}^{\mathrm{o}})^T, \tag{2.4-2}$$

where

$$\begin{aligned}(\boldsymbol{\sigma}^{\mathrm{o}})^T &= [\sigma_1,\ \sigma_2,\ \sigma_4,\ \sigma_5,\ \sigma_6],\\ (\boldsymbol{\varepsilon}^{\mathrm{o}})^T &= [\varepsilon_1,\ \varepsilon_2,\ \varepsilon_4,\ \varepsilon_5,\ \varepsilon_6],\end{aligned} \tag{2.4-3}$$

and

$$\mathbf{C}^{\mathrm{o}} = \begin{bmatrix} C_{11} & C_{12} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{24} & C_{25} & C_{26} \\ & & C_{44} & C_{45} & C_{46} \\ & & & C_{55} & C_{56} \\ & & & & C_{66} \end{bmatrix} \tag{2.4-4}$$

The matrix $\mathbf{C}^{\mathrm{o}}$ is obtained from $\mathbf{C}$ by deleting the third row and third column of $\mathbf{C}$. Since $\mathbf{C}^{\mathrm{o}}$ is a principal submatrix of $\mathbf{C}$, it is positive definite. It contains 15 independent elastic constants.

The stress-strain law $(2.3\text{-}6a)_1$ for $\varepsilon_3=0$ is

$$\varepsilon_3 = 0 = s_{3\beta}\sigma_\beta .$$

Solving for σ_3,

$$\sigma_3 = -\frac{1}{s_{33}}\sum_{\beta\neq 3} s_{3\beta}\sigma_\beta , \tag{2.4-5}$$

and substituting σ_3 into $(2.3\text{-}6a)_1$ leads to

$$\varepsilon_\alpha = \sum_{\beta\neq 3} s'_{\alpha\beta}\sigma_\beta , \tag{2.4-6}$$

where

$$s'_{\alpha\beta} = s_{\alpha\beta} - \frac{s_{\alpha 3}s_{3\beta}}{s_{33}} = s'_{\beta\alpha}. \tag{2.4-7}$$

The $s'_{\alpha\beta}$ are the *reduced elastic compliances*. It is clear that

$$s'_{\alpha 3} = 0, \quad s'_{3\beta} = 0, \quad \alpha,\beta = 1,2,\ldots,6. \tag{2.4-8}$$

Therefore there is no need to exclude $\beta=3$ in (2.4-6). Using the notation of (2.4-3), (2.4-6) can be written as

$$\boldsymbol{\varepsilon}^{\mathrm{o}} = \mathbf{s}'\boldsymbol{\sigma}^{\mathrm{o}}, \qquad \mathbf{s}' = (\mathbf{s}')^T, \tag{2.4-9}$$

where s' is the 5×5 symmetric matrix given by

$$\mathbf{s}' = \begin{bmatrix} s'_{11} & s'_{12} & s'_{14} & s'_{15} & s'_{16} \\ & s'_{22} & s'_{24} & s'_{25} & s'_{26} \\ & & s'_{44} & s'_{45} & s'_{46} \\ & & & s'_{55} & s'_{56} \\ & & & & s'_{66} \end{bmatrix}. \tag{2.4-10}$$

Like $\mathbf{C}^o$, s' contains 15 independent constants. Substitution of (2.4-9) into (2.4-2) yields

$$\boldsymbol{\sigma}^o = \mathbf{C}^o \mathbf{s}' \boldsymbol{\sigma}^o$$

which tells us that, since $\boldsymbol{\sigma}^o$ is arbitrary,

$$\mathbf{C}^o \mathbf{s}' = \mathbf{I} = \mathbf{s}' \mathbf{C}^o. \tag{2.4-11}$$

Therefore $\mathbf{C}^o$ and s' are the inverses of each other. The positive definite of $\mathbf{C}^o$ indicates that s' is also positive definite.

An alternate proof that $\mathbf{C}^o$ and s' are positive definite is to write the strain energy of (2.3-9) as, noting that $\varepsilon_3=0$,

$$W = \tfrac{1}{2}\boldsymbol{\sigma}^T \boldsymbol{\varepsilon} = \tfrac{1}{2}(\boldsymbol{\sigma}^o)^T \boldsymbol{\varepsilon}^o. \tag{2.4-12}$$

Use of (2.4-2) and (2.4-9) gives

$$W = \tfrac{1}{2}(\boldsymbol{\varepsilon}^o)^T \mathbf{C}^o \boldsymbol{\varepsilon}^o = \tfrac{1}{2}(\boldsymbol{\sigma}^o)^T \mathbf{s}' \boldsymbol{\sigma}^o. \tag{2.4-13}$$

If W is positive for any nonzero $\boldsymbol{\sigma}^o$ or $\boldsymbol{\varepsilon}^o$, the $\mathbf{C}^o$ and s' must be positive definite.

The relation (2.4-11) is a property of elastic constants. Its validity should be independent of whether $\varepsilon_3=0$ or not. To prove (2.4-11) without assuming $\varepsilon_3=0$, we write $C^o_{\alpha\beta}$ in terms of $C_{\alpha\beta}$ as

$$C^o_{\alpha\beta} = C_{\alpha\beta} - \delta_{\alpha 3}C_{3\beta} - C_{\alpha 3}\delta_{3\beta} + \delta_{\alpha 3}C_{33}\delta_{3\beta}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. From (2.4-7) and (2.3-12) it is easily verified that

$$C^o_{\alpha\beta} s'_{\beta\gamma} = \delta_{\alpha\gamma} - \delta_{\alpha 3}\delta_{3\gamma}. \tag{2.4-14}$$

Written as 6×6 matrices, $C^o_{\alpha\beta}$, $s'_{\beta\gamma}$, and $(\delta_{\alpha\gamma}-\delta_{\alpha 3}\delta_{3\gamma})$ have only zero elements in the third rows and third columns. Equation (2.4-14) is equivalent to $(2.4\text{-}11)_1$ when all matrices are reduced to 5×5 matrices by deleting the third rows and third columns.

2.5 Material Symmetry

The 6×6 matrices **C** and **s** contain 21 independent elastic constants. The number of independent constants is reduced when the material possesses a certain material symmetry.

Written as C_{ijks}, the elastic stiffnesses are components of a fourth rank tensor. Under an orthogonal transformation

$$x_i^* = \Omega_{ij}x_j \quad \text{or} \quad \mathbf{x}^* = \Omega\mathbf{x} \tag{2.5-1}$$

in which Ω is an orthogonal matrix (Section 1.1) which satisfies the relation

$$\Omega_{ij}\Omega_{kj} = \delta_{ik} = \Omega_{ji}\Omega_{jk} \tag{2.5-2a}$$

or

$$\Omega\Omega^T = \mathbf{I} = \Omega^T\Omega, \tag{2.5-2b}$$

the elastic compliances C^*_{ijks} referred to the x_i^* coordinate system are

$$C^*_{ijks} = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt}. \tag{2.5-3}$$

An identical equation can be written for S_{ijks}. When $C^*_{ijks} = C_{ijks}$, i.e.,

$$C_{ijks} = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt}, \tag{2.5-4}$$

the material is said to possess a *symmetry* with respect to Ω.

An anisotropic material possesses the symmetry of *central inversion* if (2.5-4) is satisfied for

$$\Omega = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -\mathbf{I}. \tag{2.5-5}$$

It is obvious that (2.5-4) is satisfied by the Ω given in (2.5-5) for any C_{ijks}. Therefore all anisotropic elastic materials have the symmetry of central inversion.

If Ω is a proper orthogonal, the transformation (2.5-1) represents a rigid body rotation about an axis. When (2.5-4) is satisfied by a proper orthogonal, the material possesses a *rotational symmetry*. For instance,

$$\Omega^{(r)}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.5-6}$$

represents a rotation about the x_3-axis an angle θ.

An orthogonal transformation Ω is a *reflection* if

$$\boldsymbol{\Omega} = \mathbf{I} - 2\mathbf{n}\mathbf{n}^T \tag{2.5-7a}$$

where $\mathbf{n}$ is a unit vector normal to the reflection plane. If $\mathbf{m}$ is any vector on the plane,

$$\boldsymbol{\Omega}\mathbf{n} = -\mathbf{n}, \qquad \boldsymbol{\Omega}\mathbf{m} = \mathbf{m}. \tag{2.5-7b}$$

Thus a vector normal to the reflection plane reverses its direction after the transformation while a vector on the reflection plane remains unchanged. When (2.5-4) is satisfied by the Ω of (2.5-7a), the material is said to possess a *symmetry plane*. For example, let

$$\mathbf{n}^T = [\cos\theta,\ \sin\theta,\ 0].$$

The symmetry plane contains the x_3-axis. The Ω of (2.5-7a) has the expression

$$\Omega(\theta) = \begin{bmatrix} -\cos 2\theta & -\sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad -\tfrac{\pi}{2} < \theta \le \tfrac{\pi}{2}, \tag{2.5-8}$$

which is an improper orthogonal matrix. Since θ and $\theta+\pi$ represent the same plane, θ is limited to the range shown in $(2.5\text{-}8)_2$. At $\theta=0$,

$$\Omega(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.5-9}$$

This represents a reflection about the plane $x_1=0$. When (2.5-4) is satisfied by (2.5-9), the material has a symmetry plane at $x_1=0$. If (2.5-4) is satisfied by (2.5-8) for any θ, the material is *transversely isotropic*. The x_3-axis is the *axis of symmetry*.

Two extreme cases of anisotropic elastic materials are *triclinic materials* and *isotropic materials*. A triclinic material possesses no rotational symmetry or a plane of reflection symmetry. An isotropic material possesses infinitely many rotational symmetries and planes of reflection symmetry. For isotropic materials it can be shown that

$$C_{ijks} = \lambda\delta_{ij}\delta_{ks} + \mu(\delta_{ik}\delta_{js} + \delta_{is}\delta_{jk}), \tag{2.5-10}$$

where λ and μ are the Lamé constants, satisfies (2.5-4) for any orthogonal Ω.

If we multiply both sides of (2.5-4) by

$$\Omega_{ia}\Omega_{jb}\Omega_{kc}\Omega_{sd}$$

and make use of (2.5-2a), we have

$$C_{abcd} = \Omega_{ia}\Omega_{jb}\Omega_{kc}\Omega_{sd}C_{ijks} \tag{2.5-11}$$

This implies that the material is symmetric with respect to the orthogonal matrix $\Omega^T=\Omega^{-1}$.

> **Theorem 2.5–1** If an anisotropic elastic material possesses a material symmetry with the orthogonal matrix Ω, it possesses the material symmetry with $\Omega^T=\Omega^{-1}$.

This means that, if the material has the symmetry with rotation about the x_3-axis an angle θ, it also has the symmetry with rotation about the x_3-axis an angle $-\theta$.

Let the material possess a symmetry with Ω' and Ω'', i.e.,

$$C_{ijks} = \Omega'_{ia}\Omega'_{jb}\Omega'_{kc}\Omega'_{sd}C_{abcd}, \tag{2.5-12}$$

and

$$C_{abcd} = \Omega''_{ap}\Omega''_{bq}\Omega''_{cr}\Omega''_{dt}C_{pqrt}. \tag{2.5-13}$$

Insertion of (2.5-13) into (2.5-12) leads to

$$C_{ijks} = (\Omega'_{ia}\Omega''_{ap})(\Omega'_{jb}\Omega''_{bq})(\Omega'_{kc}\Omega''_{cr})(\Omega'_{sd}\Omega''_{dt})C_{pqrt}.$$

This is in the form of (2.5-4).

> **Theorem 2.5–2** If an anisotropic elastic material possesses a symmetry with Ω' and Ω'', it possesses a symmetry with $\Omega=\Omega'\Omega''$.

The orthogonal matrix $\Omega(\theta)$ of (2.5-8) represents a reflection symmetry with a plane whose normal lies on the (x_1,x_2) plane making an angle of θ with the x_1-axis. It can be shown that

$$\begin{aligned} \Omega(-\theta) &= \Omega(0)\Omega(\theta)\Omega(0), \\ \Omega(2\theta) &= \Omega(\theta)\Omega(0)\Omega(\theta). \end{aligned} \tag{2.5-14}$$

By Theorem 2.5–2, if a material has symmetry planes at $\theta=0$ and θ_0, say, it also has symmetry planes at $\theta=-\theta_0$ and $2\theta_0$. Repeated applications of (2.5-14) show that the material has symmetry planes at $\theta=k\theta_0$ where k is any integer, positive or negative. If θ_0 is an irrational number times π, the material has symmetry plane at any θ. This suggests that, to test if a material is transversely isotropic in experiment, all one has to do is to see if the material has the reflection symmetry with the planes at $\theta=0$ and θ_0 where θ_0 is any irrational number times π. We will show in the next section that the material is transversely isotropic if it has reflection symmetry at $\theta=0$ and θ_0, where θ_0 is any angle not equal to $\pi/4$, $\pi/3$, or $\pi/2$.

Theorems 2.5–1 and 2.5–2, which remain valid for nonlinear materials (Truesdell and Noll, 1965), are useful in determining the structure of C_{ijks} when the material possesses symmetry planes.

2.6 Matrix C for Materials with Symmetry Planes

Depending on the number of rotations and/or reflection symmetry a crystal possesses, Voigt (1910) has classified crystals into 32 classes. (See also Love, 1927; Musgrave, 1970; Gurtin, 1972; Cowin and Mehrabadi, 1987; and Mehrabadi and Cowin, 1990). In terms of the 6×6 matrix **C** however there are only 8 basic groups. For a non-crystalline material the structure of **C** can also be represented by one of the 8 basic groups. We list below the 8 basic groups for **C** according to the number of symmetry planes the group has. The derivation of each group will be given later. Consideration of rotational symmetry does not change the structure of **C** in each group. Without loss in generality we choose the symmetry plane (or planes) to coincide with the coordinate planes whenever possible. If the matrix **C** referred to a different coordinate system is desired, we use (2.5-3) to obtain the new matrix **C**. (In some literature the structure of **C** referred to a rotated coordinate system is considered as a separate group.) We will therefore employ the orthogonal matrix Ω (2.5-8) which represents a reflection with respect to a plane whose normal is on the (x_1, x_2) plane making an angle θ with the x_1-axis. We will also employ the orthogonal matrix

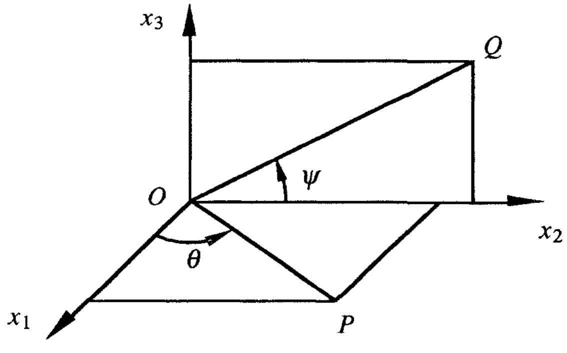

Fig. 2.1 The vector OP and OQ are, respectively, the normals to planes of reflection symmetry defined in (2.5-8) and (2.6-1).

$$\hat{\Omega}(\psi)=\begin{bmatrix}1 & 0 & 0\\ 0 & -\cos 2\psi & -\sin 2\psi\\ 0 & -\sin 2\psi & \cos 2\psi\end{bmatrix}, \qquad -\tfrac{\pi}{2}<\psi\le\tfrac{\pi}{2} \tag{2.6-1}$$

which represents a reflection with respect to a plane whose normal is on the (x_2,x_3) plane making an angle ψ with the x_2-axis, Fig. 2.1. The plane $x_2=0$ can be represented by either $\theta=\pi/2$ or $\psi=0$. Only the upper triangle of the matrix **C** is shown. The symbols employed follow those of Musgrave (1970).

$*$	indicates a possibly nonzero element
$*$—$*$	indicates the two elements are equal
$*$—$\bar{*}$	indicates the two elements are equal but opposite in signs
$\otimes$	indicates $C_{66}=\frac{1}{2}(C_{11}-C_{12})$
m	indicates the number of independent elastic constants

(I) **Triclinic Materials**. No symmetry planes exist.

$$\mathbf{C}=\begin{bmatrix} * & * & * & * & * & *\\ & * & * & * & * & *\\ & & * & * & * & *\\ & & & * & * & *\\ & & & & * & *\\ & & & & & * \end{bmatrix}, \quad m=21. \tag{2.6-2}$$

(II) **Monoclinic Materials**. One symmetry plane.

(a) Symmetry plane at $x_1=0$, i.e., $\theta=0$.

$$\mathbf{C}=\begin{bmatrix} * & * & * & * & 0 & 0\\ & * & * & * & 0 & 0\\ & & * & * & 0 & 0\\ & & & * & 0 & 0\\ & & & & * & *\\ & & & & & * \end{bmatrix}, \quad m=13. \tag{2.6-3a}$$

(b) Symmetry plane at $x_2=0$, i.e., $\theta=\pi/2$ or $\psi=0$.

$$\mathbf{C}=\begin{bmatrix} * & * & * & 0 & * & 0 \\ & * & * & 0 & * & 0 \\ & & * & 0 & * & 0 \\ & & & * & 0 & * \\ & & & & * & 0 \\ & & & & & * \end{bmatrix}, \quad \text{m=13.} \tag{2.6-3b}$$

(c) Symmetry plane at $x_3=0$, i.e., $\psi=\pi/2$.

$$\mathbf{C}=\begin{bmatrix} * & * & * & 0 & 0 & * \\ & * & * & 0 & 0 & * \\ & & * & 0 & 0 & * \\ & & & * & * & 0 \\ & & & & * & 0 \\ & & & & & * \end{bmatrix}, \quad \text{m=13.} \tag{2.6-3c}$$

(III) **Orthotropic (or Rhombic) Materials.** The three coordinate planes $\theta=0$, $\pi/2$, and $\psi=\pi/2$ are the symmetry planes.

$$\mathbf{C}=\begin{bmatrix} * & * & * & 0 & 0 & 0 \\ & * & * & 0 & 0 & 0 \\ & & * & 0 & 0 & 0 \\ & & & * & 0 & 0 \\ & & & & * & 0 \\ & & & & & * \end{bmatrix}, \quad \text{m=9.} \tag{2.6-4}$$

(IV) **Trigonal Materials.** Three symmetry planes at $\theta=0$ and $\pm\pi/3$.

$$\mathbf{C}=\begin{bmatrix} * & * & * & * & 0 & 0 \\ & * & * & \bar{*} & 0 & 0 \\ & & * & 0 & 0 & 0 \\ & & & * & 0 & 0 \\ & & & & * & * \\ & & & & & \otimes \end{bmatrix}, \quad \text{m=6.} \tag{2.6-5}$$

(V) **Tetragonal Materials.** Five symmetry planes at $\theta=0$, $\pm\pi/4$, $\pi/2$, and $\psi=\pi/2$.

$$\mathbf{C}=\begin{bmatrix} * & * & * & 0 & 0 & 0 \\ & * & * & 0 & 0 & 0 \\ & & * & 0 & 0 & 0 \\ & & & * & 0 & 0 \\ & & & & * & 0 \\ & & & & & * \end{bmatrix}, \quad m=6. \tag{2.6-6}$$

(VI) **Transversely Isotropic (or Hexagonal) Materials.** The symmetry planes are the $x_3=0$ plane and any plane that contains the x_3-axis. The x_3-axis is the axis of symmetry.

$$\mathbf{C}=\begin{bmatrix} * & * & * & 0 & 0 & 0 \\ & * & * & 0 & 0 & 0 \\ & & * & 0 & 0 & 0 \\ & & & * & 0 & 0 \\ & & & & * & 0 \\ & & & & & \otimes \end{bmatrix}, \quad m=5. \tag{2.6-7}$$

(VII) **Cubic Materials.** Nine planes of symmetry whose normals are on the three coordinate axes and on the coordinate planes making an angle $\pi/4$ with the coordinate axes.

$$\mathbf{C}=\begin{bmatrix} * & * & * & 0 & 0 & 0 \\ & * & * & 0 & 0 & 0 \\ & & * & 0 & 0 & 0 \\ & & & * & 0 & 0 \\ & & & & * & 0 \\ & & & & & * \end{bmatrix}, \quad m=3. \tag{2.6-8}$$

(VIII) **Isotropic Materials.** Any plane is a symmetry plane.

$$\mathbf{C}=\begin{bmatrix} * & * & * & 0 & 0 & 0 \\ & * & * & 0 & 0 & 0 \\ & & * & 0 & 0 & 0 \\ & & & \otimes & 0 & 0 \\ & & & & \otimes & 0 \\ & & & & & \otimes \end{bmatrix}, \quad m=2. \tag{2.6-9a}$$

If λ and μ are the *Lamé constants*, (2.6-9a) has the expression

$$\mathbf{C}=\begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}, \quad \mathrm{m}=2. \tag{2.6-9b}$$

μ is also called the *shear modulus*. While the number of nonzero elements in $\mathbf{C}$ may increase when different coordinate systems are employed, the number of independent elastic constants m does not depend on the choice of the coordinate systems.

We will now show how the structure of $\mathbf{C}$ in each group is determined. It should be noted that, by Theorem 2.5–2, the number of symmetry planes required in determining the structure of $\mathbf{C}$ is usually less than the number of the symmetry planes that exist in the material. For example, for orthotropic materials (Group III) the symmetry planes at $\theta=0$ and $\pi/2$ implies that the group also has a symmetry plane at $\psi=\pi/2$. This is so because

$$\hat{\Omega}(\tfrac{\pi}{2}) = -\Omega(0)\Omega(\tfrac{\pi}{2}).$$

The two Ω's on the right represent reflection with planes at $\theta=0$ and $\pi/2$ while the minus sign represents, by (2.5-5), the symmetry of central inversion which is possessed by all materials. In fact we will see that all materials including isotropic materials require no more than three symmetry planes to determine the structure of $\mathbf{C}$.

We begin with monoclinic materials with the symmetry plane at $x_1=0$. This means that (2.5-4) is satisfied by the Ω given in (2.5-9). If we write (2.5-9) as

$$\Omega_{ij} = \begin{cases} \delta_{ij}, & \text{if } i \neq 1 \\ -\delta_{ij}, & \text{if } i = 1 \end{cases}$$

where δ_{ij} is the Kronecker delta, (2.5-4) reduces to

$$C_{ijks} = (-1)^N C_{ijks} \tag{2.6-10}$$

in which N is the number of ones appearing in the subscripts $ijks$. We see that (2.6-10) is a trivial identity for $N=0$, 2, and 4, and

$$C_{ijks} = 0 \tag{2.6-11}$$

for $N=1$ and 3. Thus C_{ijks} vanishes when the number 1 appears once or three times in $ijks$. Due to the symmetry (2.1-2) for C_{ijks}, the following eight $ijks$ are the only possible choices for which $N=1$ or 3.

$$ijks = 3313, 3312, 2313, 2312, 1113, 2213, 1112, 2212. \tag{2.6-12}$$

Using contracted notations, (2.6-11) and (2.6-12) lead to

$$C_{35} = C_{36} = C_{45} = C_{46} = C_{15} = C_{25} = C_{16} = C_{26} = 0 \tag{2.6-13}$$

which verify the zero elements of $\mathbf{C}$ in (2.6-3a). The structure of $\mathbf{C}$ for (2.6-3b) and (2.6-3c) can be determined similarly.

We next consider the existence of a second plane of symmetry whose Ω is given by (2.5-8) with $\theta \neq 0$. By writing (2.5-8) as

$$\Omega = \Omega''\Omega',$$

$$\Omega'' = \begin{bmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\Omega' = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

it is seen that Ω' and Ω'' are both orthogonal matrices and that Ω'' is symmetric. Moreover, from the relation

$$\Omega''_{ij} = \begin{cases} \Omega'_{ij}, & \text{if } i \neq 1 \\ -\Omega'_{ij}, & \text{if } i = 1 \end{cases}$$

(2.5-4) can be written as

$$\Omega''_{ip}\Omega''_{jq}\Omega''_{kr}\Omega''_{st}C_{pqrt} = \Omega'_{ip}\Omega'_{jq}\Omega'_{kr}\Omega'_{st}C_{pqrt}$$

or

$$\Omega'_{ip}\Omega'_{jq}\Omega'_{kr}\Omega'_{st}C_{pqrt}C_{pqrt} = (-1)^N \Omega'_{ip}\Omega'_{jq}\Omega'_{kr}\Omega'_{st}C_{pqrt}$$

where N is the number of ones appearing in $ijks$. This is similar to (2.6-10). Following the same arguments in deriving (2.6-11) and (2.6-12) we obtain

$$\Omega'_{ip}\Omega'_{jq}\Omega'_{kr}\Omega'_{st}C_{pqrt} = 0 \tag{2.6-14}$$

for the eight $ijks$ given in (2.6-12).

We consider a symmetry plane at $\sin\theta \neq 0$ in addition to the symmetry plane at $\theta=0$. The latter has given us (2.6-13). For the first $ijks$ in (2.6-12), (2.6-14) reduces to

$$\Omega'_{1r}C_{33r3} = 0$$

or

$$C_{35}\cos\theta + C_{34}\sin\theta = 0.$$

Since $C_{35}=0$ by (2.6-13) and $\sin\theta \neq 0$,

$$C_{34} = 0. \tag{2.6-15}$$

For the next two sets of *ijks* in (2.6-12), (2.6-14) gives

$$(C_{13} - C_{23})\cos\theta = 0,$$
$$(C_{44} - C_{55})\cos\theta = 0.$$

These mean that

$$C_{13} = C_{23} \quad \text{and} \quad C_{44} = C_{55}, \quad \text{if} \quad \cos\theta \neq 0. \tag{2.6-16}$$

The next three sets of *ijks* in (2.6-12) lead to a system of three equations for C_{56}, C_{24}, and C_{14},

$$C_{56} = (2C_{56} - C_{24} + C_{14})\cos^2\theta,$$

$$-C_{24} = (2C_{56} - C_{24} + C_{14})\cos^2\theta,$$

$$C_{14} = (2C_{56} - C_{24} + C_{14})\cos^2\theta.$$

The right sides of the equations are identical, implying that

$$C_{56} = -C_{24} = C_{14} = \gamma, \text{ say.}$$

All three equations can then be written as

$$\gamma = 4\gamma\cos^2\theta.$$

We therefore have the results that

$$\left.\begin{aligned} C_{56} &= -C_{24} = C_{14}, &&\text{if } \cos^2\theta = \tfrac{1}{4}, \\ C_{56} &= -C_{24} = C_{14} = 0, &&\text{if } \cos^2\theta \neq \tfrac{1}{4}. \end{aligned}\right\} \tag{2.6-17}$$

The last two sets of *ijks* in (2.6-12) provide two equations which, by adding and subtracting the two equations, can be written as

$$(C_{11} - C_{22})\cos\theta = 0,$$
$$\{C_{12} + 2C_{66} - \tfrac{1}{2}(C_{11} + C_{22})\}\cos\theta\cos 2\theta = 0.$$

These two equations mean that

$$\left.\begin{aligned} &C_{11} = C_{22} \quad \text{if} \quad \cos 2\theta = 0, \\ &C_{11} = C_{22} \text{ and } C_{66} = \tfrac{1}{2}(C_{11} - C_{12}) \quad \text{if} \quad \cos\theta \neq 0 \neq \cos 2\theta. \end{aligned}\right. \tag{2.6-18}$$

The structure of **C** for Groups III–VI as shown in (2.6-4)–(2.6-7) can now be determined from (2.6-15)–(2.6-18) as follows:

Group III: θ=0 and $\pi/2$.

Group IV: θ=0 and $\pi/3$.

Group V: θ=0 and $\pi/4$.

Group VI: (i) θ=0 and θ_o, $\theta_o \neq \pi/2$, $\pi/3$, $\pi/4$, or
(ii) θ=0, $\pi/3$, and $\pi/4$, or
(iii) θ=0, $\pi/3$, and $\pi/2$, or
(iv) θ=0, $\pi/3$, and $\psi=\pi/2$.

It is seen that two planes of symmetry are sufficient to determine the structure of **C** for Groups III–VI. Group VI has four choices. The first choice (i) requires only two planes of symmetry. The second choice (ii) can be verified easily by combining the structure of **C** shown in (2.6-5) and (2.6-6). Likewise, the third choice (iii) is the result of combining (2.6-5) and (2.6-4), and the fourth choice (iv) is obtained by combining (2.6-5) and (2.6-3c).

To determine the structure of **C** for the remaining two groups we consider the transformation (2.6-1). It can be shown that

$$\hat{\Omega} = \hat{\Omega}''\hat{\Omega}'$$

where

$$\hat{\Omega}'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos\psi & -\sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix},$$

$$\hat{\Omega}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}.$$

Both $\hat{\Omega}'$ and $\hat{\Omega}''$ are orthogonal matrices and $\hat{\Omega}''$ is symmetric. Observing that

$$\hat{\Omega}''_{ij} = \begin{cases} \hat{\Omega}'_{ij}, & \text{if } i \neq 2 \\ -\hat{\Omega}'_{ij}, & \text{if } i = 2 \end{cases}$$

and following the arguments leading to (2.6-14) we have

$$\hat{\Omega}'_{ip}\hat{\Omega}'_{jq}\hat{\Omega}'_{kr}\hat{\Omega}'_{st}C_{pqrt} = 0 \tag{2.6-19}$$

whenever the number of twos appearing in *ijks* is one or three. Like (2.6-12) there are only eight possible choices which are

$$ijks = 1121, 1123, 1213, 1222, 1233, 2223, 2333, 1323. \tag{2.6-20}$$

We will assume that the material has a symmetry plane at $\theta=0$, and see what we can get when it also has a symmetry plane at a $\psi\neq0$. The derivation is similar to the one that leads to (2.6-15)–(2.6-18). It can be shown that (2.6-19) is automatically satisfied by four of the *ijks* listed in (2.6-20). The remaining four *ijks* give us

$$\begin{aligned} 2C_{14}\cos 2\psi &= (C_{12}-C_{13})\sin 2\psi, \\ 2C_{56}\cos 2\psi &= (C_{66}-C_{55})\sin 2\psi, \end{aligned} \tag{2.6-21}$$

$$\begin{aligned} 2(C_{24}+C_{34})\cos 2\psi &= (C_{22}-C_{33})\sin 2\psi, \\ 4(C_{24}-C_{34})\cos 4\psi &= (C_{12}+C_{33}-2C_{23}-4C_{44})\sin 4\psi. \end{aligned} \tag{2.6-22}$$

Equations $(2.6\text{-}21)_{1,2}$ are, respectively, from $ijks$=1123 and 1213. Equations $(2.6\text{-}22)_{1,2}$ are the results of adding and subtracting two equations obtained from applying $ijks$=2223 and 2333. Ignoring the case $\psi=0$ which is equivalent to $\theta=\pi/2$ studied earlier, we obtain from (2.6-21) and (2.6-22)

$$\begin{aligned} C_{14} = C_{24} = C_{34} = C_{56} = 0, \quad &\text{if} \quad \psi = \tfrac{1}{2}\pi, \\ C_{12} = C_{13},\ C_{55} = C_{66},\ C_{22} = C_{33},\ C_{24} = C_{34}, \quad &\text{if} \quad \psi = \tfrac{1}{4}\pi. \end{aligned} \tag{2.6-23}$$

Application of the conditions for $\psi=\pi/2$ in (2.6-23) to (2.6-5) leads to (2.6-7). This was the fourth choice of symmetry planes for Group VI discussed earlier. Use of the conditions for $\psi=\pi/4$ in (2.6-23) on (2.6-6) and (2.6-5) determines the structure of **C** for Groups VII and VIII, respectively.

Group VII: $\theta=0,\ \pi/4$, and $\psi=\pi/4$.

Group VIII: $\theta=0,\ \pi/3$, and $\psi=\pi/4$.

It is remarkable that, for isotropic materials, it takes only three planes of symmetry to reduce the number of elastic constants from 21 to 2.

2.7 Matrices s and s' for Materials with Symmetry Planes

Like C_{ijks}, the elastic compliances S_{ijks} are components of a fourth rank tensor. Under the orthogonal transformation (2.5-1) S^*_{ijks} in the new coordinate system are related to S_{ijks} by

$$S^*_{ijks} = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}S_{pqrt}. \tag{2.7-1}$$

This is identical to (2.5-3). Therefore the analysis presented in the previous section for C_{ijks} apply to S_{ijks}. The structure of the matrix **C** appearing in (2.6-2)–(2.6-9a) remains valid for the matrix **s** with the following modifications required by (2.3-8). The relation

$$C_{56} = -C_{24} = C_{14}$$

in (2.6-5) is replaced by

$$\tfrac{1}{2}s_{56} = -s_{24} = s_{14} \tag{2.7-2}$$

and the C_{66} element in (2.6-5), (2.6-7), and (2.6-9a) is replaced by

$$s_{66} = 2(s_{11} - s_{12}). \tag{2.7-3}$$

In engineering applications the matrix **s** for isotropic materials is written as

$$\mathbf{s} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & & & & 2(1+\nu) & 0 \\ & & & & & 2(1+\nu) \end{bmatrix}, \tag{2.7-4}$$

where

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)} \tag{2.7-5a}$$

are, respectively, the Young's modulus and Poisson ratio. It can be shown that

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{2.7-5b}$$

The reduced elastic compliances $s'_{\alpha\beta}$ is defined in (2.4-7) in terms of $s_{\alpha\beta}$. As shown in (2.4-10) $s'_{\alpha\beta}$ can be written as a 5×5 symmetric matrix **s'**. It is not difficult to show that in Groups IV–VIII where the relations

$$s_{11} = s_{22}, \quad s_{44} = s_{55} (= s_{66})$$

hold for **s**, the corresponding relations

$$s'_{11} = s'_{22}, \quad s'_{44} = s'_{55} (= s'_{66}) \tag{2.7-6}$$

hold for **s'**. Likewise, the relations (2.7-2) and (2.7-3), which apply to the matrix **s** for Group IV and Groups IV, VI, VIII, respectively, are replaced by the same relations

$$\tfrac{1}{2}s'_{56} = -s'_{24} = s'_{14}, \tag{2.7-7}$$

$$s'_{66} = 2(s'_{11} - s'_{12}), \tag{2.7-8}$$

for the matrix s'. The net result is that the structure of the 5×5 matrix s' for each group can be obtained from the matrix for **s** in the group by deleting the third row and third column. In addition it can be shown that $s'_{\alpha\beta} = s_{\alpha\beta}$ for certain elements of $s'_{\alpha\beta}$. For Group II, monoclinic materials,

$$s'_{\alpha\beta} = s_{\alpha\beta} \text{ for } \alpha,\beta = \begin{cases} 5 \text{ or } 6, \text{ if } x_1 = 0 \text{ is the symmetry plane,} \\ 4 \text{ or } 6, \text{ if } x_2 = 0 \text{ is the symmetry plane,} \\ 4 \text{ or } 5, \text{ if } x_3 = 0 \text{ is the symmetry plane.} \end{cases} \tag{2.7-9}$$

For Groups III–VIII,

$$s'_{\alpha\beta} = s_{\alpha\beta} \quad \text{for} \quad \alpha,\ \beta = 4,\ 5, \text{ or } 6. \tag{2.7-10}$$

The structure of **s**' for isotropic materials

$$\mathbf{s}' = \frac{1}{2\mu}\begin{bmatrix} 1-\nu & -\nu & 0 & 0 & 0 \\ & 1-\nu & 0 & 0 & 0 \\ & & 2 & 0 & 0 \\ & & & 2 & 0 \\ & & & & 2 \end{bmatrix}. \tag{2.7-11}$$

2.8 Transformation of C and s

We had stated earlier that the structure of **C** (or **s**) for each symmetry group in (2.6-2)–(2.6-8) is for the coordinate system specified in the group. For a different coordinate system we apply (2.5-3) to obtain the structure of **C** referred to the new coordinate system. The zero elements in the original matrix **C** may no longer be zero in the new matrix **C**. In applications the choice of the coordinate system is very often dictated by the boundary conditions of the problem and hence may not coincide with the symmetry planes of the material. Therefore transformation of the matrix **C** to a different coordinate system becomes necessary. The transformation law (2.5-3) is for C_{ijks}. It is not convenient for the 6×6 matrix $C_{\alpha\beta}$. We will drive the transformation law for $C_{\alpha\beta}$.

The stresses σ_{ij}, being the components of a second rank tensor, transform according to

$$\sigma^*_{ij} = \Omega_{ik}\Omega_{js}\sigma_{ks}. \tag{2.8-1}$$

With the contracted notation (2.3-1) this can be written in matrix notation as (Bond, 1943; Auld, 1973)

$$\boldsymbol{\sigma}^* = \mathbf{K}\boldsymbol{\sigma}, \tag{2.8-2}$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_1 & 2\mathbf{K}_2 \\ \mathbf{K}_3 & \mathbf{K}_4 \end{bmatrix}, \tag{2.8-3}$$

$$\mathbf{K}_1 = \begin{bmatrix} \Omega_{11}^2 & \Omega_{12}^2 & \Omega_{13}^2 \\ \Omega_{21}^2 & \Omega_{22}^2 & \Omega_{23}^2 \\ \Omega_{31}^2 & \Omega_{32}^2 & \Omega_{33}^2 \end{bmatrix}, \tag{2.8-4a}$$

$$\mathbf{K}_2 = \begin{bmatrix} \Omega_{12}\Omega_{13} & \Omega_{13}\Omega_{11} & \Omega_{11}\Omega_{12} \\ \Omega_{22}\Omega_{23} & \Omega_{23}\Omega_{21} & \Omega_{21}\Omega_{22} \\ \Omega_{32}\Omega_{33} & \Omega_{33}\Omega_{31} & \Omega_{31}\Omega_{32} \end{bmatrix}, \tag{2.8-4b}$$

$$\mathbf{K}_3 = \begin{bmatrix} \Omega_{21}\Omega_{31} & \Omega_{22}\Omega_{32} & \Omega_{23}\Omega_{33} \\ \Omega_{31}\Omega_{11} & \Omega_{32}\Omega_{12} & \Omega_{33}\Omega_{13} \\ \Omega_{11}\Omega_{21} & \Omega_{12}\Omega_{22} & \Omega_{13}\Omega_{23} \end{bmatrix}, \tag{2.8-4c}$$

$$\mathbf{K}_4 = \begin{bmatrix} \Omega_{22}\Omega_{33}+\Omega_{23}\Omega_{32} & \Omega_{23}\Omega_{31}+\Omega_{21}\Omega_{33} & \Omega_{21}\Omega_{32}+\Omega_{22}\Omega_{31} \\ \Omega_{32}\Omega_{13}+\Omega_{33}\Omega_{12} & \Omega_{33}\Omega_{11}+\Omega_{31}\Omega_{13} & \Omega_{31}\Omega_{12}+\Omega_{32}\Omega_{11} \\ \Omega_{12}\Omega_{23}+\Omega_{13}\Omega_{22} & \Omega_{13}\Omega_{21}+\Omega_{11}\Omega_{23} & \Omega_{11}\Omega_{22}+\Omega_{12}\Omega_{21} \end{bmatrix}. \tag{2.8-4d}$$

The inverse of (2.8-2) is

$$\boldsymbol{\sigma} = \mathbf{K}^{-1}\boldsymbol{\sigma}^*. \tag{2.8-5}$$

To find $\mathbf{K}^{-1}$ it is best writing (2.8-1) as

$$\sigma_{ij} = \Omega_{ki}\Omega_{sj}\sigma_{ks}^* \tag{2.8-6}$$

and cast (2.8-6) in the form of (2.8-5). It can be shown (Ting, 1987) that

$$(\mathbf{K}^{-1})^T = \begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ 2\mathbf{K}_3 & \mathbf{K}_4 \end{bmatrix}. \tag{2.8-7}$$

The transformation of the strains ε_{ij} is identical to (2.8-1) for the stresses σ_{ij}, i.e.,

$$\varepsilon_{ij}^* = \Omega_{ik}\Omega_{js}\varepsilon_{ks}. \tag{2.8-8}$$

However, the transformation of ε_{ij} in contracted notation is not

(2.8-2) due to the different laws for ε_{ij} shown in (2.3-2). It can be shown from (2.8-8) and (2.3-2) that (Lekhnitskii, 1950)

$$\boldsymbol{\varepsilon}^* = \begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ 2\mathbf{K}_3 & \mathbf{K}_4 \end{bmatrix} \boldsymbol{\varepsilon},$$

or, by (2.8-7),

$$\boldsymbol{\varepsilon}^* = (\mathbf{K}^{-1})^T \boldsymbol{\varepsilon}. \tag{2.8-9}$$

From

$$\boldsymbol{\sigma} = \mathbf{C}\,\boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \mathbf{s}\,\boldsymbol{\sigma},$$

$$\boldsymbol{\sigma}^* = \mathbf{C}^*\boldsymbol{\varepsilon}^*, \quad \boldsymbol{\varepsilon}^* = \mathbf{s}^*\boldsymbol{\sigma}^*,$$

and the transformation laws (2.8-5) and (2.8-9), it is easily shown that

$$\mathbf{C}^* = \mathbf{K}\mathbf{C}\mathbf{K}^T, \quad \mathbf{s}^* = (\mathbf{K}^{-1})^T \mathbf{s}\,\mathbf{K}^{-1}. \tag{2.8-10}$$

These are the transformation laws for **C** and **s**.

The special case in which the transformation is a rotation about the x_3-axis an angle θ,

$$\boldsymbol{\Omega} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It can be shown that

$$\mathbf{K} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & m^2-n^2 \end{bmatrix},$$

$$\mathbf{K}^{-1} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & -2mn \\ n^2 & m^2 & 0 & 0 & 0 & 2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & n & 0 \\ 0 & 0 & 0 & -n & m & 0 \\ mn & -mn & 0 & 0 & 0 & m^2-n^2 \end{bmatrix},$$

where

$$m = \cos\theta, \quad n = \sin\theta.$$

For second rank tensors such as σ_{ij} or ε_{ij}, it is known that

$$\sigma_{ii}, \quad \sigma_{ij}\sigma_{ij}, \text{ and } \quad |\sigma_{ij}|$$

are invariants under orthogonal transformations. For the fourth rank tensors C_{ijks} it is readily shown from (2.5-3) and (2.5-2a) that

$$C^*_{iikk} = C_{iikk}, \quad C^*_{ijij} = C_{ijij}.$$

Hence,

$$C_{iikk} = C_{11} + C_{22} + C_{33} + 2(C_{12} + C_{23} + C_{31}),$$

$$C_{ijij} = C_{11} + C_{22} + C_{33} + 2(C_{44} + C_{55} + C_{66}),$$

are invariants under orthogonal transformations (Hirth and Lothe, 1968; Juretschke, 1974). These are linear in $C_{\alpha\beta}$. Ting (1987) has presented several nonlinear invariants under any orthogonal transformations. When the transformation is a rotation about the x_3-axis, additional linear invariants (Lekhnitskii, 1950; Hearmon, 1961; Tsai and Pagano, 1980; Tsai and Hahn, 1980) and second order invariants (Tsai and Hahn, 1980; Ting, 1987) are obtained.

2.9 Restrictions on Elastic Constants

The strong ellipticity condition which ensures that the governing differential equations for elastostatics problems be completely elliptic is

$$C_{ijks}a_i b_j a_k b_s > 0 \tag{2.9-1}$$

for any real vectors a_i and b_i. For isotropic materials, inserting

$$C_{ijks} = \lambda\delta_{ij}\delta_{ks} + \mu(\delta_{ik}\delta_{js} + \delta_{is}\delta_{jk}) \tag{2.9-2}$$

into (2.9-1) yields

$$(\lambda + 2\mu)(\mathbf{a}\cdot\mathbf{b})^2 + \mu[(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2] > 0.$$

The coefficients of $(\lambda+2\mu)$ and μ are greater than or equal to zero. Therefore the strong elliptic conditions for isotropic materials are

$$\lambda + 2\mu > 0, \quad \mu > 0. \tag{2.9-3}$$

The strong convexity condition which is equivalent to the positive definiteness of the strain energy is

$$C_{ijks}\varepsilon_{ij}\varepsilon_{ks} > 0. \tag{2.9-4}$$

Substitution of (2.9-2) into the above leads to

$$\lambda(\varepsilon_{ii})^2 + 2\mu(\varepsilon_{ij}\varepsilon_{ij}) > 0$$

or

$$(\lambda + \tfrac{2}{3}\mu)(\varepsilon_{ii})^2 + \tfrac{2}{3}\mu[3(\varepsilon_{ij}\varepsilon_{ij}) - (\varepsilon_{ii})^2] > 0.$$

Without loss in generality, we take the principal axes of the strains to be the coordinate axes so that ε_{ij} is a diagonal matrix with the diagonal elements ε_1, ε_2, and ε_3. We then have

$$(\lambda + \tfrac{2}{3}\mu)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 + \tfrac{2}{3}\mu[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2] > 0.$$

The coefficients of $(\lambda + \frac{2}{3}\mu)$ and μ are greater than or equal to zero. Hence,

$$\infty > \lambda + \tfrac{2}{3}\mu > 0, \quad \infty > \mu > 0, \tag{2.9-5}$$

are the necessary and sufficient conditions for the strain energy to be positive for isotropic materials. The left inequalities for $(\lambda + \frac{2}{3}\mu)$ and μ are obtained from (2.7-5b) and (2.9-8) to be derived later.

We see that (2.9-5) implies (2.9-3) but the reverse does not hold. In other words, the positive definiteness of strain energy function imposes a more stringent limitation on elastic constants than the strong elliptic condition. This is true also for general anisotropic materials. To prove this statement for general anisotropic materials we first notice that (2.9-4) remains valid if ε_{ij} is not symmetric. This is so because the skew-symmetric part of ε_{ij} contributes nothing to the inequality (Section 2.1). Now, when (2.9-1) is satisfied by a choice of a_i and b_i, we take $\varepsilon_{ij}=a_i b_j$ so that (2.9-4) is satisfied. On the other hand, if (2.9-4) is satisfied by an ε_{ij}, it is not always possible to rewrite ε_{ij} as $a_i b_j$. If we regard $a_i b_j$ as a matrix, it is a singular matrix because

$$(a_i b_j)x_j = 0$$

for any vector x_i that is perpendicular to b_i. In contrast, ε_{ij} need not be singular. Therefore (2.9-4) imposes a more stringent limitation on elastic constants than (2.9-1).

In the contracted notation (2.9-4) is equivalent to (2.3-10) which implies that the 6×6 matrix **C** is positive definite. If **C** is positive definite, all its principal minors are positive (Section 1.6), i.e.,

$$C_{ii} > 0 \quad (i \text{ not summed}),$$

$$\begin{vmatrix} C_{ii} & C_{ij} \\ C_{ij} & C_{jj} \end{vmatrix} > 0 \quad (i, j, \text{ not summed}),$$

$$\begin{vmatrix} C_{ii} & C_{ij} & C_{ik} \\ C_{ij} & C_{jj} & C_{jk} \\ C_{ik} & C_{jk} & C_{kk} \end{vmatrix} > 0 \quad (i, j, k, \text{ not summed}), \tag{2.9-6}$$

$$\vdots$$

where i, j, k are distinct integers which can have any value from 1 to 6. According to Theorem 1.6–2, the *necessary and sufficient conditions* for the $\mathbf{C}$ to be positive definite are the positivity of its *leading principal minors*, i.e.,

$$C_{11} > 0,$$

$$\begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix} > 0,$$

$$\begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{vmatrix} > 0,$$

$$\vdots \tag{2.9-7}$$

$$\begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{vmatrix} > 0.$$

From (2.3-11) and the discussions in Section 2.4, the 6×6 matrix $\mathbf{s}$ and the 5×5 matrices $\mathbf{C}^o$ and $\mathbf{s}'$ are positive definite. The inequalities (2.9-6) and (2.9-7) apply to these matrices. Applications of (2.9-7) to the matrices $\mathbf{s}$ of (2.7-4) and $\mathbf{s}'$ of (2.7-11) show that these two matrices are positive definite if

$$\infty > E > 0, \qquad -1 < \nu < \tfrac{1}{2}. \tag{2.9-8}$$

It should be noted that E must be bounded. Otherwise a nonzero stress can induce zero strains and hence zero strain energy, contradicting the assumption of positive strain energy.

It should also be noted that inequalities (2.9-1) or (2.9-4) for two-dimensional deformations may yield a different set of inequalities from those for three-dimensional deformations. As an illustration take the positive strain energy condition (2.9-4) and consider a cubic material. This means that the matrix **C** of (2.6-8) is positive definite. Application of (2.9-7) leads to

$$\begin{gathered} C_{11} > 0, \quad C_{11}^2 - C_{12}^2 > 0, \\ (C_{11} - C_{12})^2 (C_{11} + 2C_{12}) > 0, \quad C_{44} > 0. \end{gathered} \tag{2.9-9}$$

They are equivalent to

$$C_{11} + 2C_{12} > 0, \quad C_{11} - C_{12} > 0, \quad C_{44} > 0, \tag{2.9-10}$$

because the first three inequalities in (2.9-9) can be replaced by

$$C_{11} + C_{12} > 0, \quad C_{11} - C_{12} > 0, \quad C_{11} + 2C_{12} > 0,$$

which in turn can be replaced by the first two inequalities in (2.9-10). For two-dimensional deformations for which $\varepsilon_3 = 0$, the third row and the third column of (2.6-9) are not needed. The reduced 5×5 matrix $\mathbf{C}^o$ is positive definite. Application of (2.9-7) yields

$$C_{11} > 0, \quad C_{11}^2 - C_{12}^2 > 0, \quad C_{44} > 0$$

which are equivalent to

$$C_{11} + C_{12} > 0, \quad C_{11} - C_{12} > 0, \quad C_{44} > 0. \tag{2.9-11}$$

The first inequalities in (2.9-10) and (2.9-11) are different. The quantities $C_{11} + 2C_{12}$ and $C_{11} + C_{12}$ can be identified, respectively, as the *bulk moduli* for three-dimensional and two-dimensional deformations. Inequalities (2.9-11) are less restrictive than those in (2.9-10).

2.10 Determination of Symmetry Planes

The classification of the materials according to the number of symmetry planes presented in Section 2.6 is based on the assumption that we have the knowledge on the number and the locations of the planes of symmetry a given material has. This is not the case when we are presented with an unknown material. We are then forced to determine elastic constants of the material referred to an arbitrarily chosen coordinate system. The result is that, if there exists a symmetry plane, it may not be one of the coordinate planes. Consequently all elements of the matrix **C** can be nonzero. The problem is to locate the symmetry planes if they exist when the matrix **C** is given.

When a plane of symmetry exists, (2.5-4) is satisfied by the $\boldsymbol{\Omega}$ given in (2.5-7a), i.e.,

$$\Omega_{ij} = \delta_{ij} - 2n_i n_j \tag{2.10-1}$$

where **n** is a unit vector normal to the plane of symmetry. The matrix Ω_{ij} has the properties given in (2.5-7b) where **m** is any vector perpendicular to **n**. It also has the properties

$$n_i \Omega_{ij} = -n_j, \quad m_i \Omega_{ij} = m_j. \tag{2.10-2}$$

Cowin and Mehrabadi (1987) have shown that a set of necessary and sufficient conditions for **n** to be a unit normal vector to a plane of symmetry is

$$C_{ijkk} n_j = (C_{pqtt} n_p n_q) n_i, \tag{2.10-3}$$

$$C_{isks} n_k = (C_{pqrq} n_p n_r) n_i, \tag{2.10-4}$$

$$C_{ijks} n_j n_s n_k = (C_{pqrt} n_p n_q n_r n_t) n_i, \tag{2.10-5}$$

$$C_{ijks} m_j m_s n_k = (C_{pqrt} n_p m_q n_r m_t) n_i. \tag{2.10-6}$$

We need (2.10-1) and (2.10-2) in deriving the four equations (2.10-3)–(2.10-6). Equation (2.10-3) is obtained by multiplying both sides of (2.5-4) by n_j and setting $s=k$. If we multiply (2.5-4) by n_k and set $j=s$, we have (2.10-4). Equation (2.10-5) and (2.10-6) are derived by multiplying (2.5-4) by $n_j n_k n_s$ and $m_j n_k, m_s$, respectively. All repeated indices imply summation in these derivations.

Cowin (1989) (see also Norris, 1989) later showed that (2.10-5) and (2.10-6) are necessary and sufficient conditions. In fact (2.10-3) and (2.10-6), or (2.10-4) and (2.10-6), are also necessary and sufficient conditions. To see this let the plane $x_1=0$ be the plane of symmetry so that

$$n_i = \delta_{i1}, \quad m_i = \delta_{i2}\cos\theta + \delta_{i3}\sin\theta, \tag{2.10-7}$$

where θ is an arbitrary constant. Substitution of $(2.10\text{-}7)_1$ into (2.10-3)–(2.10-5) leads to

$$C_{i1kk} = C_{11tt}\delta_{i1}, \quad C_{is1s} = C_{1q1q}\delta_{i1}, \quad C_{i111} = C_{1111}\delta_{i1}.$$

For $i=2, 3$ they give

$$C_{21kk} = C_{31kk} = 0, \quad C_{2s1s} = C_{3s1s} = 0, \quad C_{2111} = C_{3111} = 0,$$

or, using the contracted notation,

$$C_{16} + C_{26} + C_{36} = C_{15} + C_{25} + C_{35} = 0, \tag{2.10-8}$$

$$C_{16} + C_{26} + C_{45} = C_{15} + C_{46} + C_{35} = 0, \tag{2.10-9}$$

$$C_{16} = C_{15} = 0. \tag{2.10-10}$$

Substitution of $(2.10\text{-}7)_2$ into (2.10-6) with (i) θ=0, (ii) $\theta=\pi/2$, and (iii) arbitrary θ yields, respectively,

$$C_{i212} = C_{1212}\delta_{i1}, \quad C_{i313} = C_{1313}\delta_{i1},$$

$$C_{i213} + C_{i312} = (C_{1213} + C_{1312})\delta_{i1}.$$

For i=2 ,3, and using the contracted notation we obtain

$$C_{26} = C_{46} = C_{45} = C_{35} = C_{25} = C_{36} = 0. \tag{2.10-11}$$

Equations (2.10-8)–(2.10-11) are, respectively, a specialization of (2.10-3)–(2.10-6) when the unit normal **n** is along the x_1-axis. If x_1=0 is a plane of symmetry, we must have by (2.6-3a)

$$C_{15} = C_{16} = C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = 0. \tag{2.10-12}$$

It is clear that (2.10-10) and (2.10-11) are equivalent to (2.10-12), so are (2.10-8) and (2.10-11), and (2.10-9) and (2.10-11). However, (2.10-8)–(2.10-10) do not imply (2.10-12).

Equations (2.10-3)–(2.10-6) tell us that **n** is an eigenvector of the 3×3 symmetric matrices **U**, **V**, **Q**(**n**), and **Q**(**m**) whose elements are

$$U_{ij} = C_{ijkk}, \quad V_{ik} = C_{isks}, \quad Q_{ik}(\mathbf{n}) = C_{ijks}n_j n_s. \tag{2.10-13}$$

We may therefore state a modified Cowin–Mehrabadi theorem as follows.

> **Theorem 2.10–1** An anisotropic elastic material with given elastic compliances C_{ijks} has a plane of symmetry if and only if **n** is an eigenvector of (i) **Q**(**n**) and **Q**(**m**), (ii) **U** and **Q**(**m**), or (iii) **V** and **Q**(**m**). The vector **n** is normal to the plane of symmetry while **m** is any vector on the plane of symmetry.

The conditions in (i) are due to Cowin (1989). The tensor **Q**(**n**) is the acoustic tensor for waves propagating in the direction of **n**. When **n** is an eigenvector of **Q**(**n**) the wave is *longitudinal*, with **n** being the *specific direction*. Kolodner (1966) has shown that there exist at least three distinct specific directions in an anisotropic elastic material. If **n** is an eigenvector of **Q**(**m**) the wave is *transversal*, with **m** being the propagation direction and **n** the *specific axis*. The conditions in (i) mean that necessary and sufficient conditions for $\mathbf{n}$ to be normal to a plane of symmetry are that it be a specific direction and a specific axis. Physical interpretations of the plane of symmetry can also be made in terms of static loading (Hayes and Norris, 1992).

Theorem 2.10–1 is not suitable for determining $\mathbf{n}$ because the matrix $\mathbf{Q}(\mathbf{m})$ depends on $\mathbf{m}$ which, in turns, depends on $\mathbf{n}$. The following theorem is more useful for computing $\mathbf{n}$.

> **Theorem 2.10–2** An anisotropic elastic material has a plane of symmetry if and only if the normal $\mathbf{n}$ to the plane of symmetry is a common eigenvector of $\mathbf{U}$ and $\mathbf{V}$, and satisfies
>
> $$C_{ijks} m_i n_j n_k n_s = 0, \tag{2.10-14}$$
>
> $$C_{ijks} m_i m_j m_k n_s = 0, \tag{2.10-15}$$
>
> for any two independent vectors $\mathbf{m}^{(\alpha)}$ (α=1,2) on the plane of symmetry that do not form an angle a multiple of $\pi/3$.

In the above, (2.10-14) and (2.10-15) need to be satisfied by only two independent vectors on the plane of symmetry that do not form an angle a multiple of $\pi/3$. To prove the theorem let $\mathbf{n}$ be along the x_1-axis so that $n_i = \delta_{i1}$ and let the two $\mathbf{m}$'s be given by

$$\left.\begin{aligned} m_i^{(1)} &= \delta_{i2}\cos\theta + \delta_{i3}\sin\theta, \\ m_i^{(2)} &= \delta_{i2}\cos\psi + \delta_{i3}\sin\psi, \end{aligned}\right\} \tag{2.10-16}$$

where θ and ψ are arbitrary constants subjected to the condition

$$\theta - \psi \neq \tfrac{1}{3}k\pi, \quad k \text{ integer.} \tag{2.10-17}$$

Substitution of (2.10-16) into (2.10-14) yields

$$\begin{aligned} C_{16}\cos\theta + C_{15}\sin\theta = 0, \\ C_{16}\cos\psi + C_{15}\sin\psi = 0. \end{aligned}$$

Since the determinant of the coefficient matrix for $[C_{15}, C_{16}]$ is $\sin(\theta-\psi)$ which is nonzero in view of (2.10-17),

$$C_{15} = C_{16} = 0. \tag{2.10-18}$$

When $n_i = \delta_{i1}$ is a common eigenvector of $\mathbf{U}$ and $\mathbf{V}$, (2.10-8) and (2.10-9) apply. Together with (2.10-18) we have

$$C_{36} = C_{45} = -C_{26}, \quad C_{25} = C_{46} = -C_{35}. \tag{2.10-19}$$

Finally, application of (2.10-16) to (2.10-15) and making use of (2.10-19) lead to

$$\cos\theta(2\cos 2\theta - 1)C_{26} - \sin\theta(2\cos 2\theta + 1)C_{35} = 0,$$
$$\cos\psi(2\cos 2\psi - 1)C_{26} - \sin\psi(2\cos 2\psi + 1)C_{35} = 0.$$

The determinant of the coefficient matrix of $[C_{26}, C_{35}]$ can be shown to be

$$\sin(\theta - \psi)[4\cos^2(\theta - \psi) - 1]$$

which is nonzero in view of (2.10-17). Hence

$$C_{26} = C_{35} = 0. \tag{2.10-20}$$

Equations (2.10-18)–(2.10-20) are equivalent to (2.10-12). This completes the proof.

In the contracted notation **U** and **V** have the expressions

$$\mathbf{U} = \begin{bmatrix} C_{11}+C_{12}+C_{13} & C_{16}+C_{26}+C_{36} & C_{15}+C_{25}+C_{35} \\ & C_{12}+C_{22}+C_{23} & C_{14}+C_{24}+C_{34} \\ & & C_{13}+C_{23}+C_{33} \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} C_{11}+C_{55}+C_{66} & C_{16}+C_{26}+C_{45} & C_{15}+C_{35}+C_{46} \\ & C_{22}+C_{44}+C_{66} & C_{24}+C_{34}+C_{56} \\ & & C_{33}+C_{44}+C_{55} \end{bmatrix} \tag{2.10-21}$$

Only the upper triangles are shown for **U** and **V** because they are symmetric. We will show that there is no need to find the eigenvectors of **U** and **V** in order to see if **U** and **V** have an eigenvector in common. If **n** is an eigenvector of **U** and **V**, we write

$$\mathbf{U}\mathbf{n} = u\mathbf{n}, \quad \mathbf{V}\mathbf{n} = v\mathbf{n}, \tag{2.10-22}$$

where u and v are the corresponding eigenvalues. This means that

$$\mathbf{V}\mathbf{U}\mathbf{n} = uv\mathbf{n}, \quad \mathbf{U}\mathbf{V}\mathbf{n} = uv\mathbf{n}. \tag{2.10-23}$$

Subtracting the two equations leads to

$$\mathbf{W}\mathbf{n} = \mathbf{0}, \tag{2.10-24}$$

where, noticing that **U** and **V** are symmetric,

$$\mathbf{W} = \mathbf{U}\mathbf{V} - \mathbf{V}\mathbf{U} = \mathbf{U}\mathbf{V} - (\mathbf{U}\mathbf{V})^T. \tag{2.10-25}$$

Thus **n** is a null vector of the skew-symmetric matrix **W**. When **UV** is symmetric, we have $\mathbf{W}=\mathbf{0}$ or $\mathbf{UV}=\mathbf{VU}$, implying that the product of $\mathbf{U}$ and $\mathbf{V}$ commutes. By Theorem 1.5–2, $\mathbf{U}$ and $\mathbf{V}$ share the same set of eigenvectors. If $\mathbf{UV}$ is not symmetric, let

$$\mathbf{W}=\begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{bmatrix}.$$

The unnormalized null vector $\mathbf{n}$ can be taken as

$$\mathbf{n}^T=[w_1,\ w_2,\ w_3]. \tag{2.10-26}$$

If the $\mathbf{n}$ given by (2.10-26) is an eigenvector of $\mathbf{U}$, it is an eigenvector of $\mathbf{V}$, and vice versa. To see this, suppose that the $\mathbf{n}$ of (2.10-26) satisfies $(2.10\text{-}22)_1$. Premultiplied by $\mathbf{V}$ leads to

$$\mathbf{VUn}=u\mathbf{Vn}, \quad \text{or} \quad (\mathbf{UV}-\mathbf{W})\mathbf{n}=u\mathbf{Vn},$$

by (2.10-25). In view of (2.10-24) we have

$$\mathbf{U}(\mathbf{Vn})=u(\mathbf{Vn}).$$

Comparison with $(2.10\text{-}22)_1$ tells us that $\mathbf{Vn}$ is proportional to $\mathbf{n}$, which is exactly $(2.10\text{-}22)_2$.

> **Theorem 2.10–3** If $\mathbf{UV}$ is symmetric, $\mathbf{U}$ and $\mathbf{V}$ share the same set of eigenvectors. If $\mathbf{UV}$ is not symmetric, $\mathbf{U}$ and $\mathbf{V}$ have one common eigenvector when the $\mathbf{n}$ of (2.10-26) is an eigenvector of $\mathbf{U}$ or $\mathbf{V}$. Otherwise there is no common eigenvector.

In conjunction with Theorem 2.10–2 it is clear that $\mathbf{UV}$ must be symmetric if a material possesses more than one symmetry plane. If $\mathbf{UV}$ is not symmetric, the material can have at most one symmetry plane. It should be noted that $\mathbf{UV}$ can be symmetric for materials with one or no symmetry plane.

In closing this chapter we note that the transformation of coordinate systems employed here is restricted to orthogonal coordinate systems. If one allows the transformation for the full linear group, Olver (1988, 1992) showed that all anisotropic materials are equivalent to an orthotropic material.