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A. N. Stroh

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Dislocations and Cracks in Anisotropic Elasticity†

By A. N. Stroh
Department of Physics, University of Sheffield

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Abstract

The solution of the elastic equations is considered for the case in which the state of the solid is independent of one of the three Cartesian coordinates. The stresses due to a dislocation, a wall of parallel dislocations, and a crack in an arbitrary non-uniform stress field are obtained. The results hold for the most general anisotropy in which no symmetry elements of the crystal are assumed.

§ 1. Introduction

Eshelby *et al.* (1953) have developed the theory of anisotropic elasticity for a three dimensional state of stress in which the stress is independent of one of the Cartesian coordinates, and have applied this to find the stress field of a dislocation. In the present paper, which follows their treatment, the stresses due to a dislocation are treated more fully, and the interactions of dislocations considered; also, the stresses round a crack subjected to an arbitrary non-uniform applied stress are obtained. The object will be to present the results in a form which is, analytically, as simple as possible. It is hoped that, in applications of the theory, this will often allow of the properties of the system studied to be deduced without the need for numerical computation, and that when such computation is unavoidable, as when definite numerical values are required, the labour involved will be reduced to a minimum. For this purpose, the properties of a number of constants introduced in the theory and which are related to the elastic constants are investigated in some detail (§ 3). In § 2 some general relations are considered; most of these are given in Eshelby *et al.* but are included here, both for ease of reference, and so that the whole theory may be presented in a uniform notation.

§ 2. General Equations

The stresses σ_{ij} are related to the elastic displacements u_k by the equations

$$\sigma_{ij} = c_{ijkl} \partial u_k / \partial x_l, \qquad (1)$$

where $i, j, k, l = 1, 2, 3$ and the convention of summing over a repeated Latin suffix is used. The elastic moduli c_{ijkl} have the symmetry properties

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \qquad (2)$$

† Communicated by the Author.

It is often convenient to replace the pairs of suffices (i, j) and (k, l) by single suffices M and N according to the scheme that 11 corresponds to 1, 22 to 2, 33 to 3, 23 to 4, 31 to 5, and 12 to 6. Besides securing some brevity in writing, this has the advantage that it enables the elastic constants to be considered as the elements of a matrix, but has the disadvantage that it does not make the tensor character of the constants apparent. In the sequal either four suffix notation c_{ijkl} or two suffix notation c_{MN} will be used according as to which is the more convenient in each case. In addition to the symmetry relations (2) the elastic moduli are subjected to further restrictions which arise because the elastic energy density must be everywhere positive. If the elastic strains are $e_1 = \partial u_1/\partial x_1, \ldots\ldots, e_6 = \partial u_1/\partial x_2 + \partial u_2/\partial x_1$, the energy density is

$$\tfrac{1}{2} c_{MN} e_M e_N > 0, \qquad (3)$$

provided not all the e_M are zero. The condition for (3) is that the determinant $|c_{MN}|$ and its principal minors of all orders should be positive.

On substituting (1) in the equilibrium equations

$$\partial \sigma_{ij}/\partial x_j = 0, \qquad (4)$$

we obtain

$$c_{ijkl} \partial^2 u_k/\partial x_j \partial x_l = 0. \qquad (5)$$

Now we suppose that u_k is independent of x_3, and, following Eshelby *et al.* take

$$u_k = A_k f(x_1 + p x_2), \qquad (6)$$

where $f(z)$ is an analytic function of the complex variable z; (6) is a solution of eqns. (5) provided the constant vector A_k satisfies the equations

$$(c_{i1k1} + p c_{i1k2} + p c_{i2k1} + p^2 c_{i2k2}) A_k = 0. \qquad (7)$$

Values of A_k, not identically zero, can be found to satisfy these equations if p is a root of the sextic equation

$$|c_{i1k1} + p c_{i1k2} + p c_{i2k1} + p^2 c_{i2k2}| = 0. \qquad (8)$$

Eshelby *et al.* have proved that eqn. (8) has no real root, so that the roots occur in complex conjugate pairs. The three roots with positive imaginary part will be denoted by $p_\alpha (\alpha = 1, 2, 3)$ with complex conjugates $\bar{p}_\alpha$; the corresponding values of A_k obtained from eqns. (7) are $A_{k\alpha}$ and $\bar{A}_{k\alpha}$. Summation over α, which of course is not a tensor suffix, and generally over Greek suffices will always be indicated explicitly. It will be assumed that the roots p_α are all distinct; equal roots may be regarded as the limiting case of distinct roots. A general expression for the displacement may then be written

$$u_k = \sum_\alpha A_{k\alpha} f_\alpha(z_\alpha) + \sum_\alpha \bar{A}_{k\alpha} \overline{f_\alpha(z_\alpha)}, \qquad (9)$$

where $z_\alpha = x_1 + p_\alpha x_2$.

It is convenient to express the stresses in terms of a vector, the components of which will be denoted by ϕ_i; with the stresses independent of x_3, eqns. (4) are satisfied identically if the stresses are derived from the ϕ_i by the relations

$$\sigma_{i1} = -\partial \phi_i/\partial x_2, \quad \sigma_{i2} = \partial \phi_i/\partial x_1. \qquad (10)$$

These equations determine in terms of the ϕ_i all the components of the stress except σ_{33}. But since in the present case σ_{33} is linearly dependent on the other stress components (the equation showing this is just the condition, expressed in terms of the stresses, that the strain $e_3=\partial u_3/\partial x_3$ be zero), we may regard the problem as solved when the remaining five components of stress, or equivalently the functions ϕ_i, are determined. Since

$$\partial\phi_1/\partial x_1=\sigma_{12}=-\partial\phi_2/\partial x_2, \qquad (11)$$

ϕ_1 and ϕ_2 may be expressed in terms of a single function χ by

$$\phi_1=-\partial\chi/\partial x_2, \quad \phi_2=\partial\chi/\partial x_1; \qquad (12)$$

however the theory takes on a more symmetrical form if this is not done, and we shall generally prefer not to introduce χ. Eshelby *et al.* have shown that the resultant force acting across a curve C is $\Delta\phi_i$, where $\Delta\phi_i$ is the change in ϕ_i on going along the curve. It may also be shown that the moment about the x_3 axis of the forces acting across C is

$$\Delta(x_2\phi_1-x_1\phi_2+\chi).$$

From eqns. (1), (9) and (10) we obtain

$$\phi_i=\sum_\alpha L_{i\alpha}f_\alpha(z_\alpha)+\sum_\alpha \bar{L}_{i\alpha}\overline{f_\alpha(z_\alpha)}, \qquad (13)$$

where the three vectors $L_{i\alpha}$ are defined by

$$L_{i\alpha}=(c_{i2k1}+p_\alpha c_{i2k2})A_{k\alpha}, \qquad (14)$$

or alternatively,

$$L_{i\alpha}=-(p_\alpha^{-1}c_{i1k1}+c_{i1k2})A_{k\alpha}, \qquad (15)$$

these expressions being equivalent by eqns. (7). For (11) to hold, we must have

$$L_{1\alpha}+p_\alpha L_{2\alpha}=0, \qquad (16)$$

a relation which also follows directly from (14) and (15). Use of eqn. (16) will simplify the expressions for the $L_{i\alpha}$ and so we shall prefer to express our results in terms of the $L_{i\alpha}$ rather than the $A_{k\alpha}$; as will be seen in § 7 the $L_{i\alpha}$ can be obtained in terms of the p_α without first determining the $A_{k\alpha}$. Nevertheless, in the general case, these expressions still remain rather cumbersome, and we proceed first to develop the theory as far as possible without introducing the explicit values of the $A_{k\alpha}$ or $L_{i\alpha}$.

§ 3. Properties of $A_{k\alpha}$, $L_{i\alpha}$ and some Related Constants

From the vectors $\bar{A}_{i\alpha}$ and $L_{i\alpha}$ we may form the 3×3 matrix whose elements are $\bar{A}_{i\beta}L_{i\alpha}$; this matrix we now prove has skew-Hermitian symmetry. For from (14) we have

$$\bar{p}_\beta\bar{A}_{i\beta}L_{i\alpha}=\bar{p}_\beta\bar{A}_{i\beta}c_{i2k1}A_{k\alpha}+\bar{p}_\beta\bar{A}_{i\beta}c_{i2k2}p_\alpha A_{k\alpha},$$

and from (15)

$$\bar{A}_{i\beta}L_{i\alpha}p_\alpha=-\bar{A}_{i\beta}c_{i1k1}A_{k\alpha}-\bar{A}_{i\beta}c_{i1k2}p_\alpha A_{k\alpha};$$

subtracting these two equations,

$$\bar{A}_{i\beta}L_{i\alpha}(\bar{p}_\beta-p_\alpha)=\bar{A}_{i\beta}c_{i1k1}A_{k\alpha}+\bar{p}_\beta\bar{A}_{i\beta}c_{i2k2}p_\alpha A_{k\alpha}+(\bar{p}_\beta\bar{A}_{i\beta}c_{i2k1}A_{k\alpha} + \bar{A}_{i\beta}c_{i1k2}p_\alpha A_{k\alpha}). \qquad (17)$$

Now each of the three terms on the right of (17) has Hermitian symmetry, and so the left side must also; that is

$$\bar{A}_{i\beta}L_{i\alpha}(\bar{p}_\beta - p_\alpha) = A_{i\alpha}\bar{L}_{i\beta}(p_\alpha - \bar{p}_\beta),$$

or since $(\bar{p}_\beta - p_\alpha)$ has negative imaginary part and so is not zero

$$\bar{A}_{i\beta}L_{i\alpha} = -A_{i\alpha}\bar{L}_{i\beta}, \qquad \ldots \ldots \quad (18)$$

which is the required relation.

Since $A_{k\alpha}$, $\bar{A}_{k\alpha}$ form a set of six vectors in three dimensional space, not more than three of them can be linearly independent; the three vectors $A_{k\alpha}$, corresponding to the three roots p_α with positive imaginary part, do however form such a linearly independent set. For suppose that this were not so and that a relation

$$\sum_\alpha \xi_\alpha A_{k\alpha} = 0 \qquad \ldots \ldots \quad (19)$$

existed with not all the ξ_α zero. Multiply eqn. (17) by $\xi_\alpha\bar{\xi}_\beta$ and sum over α and β. The left hand side becomes

$$\sum_\alpha\sum_\beta \bar{\xi}_\beta \bar{A}_{j\beta}L_{j\alpha}(\bar{p}_\beta - p_\alpha)\xi_\alpha = -\sum_\beta \bar{L}_{j\beta}\bar{p}_\beta\bar{\xi}_\beta \sum_\alpha A_{j\alpha}\xi_\alpha - \sum_\beta \bar{A}_{j\beta}\bar{\xi}_\beta \sum_\alpha L_{j\alpha}p_\alpha\xi_\alpha = 0$$

on using (18) and (19); also after using (19) and its complex conjugate the only term that remains on the right-hand side is

$$\left(\sum_\beta \bar{\xi}_\beta\bar{p}_\beta\bar{A}_{i\beta}\right)c_{i2k2}\left(\sum_\alpha \xi_\alpha p_\alpha A_{k\alpha}\right) = 0.$$

But $[c_{i2k2}]$, being a principal minor of the 6×6 matrix $[c_{MN}]$, is by (3) positive definite; hence

$$\sum_\alpha \xi_\alpha p_\alpha A_{k\alpha} = 0. \qquad \ldots \ldots \quad (20)$$

From eqn. (7) we obtain

$$c_{i1k1}\sum_\alpha \xi_\alpha p_\alpha^{-1}A_{k\alpha} + (c_{i1k2} + c_{i2k1})\sum_\alpha \xi_\alpha A_{k\alpha} + c_{i2k2}\sum_\alpha \xi_\alpha p_\alpha A_{k\alpha} = 0,$$

or using (19) and (20)

$$c_{i1k1}\sum_\alpha \xi_\alpha p_\alpha^{-1}A_{k\alpha} = 0;$$

again by (3) $|c_{i1k1}| \neq 0$, so that

$$\sum_\alpha \xi_\alpha p_\alpha^{-1}A_{k\alpha} = 0. \qquad \ldots \ldots \quad (21)$$

Now for a fixed value of k, (19), (20) and (21) may be regarded as a set of three equations in the three unknowns $\xi_\alpha A_{k\alpha}$; the determinant of these equations is equal to $(p_1-p_2)(p_2-p_3)(p_3-p_1)/p_1p_2p_3$, which is not zero if all the p_α are distinct; hence the only solution is $\xi_\alpha A_{k\alpha}=0$. But $A_{k\alpha}$ is not identically zero and so we must have $\xi_\alpha = 0$, or no relation of the form (19) with non-zero ξ_α is possible.

Next we prove that the linear independence of the $L_{i\alpha}$ follows from that of the $A_{k\alpha}$; in the course of establishing this result some constants which will also be needed later are introduced. Again we suppose that the $L_{i\alpha}$ are not linearly independent so that we can write

$$\sum_\alpha \xi_\alpha L_{i\alpha} = 0 \qquad \ldots \ldots \quad (22)$$

with not all the ξ_α zero. From (14) we have

$$\sum_\alpha\sum_\beta \bar{\xi}_\beta \bar{A}_{i\beta} L_{i\alpha} p_\alpha^{-1}\xi_\alpha = \sum_\beta \bar{\xi}_\beta \bar{A}_{i\beta} c_{i2k1} \sum_\alpha p_\alpha^{-1}\xi_\alpha A_{k\alpha} + \sum_\beta \bar{\xi}_\beta \bar{A}_{i\beta} c_{i2k2} \sum_\alpha \xi_\alpha A_{k\alpha}, \quad (23)$$

and the left-hand side is seen to be zero on using (18) and the conjugate of (22). Also, from (15) we have

$$c_{i1k1}\sum_\alpha p_\alpha^{-1} A_{k\alpha}\xi_\alpha + c_{i1k2}\sum_\alpha A_{k\alpha}\xi_\alpha = -\sum_\alpha L_{i\alpha}\xi_\alpha = 0. \quad . \quad . \quad . \quad (24)$$

Now the determinant $|c_{i1k1}|$ is by (3) positive and not zero, and hence the matrix $[s'_{ik}]$ reciprocal to $[c_{i1k1}]$ exists. Then from (24)

$$\sum_\alpha p_\alpha^{-1} A_{k\alpha}\xi_\alpha = -s'_{ki} c_{i1l2} \sum_\alpha \xi_\alpha A_{l\alpha},$$

and substituting this in (23), we obtain

$$(\sum_\beta \bar{\xi}_\beta \bar{A}_{i\beta}) \beta_{il} (\sum_\alpha \xi_\alpha A_{l\alpha}) = 0, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where

$$\beta_{il} = c_{i2l2} - c_{i2k1} s'_{kj} c_{j1l2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The expression

$$\beta_{il} \cdot |c_{j1k1}| = \begin{vmatrix} c_{1111} & c_{1121} & c_{1131} & c_{11l2} \\ c_{2111} & c_{2121} & c_{2131} & c_{21l2} \\ c_{3111} & c_{3121} & c_{3131} & c_{31l2} \\ c_{i211} & c_{i221} & c_{i231} & c_{i2l2} \end{vmatrix}$$

is easily verified by expanding the determinant on the right by its 4th row and 4th column. With

$$\Delta_1 = |c_{j1k1}| = \begin{vmatrix} c_{11} & c_{15} & c_{16} \\ c_{15} & c_{55} & c_{56} \\ c_{16} & c_{56} & c_{66} \end{vmatrix}, \quad . \quad . \quad . \quad . \quad (27)$$

we have

$$\left.\begin{aligned} \Delta_1\beta_{22} &= \begin{vmatrix} c_{11} & c_{12} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{25} & c_{26} \\ c_{15} & c_{25} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{56} & c_{66} \end{vmatrix}, \\ \Delta_1\beta_{33} &= \begin{vmatrix} c_{11} & c_{14} & c_{15} & c_{16} \\ c_{14} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{46} & c_{56} & c_{66} \end{vmatrix}, \\ \Delta_1\beta_{23} = \Delta_1\beta_{32} &= \begin{vmatrix} c_{11} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{24} & c_{25} & c_{26} \\ c_{15} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{46} & c_{56} & c_{66} \end{vmatrix}, \end{aligned}\right\} \quad . \quad . \quad . \quad (28)$$

and the remaining elements of β_{il} are zero. It follows from (3) that

$$\beta_{22} > 0, \text{ and } \beta_{33} > 0. \qquad (29)$$

Further, consider the determinant

$$\Delta = \begin{vmatrix} c_{11} & c_{12} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{24} & c_{25} & c_{26} \\ c_{14} & c_{24} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{46} & c_{56} & c_{66} \end{vmatrix}; \qquad (30)$$

$\beta_{22}\Delta_1$ is the cofactor of c_{44} in Δ, $\beta_{33}\Delta_1$ of c_{22}, and $-\beta_{23}\Delta_1$ of c_{24}. Hence it follows from Jacobi's theorem that

$$\beta_{22}\beta_{33} - \beta_{23}{}^2 = \Delta/\Delta_1 > 0. \qquad (31)$$

On account of (29) and (31) eqn. (25) can be true only if

$$\sum_\alpha \xi_\alpha A_{2\alpha} = 0 \text{ and } \sum_\alpha \xi_\alpha A_{3\alpha} = 0. \qquad (32)$$

The third equation,

$$\sum_\alpha \xi_\alpha A_{1\alpha} = 0, \qquad (33)$$

does not follow from (25); it may however be deduced in a similar way but starting from the equation formed from (15) in a manner analogous to that by which (23) was formed from (14). Then (32) and (33) show that the linear dependance of the $L_{i\alpha}$ implies that of the $A_{k\alpha}$ which has already been proved untrue; hence the linear independence of the $L_{i\alpha}$ is established.

In deriving (33) we would have to introduce constants γ_{il} analogous to β_{il} and which will also be needed later. The non-zero components of γ_{il} are

$$\left.\begin{aligned} \Delta_2\gamma_{11} &= \begin{vmatrix} c_{11} & c_{12} & c_{14} & c_{16} \\ c_{12} & c_{22} & c_{24} & c_{26} \\ c_{14} & c_{24} & c_{44} & c_{46} \\ c_{16} & c_{26} & c_{46} & c_{66} \end{vmatrix}, \\ \Delta_2\gamma_{33} &= \begin{vmatrix} c_{22} & c_{24} & c_{25} & c_{26} \\ c_{24} & c_{44} & c_{45} & c_{46} \\ c_{25} & c_{45} & c_{55} & c_{56} \\ c_{26} & c_{46} & c_{56} & c_{66} \end{vmatrix}, \\ \Delta_2\gamma_{13} = \Delta_2\gamma_{31} &= \begin{vmatrix} c_{12} & c_{14} & c_{15} & c_{16} \\ c_{22} & c_{24} & c_{25} & c_{26} \\ c_{24} & c_{44} & c_{45} & c_{46} \\ c_{26} & c_{46} & c_{56} & c_{66} \end{vmatrix}, \end{aligned}\right\} \qquad (34)$$

where

$$\Delta_2 = |c_{j2k2}| = \begin{vmatrix} c_{22} & c_{24} & c_{26} \\ c_{24} & c_{44} & c_{46} \\ c_{26} & c_{46} & c_{66} \end{vmatrix}. \qquad (35)$$

Also

$$\gamma_{11} > 0, \; \gamma_{33} > 0, \text{ and } \gamma_{11}\gamma_{33} - \gamma_{13}{}^2 = \Delta/\Delta_2 > 0. \qquad (36)$$

Now having established that the $L_{i\alpha}$ are linearly independent, we are able to introduce three vectors $M_{\alpha i}$, reciprocal to the $L_{i\alpha}$, and which are defined by

$$M_{\alpha i}L_{i\beta}=\delta_{\alpha\beta}. \qquad (37)$$

From (37) the relation

$$\sum_{\alpha}L_{i\alpha}M_{\alpha j}=\delta_{ij} \qquad (38)$$

follows. Also eqn. (18) may now be written

$$\sum_{\alpha}\bar{A}_{j\alpha}\bar{M}_{\alpha i}=-\sum_{\alpha}A_{i\alpha}M_{\alpha j}. \qquad (39)$$

Since, for each α, eqn. (7) determine only the ratios of the components, $A_{i\alpha}$ (and hence $L_{i\alpha}$ and $M_{\alpha i}$) contains an arbitrary factor independent of i but which may depend on α. However this factor does not appear in the product $A_{i\alpha}M_{\alpha j}$ so that $\sum A_{i\alpha}M_{\alpha j}$ is unambiguously defined. A related tensor which will play an important role in the theory is

$$B_{ij}=\tfrac{1}{2}i\sum_{\alpha}(A_{i\alpha}M_{\alpha j}-\bar{A}_{i\alpha}\bar{M}_{\alpha j}); \qquad (40)$$

B_{ij} is clearly real, and being equal to the imaginary part of a skew-Hermitian matrix (cf. eqn. (39)) is itself symmetric.

Finally we seek a relation between B_{ij} and $L_{i\alpha}$, $M_{\alpha i}$ not involving the $A_{i\alpha}$ explicitly. From eqns. (15) and (38) we have

$$\delta_{ij}=\sum_{\alpha}L_{i\alpha}M_{\alpha j}=-c_{i1k1}\sum_{\alpha}A_{k\alpha}p_{\alpha}^{-1}M_{\alpha j}-c_{i1k2}\sum_{\alpha}A_{k\alpha}M_{\alpha j},$$

and from (14),

$$\sum_{\alpha}L_{i\alpha}p_{\alpha}^{-1}M_{\alpha j}=c_{i2k1}\sum_{\alpha}A_{k\alpha}p_{\alpha}^{-1}M_{\alpha j}+c_{i2k2}\sum_{\alpha}A_{k\alpha}M_{\alpha j}.$$

From these two equations we obtain, on using (26) and (40),

$$\frac{1}{2i}(Q_{ij}-\bar{Q}_{ij})=-\beta_{ik}B_{kj}, \qquad (41)$$

where

$$Q_{ij}=\sum_{\alpha}L_{i\alpha}p_{\alpha}^{-1}M_{\alpha j}. \qquad (42)$$

In the same way with

$$P_{ij}=\sum_{\alpha}L_{i\alpha}p_{\alpha}M_{\alpha j}, \qquad (43)$$

we obtain

$$\frac{1}{2i}(P_{ij}-\bar{P}_{ij})=\gamma_{ik}B_{kj}. \qquad (44)$$

The matrices $[P_{ij}]$ and $[Q_{ij}]$ are reciprocals of one another, as is easily seen on forming their product and using (37) and (38).

Since $[\beta_{ik}]$ and $[\gamma_{ik}]$ are singular matrices, neither (41) nor (44) alone can be solved for B_{kj}. However taken together (41) and (44) are sufficient to determine B_{kj} and when numerical values are required it will usually be easier to solve them than first to determine the $A_{i\alpha}$ and to use the definition (40).

§ 4. Dislocations

Consider the following displacements, which are of the form (9):

$$u_k = \frac{1}{2\pi i}\sum_\alpha A_{k\alpha} D_\alpha \log z_\alpha - \frac{1}{2\pi i}\sum_\alpha \bar{A}_{k\alpha}\bar{D}_\alpha \log \bar{z}_\alpha. \quad . \quad . \quad . \quad (45)$$

Along a closed path encircling the x_3 axis, u_k changes by an amount

$$b_k = \sum_\alpha (A_{k\alpha} D_\alpha + \bar{A}_{k\alpha}\bar{D}_\alpha). \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Also from (13) the stress functions corresponding to the displacements (45) are

$$\phi_i = \frac{1}{2\pi i}\sum_\alpha L_{i\alpha} D_\alpha \log z_\alpha - \frac{1}{2\pi i}\sum_\alpha \bar{L}_{i\alpha}\bar{D}_\alpha \log \bar{z}_\alpha; \quad . \quad . \quad . \quad (47)$$

the change in ϕ_i along a closed path about the x_3 axis is

$$\Delta\phi_i = \sum_\alpha (L_{i\alpha} D_\alpha + \bar{L}_{i\alpha}\bar{D}_\alpha), \quad . \quad . \quad . \quad . \quad (48)$$

and it has been noted (§ 2) that this change in ϕ_i just represents the resultant force F_i acting.

Thus in general (45) represents the displacements due to a dislocation with Burgers vector b_i, together with a line of body force F_i, along the x_3 axis. Since the D_α and $\bar{D}_\alpha$ constitute a set of six independent constants we may expect that it will always be possible to choose them to correspond to any given values of b_i and F_i. We now determine these constants so that we have a pure dislocation without a line of force along its axis, that is we take $F_i = 0$.

It is always possible to express D_α in terms of new constants d_j by the relation

$$D_\alpha = \tfrac{1}{2} i M_{\alpha j} d_j;$$

then substituting this in (48) we see that $F_i = 0$ if d_j is real. Equations (46), (47) and (40) now show that the stress functions

$$\phi_i = \frac{1}{4\pi}\sum_\alpha (L_{i\alpha} M_{\alpha j} \log z_\alpha + \bar{L}_{i\alpha}\bar{M}_{\alpha j} \log \bar{z}_\alpha)\, d_j \quad . \quad . \quad . \quad (49)$$

represent a dislocation with Burgers vector

$$b_i = B_{ij}\, d_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

for all real values of the vector d_j. It will be seen later that the determinant $|B_{ij}|$ is not zero; hence given the value of b_i it is always possible to solve eqns. (50) for d_j, so that (49) can represent a dislocation with arbitrary Burgers vector. However the stresses are more closely related to the vector d_j than to b_i and so we shall usually prefer to express our results in terms of the former. (The nature of d_j may perhaps be made clearer by reference to the isotropic case where d_j are the components of $G\mathbf{b}$ for a screw dislocation and of $G\mathbf{b}/(1-\nu)$ for an edge, with G the rigidity and ν Poisson's ratio). The stresses derived from (49) by eqns. (10) are

$$\sigma_{i1} = -(1/4\pi)\sum_\alpha (L_{i\alpha} p_\alpha z_\alpha^{-1} M_{\alpha j} + \bar{L}_{i\alpha}\bar{p}_\alpha \bar{z}_\alpha^{-1}\bar{M}_{\alpha j})\, d_j, \quad . \quad (51)$$

and

$$\sigma_{i2} = (1/4\pi)\sum (L_{i\alpha} z_\alpha^{-1} M_{\alpha j} + \bar{L}_{i\alpha}\bar{z}_\alpha^{-1}\bar{M}_{\alpha j})\, d_j. \quad . \quad . \quad . \quad (52)$$

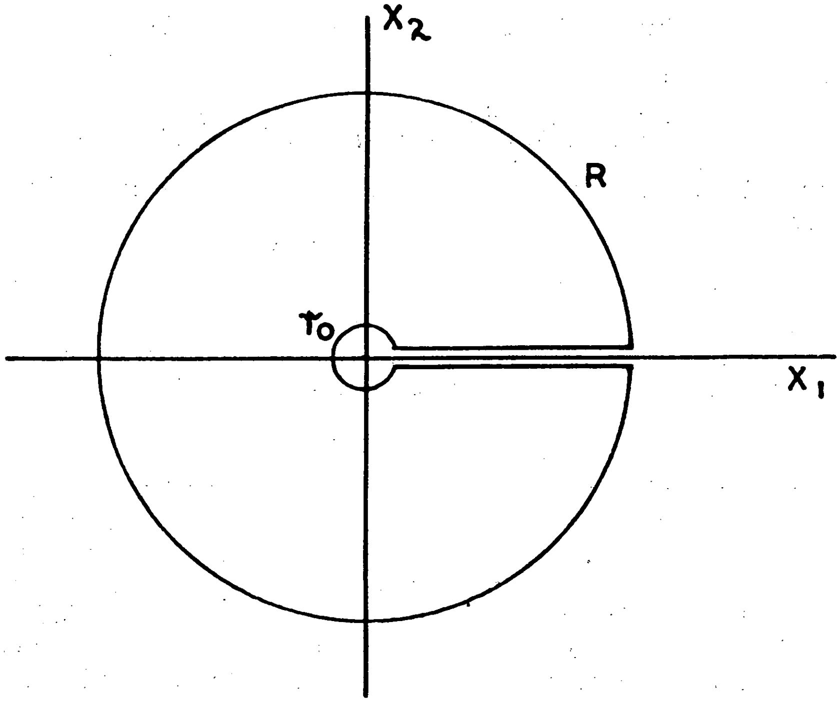
The elastic energy per unit length of the material lying inside a cylinder with generators parallel to the x_3 axis, and which meets the plane $x_3=0$ in a curve C, can quite generally be written

$$U=\tfrac{1}{2}\int_c u_i\sigma_{ij}n_j\,ds, \qquad \qquad (53)$$

where n_j are the direction cosines of the outward normal to C. On using (10) this becomes

$$U=-\tfrac{1}{2}\int_c u_i(d\phi_i/ds)\,ds. \qquad \qquad (54)$$

Fig. 1



We apply (53) and (54) to calculate contribution to the energy of a dislocation from the material between radii R and r_0 ($R>r_0$) from the x_3 axis (fig. 1). This region may be made simply connected by making a cut in the plane $x_2=0$ from $x_1=r_0$ to $x_1=R$; in the cut region we take the displacement u_i to be single valued but discontinuous across the cut. On the surface of the cylinder radius R, (54) gives a contribution to the energy of

$$U_R=-\tfrac{1}{2}\int_0^{2\pi} u_i(d\phi_i/d\theta)\,d\theta,$$

where we have taken $z_\alpha=x_1+p_\alpha x_2=R(\cos\theta+p_\alpha\sin\theta)$. From (49) we see that $d\phi_i/d\theta$ is independent of R, and from (45) that u_i is of the form

$$u_i=k\log R+g(\theta),$$

where k is a constant and $g(\theta)$ is independent of R. Thus the part of U_R involving R is

$$-\tfrac{1}{2}k\log R\int_0^{2\pi}(d\phi_i/d\theta)\,d\theta=0$$

since ϕ_i is single valued; U_R is thus independent of the radius R. From the surface of the inner cylinder of radius r_0 we obtain a similar contribution which will be of the same magnitude but, since the direction of the normal n_j is now reversed, opposite in sign. Thus the net contribution to the energy from the two curved surfaces is zero.

The displacements on the two sides of the cut differ by b_i, and the stress across the plane of the cut $x_2=0$ is, from (52),

$$\sigma_{12}=(1/4\pi x_1)\sum_\alpha(L_{i\alpha}M_{\alpha j}+\bar{L}_{i\alpha}\bar{M}_{\alpha j})d_j=d_i/2\pi x_1.$$

Hence (53) gives the energy as

$$U=(b_i d_i/4\pi)\int_{r_0}^{R}dx_1/x_1=(b_i d_i/4\pi)\log(R/r_0)\,; \quad . \quad . \quad . \quad (55)$$

using (50) we have the alternate expression for the energy of the dislocation

$$U=(B_{ij}d_i d_j/4\pi)\log(R/r_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Since U must be positive for all non-zero values of d_i, we see that $[B_{ij}]$ is positive definite. In particular the determinant $|B_{ij}|$ is positive and not zero; this establishes the statement made previously that eqn. (50) can always be solved to give d_j in terms of b_i.

§ 5. Interaction of Two Dislocations

The energy of interaction of two dislocations can most easily be found by the method due to Cottrell (1949) of forming one dislocation in the presence of the other. Consider a dislocation with Burgers vector $b_i^{(1)}$ lying along the x_3 axis and suppose a second dislocation with Burgers vector $b_i^{(2)}$ is formed parallel to the first and through the point $(X_1, X_2, 0)$. This dislocation may be formed by making a cut along the plane $x_2=X_2$ from $x_1=X_1$ to $x_1=\infty$, and giving the surfaces of the cut a relative displacement $b_i^{(2)}$. The work done in this process against the stresses of the first dislocation is

$$V=b_i^{(2)}\int_{X_1}^{\infty}\sigma_{i2}^{(1)}\,dx_1\,;$$

using (52) this becomes

$$V=-(b_i^{(2)}/4\pi)\sum_\alpha(L_{i\alpha}M_{\alpha j}\log Z_\alpha+\bar{L}_{i\alpha}\bar{M}_{\alpha j}\log\bar{Z}_\alpha)\,d_j^{(1)}, \quad . \quad . \quad (57)$$

where now $Z_\alpha=X_1+p_\alpha X_2$. Introducing polar coordinates (r, θ) (fig. 2) we have

$$\log Z_\alpha=\log r+\log(\cos\theta+p_\alpha\sin\theta).$$

Then from (57) and (38) we obtain the radial force

$$F_r=-\partial V/\partial r=b_i^{(2)}d_i^{(1)}/2\pi r\,; \quad . \quad . \quad . \quad . \quad (58)$$

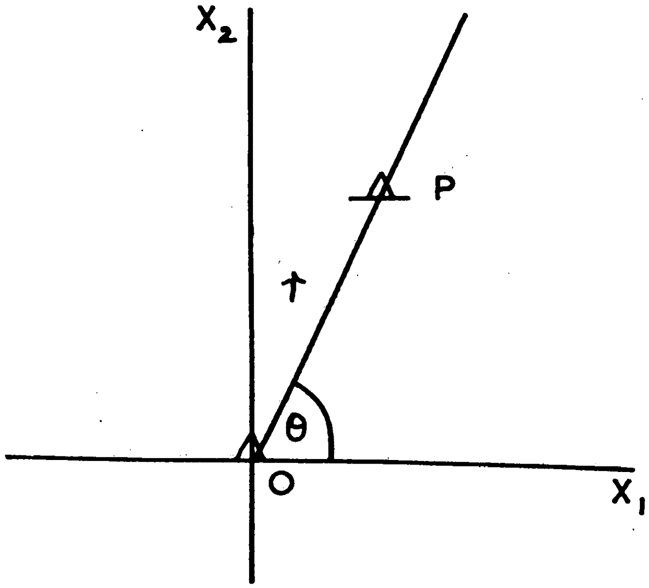
by using (50) this may also be written in the symmetrical form

$$F_r = B_{ij}\, d_i^{(1)} d_j^{(2)}/2\pi r. \qquad \ldots\ldots \quad (59)$$

The expression for the tangential component of the force is rather more complicated and is given by

$$F_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\frac{b_i^{(2)} d_j^{(1)}}{4\pi r}\sum_\alpha \left(L_{i\alpha} M_{\alpha j}\frac{p_\alpha \cos\theta - \sin\theta}{\cos\theta + p_\alpha \sin\theta} + \bar{L}_{i\alpha}\bar{M}_{\alpha j}\frac{\bar{p}_\alpha \cos\theta - \sin\theta}{\cos\theta + \bar{p}_\alpha \sin\theta}\right). \qquad \ldots \quad (60)$$

Fig. 2



An important special case of the forces between dislocations is that of two like parallel edge dislocations in different slip planes; the well-known result of isotropic theory that the dislocations have an equilibrium position with the line joining them normal to the slip will not be true in general. For suppose the dislocations have Burgers vectors

$$b_i = (b,\, 0,\, 0) \qquad \ldots\ldots\ldots \quad (61)$$

so that the slip planes are parallel to the plane $x_2 = 0$, and let the dislocations pass through the origin and the point $(0, x_2, 0)$, the component of force in the slip plane may be found from (58) and (60), or more simply by noting that this force is

$$F_1 = b_i \sigma_{i2} = b\sigma_{12};$$

on using (51) and (38) this becomes

$$F_1 = -(b/4\pi x_2)\sum_\alpha (L_{2\alpha} M_{\alpha j} + \bar{L}_{2\alpha}\bar{M}_{\alpha j})\, d_j = -b\, d_2/2\pi x_2. \qquad . \quad (62)$$

The value of d_2 used here is to be obtained by solving eqns. (50) with b_i given by (61); it is shown in § 7, by consideration of a special case, that in general d_2 will not be zero, and so the dislocations will not be in equilibrium.

It is easily seen from (52) that σ_{12} changes sign as x_1 varies from $-\infty$ to $+\infty$, so that it will vanish at an odd number of points; that is two parallel edge dislocations have an odd number of equilibrium positions. It is convenient to characterize these by the angle θ between the line joining the dislocations and the slip plane, since, from the form of the forces (58) and (60), the equilibrium positions depend on θ but not on r. If, then, the dislocations at O and P in fig. 2 are in equilibrium, we can add any number of similar dislocations at different points along the line OP; every pair of dislocations, and hence the system as a whole, will be in equilibrium. Thus the values of θ corresponding to the equilibrium of a single pair of dislocations define possible orientations in which dislocation walls may be formed; moreover it is easily seen that these are the only orientations in which the dislocations forming a wall will be in equilibrium. In particular we are led to the conclusion that a wall of edge dislocations normal to the slip plane (a simple tilt boundary) will not be possible unless $\theta=\frac{1}{2}\pi$ is an equilibrium position for two dislocations, which, we have seen, is not always the case.

This last conclusion appears inconsistent with that obtained from the treatment of Frank (1950); in this the dislocations walls of least energy (and which are therefore the most stable) are obtained as those for which the material is unstrained at a distance from the walls. This gives the simple tilt boundary normal to the slip plane as the most stable arrangement which can be formed from edge dislocations all of the same kind; Frank's arguments are very general and should apply quite independently of the crystal symmetry. We therefore consider, in the next section, the stresses due to a wall of dislocations and verify that the remote stress field leads to results in agreement with Frank.

§ 6. Dislocation Walls

Consider a wall of like parallel dislocations, with arbitrary Burgers vector b_i; the dislocations are to be equally spaced in the wall, the distance between neighbouring dislocations being h. We take x_1 axis normal to the wall, and, as usual, x_3 axis parallel to the dislocation lines. The stresses due to the wall can be found by summing the stresses (51) and (52) over all the dislocations in the wall. Thus the stresses will be given by two expressions of the form (51) and (52) but with $z_\alpha{}^{-1}=(x_1+p_\alpha x_2)^{-1}$ replaced by

$$\lim_{N\to\infty}\sum_{\nu=-N}^{N}\{x_1+p_\alpha(x_2-\nu h)\}^{-1}=\frac{\pi}{p_\alpha h}\cot\frac{\pi}{p_\alpha h}(x_1+p_\alpha x_2). \qquad (63)$$

When x_1 is large (63) reduces to $i\pi/p_\alpha h$; then from (51) and (38) we find

$$\sigma_{i1}=0,$$

and from (52), (41), (42) and (50)

$$\sigma_{i2} = (i/4h) \sum_{\alpha} (L_{i\alpha} p_\alpha^{-1} M_{\alpha j} - \bar{L}_{i\alpha} \bar{p}_\alpha^{-1} \bar{M}_{\alpha j})\, d_j$$
$$= \beta_{ik} B_{kj}\, d_j/2h = \beta_{ik} b_k/2h.$$

Since the only elements of β_{ik} which are not zero are those given in (28), the non-vanishing stresses are

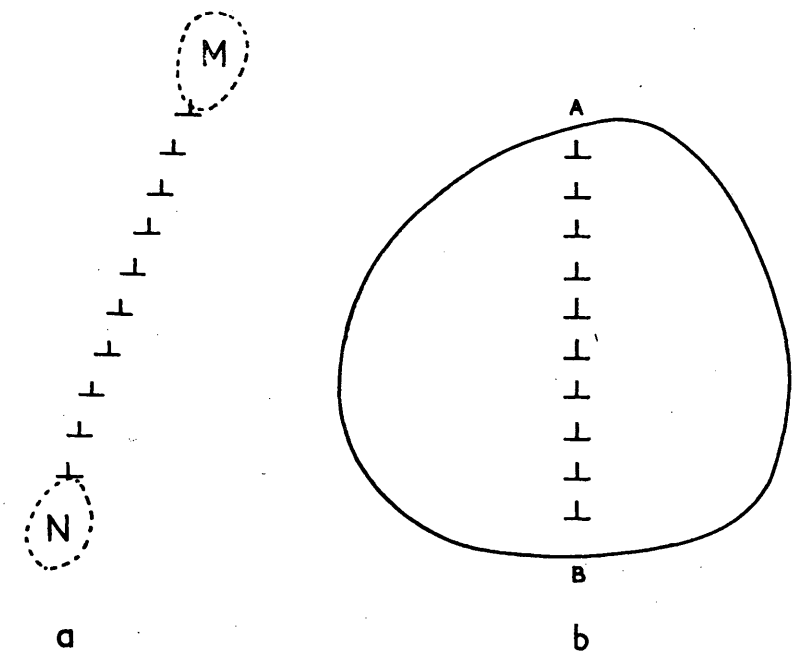
$$\sigma_{22} = (\beta_{22} b_2 + \beta_{23} b_3)/2h,$$

and

$$\sigma_{23} = (\beta_{23} b_2 + \beta_{33} b_3)/2h.$$

These stresses clearly vanish if b_i is given by (61), and by (31) this is the only value of b_i for which they vanish. Thus the simple tilt boundary normal to the slip plane is the only type of boundary which can be formed from dislocations all of the same kind, and for which the remote stress field is zero. This agrees with Frank's results.

Fig. 3



It remains to reconcile the treatment here with that of the previous section. Consider a wall built up of a finite number of dislocations. First suppose that the wall occurs in an infinite crystal (fig. 3 (*a*)); the treatment of § 5 will now be valid; but the present section fails to consider the regions of high stress concentration M and N at the ends of the wall, and so cannot give any information of the total energy and the equilibrium orientation of the wall in this case. Secondly, consider a finite crystal with the wall extending right across it (fig. 3 (*b*)); now the surface of the crystal must be stress free and, to secure this, additional stresses (which possibly alter the equilibrium positions of the dislocations) must be added to the stresses of § 5. On the other hand suppose the stresses in the crystal are those obtained in this section, that is they are given by eqns. (51) and (52) with z_α^{-1} replaced by the expression (63); with the Burgers vector given by eqn. (61) the stress on the surface of the crystal is then negligibly small except within a distance of order h from the points A and B where the surface is met by the wall, and the stresses here are easily shown to give zero resultant force and moment; hence if the stresses on the surface are relaxed the change in the stress field will, by Saint-Venant's principle be confined to regions at the ends of the wall with dimensions of order h. Also the stress σ_{12} on each dislocation due to the others may be shown to be zero, and so the dislocations are in equilibrium. Thus the stress field has essentially been found.

We see then that there are two distinct physical situations: if the wall terminates inside the crystal, its orientation must be determined by the considerations of § 5; while if the wall extends right across the grain, the treatment of the present section will apply, and the wall will be normal to the slip plane.

However if the angle θ (fig. 2) defining the equilibrium between a pair of edge dislocations is a right angle, then incomplete walls will also be formed normal to the slip plane and the distinction between two cases largely disappears. Owing to the high symmetry of the lattices commonly occurring in metals, this will be so in a number of important instances. For if either the x_1 or the x_2 axis, i.e. either the slip direction or the normal to the slip plane, is a two fold axis, then the equilibrium positions of the dislocation at P must be arranged symmetrically on either side of the plane $x_1=0$; also the total number of equilibrium positions was seen in § 5 to be odd; hence an equilibrium position must occur on $x_1=0$. The only important case not covered by these symmetry requirements seems to be that of the body-centred cubic lattice with slip direction [111] and slip plane $(11\bar{2})$ or $(12\bar{3})$ (but not $1\bar{1}0$)). Here dislocation walls may be expected in different orientations according as they are complete or incomplete.

On the other hand dislocation walls terminating inside the crystal may not occur with any great frequency as, owing to the high stress concentrations at their ends, they represent a state of high energy. They are unlikely to survive any prolonged heat treatment in which dislocation climb can occur; while even if only glide occurs, the nature of the stresses at the ends of the wall is such that dislocations will tend to be captured, and in this way the wall can extend across the crystal.

§ 7. Evaluation of the Constants

In this section we consider how the various constants we have introduced, and particularly the $L_{i\alpha}$, may be evaluated. It will be convenient to use instead of the elastic moduli c_{MN}, the elastic coefficients s_{MN}.

This also has the advantages for numerical work that compilations of experimental values, for example those of Schmid and Boas (1950), and of Hearmon (1946), usually give the values of s_{MN} rather than of c_{MN}.

The relation between stress and strain is then

$$e_M = s_{MN}\sigma_N. \qquad (64)$$

In the present case the strain $e_3 = \partial u_3/\partial x_3$ is identically zero, and we may use this to eliminate the stress component $\sigma_3 = \sigma_{33}$ from eqns. (64); we obtain

$$e_M = S_{MN}\sigma_N, \qquad (65)$$

where

$$S_{MN} = s_{MN} - s_{M3}s_{N3}/s_{33}. \qquad (66)$$

Equation (66) gives $S_{M3} = S_{3N} = 0$, and omitting these elements from the matrix $[S_{MN}]$, we may show that the remaining 5×5 matrix is reciprocal to (30).

The constants β_{ij} and γ_{ij} are now easily expressed in terms of the S_{MN}; from Jacobi's theorem we obtain the non-zero elements in eqns. (28) and (34)

$$\left.\begin{aligned} \beta_{22} &= S_{44}/(S_{22}S_{44} - S_{24}^2), \\ \beta_{33} &= S_{22}/(S_{22}S_{44} - S_{24}^2), \\ \beta_{23} &= \beta_{32} = -S_{24}/(S_{22}S_{44} - S_{24}^2), \end{aligned}\right\} \qquad (67)$$

and

$$\left.\begin{aligned} \gamma_{11} &= S_{55}/(S_{11}S_{55} - S_{15}^2), \\ \gamma_{33} &= S_{11}/(S_{11}S_{55} - S_{15}^2), \\ \gamma_{13} &= \gamma_{31} = -S_{15}/(S_{11}S_{55} - S_{15}^2). \end{aligned}\right\} \qquad (68)$$

We now consider solutions of the elastic equations which involve only one of the roots p_α, that is we take the displacement to be of the simple form (6) rather than the more general form (9). The stress functions ϕ_i may then be written, on using (16), as

$$\left.\begin{aligned} \phi_1 &= -p_\alpha L_{2\alpha} f(x_1 + p_\alpha x_2), \\ \phi_2 &= L_{2\alpha} f(x_1 + p_\alpha x_2), \\ \phi_3 &= L_{3\alpha} f(x_1 + p_\alpha x_2). \end{aligned}\right\} \qquad (69)$$

The stresses deduced from these functions must satisfy the compatibility conditions. Expressed in terms of the strains, the compatibility equations reduce, in the present case, to

$$\frac{\partial^2 e_1}{\partial x_2^2} + \frac{\partial^2 e_2}{\partial x_1^2} = \frac{\partial^2 e_6}{\partial x_1 \partial x_2}, \qquad (70)$$

and

$$\frac{\partial e_4}{\partial x_1} - \frac{\partial e_5}{\partial x_2} = 0. \qquad (71)$$

Inserting (65) in (70) and (71) and using (10) and (69) we obtain

$$L_{2\alpha}[p_\alpha^4S_{11}-2p_\alpha^3S_{16}+p_\alpha^2(2S_{12}+S_{66})-2p_\alpha S_{26}+S_{22}]$$
$$-L_{3\alpha}[p_\alpha^3S_{15}-p_\alpha^2(S_{14}+S_{56})+p_\alpha(S_{25}+S_{46})-S_{24}]=0, \quad . \quad (72)$$

and

$$L_{2\alpha}[p_\alpha^3S_{15}-p_\alpha^2(S_{14}+S_{56})+p_\alpha(S_{25}+S_{46})-S_{24}]$$
$$-L_{3\alpha}[p_\alpha^2S_{55}-2p_\alpha S_{45}+S_{44}]=0. \quad . \quad (73)$$

Eliminating $L_{2\alpha}$ and $L_{3\alpha}$ from (72) and (73) we find that p_α must be a root of the sextic equation

$$[p^4S_{11}-2p^3S_{16}+p^2(2S_{12}+S_{66})-2pS_{26}+S_{22}][p^2S_{55}-2pS_{45}+S_{44}]$$
$$-[p^3S_{15}-p^2(S_{14}+S_{56})+p(S_{25}+S_{46})-S_{24}]^2=0. \quad . \quad (74)$$

Now eqns. (8) and (74) both just determine the values of p for which a solution with the displacements of the form (6) exists. Thus these two equations must be identical and we have simply expressed the coefficients in terms of the S_{MN} instead of the c_{MN}.

Now we recall that the vectors $L_{i\alpha}$ are undefined to the extent of an arbitrary constant factor (for each α); if $L_{2\alpha}\neq 0$ we choose this factor so that $L_{2\alpha}=1$. Then $L_{i\alpha}=(-p_\alpha, 1, l_\alpha)$, where l_α is a constant which has still to be determined. Substituting this value of $L_{i\alpha}$ in (73) we find

$$l_\alpha=[p_\alpha^3S_{15}-p_\alpha^2(S_{14}+S_{56})+p_\alpha(S_{26}+S_{46})-S_{24}]/[p_\alpha^2S_{55}-2p_\alpha S_{45}+S_{44}], \quad (75)$$

and l_α is determined provided p_α does not satisfy the equation

$$p^2S_{55}-2pS_{45}+S_{44}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

Then if none of the roots of (74) also satisfy (76), we may write

$$[L_{i\alpha}]=\begin{bmatrix}-p_1 & -p_2 & -p_3\\ 1 & 1 & 1\\ l_1 & l_2 & l_3\end{bmatrix}; \quad . \quad . \quad . \quad . \quad . \quad (77)$$

and

$$[M_{\alpha i}]=L^{-1}\begin{bmatrix}l_3-l_2 & l_3p_2-l_2p_3 & p_3-p_2\\ l_1-l_3 & l_1p_3-l_3p_1 & p_1-p_3\\ l_2-l_1 & l_2p_1-l_1p_2 & p_2-p_1\end{bmatrix}, \quad . \quad . \quad . \quad (78)$$

where L denotes the value of the determinant $|L_{i\alpha}|$. To determine B_{ij}, we must first evaluate P_{ij} and Q_{ij} defined in eqns. (42) and (43) and then solve (41) and (44). However, the expressions become increasingly complicated, and it is probably best to procede numerically at this point.

If eqns. (74) and (76) have a root in common we choose the labelling α so that this root is p_3. Comparing (74) and (76) we see that we must have

$$p_3^3S_{15}-p_3^2(S_{14}+S_{56})+p_3(S_{25}+S_{46})-S_{24}=0, \quad . \quad . \quad . \quad (79)$$

and (72) and (73) will be satisfied if L_{i3} is of the form (0, 0, 1). It is of course possible for eqn. (79) to be true without the coefficients vanishing identically, but this does not appear to be of much physical significance. A more important case is when (79) holds because the coefficients vanish identically; a sufficient condition for this is that the x_3 axis should be a two-fold symmetry axis. Then (74) factorizes into a quartic and quadratic equation, and also (72) and (73) give $l_1 = l_2 = 0$. Thus

$$[L_{i\alpha}] = \begin{bmatrix} -p_1 & -p_2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (80)$$

and

$$[M_{\alpha i}] = \begin{bmatrix} (p_2-p_1)^{-1} & p_2(p_2-p_1)^{-1} & 0 \\ -(p_2-p_1)^{-1} & -p_1(p_2-p_1)^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (81)$$

On account of the simple form of these matrices, the remaining constants are readily evaluated explicitly. From (43)

$$[P_{ij}] = \begin{bmatrix} p_1+p_2 & p_1p_2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & p_3 \end{bmatrix}, \qquad (82)$$

and from (42)

$$[Q_{ij}] = \begin{bmatrix} 0 & -1 & 0 \\ 1/p_1p_2 & 1/p_1+1/p_2 & 0 \\ 0 & 0 & 1/p_3 \end{bmatrix}. \qquad (83)$$

Also in the present case, the non-zero elements of β_{ij} and γ_{ij} reduce, by (67) and (68) to

$$\left.\begin{aligned} \beta_{22} &= S_{22}{}^{-1}, & \beta_{33} &= S_{44}{}^{-1}, \\ \gamma_{11} &= S_{11}{}^{-1}, & \gamma_{33} &= S_{55}{}^{-1}, \end{aligned}\right\} \qquad (84)$$

and

since $S_{24} = S_{15} = 0$. Then substituting (82), (83) and (84) in (41) and (44), we find

$$[B_{ij}] = \frac{1}{2i}\begin{bmatrix} S_{11}(p_1+p_2-\bar{p}_1-\bar{p}_2) & S_{11}(p_1p_2-\bar{p}_1\bar{p}_2) & 0 \\ S_{11}(p_1p_2-\bar{p}_1\bar{p}_2) & -S_{22}(1/p_1+1/p_2-1/\bar{p}_1-1/\bar{p}_2) & 0 \\ 0 & 0 & S_{55}(p_3-\bar{p}_3) \end{bmatrix} \qquad (85)$$

By substituting this value of B_{ij} in eqns. (50), d_i may be expressed in terms of the Burgers vector, and hence the stresses of the dislocation may be obtained from (51) and (52) by a straight forward substitution.

In particular we note that if $b_i = (b, 0, 0)$, then $d_2 = 0$ only if $p_1p_2 = \bar{p}_1\bar{p}_2$; but this would imply a relation between the elastic coefficients which will not in general hold. The statement following eqn. (62) is thus verified.

§ 8. Cracks: The General Solution

Suppose we have an infinite crystal in which there is a crack along that portion of the plane $x_2 = 0$ which is given by $-1 \leqslant x_1 \leqslant 1$. An arbitrary stress is applied to the crystal such that in the absence of any stress relaxation due to the crack there would be a traction of $\tau_i(x_1)$ over the surface of the crack. We must therefore add a stress field which produces a traction of $-\tau_i(x_1)$ over the surface of the crack and which vanishes at infinity.

We consider the semi-infinite regions $x_2 > 0$, and $x_2 < 0$ separately. In $x_2 > 0$ a displacement which has the form (9) and which remains finite at infinity is

$$u_k = \sum_\alpha A_{k\alpha} \int_0^\infty d\rho F_\alpha{}^+(\rho) \exp(i\rho z_\alpha) + \sum_\alpha \bar{A}_{k\alpha} \int_0^\infty d\rho \overline{F_\alpha{}^+(\rho)} \exp(-i\rho\bar{z}_\alpha). \quad (86)$$

The corresponding stress functions are

$$\phi_k = \sum_\alpha L_{k\alpha} \int_0^\infty d\rho F_\alpha{}^+(\rho) \exp(i\rho z_\alpha) + \sum_\alpha \bar{L}_{k\alpha} \int_0^\infty d\rho \overline{F_\alpha{}^+(\rho)} \exp(-i\rho\bar{z}_\alpha). \quad (87)$$

In the region $x_2 < 0$, we may take the stress functions to be

$$\phi_k = \sum_\alpha L_{k\alpha} \int_0^\infty d\rho F_\alpha{}^-(\rho) \exp(-i\rho z_\alpha) + \sum_\alpha \bar{L}_{k\alpha} \int_0^\infty d\rho \overline{F_\alpha{}^-(\rho)} \exp(i\rho\bar{z}_\alpha). \quad (88)$$

We have now to relate the solutions in the two regions. In the part of the plane $x_2 = 0$ outside the crack the material must be joined together, and the boundary condition on the stress is then that the components σ_{i2} should be continuous. In the crack the components take on prescribed values and so are a fortiore also continuous here. Hence $\sigma_{i2} = \partial\phi_i/\partial x_1$, is continuous over the whole plane $x_2 = 0$. Now an arbitrary constant can be added to ϕ_i without affecting the stresses, and if this constant is chosen suitably then ϕ_i will also be continuous across $x_2 = 0$. Putting $x_2 = 0$ in (87) and (88) and comparing the resulting equations we obtained the boundary conditions

$$\sum_\alpha L_{i\alpha} F_\alpha{}^+(\rho) = \sum_\alpha \bar{L}_{i\alpha} \overline{F_\alpha{}^-(\rho)};$$

denoting these expressions by $\psi_i(\rho)$, we have

$$\left.\begin{aligned} F_\alpha{}^+(\rho) &= M_{\alpha i}\psi_i(\rho), \\ F_\alpha{}^-(\rho) &= M_{\alpha i}\overline{\psi_i(\rho)}. \end{aligned}\right\} \quad (89)$$

On substituting from (89) in (86) we obtain the displacement in the region $x_2 > 0$

$$u_k = \sum_\alpha A_{k\alpha} M_{\alpha i} \int_0^\infty \psi_i(\rho) \exp(i\rho z_\alpha)\, d\rho + \sum_\alpha \bar{A}_{k\alpha} \bar{M}_{\alpha i} \int_0^\infty \overline{\psi_i(\rho)} \exp(-i\rho\bar{z}_\alpha)\, d\rho. \quad (90)$$

In the region $x_2 < 0$, the displacement will be

$$u_k = \sum_\alpha A_{k\alpha} M_{\alpha i} \int_0^\infty \overline{\psi_i(\rho)} \exp(-i\rho z_\alpha)\, d\rho + \sum_\alpha \bar{A}_{k\alpha} \bar{M}_{\alpha i} \int_0^\infty \psi_i(\rho) \exp(i\rho\bar{z}_\alpha)\, d\rho. \quad (91)$$

Subtracting (90) and (91) we find that the difference in displacement on either side of the plane $x_2 = 0$ is

$$\Delta u_k = -2iB_{ki} \int_0^\infty \{\psi_i(\rho) \exp(i\rho x_1) - \overline{\psi_i(\rho)} \exp(-i\rho x_1)\}\, d\rho, \quad (92)$$

where B_{ki} is defined in eqn. (40). Outside the crack the displacement must be continuous and so $\Delta u_k = 0$. Since $|B_{ki}| \neq 0$, this gives

$$\int_0^\infty \{\psi_i(\rho)\exp(i\rho x_1) - \overline{\psi_i(\rho)}\exp(-i\rho x_1)\}\,d\rho = 0, \quad |x_1| > 1. \quad . \quad (93)$$

Also substituting (89) in (87) we obtain the stress functions in the region $x_2 > 0$

$$\phi_k = \sum_\alpha L_{k\alpha} M_{\alpha i} \int_0^\infty \psi_i(\rho)\exp(i\rho z_\alpha)\,d\rho + \sum_\alpha \bar{L}_{k\alpha}\bar{M}_{\alpha i}\int_0^\infty \overline{\psi_i(\rho)}\exp(-i\rho\bar{z}_\alpha)\,d\rho. \quad (94)$$

Now on the surface of the crack the stresses $\sigma_{k2} = \partial\phi_k/\partial x_1$, must be equal to $-\tau_k(x_1)$. Hence from (94) we obtain

$$i\int_0^\infty \{\psi_i(\rho)\exp(i\rho x_1) - \overline{\psi_i(\rho)}\exp(-i\rho x_1)\}\rho\,d\rho = -\tau_i(x_1), \quad -1 < x_1 < 1. \quad (95)$$

If we write $\psi_i = \psi_i' + i\psi_i''$ where ψ_i' and ψ_i'' are real, eqns. (93) and (95) become

$$\int_0^\infty \{\psi_i' \sin\rho x_1 + \psi_i'' \cos\rho x_1\}\,d\rho = 0 \qquad |x_1| > 1,$$

and

$$\int_0^\infty \{\psi_i' \sin\rho x_1 + \psi_i'' \cos\rho x_1\}\rho\,d\rho = \tfrac{1}{2}\tau_i(x_1) \qquad -1 < x_1 < 1.$$

These are equivalent to the two pairs of dual integral equations

$$\left.\begin{aligned} &\int_0^\infty \psi_i'(\rho)\sin\rho x_1\,d\rho = 0 && x_1 > 1, \\ &\int_0^\infty \psi_i'(\rho)\sin\rho x_1\,.\,\rho\,d\rho = \tfrac{1}{4}\{\tau_i(x_1) - \tau_i(-x_1)\}, && 0 < x_1 < 1 \end{aligned}\right\} \quad . \quad (96)$$

and

$$\left.\begin{aligned} &\int_0^\infty \psi_i''(\rho)\cos\rho x_1\,d\rho = 0 && x_1 > 0, \\ &\int_0^\infty \psi_i''(\rho)\cos\rho x_1\,.\,\rho\,d\rho = \tfrac{1}{4}\{\tau_i(x_1) + \tau_i(-x_1)\} && 0 < x_1 < 1. \end{aligned}\right\} \quad . \quad (97)$$

Equations (96) and (97) are special cases of a general set of dual integral equations studied by Titchmarsh (1937) and Busbridge (1938). Their general solution leads to

$$\psi_i'(\rho) = \frac{1}{2\pi}\int_0^1 \mu J_1(\mu\rho)\,d\mu \int_{-1}^1 \tau_i(\mu\xi)(1-\xi^2)^{-1/2}\xi\,d\xi, \quad . \quad . \quad (98)$$

and

$$\psi_i''(\rho) = \frac{1}{2\pi}\int_0^1 \mu J_0(\mu\rho)\,d\mu \int_{-1}^1 \tau_i(\mu\xi)(1-\xi^2)^{-1/2}\,d\xi. \quad . \quad . \quad (99)$$

On substituting (98) and (99) in eqn. (90) we obtain the displacements

$$\begin{aligned} u_k = {} & \frac{1}{2\pi}\sum_\alpha (A_{k\alpha}M_{\alpha j} + \bar{A}_{k\alpha}\bar{M}_{\alpha j})\int_0^1 d\mu \int_{-1}^1 \tau_j(\mu\xi)\xi(1-\xi^2)^{-1/2}\,d\xi \\ & - \frac{1}{2\pi}\sum_\alpha A_{k\alpha}M_{\alpha j}\int_0^1 d\mu(z_\alpha^2 - \mu^2)^{-1/2}\int_{-1}^1 \tau_j(\mu\xi)(\mu + z_\alpha\xi)(1-\xi^2)^{-1/2}\,d\xi \\ & - \frac{1}{2\pi}\sum_\alpha \bar{A}_{k\alpha}\bar{M}_{\alpha j}\int_0^1 d\mu(\bar{z}_\alpha^2 - \mu^2)^{-1/2}\int_{-1}^1 \tau_j(\mu\xi)(\mu + \bar{z}_\alpha\xi)(1-\xi^2)^{-1/2}\,d\xi. \quad (100) \end{aligned}$$

This expression will be valid both in the region $x_2 > 0$, and in $x_2 < 0$ provided we choose the sign of $(z_\alpha^2 - \mu^2)^{1/2}$ so that

$$\arg (z_\alpha^2 - \mu^2)^{1/2} \to \arg z_\alpha \quad \text{as} \quad |z_\alpha| \to \infty.$$

The stress functions ϕ_k will be given by an expression of the form (100) but in which $A_{k\alpha}$ is replaced by $L_{k\alpha}$. Once the ϕ_k are determined the stresses are readily obtained by differentiation according to eqns. (10); it is to be remembered that the stresses obtained in this way are to be added to the stresses which would occur in the infinite material if the crack were absent.

A particular case of some importance is that in which the applied stress field is uniform. With τ_j constant, the equation corresponding to (100) for the stress functions reduces to

$$\phi_k = \tfrac{1}{2} \sum_\alpha \{L_{k\alpha} M_{\alpha j}[(z_\alpha^2 - 1)^{1/2} - z_\alpha] + \bar{L}_{k\alpha} \bar{M}_{\alpha j}[(\bar{z}_\alpha^2 - 1)^{1/2} - \bar{z}_\alpha]\}\tau_j.$$

From this we obtain the stresses

$$\sigma_{k1} = -\tfrac{1}{2} \sum_\alpha \{L_{k\alpha} M_{\alpha j} p_\alpha [z_\alpha (z_\alpha^2 - 1)^{-1/2} - 1] + \bar{L}_{k\alpha} \bar{M}_{\alpha j} \bar{p}_\alpha [\bar{z}_\alpha (\bar{z}_\alpha^2 - 1)^{-1/2} - 1]\}\tau_j,$$

$$\sigma_{k2} = \tfrac{1}{2} \sum_\alpha \{L_{k\alpha} M_{\alpha j} z_\alpha (z_\alpha^2 - 1)^{-1/2} + \bar{L}_{k\alpha} \bar{M}_{\alpha j} \bar{z}_\alpha (\bar{z}_\alpha^2 - 1)^{-1/2}\}\tau_j - \tau_k.$$

§ 9. The Stresses near the Tip of a Crack

The stresses near the tip of the crack are of special importance since there will be a large stress concentration here, and we shall consider now the nature of these stresses.

First we reduce the repeated integral in (100) to a single integral. This may be accomplished by replacing the variable ξ by $\eta = \xi\mu$ and changing the order of integration; we obtain

$$\phi_k = -\frac{1}{2\pi} \sum_\alpha \left\{ L_{k\alpha} M_{\alpha j} \int_{-1}^{1} \tau_j(\xi)\, d\xi \cos^{-1} \frac{z_\alpha \xi - 1}{z_\alpha - \xi} + \bar{L}_{k\alpha} \bar{M}_{\alpha j} \int_{-1}^{1} \tau_j(\xi)\, d\xi \cos^{-1} \frac{\bar{z}_\alpha \xi - 1}{\bar{z}_\alpha - \xi} \right\} \quad . \quad . \quad . \quad (101)$$

on dropping a constant term which can contribute nothing to the stresses. Then from (101) we obtain the stresses

$$\sigma_{k1} = -\frac{1}{2\pi} \sum_\alpha \left\{ L_{k\alpha} p_\alpha M_{\alpha j} (z_\alpha^2 - 1)^{-1/2} \int_{-1}^{1} \tau_j(\xi)\, d\xi (1 - \xi^2)^{1/2} (z_\alpha - \xi)^{-1} + \bar{L}_{k\alpha} \bar{p}_\alpha \bar{M}_{\alpha j} (\bar{z}_\alpha^2 - 1)^{-1/2} \int_{-1}^{1} \tau_j(\xi)\, d\xi (1 - \xi^2)^{1/2} (\bar{z}_\alpha - \xi)^{-1} \right\}, \quad . \quad (102)$$

and

$$\sigma_{k2} = \frac{1}{2\pi} \sum_\alpha \left\{ L_{k\alpha} M_{\alpha j} (z_\alpha^2 - 1)^{-1/2} \int_{-1}^{1} \tau_j(\xi)\, d\xi (1 - \xi^2)^{1/2} (z_\alpha - \xi)^{-1} + \bar{L}_{k\alpha} \bar{M}_{\alpha j} (\bar{z}_\alpha^2 - 1)^{-1/2} \int_{-1}^{1} \tau_j(\xi)\, d\xi (1 - \xi^2)^{1/2} (\bar{z}_\alpha - \xi)^{-1} \right\}. \quad . \quad (103)$$

Now we are interested in points near the tip of the crack where $z_\alpha = 1$; hence we write $z_\alpha = 1 + \zeta_\alpha$ and take $|\zeta_\alpha|$ small compared with unity. If r and θ are defined as in fig. 4 we have

$$\zeta_\alpha = r(\cos\theta + p_\alpha \sin\theta). \qquad (104)$$

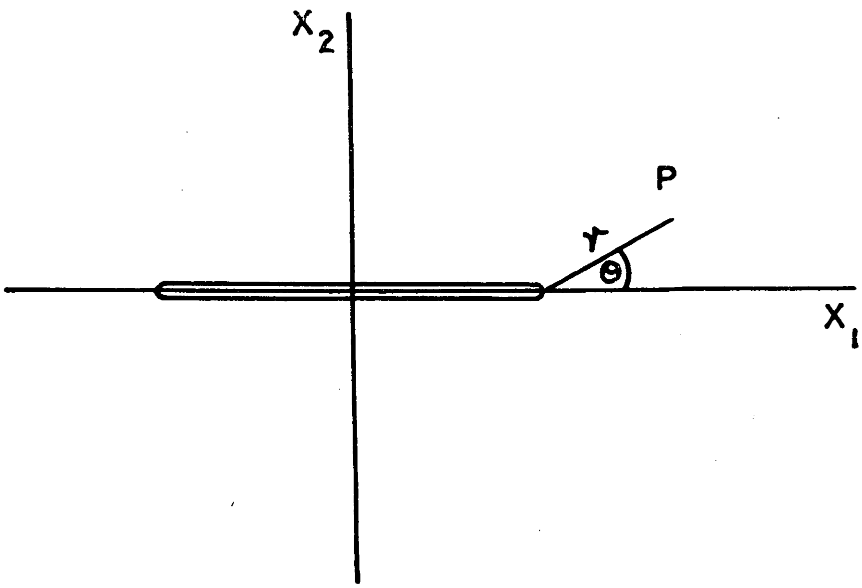
Then

$$\frac{1}{\pi}\int_{-1}^{1} \tau_j(\xi)(1-\xi^2)^{1/2}(z_\alpha - \xi)^{-1}\,d\xi \simeq \frac{1}{\pi}\int_{-1}^{1} \tau_j(\xi)(1+\xi)^{1/2}(1-\xi)^{-1/2}\,d\xi,$$
$$= T_j, \text{ say.} \qquad (105)$$

T_j may be regarded as a suitably weighted average value of τ_j over the crack; it has been defined so that if τ_j is constant $T_j = \tau_j$. Substituting (105) in (102) and (103) and remembering that ζ_α is small, we obtain

$$\left.\begin{aligned} \sigma_{k1} &= -\tfrac{1}{2}\sum_\alpha \{L_{k\alpha} p_\alpha M_{\alpha j}(2\zeta_\alpha)^{-1/2} + \bar{L}_{k\alpha}\bar{p}_\alpha \bar{M}_{\alpha j}(2\bar{\zeta}_\alpha)^{-1/2}\} T_j, \\ \sigma_{k2} &= \tfrac{1}{2}\sum_\alpha \{L_{k\alpha} M_{\alpha j}(2\zeta_\alpha)^{-1/2} + \bar{L}_{k\alpha}\bar{M}_{\alpha j}(2\bar{\zeta}_\alpha)^{-1/2}\} T_j. \end{aligned}\right\} \qquad (106)$$

Fig. 4



In particular we note that these stresses decrease with the distance from the tip of the crack as $r^{-1/2}$, just as in the isotropic case.

In the plane of the crack we have $\theta = 0$ and $\zeta_\alpha = r$, and so from (106) the stresses acting across this plane are

$$\sigma_{k2} = (2r)^{-1/2}\, T_k; \qquad (107)$$

it is interesting to note that these stresses depend neither on the symmetry nor the elastic constants of the material.

If the length of the crack is $2c$ then we have only to replace r in eqns. (104) and (107) by r/c.

§ 10. The Energy of a Crack

If the two sides of the crack undergo a relative displacement Δu_k when the surface tractions τ_k are relaxed, the elastic energy of the material will change by an amount

$$U=\tfrac{1}{2}\int_{-1}^{1}\tau_k(x_1)\Delta u_k\,dx_1. \qquad (108)$$

Now Δu_k is given by eqn. (92) $(-1<x_1<1)$, and substituting from (98) and (99) in this we obtain

$$\Delta u_k=\frac{2}{\pi}B_{kj}\int_{|x_1|}^{1}\frac{d\mu}{(\mu^2-x_1{}^2)^{1/2}}\int_{-1}^{1}\tau_j(\mu\xi)(\mu+x_1\xi)\frac{d\xi}{(1-\xi^2)^{1/2}}. \qquad (109)$$

From (108) and (109) the energy of the crack may be written as

$$U=\tfrac{1}{2}\pi B_{jk}\int_{-1}^{1}|\mu|\,d\mu T_j(\mu)T_k(\mu), \qquad (110)$$

where

$$T_j(\mu)=\pi^{-1}\int_{-\pi/2}^{\pi/2}\tau_j(\mu\sin\theta)(1+\sin\theta)\,d\theta. \qquad (111)$$

T_j as defined in eqn. (105) is related to the present $T_j(\mu)$ by

$$T_j=T_j(1).$$

In the particular case in which the applied stresses are uniform, $T_j(\mu)=\tau_j$ and the energy (110) becomes

$$U=\tfrac{1}{2}\pi B_{jk}\tau_j\tau_k;$$

or if the crack has length $2c$ then this energy is

$$U=\tfrac{1}{2}\pi B_{jk}\tau_j\tau_k c^2. \qquad (112)$$

Similarly, in the general case a factor c^2 must be inserted in eqn. (110).

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