Chapter 5

THE STROH FORMALISM

In this chapter we study Stroh's sextic formalism for two-dimensional deformations of an anisotropic elastic body. The Stroh formalism can be traced to the work of Eshelby, Read, and Shockley (1953). We therefore present the latter first. Not all results presented in this chapter are due to Stroh (1958, 1962). Nevertheless we name the sextic formalism after Stroh because he laid the foundations for researchers who followed him. The derivation of Stroh's formalism is rather simple and straightforward. The general solution resembles that obtained by the Lekhnitskii formalism. However, the resemblance between the two formalisms stops there. As we will see in the rest of the book, the Stroh formalism is indeed mathematically elegant and technically powerful in solving two-dimensional anisotropic elasticity problems. The possibility of extending the formalism to three-dimensional deformations is explored in Chapter 15.

5.1 The Eshelby–Reid–Shockley Formalism

In a fixed rectangular coordinate system x_i (i=1,2,3) let u_i and σ_{ij} be, respectively, the displacement and stress in an anisotropic elastic material. The stress-strain laws and the equations of equilibrium are

$$\sigma_{ij} = C_{ijks} u_{k,s}, \tag{5.1-1}$$

$$C_{ijks} u_{k,sj} = 0, \tag{5.1-2}$$

in which a comma denotes differentiation, repeated indices imply summation and C_{ijks} are the elastic stiffnesses which are assumed to possess the full symmetry

$$C_{ijks} = C_{jiks} = C_{ksij} = C_{ijsk}. \tag{5.1-3}$$

It is shown in Section 2.1 that $(5.1\text{-}3)_1$ and $(5.1\text{-}3)_2$ imply $(5.1\text{-}3)_3$. For two-dimensional deformations in which u_i (i=1,2,3) depends on x_1 and x_2 only, (5.1-2) is a homogeneous second-order differential equation in two independent variables x_1 and x_2. A general solution for u_i depends on one composite variable that is a linear combination of x_1 and x_2. Without loss in generality we choose the

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coefficient of x_1 in the linear combination to be unity and let

$$u_i = a_i f(z), \quad \text{or} \quad \mathbf{u} = \boldsymbol{a} f(z) \tag{5.1-4}$$

where

$$z = x_1 + p x_2. \tag{5.1-5}$$

In the above f is an arbitrary function of z, and p and a_i are unknown constants to be determined. Differentiation of (5.1-4) with x_s gives

$$u_{k,s} = (\delta_{s1} + p\delta_{s2}) a_k f'(z) \tag{5.1-6}$$

in which the prime denotes differentiation with the argument z and δ_{si} is the Kronecker delta. Differentiating once more with x_j we find that (5.1-2) is satisfied if

$$C_{ijks}(\delta_{j1} + p\delta_{j2})(\delta_{s1} + p\delta_{s2}) a_k = 0 \tag{5.1-7}$$

or

$$\left\{C_{i1k1} + p(C_{i1k2} + C_{i2k1}) + p^2 C_{i2k2}\right\} a_k = 0.$$

This can be written in matrix notation as

$$\left\{\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2 \mathbf{T}\right\} \boldsymbol{a} = \mathbf{0} \tag{5.1-8}$$

where $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$ are 3×3 matrices whose components are

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}. \tag{5.1-9}$$

It is seen that $\mathbf{Q}$ and $\mathbf{T}$ are symmetric and positive definite if the strain energy is positive (Section 2.9). For a nontrivial solution of $\boldsymbol{a}$ we must have

$$\left|\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2 \mathbf{T}\right| = 0 \tag{5.1-10}$$

which gives six roots for the eigenvalue p. The associated eigenvector $\boldsymbol{a}$ is determined from (5.1-8). The stresses σ_{ij} are obtained by inserting (5.1-6) into (5.1-1). Making use of (5.1-9) we obtain

$$\left.\begin{aligned} \sigma_{i1} &= (Q_{ik} + pR_{ik}) a_k f'(z), \\ \sigma_{i2} &= (R_{ki} + pT_{ik}) a_k f'(z). \end{aligned}\right\} \tag{5.1-11}$$

The p of (5.1-10) cannot be real. If p is real, (5.1-7) multiplied by a_i and summed with i leads to

$$C_{ijks}(\delta_{j1} + p\delta_{j2}) a_i (\delta_{s1} + p\delta_{s2}) a_k = 0.$$

By choosing

$$u_{i,j} = (\delta_{j1} + p\delta_{j2}) a_i, \quad u_{k,s} = (\delta_{s1} + p\delta_{s2}) a_k,$$

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we obtain

$$C_{ijks}u_{i,j}u_{k,s} = 0$$

which violates the condition that the strain energy is positive definite (Section 2.1).

Since the coefficients of the sextic equation for p arising from (5.1-10) are real there are three pairs of complex conjugates for p. If p_α, $\mathbf{a}_\alpha$ (α=1,2,...,6) are the eigenvalues and the associated eigenvectors, we let

$$\operatorname{Im} p_\alpha > 0, \quad p_{\alpha+3} = \bar{p}_\alpha, \quad \mathbf{a}_{\alpha+3} = \bar{\mathbf{a}}_\alpha \quad (\alpha = 1,2,3), \tag{5.1-12}$$

where Im stands for the imaginary part and the overbar denotes the complex conjugate. Assuming that p are distinct, the general solution obtained by superposing six solutions of the form (5.1-4) is

$$\mathbf{u} = \sum_{\alpha=1}^{3} \left\{ \mathbf{a}_\alpha f_\alpha(z_\alpha) + \bar{\mathbf{a}}_\alpha f_{\alpha+3}(\bar{z}_\alpha) \right\}, \tag{5.1-13}$$

where f_α (α=1,2,...,6) are arbitrary functions of their arguments and

$$z_\alpha = x_1 + p_\alpha x_2. \tag{5.1-14}$$

Likewise, the general solution for the stresses obtained from (5.1-11) can be written as

$$\begin{aligned} \mathbf{t}_1 &= \sum_{\alpha=1}^{3} \left\{ (\mathbf{Q} + p_\alpha \mathbf{R})\, \mathbf{a}_\alpha\, f'_\alpha(z_\alpha) + (\mathbf{Q} + \bar{p}_\alpha \mathbf{R})\, \bar{\mathbf{a}}_\alpha\, f'_{\alpha+3}(\bar{z}_\alpha) \right\} \\ \mathbf{t}_2 &= \sum_{\alpha=1}^{3} \left\{ (\mathbf{R}^T + p_\alpha \mathbf{T})\, \mathbf{a}_\alpha\, f'_\alpha(z_\alpha) + (\mathbf{R}^T + \bar{p}_\alpha \mathbf{T})\, \bar{\mathbf{a}}_\alpha\, f'_{\alpha+3}(\bar{z}_\alpha) \right\} \end{aligned} \tag{5.1-15}$$

in which the components of the vectors $\mathbf{t}_1$, $\mathbf{t}_2$ are

$$(\mathbf{t}_1)_i = \sigma_{i1}, \quad (\mathbf{t}_2)_i = \sigma_{i2}. \tag{5.1-16}$$

The stress component σ_{33} is determined from (2.4-5), which is the condition for ε_{33}=0.

5.2 Eigenvalues p

Employing the contracted notation for C_{ijks} introduced in Section 2.3 the three matrices **Q**, **R**, and **T** defined in (5.1-9) have the expressions

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$$\mathbf{Q}=\begin{bmatrix} C_{11} & C_{16} & C_{15} \\ C_{16} & C_{66} & C_{56} \\ C_{15} & C_{56} & C_{55} \end{bmatrix}, \quad \mathbf{R}=\begin{bmatrix} C_{16} & C_{12} & C_{14} \\ C_{66} & C_{26} & C_{46} \\ C_{56} & C_{25} & C_{45} \end{bmatrix}, \quad \mathbf{T}=\begin{bmatrix} C_{66} & C_{26} & C_{46} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44} \end{bmatrix}. \tag{5.2-1}$$

Hence (5.1-8) can be written as

$$\begin{bmatrix} C_{11}+2pC_{16}+p^2C_{66} & C_{16}+p(C_{12}+C_{66})+p^2C_{26} & C_{15}+p(C_{14}+C_{56})+p^2C_{46} \\ C_{16}+p(C_{12}+C_{66})+p^2C_{26} & C_{66}+2pC_{26}+p^2C_{22} & C_{56}+p(C_{46}+C_{25})+p^2C_{24} \\ C_{15}+p(C_{14}+C_{56})+p^2C_{46} & C_{56}+p(C_{46}+C_{25})+p^2C_{24} & C_{55}+2pC_{45}+p^2C_{44} \end{bmatrix}\boldsymbol{a}=\mathbf{0}. \tag{5.2-2}$$

A sextic equation for the eigenvalues p is obtained by setting the determinant of the 3×3 matrix on the left to zero.

The sextic equation for p is simplified for special materials such as monoclinic materials with the symmetry plane at one of the coordinate planes. We consider these and other special materials below.

(1) **Monoclinic Materials with the Symmetry Plane at x_1=0.** For this case (5.2-2) reduces to

$$\begin{bmatrix} C_{11}+p^2C_{66} & p(C_{12}+C_{66}) & p(C_{14}+C_{56}) \\ p(C_{12}+C_{66}) & C_{66}+p^2C_{22} & C_{56}+p^2C_{24} \\ p(C_{14}+C_{56}) & C_{56}+p^2C_{24} & C_{55}+p^2C_{44} \end{bmatrix}\boldsymbol{a}=\mathbf{0}. \tag{5.2-3}$$

The sextic equation is a polynomial of degree three in p^2. If the three roots of p^2 are all real, they must be negative since p cannot be real. We then have

$$p_1=i\beta_1, \quad p_2=i\beta_2, \quad p_3=i\beta_3, \tag{5.2-4}$$

where β_1, β_2, and β_3 are real and positive. If only one root of p^2, say p_3^2, is real the other two roots p_1^2 and p_2^2 must be complex conjugates. We then have

$$p_1=\alpha+i\beta, \quad p_2=-\alpha+i\beta, \quad p_3=i\beta_3, \tag{5.2-5}$$

in which α is real, and β and β_3 are real and positive.

(2) **Monoclinic Materials with the Symmetry Plane at x_2=0.** In this case (5.2-2) reduces to

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$$\begin{bmatrix} C_{11}+p^2C_{66} & p(C_{12}+C_{66}) & C_{15}+p^2C_{46} \\ p(C_{12}+C_{66}) & C_{66}+p^2C_{22} & p(C_{46}+C_{25}) \\ C_{15}+p^2C_{46} & p(C_{46}+C_{25}) & C_{55}+p^2C_{44} \end{bmatrix} \mathbf{a} = \mathbf{0}. \tag{5.2-6}$$

The sextic equation is again a polynomial of degree three in p^2. Hence the eigenvalues have the representations (5.2-4) or (5.2-5).

(3) **Monoclinic Materials with the Symmetry Plane at x_3=0.** Equation (5.2-2) now becomes

$$\begin{bmatrix} C_{11}+2pC_{16}+p^2C_{66} & C_{16}+p(C_{12}+C_{66})+p^2C_{26} & 0 \\ C_{16}+p(C_{12}+C_{66})+p^2C_{26} & C_{66}+2pC_{26}+p^2C_{22} & 0 \\ 0 & 0 & C_{55}+2pC_{45}+p^2C_{44} \end{bmatrix} \mathbf{a} = \mathbf{0}. \tag{5.2-7}$$

The sextic equation for p decomposes into a quadratic equation

$$C_{55} + 2pC_{45} + p^2C_{44} = 0 \tag{5.2-8}$$

and a quartic equation

$$\left(C_{11}+2pC_{16}+p^2C_{66}\right)\left(C_{11}+2pC_{16}+p^2C_{66}\right) - \left[C_{16}+p(C_{12}+C_{66})+p^2C_{26}\right]^2 = 0. \tag{5.2-9}$$

If p_3 denotes the root of (5.2-8) with positive imaginary part,

$$p_3 = \frac{1}{C_{44}}\left\{-C_{45} + i\sqrt{C_{44}C_{55} - C_{45}^2}\right\}. \tag{5.2-10}$$

The roots p_1, p_2 of (5.2-9) with positive imaginary parts have the representations

$$p_1 = \alpha_1 + i\beta_1, \quad p_2 = \alpha_2 + i\beta_2, \tag{5.2-11}$$

where α_1 and α_2 are real, and β_1 and β_2 are real and positive.

(4) **Transversely Isotropic Materials with the Axis of Symmetry at the x_3-axis.** For this case we have

$$\begin{bmatrix} \lambda+2\mu+\mu p^2 & (\lambda+\mu)p & 0 \\ (\lambda+\mu)p & \mu+(\lambda+2\mu)p^2 & 0 \\ 0 & 0 & C_{44}(1+p^2) \end{bmatrix} \mathbf{a} = \mathbf{0} \tag{5.2-12}$$

where λ and μ are constants. It is easily verified that

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$$p_1 = p_2 = p_3 = i \tag{5.2-13}$$

and that, at $p=i$, (5.2-12) becomes

$$\begin{bmatrix} (\lambda+\mu) & (\lambda+\mu)i & 0 \\ (\lambda+\mu)i & -(\lambda+\mu) & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{a} = \mathbf{0}. \tag{5.2-14}$$

The three columns of the 3×3 matrix are proportional to each other, yielding only two independent eigenvectors $\boldsymbol{a}$ associated with the triple eigenvalues (5.2-13). The system is mathematically degenerate.

(5) **Isotropic Materials**. Equation (5.2-12) applies to isotropic materials when λ and μ are identified as the Lamé constants and $C_{44}=\mu$. Equations (5.2-13) and (5.2-14) remain valid for isotropic materials. The eigenvectors $\boldsymbol{a}$ for isotropic materials are identical to those for transversely isotropic materials with the axis of symmetry at the x_3-axis. In fact the solutions to two-dimensional elasticity problems for isotropic materials and transversely isotropic materials with the axis of symmetry at the x_3-axis are identical except that the constant C_{44} is μ for the former and arbitrary for the latter.

It should be pointed out that the triple eigenvalues (5.2-13) are not limited to isotropic materials and transversely isotropic materials with the axis of symmetry at the x_3-axis. There are infinitely many anisotropic materials whose eigenvalues are given by (5.2-13) (Ting, 1994; see also Section 4.5).

For the special materials considered above explicit solutions of p in radicals can be obtained. For general anisotropic materials or for materials with the symmetry planes that are not on the coordinate planes, the sextic equation for p is not solvable in radical, not even for cubic materials which have only three elastic constants (Head, 1979a,b).

5.3 The Sextic Formalism of Stroh

The stresses shown in (5.1-11) can be rewritten as

$$\sigma_{i1} = -pb_i f'(z), \qquad \sigma_{i2} = b_i f'(z), \tag{5.3-1}$$

where, in matrix notation,

$$\boldsymbol{b} = (\mathbf{R}^T + p\mathbf{T})\boldsymbol{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\boldsymbol{a}. \tag{5.3-2}$$

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The second equality in (5.3-2) follows from (5.1-8). Introducing the stress function

$$\varphi_i = b_i f(z) \quad \text{or} \quad \boldsymbol{\phi} = \boldsymbol{b} f(z), \tag{5.3-3}$$

(5.3-1) are equivalent to

$$\sigma_{i1} = -\varphi_{i,2}, \quad \sigma_{i2} = \varphi_{i,1}. \tag{5.3-4}$$

It is sufficient therefore to consider the stress function $\boldsymbol{\phi}$ because the stresses can be obtained from (5.3-4) by differentiation. The two equations in (5.3-4) can be combined into one equation as

$$\sigma_{ij} = \delta_{3kj}\varphi_{i,k}.$$

The fact that $\sigma_{21}=\sigma_{12}$ means that, by (5.3-4),

$$\varphi_{1,1} + \varphi_{2,2} = 0 \tag{5.3-5}$$

and, by (5.3-3),

$$b_1 + pb_2 = 0. \tag{5.3-6}$$

Indeed, (5.3-6) can be verified directly by using (5.2-1) when b_1 is determined from $(5.3\text{-}2)_1$ and b_2 from $(5.3\text{-}2)_2$.

The general solution for the stress function $\boldsymbol{\phi}$ is obtained by superposing six solutions of the form (5.3-3) associated with six eigenvalues p_α. Together with (5.1-13) we have

$$\begin{aligned}
\mathbf{u} &= \sum_{\alpha=1}^{3}\left\{\boldsymbol{a}_\alpha f_\alpha(z_\alpha) + \bar{\boldsymbol{a}}_\alpha f_{\alpha+3}(\bar{z}_\alpha)\right\}, \\
\boldsymbol{\phi} &= \sum_{\alpha=1}^{3}\left\{\boldsymbol{b}_\alpha f_\alpha(z_\alpha) + \bar{\boldsymbol{b}}_\alpha f_{\alpha+3}(\bar{z}_\alpha)\right\}.
\end{aligned} \tag{5.3-7}$$

In the above $\boldsymbol{b}_\alpha$ is related to $\boldsymbol{a}_\alpha$ through (5.3-2) and

$$\boldsymbol{b}_{\alpha+3} = \bar{\boldsymbol{b}}_\alpha \quad (\alpha = 1,2,3). \tag{5.3-8}$$

Equations (5.3-7) are the *sextic formalism* due to Stroh (1958, 1962). The vectors $\boldsymbol{a}_\alpha$, $\boldsymbol{b}_\alpha$ are the Stroh eigenvectors. The displacements are given by $(5.3\text{-}7)_1$ while the stresses are obtained from $(5.3\text{-}7)_2$ and (5.3-4) by differentiation. The only stress component missing is σ_{33}. It is determined in terms of other stress components using (2.4-5), which is the condition for $\varepsilon_{33}=0$.

In most applications the arbitrary functions f_α appearing in (5.3-7) have the same function form. We may therefore let

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$$f_\alpha(z_\alpha) = f(z_\alpha)q_\alpha, \quad f_{\alpha+3}(\bar{z}_\alpha) = \bar{f}(\bar{z}_\alpha)\bar{q}_\alpha, \tag{5.3-9}$$

(α=1,2,3) where q_α are arbitrary complex constants. The second equation is useful for obtaining real form solutions for $\mathbf{u}$ and $\boldsymbol{\phi}$. Equations (5.3-7) can then be written as

$$\mathbf{u} = 2\,\mathrm{Re}\{\mathbf{A}\langle f(z_*)\rangle\mathbf{q}\}, \quad \boldsymbol{\phi} = 2\,\mathrm{Re}\{\mathbf{B}\langle f(z_*)\rangle\mathbf{q}\}. \tag{5.3-10}$$

In the above Re stands for the real part, $\mathbf{A}$ and $\mathbf{B}$ are 3×3 complex matrices defined by

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3] \tag{5.3-11}$$

and $\langle f(z_*)\rangle$ is the diagonal matrix

$$\langle f(z_*)\rangle = \mathrm{diag}[f(z_1), f(z_2), f(z_3)]. \tag{5.3-12}$$

When $\mathbf{q}$ is replaced by $-i\mathbf{q}$, (5.3-10) leads to the alternate form

$$\mathbf{u} = 2\,\mathrm{Im}\{\mathbf{A}\langle f(z_*)\rangle\mathbf{q}\}, \quad \boldsymbol{\phi} = 2\,\mathrm{Im}\{\mathbf{B}\langle f(z_*)\rangle\mathbf{q}\}. \tag{5.3-13}$$

For a given problem all one has to do is to determine the unknown function $f(z_*)$ and the complex vector $\mathbf{q}$.

Other representations of the general solutions are obtained as follows. Let the 3-vectors $[f(z)]$, $[\hat{f}(\bar{z})]$ be defined by

$$[f(z)] = \begin{bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \end{bmatrix}, \quad [\hat{f}(\bar{z})] = \begin{bmatrix} f_4(\bar{z}_1) \\ f_5(\bar{z}_2) \\ f_6(\bar{z}_3) \end{bmatrix}. \tag{5.3-14}$$

Equation (5.3-7) can be written as

$$\begin{bmatrix} \mathbf{u} \\ \boldsymbol{\phi} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} \begin{bmatrix} [f(z)] \\ [\hat{f}(\bar{z})] \end{bmatrix}. \tag{5.3-15}$$

When the columns of $\mathbf{A}$ and $\mathbf{B}$ are properly normalized we will show in Section 5.5 that

$$\begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \bar{\mathbf{B}}^T & \bar{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

Hence (5.3-15) is equivalent to

$$\begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \bar{\mathbf{B}}^T & \bar{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\phi} \end{bmatrix} = \begin{bmatrix} [f(z)] \\ [\hat{f}(\bar{z})] \end{bmatrix} \tag{5.3-16}$$

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from which we have the representation (Yeh et al., 1993a,b)

$$\mathbf{A}^T\boldsymbol{\phi} + \mathbf{B}^T\mathbf{u} = [f(z)]. \tag{5.3-17}$$

We also have

$$\boldsymbol{\phi} + i\mathbf{M}^T\mathbf{u} = (\mathbf{A}^T)^{-1}[f(z)] \tag{5.3-18}$$

where

$$\mathbf{M} = -i\mathbf{B}\mathbf{A}^{-1}.$$

When the explicit expressions for **A**, **B**, and **M** to be derived in (6.3-7), (6.3-8), (6.4-4) are employed for special anisotropic materials under antiplane deformations, (5.3-17) and (5.3-18) both reduce to the general solution (3.3-14) for antiplane deformations.

5.4 The Stress Function $\boldsymbol{\phi}$ and the Airy Function χ

Let **t** be the traction vector at a point on the boundary Γ of an anisotropic elastic material and **n** be the unit outward normal to Γ so that (Fig. 5.1)

$$t_i = \sigma_{ij} n_j .$$

If s is the arc length measured along the boundary Γ as shown,

$$n_1 = -\frac{dx_2}{ds}, \quad n_2 = \frac{dx_1}{ds}, \quad n_3 = 0. \tag{5.4-1}$$

Using (5.3-4) and (5.4-1) differentiation of $\varphi_i(x_1,x_2)$ along the boundary Γ is

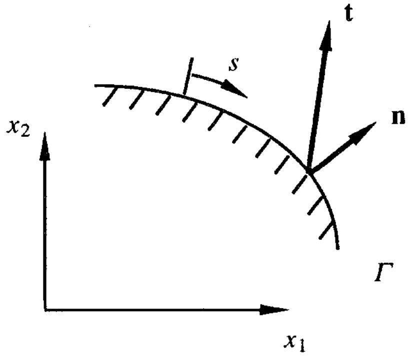

Fig. 5.1 The surface traction **t** on a curved boundary Γ with a unit outward normal vector **n**. It is related to the stress function $\boldsymbol{\phi}$ by (5.4-2).

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$$\frac{d}{ds}\varphi_i = \varphi_{i,1}\frac{dx_1}{ds} + \varphi_{i,2}\frac{dx_2}{ds}$$
$$= \sigma_{i1}n_1 + \sigma_{i2}n_2 = \sigma_{ij}n_j.$$

Hence

$$t_i = \frac{d}{ds}\varphi_i \quad \text{or} \quad \mathbf{t} = \frac{d}{ds}\boldsymbol{\phi}. \tag{5.4-2}$$

Equation (5.4-2) reduces to $(5.3\text{-}4)_{1,2}$ when Γ is a plane parallel to the $x_1=0$ plane or the $x_2=0$ plane. In applying (5.4-2), s should be chosen such that when one faces the direction of increasing s the material lies on the right side.

Let $s_2>s_1$ be two points on Γ. The difference $\Delta\boldsymbol{\phi}$ between $\boldsymbol{\phi}(s_2)$ and $\boldsymbol{\phi}(s_1)$ is

$$\Delta\boldsymbol{\phi} = \boldsymbol{\phi}(s_2) - \boldsymbol{\phi}(s_1) = \int_{s_1}^{s_2}\mathbf{t}(s)ds, \tag{5.4-3}$$

which represents the resultant force of surface tractions acting on Γ between s_1 and s_2. If Γ encloses a region and there is a concentrated force $\mathbf{f}$ inside the region, the equilibrium of the body demands that

$$\Delta\boldsymbol{\phi} = \int_{\Gamma}\mathbf{t}(s)ds = -\mathbf{f}. \tag{5.4-4}$$

This means that if we traverse Γ counter-clockwise one complete circle, $\boldsymbol{\phi}$ increases by the amount $\mathbf{f}$. Therefore $\boldsymbol{\phi}$ must be a multiple-valued function. A Riemann surface with a proper cut has to be introduced to maintain a unique solution for $\boldsymbol{\phi}$. If no concentrated forces are inside the region of Γ, $\boldsymbol{\phi}$ is a single-valued function.

Equation (5.3-5) is in the form of (3.14-1). According to Theorem 3.14–1 there exists a function, say χ, such that

$$\varphi_1 = -\chi_{,2}, \quad \varphi_2 = \chi_{,1}. \tag{5.4-5}$$

Hence

$$\chi(z) = \int^z \varphi_2(\lambda)d\lambda = b_2\int^z f(\lambda)d\lambda. \tag{5.4-6}$$

It is readily shown from (5.4-5) and (5.3-4) that χ is the *Airy function* (Section 4.1).

The moment about the x_3-axis due to the traction $\mathbf{t}$ on Γ is, using (5.4-2) and (5.4-5),

$$\int_\Gamma (x_1t_2 - x_2t_1)ds = \int_\Gamma x_1 d\varphi_2 - x_2 d\varphi_1$$

$$= \int_\Gamma d(x_1\varphi_2 - x_2\varphi_1) - \int_\Gamma (\varphi_2 dx_1 - \varphi_1 dx_2)$$

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$$= \int_\Gamma d(x_1\varphi_2 - x_2\varphi_1 - \chi)$$

Thus the moment due to surface tractions acting between z_1 and z_2 is

$$\Delta(x_1\varphi_2 - x_2\varphi_1 - \chi) = \Delta(x_1\varphi_2 - x_2\varphi_1) - \int_{z_1}^{z_2} \varphi_2(\lambda)d\lambda. \qquad (5.4\text{-}7)$$

The general solution for the displacement u_i given by (5.1-4) tells us that it is codirectional with the vector a_i. The general solution φ_i given by $(5.3\text{-}3)_1$ together with $(5.4\text{-}2)_1$ implies that the surface traction t_i is codirectional with the vector b_i. Hence the Stroh eigenvectors a_i, b_i represent the directions of the displacement u_i and traction t_i, respectively. Although a_i and b_i are complex they do have a real physical interpretation (Section 5.8).

5.5 Orthogonality and Closure Relations

What distinguishes the Stroh formalism from others is that the vectors $\boldsymbol{a}_\alpha$ and $\boldsymbol{b}_\alpha$ for different α are related. The complex matrices **A** and **B** defined in (5.3-11) possess certain properties that are important in applications.

The two equations in (5.3-2) can be cast in the form

$$\begin{bmatrix} -\mathbf{Q} & \mathbf{0} \\ -\mathbf{R}^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = p \begin{bmatrix} \mathbf{R} & \mathbf{I} \\ \mathbf{T} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}, \qquad (5.5\text{-}1)$$

where **I** is the 3×3 identity matrix. Noting that **T** is positive definite and $\mathbf{T}^{-1}$ exists, it is easily shown that

$$\begin{bmatrix} \mathbf{0} & \mathbf{T}^{-1} \\ \mathbf{I} & -\mathbf{R}\mathbf{T}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{I} \\ \mathbf{T} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \qquad (5.5\text{-}2)$$

Therefore pre-multiplication on both sides of (5.5-1) by the first matrix on the left of (5.5-2) reduces (5.5-1) to the following standard eigenrelation

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \qquad (5.5\text{-}3)$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \qquad \boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}, \qquad (5.5\text{-}4)$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}. \qquad (5.5\text{-}5)$$

It is seen that $\mathbf{N}_2$ and $\mathbf{N}_3$ are symmetric and that $\mathbf{N}_2$ is positive definite. We will show in Section 5.3 that $-\mathbf{N}_3$ is positive semi-

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definite. The 6×6 matrix **N** is the *fundamental elasticity matrix* first introduced by Ingebrigtsen and Tonning (1969). It should be noted that $\mathbf{N}_3$ has the dimensions of stress, $\mathbf{N}_2$ has the dimension of compliance and $\mathbf{N}_1$ is dimensionless.

Since **N** is not symmetric the ξ in (5.5-3) is a right eigenvector. The left eigenvector denoted by η satisfies the eigenrelation

$$\mathbf{N}^T\eta = p\eta. \tag{5.5-6}$$

Introducing the 6×6 constant matrix

$$\hat{\mathbf{I}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \qquad \hat{\mathbf{I}} = \hat{\mathbf{I}}^T = \hat{\mathbf{I}}^{-1}, \tag{5.5-7}$$

it is easily shown that $\hat{\mathbf{I}}\mathbf{N}$ is symmetric, i.e.,

$$\hat{\mathbf{I}}\mathbf{N} = (\hat{\mathbf{I}}\mathbf{N})^T = \mathbf{N}^T\hat{\mathbf{I}}. \tag{5.5-8}$$

From (5.5-3)

$$\hat{\mathbf{I}}\mathbf{N}\xi = p\hat{\mathbf{I}}\xi$$

and, by (5.5-8),

$$\mathbf{N}^T(\hat{\mathbf{I}}\xi) = p(\hat{\mathbf{I}}\xi).$$

The left eigenvector η can therefore assume the form

$$\eta = \hat{\mathbf{I}}\xi = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{bmatrix}. \tag{5.5-9}$$

Thus the Stroh eigenvectors $\boldsymbol{a}$ and $\boldsymbol{b}$ which form the right eigenvector ξ and the left eigenvector η of the fundamental elasticity matrix **N** are in the reverse order.

It is known that the left and right eigenvectors associated with different eigenvalues are orthogonal to each other, i.e.,

$$\eta_\alpha \cdot \xi_\beta = 0, \qquad \text{if } p_\alpha \neq p_\beta. \tag{5.5-10}$$

The vector ξ given by (5.5-3), and hence the vector η by (5.5-9), is unique up to an arbitrary multiplier. Assuming that p_α are distinct, we normalize ξ_α such that

$$\eta_\alpha \cdot \xi_\beta = \delta\alpha\beta \tag{5.5-11}$$

or, by $(5.5\text{-}4)_2$ and (5.5-9),

$$\boldsymbol{b}_\alpha^T\boldsymbol{a}_\beta + \boldsymbol{a}_\alpha^T\boldsymbol{b}_\beta = \delta_{\alpha\beta} \tag{5.5-12}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. In view of $(5.1\text{-}12)_3$, (5.3-8),

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and the matrices $\mathbf{A}$, $\mathbf{B}$ defined in (5.3-11), (5.5-12) is written as

$$\begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \overline{\mathbf{B}}^T & \overline{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{5.5-13}$$

or

$$\begin{aligned} &\mathbf{B}^T\mathbf{A} + \mathbf{A}^T\mathbf{B} = \mathbf{I} = \overline{\mathbf{B}}^T\overline{\mathbf{A}} + \overline{\mathbf{A}}^T\overline{\mathbf{B}}, \\ &\mathbf{B}^T\overline{\mathbf{A}} + \mathbf{A}^T\overline{\mathbf{B}} = \mathbf{0} = \overline{\mathbf{B}}^T\mathbf{A} + \overline{\mathbf{A}}^T\mathbf{B}. \end{aligned} \tag{5.5-14}$$

These are the *orthogonality* relations. The two 6×6 matrices on the left of (5.5-13) are the inverses of each other, and hence their products commute, i.e.,

$$\begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \overline{\mathbf{B}}^T & \overline{\mathbf{A}}^T \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{5.5-15}$$

or

$$\begin{aligned} &\mathbf{A}\mathbf{B}^T + \overline{\mathbf{A}}\overline{\mathbf{B}}^T = \mathbf{I} = \mathbf{B}\mathbf{A}^T + \overline{\mathbf{B}}\overline{\mathbf{A}}^T, \\ &\mathbf{A}\mathbf{A}^T + \overline{\mathbf{A}}\overline{\mathbf{A}}^T = \mathbf{0} = \mathbf{B}\mathbf{B}^T + \overline{\mathbf{B}}\overline{\mathbf{B}}^T. \end{aligned} \tag{5.5-16}$$

These are the *closure* relations. Equations $(5.5\text{-}16)_{1,2}$ tell us that the real part of $\mathbf{AB}^T$ is $\frac{1}{2}\mathbf{I}$, while $(5.5\text{-}16)_{3,4}$ imply that $\mathbf{AA}^T$ and $\mathbf{BB}^T$ are purely imaginary. Hence the three matrices $\mathbf{S}$, $\mathbf{H}$, and $\mathbf{L}$ defined by

$$\mathbf{S} = i(2\mathbf{AB}^T - \mathbf{I}), \quad \mathbf{H} = 2i\mathbf{AA}^T, \quad \mathbf{L} = -2i\mathbf{BB}^T \tag{5.5-17}$$

are real. We will show in Section 7.2 that they are tensors of rank two when the transformation is a rotation about the x_3-axis. It is clear that $\mathbf{H}$ and $\mathbf{L}$ are symmetric. It will be shown (Section 7.6) that they are positive definite. Therefore the matrices $\mathbf{A}$ and $\mathbf{B}$ are nonsingular (see also Section 12.4).

The matrix products $\mathbf{AB}^T$, $\mathbf{AA}^T$, and $\mathbf{BB}^T$ appear often in the final solutions to two-dimensional anisotropic elasticity problems. They can be replaced in terms of the real tensors $\mathbf{S}$, $\mathbf{H}$, and $\mathbf{L}$ through (5.5-17). Therefore it is possible to obtain the solutions in a real form for many problems. As will be shown in Sections 6.4 and 7.6 there are means to compute $\mathbf{S}$, $\mathbf{H}$, and $\mathbf{L}$ explicitly.

The eigenrelation (5.5-3) for $p = p_1, p_2, p_3$ can be combined into one equation as

$$\mathbf{N}\begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{P}} \end{bmatrix} \tag{5.5-18}$$

where

$$\mathbf{P} = \langle p_* \rangle = \mathrm{diag}[p_1, p_2, p_3]. \tag{5.5-19}$$

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By (5.5-15) we have

$$\mathbf{N}=\begin{bmatrix}\mathbf{A} & \overline{\mathbf{A}}\\ \mathbf{B} & \overline{\mathbf{B}}\end{bmatrix}\begin{bmatrix}\mathbf{P} & \mathbf{0}\\ \mathbf{0} & \overline{\mathbf{P}}\end{bmatrix}\begin{bmatrix}\mathbf{B}^T & \mathbf{A}^T\\ \overline{\mathbf{B}}^T & \overline{\mathbf{A}}^T\end{bmatrix} \tag{5.5-20}$$

which is the diagonalization of **N**.

The derivations presented so far assume that the eigenvalues p_α are distinct or, if they are not, there exist six independent eigenvectors ξ_α. In other words the derivations are valid if the 6×6 matrix **N** in (5.5-3) is simple or semisimple (Section 1.3). A semisimple matrix **N** can be found for certain transversely isotropic materials whose axis of symmetry is not on the x_3-axis (Tanuma, 1996). If **N** is nonsemisimple, the general solutions (5.3-7), (5.3-10), and (5.3-13) need to be modified. The orthogonality relations (5.5-13) and the closure relations (5.5-15) are not valid. Anisotropic elastic materials for which **N** is non-semisimple are called *degenerate materials*. They are degenerate in the mathematical sense, not necessarily in the physical sense. Isotopic materials are a special group of degenerate materials for which $p=i$ is an eigenvalue of multiplicity three but there are only two independent eigenvectors for ξ (Section 5.2). Although **S**, **H**, and **L** defined in (5.5-17) are not valid for degenerate materials, Barnett and Lothe (1973) devised an integral formalism to compute these matrices directly from the elastic stiffnesses C_{ijks} (Section 7.6). Thus the need for calculating the eigenvalues p_α and the eigenvectors ξ_α is circumvented, and the problems associated with degenerate materials disappear. **S**, **H**, and **L** are therefore called the *Barnett–Lothe tensors*. The tensorial nature of **S**, **H**, and **L** will be studied in Section 7.2.

It should be pointed out that the problems associated with degenerate materials are not unique with the Stroh formalism. Other formalisms have the same problems. Modifications required for the general solutions and for the Barnett–Lothe tensors **S**, **H**, **L** will be discussed in Sections 13.2 and 5.9, respectively.

5.6 Positive Definite Hermitian Matrices

We have pointed out that the products

$$2i\mathbf{A}\mathbf{A}^T \quad \text{and} \quad -2i\mathbf{B}\mathbf{B}^T$$

appearing in (5.5-17) are real, symmetric, and positive definite. They are special cases of positive definite Hermitian matrices in that they are real, not complex. From $(5.5\text{-}14)_4$ Stroh (1958) observed that $\overline{\mathbf{B}}^T\mathbf{A}$ is skew-Hermitian (Section 1.7). This means that

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$$i\overline{\mathbf{B}}^T\mathbf{A} = -i\overline{\mathbf{A}}^T\mathbf{B}$$

is Hermitian. We will show that it is also positive definite, and that it is but one of several positive definite Hermitian matrices that involve products of matrices associated with **A** and **B**.

The *impedance tensor* **M** and its inverse $\mathbf{M}^{-1}$ defined by (Ingebrigtsen and Tonning, 1969; Lothe and Barnett, 1976a)

$$\mathbf{M} = -i\mathbf{B}\mathbf{A}^{-1}, \quad \mathbf{M}^{-1} = i\mathbf{A}\mathbf{B}^{-1}, \tag{5.6-1}$$

are Hermitian because (5.6-1) are equivalent to

$$\mathbf{M} = \mathbf{B}(i\overline{\mathbf{B}}^T\mathbf{A})^{-1}\overline{\mathbf{B}}^T, \quad \mathbf{M}^{-1} = \mathbf{A}(-i\overline{\mathbf{A}}^T\mathbf{B})^{-1}\overline{\mathbf{A}}^T. \tag{5.6-2}$$

Since $i\overline{\mathbf{B}}^T\mathbf{A} = -i\overline{\mathbf{A}}^T\mathbf{B}$ are Hermitian, so are their inverses. Hence **M** and its inverse $\mathbf{M}^{-1}$ are Hermitian. We will show in Section 6.6 that **M** and $\mathbf{M}^{-1}$ are positive definite. By (5.6-2), $(i\overline{\mathbf{B}}^T\mathbf{A})^{-1}$ is also positive definite because **B** is nonsingular. Thus $i\overline{\mathbf{B}}^T\mathbf{A}$ is positive definite Hermitian as we stated earlier.

Equation $(5.6\text{-}1)_1$ means

$$\mathbf{B} = i\mathbf{M}\mathbf{A}$$

or

$$\boldsymbol{b}_\alpha = i\mathbf{M}\boldsymbol{a}_\alpha \quad (\alpha=1,2,3). \tag{5.6-3}$$

The relation between $\boldsymbol{b}_\alpha$ and $\boldsymbol{a}_\alpha$ is independent of α. It offers an alternative to (5.3-2) in which the relation between $\boldsymbol{b}_\alpha$ and $\boldsymbol{a}_\alpha$ depends on α. As indicated in (5.6-3) **M** effectively transforms the displacement related vector $\boldsymbol{a}_\alpha$ to the traction related vector $\boldsymbol{b}_\alpha$, justifying the term impedance tensor for **M**.

The matrices

$$i\mathbf{B}^{-1}\overline{\mathbf{B}} \quad \text{and} \quad -i\mathbf{A}^{-1}\overline{\mathbf{A}} \tag{5.6-4}$$

are positive definite Hermitian. In fact they are orthogonal matrices (Ting, 1993). From

$$i\mathbf{B}^{-1}\overline{\mathbf{B}} = i\mathbf{B}^T(\mathbf{B}\mathbf{B}^T)^{-1}\overline{\mathbf{B}} = 2\mathbf{B}^T\mathbf{L}^{-1}\overline{\mathbf{B}}$$

and, employing $(5.5\text{-}16)_4$,

$$(i\mathbf{B}^{-1}\overline{\mathbf{B}})(i\mathbf{B}^{-1}\overline{\mathbf{B}})^T = -\mathbf{B}^{-1}\overline{\mathbf{B}}\overline{\mathbf{B}}^T(\mathbf{B}^{-1})^T = \mathbf{B}^{-1}\mathbf{B}\mathbf{B}^T(\mathbf{B}^{-1})^T = \mathbf{I},$$

the positive definite and orthogonal properties of $i\mathbf{B}^{-1}\overline{\mathbf{B}}$ are verified. Following the same procedures, $-i\mathbf{A}^{-1}\overline{\mathbf{A}}$ can be shown to be positive definite orthogonal Hermitian.

Making use of (5.5-17) and the identities

$$\mathbf{B}\mathbf{A}^{-1} = (\mathbf{A}\mathbf{B}^T)^T(\mathbf{A}\mathbf{A}^T)^{-1}, \quad \mathbf{A}\mathbf{B}^{-1} = (\mathbf{A}\mathbf{B}^T)(\mathbf{B}\mathbf{B}^T)^{-1}, \tag{5.6-5}$$

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which can be verified easily, (5.6-1) is written as

$$\mathbf{M} = -i\mathbf{B}\mathbf{A}^{-1} = \mathbf{H}^{-1} + i\mathbf{H}^{-1}\mathbf{S},$$
$$\mathbf{M}^{-1} = i\mathbf{A}\mathbf{B}^{-1} = \mathbf{L}^{-1} - i\mathbf{S}\mathbf{L}^{-1}. \tag{5.6-6}$$

Since the imaginary part of a Hermitian is skew-symmetric we have

$$\mathbf{H}^{-1}\mathbf{S} + \mathbf{S}^T\mathbf{H}^{-1} = \mathbf{0} = \mathbf{S}\mathbf{L}^{-1} + \mathbf{L}^{-1}\mathbf{S}^T. \tag{5.6-7}$$

In fact we had used $(5.6\text{-}7)_1$ in writing $(5.6\text{-}6)_2$. By inserting (5.6-6) into the relation $\mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$ we obtain

$$(\mathbf{H}^{-1} + i\mathbf{H}^{-1}\mathbf{S})(\mathbf{L}^{-1} - i\mathbf{S}\mathbf{L}^{-1}) = \mathbf{I}.$$

The imaginary parts cancel each other while the real parts give

$$\mathbf{H}^{-1}\mathbf{L}^{-1} + \mathbf{H}^{-1}\mathbf{S}\mathbf{S}\mathbf{L}^{-1} = \mathbf{I}$$

or

$$\mathbf{H}\mathbf{L} - \mathbf{S}\mathbf{S} = \mathbf{I}. \tag{5.6-8}$$

Thus the three real tensors **S**, **H**, and **L** are related. More identities connecting **S**, **H**, and **L** will be presented in Section 6.5. It should be noted that the impedance tensor **M** alone is sufficient to determine **S**, **H**, and **L**. When **M** is given **H** and **S** are computed from the real and imaginary parts of **M** using $(5.6\text{-}6)_2$. **L** is then deduced from (5.6-8).

The orthogonality relations (5.5-13) and the closure relations (5.5-15) are useful in applications. So are the positive definite Hermitian matrices and the identities (5.6-6)–(5.6-8) presented here. These features do not exist in other formalisms.

5.7 The Matrix Differential Equation

For two-dimensional deformations studied here the equations of equilibrium are

$$\sigma_{i1,1} + \sigma_{i2,2} = 0. \tag{5.7-1}$$

From Theorem 3.14–1 there exists a stress function φ_i such that

$$\sigma_{i1} = -\varphi_{i,2}, \quad \sigma_{i2} = \varphi_{i,1}, \tag{5.7-2}$$

which were obtained in (5.3-4) by a different approach. Since

$$\sigma_{i1} = C_{i1k1}u_{k,1} + C_{i1k2}u_{k,2} = Q_{ik}u_{k,1} + R_{ik}u_{k,2},$$
$$\sigma_{i2} = C_{i2k1}u_{k,1} + C_{i2k2}u_{k,2} = R_{ki}u_{k,1} + T_{ik}u_{k,2},$$

(5.7-2) can be written as

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$$\begin{aligned} \mathbf{Q}\,\mathbf{u}_{,1} + \mathbf{R}\mathbf{u}_{,2} &= -\boldsymbol{\phi}_{,2}, \\ \mathbf{R}^T\mathbf{u}_{,1} + \mathbf{T}\mathbf{u}_{,2} &= \boldsymbol{\phi}_{,1}, \end{aligned} \tag{5.7-3a}$$

or

$$\begin{bmatrix} -\mathbf{Q} & \mathbf{0} \\ -\mathbf{R}^T & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{u}_{,1} \\ \boldsymbol{\phi}_{,1} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{I} \\ \mathbf{T} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{u}_{,2} \\ \boldsymbol{\phi}_{,2} \end{bmatrix}. \tag{5.7-3b}$$

Employing the identity (5.5-2) we obtain (Malen, 1971; Barnett and Lothe, 1973; Chadwick and Smith, 1977)

$$\mathbf{N}\boldsymbol{\varpi}_{,1} = \boldsymbol{\varpi}_{,2}, \quad \boldsymbol{\varpi} = \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\phi} \end{bmatrix}, \tag{5.7-4}$$

or

$$\begin{aligned} \mathbf{u}_{,2} &= \mathbf{N}_1\mathbf{u}_{,1} + \mathbf{N}_2\boldsymbol{\phi}_{,1}, \\ \boldsymbol{\phi}_{,2} &= \mathbf{N}_3\mathbf{u}_{,1} + \mathbf{N}_1^T\boldsymbol{\phi}_{,1}. \end{aligned} \tag{5.7-5}$$

This is the *matrix differential equation* for two-dimensional deformations of an anisotropic elastic solid. A general solution for $\boldsymbol{\varpi}$ is

$$\boldsymbol{\varpi} = \boldsymbol{\xi} f(z), \quad z = x_1 + p x_2. \tag{5.7-6}$$

On substituting (5.7-6) into $(5.7\text{-}4)_1$ we recover the eigenrelation (5.5-3).

The matrix differential equation (5.7-4) is obtained without the differentiation of elastic constants C_{ijks}. Therefore (5.7-4) remains valid for inhomogeneous materials for which C_{ijks} can depend on x_1 and x_2. Of course the general solution (5.7-6) does not apply to inhomogeneous materials.

When $x_2=0$ is an interface between two materials $(5.7\text{-}4)_1$ or (5.7-5) allows us to determine the *anti-plane* differentiation $\boldsymbol{\varpi}_{,2}$ in terms of the *in-plane* differentiation $\boldsymbol{\varpi}_{,1}$ (Hill, 1983). If $\mathbf{u}$ and $\boldsymbol{\phi}$ are continuous across the interface so are $\mathbf{u}_{,1}$ and $\boldsymbol{\phi}_{,1}$. Equations (5.7-5) then provide us $\mathbf{u}_{,2}$ and $\boldsymbol{\phi}_{,2}$ in one material when $\mathbf{u}_{,1}$ and $\boldsymbol{\phi}_{,1}$ in the other material are known. From (5.3-4) and (5.1-16) we rewrite (5.7-5) as

$$\begin{aligned} \mathbf{u}_{,2} &= \mathbf{N}_1\mathbf{u}_{,1} + \mathbf{N}_2\mathbf{t}_2, \\ -\mathbf{t}_1 &= \mathbf{N}_3\mathbf{u}_{,1} + \mathbf{N}_1^T\mathbf{t}_2. \end{aligned} \tag{5.7-7}$$

The vector $\mathbf{t}_2$ is the traction vector on the surface $x_2=0$ while $\mathbf{t}_1$ is the *hoop stress vector*, which is the traction vector acting on the surface perpendicular to $x_2=0$. For an interface for which $\mathbf{u}$ and $\mathbf{t}_2$ are continuous (5.7-7) leads to

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$$\begin{aligned}[\mathbf{u}_{,2}] &= [\mathbf{N}_1]\mathbf{u}_{,1} + [\mathbf{N}_2]\mathbf{t}_2, \\ -[\mathbf{t}_1] &= [\mathbf{N}_3]\mathbf{u}_{,1} + [\mathbf{N}_1^T]\mathbf{t}_2,\end{aligned} \tag{5.7-8}$$

in which the square brackets denote the discontinuity in the quantity across the interface $x_2=0$.

If there is only one material occupying the space $x_2 > 0$ for which $x_2=0$ is a *free surface* or a *rigid (clamped) surface*, we have

$$\mathbf{t}_1 = -\mathbf{N}_3\mathbf{u}_{,1}, \quad \text{for a free surface,} \tag{5.7-9}$$

$$\mathbf{t}_1 = -\mathbf{N}_1^T\mathbf{t}_2, \quad \text{for a rigid surface.} \tag{5.7-10}$$

Equation (5.7-9) follows from $(5.7\text{-}7)_2$ by setting $\mathbf{t}_2=\mathbf{0}$ while (5.7-10) is obtained from $(5.7\text{-}7)_2$ by applying (5.7-14) derived below. Equations (5.7-9) and (5.7-10) are useful in obtaining the hoop stress vector $\mathbf{t}_1$ when $\mathbf{u}$ along the free surface, or $\mathbf{t}_2$ on the rigid surface, is known. When $x_2=0$ is a traction free surface, the stress component σ_{12} and σ_{21} must vanish. This means that the second component of the hoop stress vector $\mathbf{t}_1$ in (5.7-9) should vanish for any $\mathbf{u}_{,1}$. This is assured because the matrix $\mathbf{N}_3$ has the structure (Section 6.1)

$$\mathbf{N}_3 = \begin{bmatrix} * & 0 & * \\ 0 & 0 & 0 \\ * & 0 & * \end{bmatrix}, \tag{5.7-11}$$

where the asterisk $*$ denotes a possibly nonzero element.

The fact that $x_2=0$ is a rigid surface does not necessarily mean that $\mathbf{u}=\mathbf{0}$ at $x_2=0$. The displacement at $x_2=0$ can have a rigid body translation and rotation. Assuming that $x_2 < 0$ consists of a rigid body, the displacement in $x_2 < 0$ is

$$\mathbf{u} = \mathbf{u}_o + \omega \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} \tag{5.7-12}$$

where ω and $\mathbf{u}_o$ are constants. Therefore

$$\mathbf{u}_{,1} = \omega \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{5.7-13}$$

which applies to $x_2 \le 0$. With (5.7-11) and (5.7-13) it is clear that

$$\mathbf{N}_3\mathbf{u}_{,1} = \mathbf{0}, \tag{5.7-14}$$

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proving that $(5.7\text{-}7)_2$ leads to (5.7-10).

5.8 Physical Meanings of p, a, and b

The general solutions for the displacement $\mathbf{u}$ and the stress function ϕ as given in (5.3-7) can be made real by letting $f_{\alpha+3}$ the complex conjugate of f_α (α=1,2,3). We then have

$$\mathbf{u} = \sum_{\alpha=1}^{3}\left\{\boldsymbol{a}_\alpha f_\alpha(z_\alpha)+\bar{\boldsymbol{a}}_\alpha \bar{f}_\alpha(\bar{z}_\alpha)\right\}, \quad \phi = \sum_{\alpha=1}^{3}\left\{\boldsymbol{b}_\alpha f_\alpha(z_\alpha)+\bar{\boldsymbol{b}}_\alpha \bar{f}_\alpha(\bar{z}_\alpha)\right\}, \quad (5.8\text{-}1)$$

where

$$z_\alpha = x_1 + p_\alpha x_2.$$

The eigenvalues p_α and the eigenvectors $\boldsymbol{a}_\alpha$ and $\boldsymbol{b}_\alpha$ depend on the elastic stiffnesses C_{ijks} only. Once they are determined C_{ijks} are no longer needed. Therefore p_α, $\boldsymbol{a}_\alpha$, and $\boldsymbol{b}_\alpha$ can be regarded as material constants even though they are complex-valued.

The general solutions (5.8-1) consist of three *independent* real solutions of the form

$$\mathbf{u} = \boldsymbol{a}\, f(z)+\bar{\boldsymbol{a}}\, \bar{f}(\bar{z}), \quad \phi = \boldsymbol{b}\, f(z)+\bar{\boldsymbol{b}}\, \bar{f}(\bar{z}). \qquad (5.8\text{-}2)$$

Let

$$\boldsymbol{a} = \boldsymbol{a}' + i\boldsymbol{a}'', \quad \boldsymbol{b} = \boldsymbol{b}' + i\boldsymbol{b}'',$$

where $\boldsymbol{a}'$, $\boldsymbol{a}''$, $\boldsymbol{b}'$, and $\boldsymbol{b}''$ are real. The real vectors $\boldsymbol{a}'$ and $\boldsymbol{a}''$ span a plane if $\boldsymbol{a}$ is *genuinely complex*, i.e., if $\boldsymbol{a}'$ and $\boldsymbol{a}''$ are not proportional to each other. The plane spanned by $\boldsymbol{a}'$ and $\boldsymbol{a}''$ is called an *eigenplane* (Section 1.8) or *the plane* $\boldsymbol{a}$. It is clear that the plane $\boldsymbol{a}$ and the plane $\bar{\boldsymbol{a}}$ are the same plane. Likewise, the plane $\boldsymbol{b}$ and the plane $\bar{\boldsymbol{b}}$ are the same plane. Equations (5.8-2) tell us that the displacement $\mathbf{u}$ at any point (x_1,x_2) is polarized on the plane $\boldsymbol{a}$ while the stress function ϕ is polarized on the plane $\boldsymbol{b}$. The latter means that, from (5.4-2), the surface traction vector $\mathbf{t}$ is polarized on the plane $\boldsymbol{b}$.

For general anisotropic elastic materials there are three polarization planes (the planes $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$) for the displacement $\mathbf{u}$ and three polarization planes (the planes $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$) for the surface traction $\mathbf{t}$. If one of complex vectors, say $\boldsymbol{a}_3$, is not genuinely complex, $\boldsymbol{a}_3$ can be taken as a real vector. The displacement $\mathbf{u}$ associated with $\boldsymbol{a}_3$ is then polarized along this real vector. This happens to monoclinic materials with the symmetry plane at x_3=0. The vectors $\boldsymbol{a}_3$ and $\boldsymbol{b}_3$ are both real and are both in the direction parallel to the x_3-axis. The displacement associated with $\boldsymbol{a}_3$ is the antiplane displacement and the stresses associated with $\boldsymbol{b}_3$ are the antiplane stresses. As to

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the planes $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ for monoclinic materials with the symmetry plane at $x_3=0$, they coincide with the plane $x_3=0$, so do the planes $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. Therefore the displacements associated with $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ are inplane displacements and the stresses associated with $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ are inplane stresses.

As to the complex eigenvalues p_α, no physical meanings have been found that can be attributed directly to p_α. However it has been shown in Sections 3.5, 3.6, 3.8–3.10, and will be shown in Chapters 8 and 10 that the complex eigenvalues p_α are solely responsible for the locations of the image singularities for Green's functions for half-spaces, bimaterials, and elliptic inclusions. The locations of the image singularities do not depend on the nature of the singularity prescribed, nor on the boundary conditions.

5.9 Nonsemisimple N

When $\mathbf{N}$ is nonsemisimple, say $p_1=p_2$, there is only one eigenvector ξ_1 associated with $p_1=p_2$. The eigenrelations for $p_1=p_2$ are

$$\begin{aligned} \mathbf{N}\xi_1 &= p_1\xi_1, \\ \mathbf{N}\xi_2 &= p_1\xi_2 + \xi_1, \end{aligned} \tag{5.9-1}$$

where ξ_2 is the generalized eigenvector (Section 1.4). The left eigenvectors associated with $p_1=p_2$ satisfy the eigenrelations

$$\begin{aligned} \mathbf{N}^T\eta_1 &= p_1\eta_1 + \eta_2, \\ \mathbf{N}^T\eta_2 &= p_1\eta_2, \end{aligned} \tag{5.9-2}$$

in which η_1, not η_2, is the generalized eigenvector. The ξ_1, ξ_2, η_1, and η_2 are not unique. We will show how ξ_1, ξ_2, η_1, and η_2 can be chosen such that the orthogonality relations

$$\eta_2^T\xi_1 = 0, \quad \eta_1^T\xi_2 = 0, \quad \eta_1^T\xi_1 = 1, \quad \eta_2^T\xi_2 = 1, \tag{5.9-3}$$

are satisfied. When $(5.9\text{-}1)_{1,2}$ are multiplied by η_1^T and use is made of (5.9-2) we obtain $(5.9\text{-}3)_1$ and

$$\eta_2^T\xi_2 = \eta_1^T\xi_1.$$

Therefore we only have to satisfy $(5.9\text{-}3)_2$ and $(5.9\text{-}3)_3$ or $(5.9\text{-}3)_4$. Multiplication of (5.9-1) by $\hat{\mathbf{I}}$ defined in (5.5-7) and with the aid of the relation (5.5-8) we have

$$\begin{aligned} \mathbf{N}^T(\hat{\mathbf{I}}\xi_1) &= p_1(\hat{\mathbf{I}}\xi_1), \\ \mathbf{N}^T(\hat{\mathbf{I}}\xi_2) &= p_1(\hat{\mathbf{I}}\xi_2) + (\hat{\mathbf{I}}\xi_1). \end{aligned}$$

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Comparison with (5.9-2) suggests that we may set

$$\eta_1 = \hat{\mathbf{I}}\xi_2, \quad \eta_2 = \hat{\mathbf{I}}\xi_1, \tag{5.9-4}$$

so that (5.9-3) are written as

$$\xi_1^T\hat{\mathbf{I}}\xi_1 = 0, \quad \xi_2^T\hat{\mathbf{I}}\xi_2 = 0, \quad \xi_2^T\hat{\mathbf{I}}\xi_1 = 1, \quad \xi_1^T\hat{\mathbf{I}}\xi_2 = 1. \tag{5.9-5}$$

It is seen that $(5.9\text{-}5)_3$ and $(5.9\text{-}5)_4$ are equivalent. We need to satisfy $(5.9\text{-}5)_2$ and $(5.9\text{-}5)_4$. Noting that if ξ_1^o and ξ_2^o are solutions of (5.9-1), so are

$$\xi_1 = k_1\xi_1^o, \quad \xi_2 = k_1\xi_2^o + k_2\xi_1^o, \tag{5.9-6}$$

where k_1 and k_2 are constants. Upon substitution into $(5.9\text{-}5)_4$ and by virtue of $(5.9\text{-}5)_1$ which applies to ξ_1^o and ξ_2^o one obtains

$$k_1^2 = \left((\xi_1^o)^T\hat{\mathbf{I}}\xi_2^o\right)^{-1}. \tag{5.9-7}$$

Insertion of $(5.9\text{-}6)_2$ into $(5.9\text{-}5)_2$ and with the aid of $(5.9\text{-}5)_1$ and (5.9-7),

$$k_2 = -\tfrac{1}{2}k_1^3\left((\xi_2^o)^T\hat{\mathbf{I}}\xi_2^o\right). \tag{5.9-8}$$

This completes the normalization of ξ_1 and ξ_2. The η_1 and η_2 are determined by (5.9-4).

The eigenrelations for $p_1 = p_2$ and p_3 can be written as

$$\mathbf{N}[\xi_1, \xi_2, \xi_3] = [\xi_1, \xi_2, \xi_3]\mathbf{J}, \tag{5.9-9}$$

$$\mathbf{N}^T[\eta_1, \eta_2, \eta_3] = [\eta_1, \eta_2, \eta_3]\mathbf{J}^T, \tag{5.9-10}$$

where ξ_2 and η_1 are the generalized eigenvectors and

$$\mathbf{J} = \begin{bmatrix} p_1 & 1 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \tag{5.9-11}$$

is a Jordan canonical form (Section 1.4). By (5.9-4),

$$[\eta_1, \eta_2, \eta_3] = \hat{\mathbf{I}}[\xi_2, \xi_1, \xi_3] = \hat{\mathbf{I}}[\xi_1, \xi_2, \xi_3]\Gamma, \tag{5.9-12}$$

where

$$\Gamma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \Gamma^T. \tag{5.9-13}$$

If we express the 6×1 vector ξ by the two 3×1 vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and

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employ the notation (5.3-11) we have

$$[\xi_1, \xi_2, \xi_3] = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}, \quad [\eta_1, \eta_2, \eta_3] = \begin{bmatrix} \mathbf{B}\Gamma \\ \mathbf{A}\Gamma \end{bmatrix}. \tag{5.9-14}$$

Similar expressions can be written for the eigenvectors associated with $p_4 = p_5$ and p_6 which are complex conjugates of $p_1 = p_2$ and p_3. The orthogonality relations between ξ_α and η_α now take the form

$$\begin{bmatrix} \Gamma\mathbf{B}^T & \Gamma\mathbf{A}^T \\ \Gamma\overline{\mathbf{B}}^T & \Gamma\overline{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} = \mathbf{I}. \tag{5.9-15}$$

The two 6×6 matrices on the left are the inverses of each other, and hence

$$\begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \Gamma\mathbf{B}^T & \Gamma\mathbf{A}^T \\ \Gamma\overline{\mathbf{B}}^T & \Gamma\overline{\mathbf{A}}^T \end{bmatrix} = \mathbf{I} \tag{5.9-16}$$

or

$$\begin{aligned} \mathbf{A}\Gamma\mathbf{B}^T + \overline{\mathbf{A}}\Gamma\overline{\mathbf{B}}^T &= \mathbf{I} = \mathbf{B}\Gamma\mathbf{A}^T + \overline{\mathbf{B}}\Gamma\overline{\mathbf{A}}^T, \\ \mathbf{A}\Gamma\mathbf{A}^T + \overline{\mathbf{A}}\Gamma\overline{\mathbf{A}}^T &= \mathbf{I} = \mathbf{B}\Gamma\mathbf{B}^T + \overline{\mathbf{B}}\Gamma\overline{\mathbf{B}}^T. \end{aligned} \tag{5.9-17}$$

These are the closure relations. Using the arguments following (5.5-16) the three matrices **S**, **H**, and **L** defined by

$$\mathbf{S} = i(2\mathbf{A}\Gamma\mathbf{B}^T - \mathbf{I}), \quad \mathbf{H} = 2i\mathbf{A}\Gamma\mathbf{A}^T, \quad \mathbf{L} = -2i\mathbf{B}\Gamma\mathbf{B}^T, \tag{5.9-18}$$

are real. It is clear that **H** and **L** are symmetric. We will show in Section 7.6 that they are positive definite.

From (5.9-9) and $(5.9\text{-}14)_1$

$$\mathbf{N}\begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{J}} \end{bmatrix} \tag{5.9-19}$$

or, by (5.9-16),

$$\mathbf{N} = \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{J}} \end{bmatrix} \begin{bmatrix} \Gamma\mathbf{B}^T & \Gamma\mathbf{A}^T \\ \Gamma\overline{\mathbf{B}}^T & \Gamma\overline{\mathbf{A}}^T \end{bmatrix}. \tag{5.9-20}$$

This is the diagonalization of **N** when it is nonsemisimple.

The above derivations are taken from Ting and Hwu (1988) (see also Sections 13.1 and 13.2) where a more general situation was investigated. The matrix **N** was assumed to be almost nonsemisimple, i.e., p_1 and p_2 are almost identical, so are ξ_1 and ξ_2. This causes a problem in that the magnitudes of the normalized eigenvectors ξ_1 and ξ_2 are very large, and become infinity at $p_1 = p_2$. When

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$p_1 \neq p_2$, the eigenrelations for p_1 and p_2 are

$$\mathbf{N}\xi_1 = p_1\xi_1, \quad \mathbf{N}\xi_2 = p_2\xi_2, \tag{5.9-21}$$

from which we have

$$\mathbf{N}(\xi_2 - \xi_1) = p_2\xi_2 - p_1\xi_1 = p_2(\xi_2 - \xi_1) + (p_2 - p_1)\xi_1$$

or

$$\mathbf{N}\xi' = p_2\xi' + \xi_1, \tag{5.9-22}$$

$$\xi' = \frac{\xi_2 - \xi_1}{p_2 - p_1}. \tag{5.9-23}$$

The vector ξ' is bounded. As $p_2 \to p_1$ and $\xi_2 \to \xi_1$, (5.9-22) becomes

$$\mathbf{N}\xi' = p_1\xi' + \xi_1 \tag{5.9-24}$$

which is identical to $(5.9\text{-}1)_2$ if we identify ξ' as the generalized eigenvector ξ_2. Notice that (5.9-24) is the differentiation of $(5.9\text{-}21)_1$ with p_1, assuming that ξ_1 depends on p_1. A similar operation on (5.1-8) and (5.3-2) gives

$$\begin{gathered}\left\{\mathbf{Q} + p_2(\mathbf{R} + \mathbf{R}^T) + p_2^2\mathbf{T}\right\}\boldsymbol{a}' = -\left\{(\mathbf{R} + \mathbf{R}^T) + (p_1 + p_2)\mathbf{T}\right\}\boldsymbol{a}_1,\\ \boldsymbol{b}' = (\mathbf{R}^T + p_2\mathbf{T})\,\boldsymbol{a}' + \mathbf{T}\,\boldsymbol{a}_1 = -\frac{1}{p_2}(\mathbf{Q} + p_1\mathbf{R})\,\boldsymbol{a}' + \frac{1}{p_1p_2}\mathbf{Q}\,\boldsymbol{a}_1.\end{gathered} \tag{5.9-25}$$

At $p_2 = p_1$, (5.9-25) provide an alternate to $(5.9\text{-}1)_2$ for determining the generalized eigenvector $\xi_2 = (\mathbf{a}', \mathbf{b}')$. The presentation by Ting and Hwu allows the matrix $\mathbf{N}$ to be simple or nonsemisimple so that there is a smooth transition from a simple $\mathbf{N}$ to a nonsemisimple $\mathbf{N}$.

We will not discuss the degenerate case in which $p_1 = p_2 = p_3$ with only one eigenvector ξ_1. The question is open whether there exists an anisotropic material that possesses such a degeneracy. Even for isotropic materials for which $p_1 = p_2 = p_3 = i$, there exists two independent eigenvectors ξ_1 and ξ_3 associated with the triple eigenvalues.

5.10 Dependence of Solutions on Elastic Constants

The first part of this section complement the first part of Section 4.7. The second part gives additional information on the dependence of stress solutions on elastic constants.

The general solutions for generalized plane strain deformations presented in (5.1-13) and (5.1-15) depend on the 15 elastic stiffnesses shown in (5.2-1). They can be represented by the 5×5 sym

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metric matrix $\mathbf{C}^{o}$ given in (2.4-4). They do not depend on $C_{3\alpha}$ (α=1,2,...,6). Therefore any solution of generalized plane strain deformations applies to a group of anisotropic elastic materials whose $\mathbf{C}^{o}$ is identical but whose $C_{3\alpha}$ are different. If $\mathbf{C}^{o}$ is positive definite, we may choose $C_{3\alpha}$ such that $\mathbf{C}$ is positive definite. To do this, first observe that the determinant of $\mathbf{C}$ remains the same if we move the third column to sixth column and the third row to sixth row. The re-arranged matrix $\mathbf{C}$ can be written as

$$\mathbf{C}=\begin{bmatrix}\mathbf{C}^{o} & \mathbf{h}\\ \mathbf{h}^{T} & C_{33}\end{bmatrix}=\begin{bmatrix}\mathbf{C}^{o} & \mathbf{0}\\ \mathbf{h}^{T} & 1\end{bmatrix}\begin{bmatrix}\mathbf{I} & (\mathbf{C}^{o})^{-1}\mathbf{h}\\ \mathbf{0} & C_{33}-\mathbf{h}^{T}(\mathbf{C}^{o})^{-1}\mathbf{h}\end{bmatrix} \tag{5.10-1}$$

where

$$\mathbf{h}^{T}=(C_{13},\ C_{23},\ C_{34},\ C_{35},\ C_{36}).$$

Taking the determinant on both sides of (5.10-1) leads to

$$|\mathbf{C}|=|\mathbf{C}^{o}|\{C_{33}-\mathbf{h}^{T}(\mathbf{C}^{o})^{-1}\mathbf{h}\}. \tag{5.10-2}$$

If $\mathbf{C}^{o}$ is positive definite, so is $\mathbf{C}$ provided $|\mathbf{C}|>0$ (Theorem 1.6–2). Therefore if we choose

$$C_{33}>\mathbf{h}^{T}(\mathbf{C}^{o})^{-1}\mathbf{h}=\mathbf{h}^{T}\mathbf{s}'\mathbf{h} \tag{5.10-3}$$

where $\mathbf{s}'$ is the 5×5 matrix given in (2.4-10) and (2.4-11), we have a positive definite $\mathbf{C}$.

A particularly interesting special case is when $\mathbf{C}^{o}$ is identical to that of isotropic materials. We then have

$$\mathbf{C}=\begin{bmatrix}\lambda+2\mu & \lambda & C_{13} & 0 & 0 & 0\\ & \lambda+2\mu & C_{23} & 0 & 0 & 0\\ & & C_{33} & C_{34} & C_{35} & C_{36}\\ & & & \mu & 0 & 0\\ & & & & \mu & 0\\ & & & & & \mu\end{bmatrix} \tag{5.10-4}$$

where λ and μ are Lamé constants and $C_{3\alpha}$ (α=1,2,...,6) are arbitrary real constants (see also Section 4.7). It can be shown that (5.10-3) for the $\mathbf{C}$ given by (5.10-4) becomes

$$C_{33}>\frac{1}{4\mu(\lambda+\mu)}\left\{\lambda(C_{13}-C_{23})^{2}+2\mu(C_{13}^{2}+C_{23}^{2})\right\}+\frac{1}{\mu}\left\{C_{34}^{2}+C_{35}^{2}+C_{36}^{2}\right\}. \tag{5.10-5}$$

Thus if λ and μ satisfy the conditions (2.9-5) and $C_{3\alpha}$ (α=1,2,...,6) are chosen such that (5.10-5) is satisfied, the $\mathbf{C}$ in (5.10-4) is posi

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tive definite. The materials represented by (5.10-4) can be completely anisotropic without a plane of symmetry. Nevertheless the solutions to two-dimensional elastostatics problems and steady-state motions such as surface waves for the materials given by (5.10-4) are identical to the solutions for isotropic materials.

From the fact that **C** in (5.10-4) and $s_{\alpha\beta}$ in (4.7-3) are the inverses of each other, $C_{3\alpha}$ can be determined in terms of γ_α as (Ting, 1994a)

$$C_{13} = -\frac{1}{\gamma_3}\{\lambda(\gamma_1+\gamma_2)+2\mu\gamma_1\},$$

$$C_{23} = -\frac{1}{\gamma_3}\{\lambda(\gamma_1+\gamma_2)+2\mu\gamma_2\},$$

$$C_{33} = \frac{1}{\gamma_3^2}\{2\mu(1+\gamma_1^2+\gamma_2^2)+\mu(\gamma_4^2+\gamma_5^2+\gamma_6^2)+\lambda(\gamma_1+\gamma_2)^2\}, \tag{5.10-6}$$

$$C_{33} = \frac{1}{\gamma_3^2}\{2\mu(1+\gamma_1^2+\gamma_2^2)+\mu(\gamma_4^2+\gamma_5^2+\gamma_6^2)+\lambda(\gamma_1+\gamma_2)^2\},$$

$$C_{3\alpha} = -\mu\frac{\gamma_\alpha}{\gamma_3}, \qquad \alpha = 4,5,6.$$

Since $s_{\alpha\beta}$ in (4.7-3) is positive definite for arbitrary γ_α with $\gamma_3 \neq 0$, the same can be said about the matrix **C** of (5.10-4) with $C_{3\alpha}$ given by (5.10-6).

The solution for the stress does not depend on elastic constants as much as the solution for the displacement does. It is known for isotropic materials that the solution for the stress does not depend on elastic constants if

(i) the deformation is plane strain, i.e., u_1, u_2 depend on x_1, x_2 only while u_3=0,

(ii) the boundary conditions are prescribed in terms of tractions, and

(iii) the Michell (1899) condition is satisfied, i.e., the integration of the surface tractions on any closed curve vanishes.

Condition (i) is possible for monoclinic materials with the symmetry plane at x_3=0. Only six out of thirteen elastic constants are needed for these materials under a plane strain deformation. The plane strain solution is in terms of two complex variables $z_1=x_1+p_1x_2$ and $z_2=x_1+p_2x_2$. The real and imaginary parts of p_1, p_2 account for four elastic constants. It can be shown (Ting, 1995a) that the solution for the stress does not depend on elastic constants other than p_1 and p_2 when conditions (ii) and (iii) are satisfied.

Consider now a bimaterial that consists of two dissimilar homogeneous isotropic media bonded together along their interface. The

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geometry of the interface can be rather arbitrary. There are two elastic constants each in the two materials, resulting in a total of four elastic constants. The solution for the stress depends on these four elastic constants, but they can be reduced to three by a dimensional analysis. Dundurs (1967a,b, 1970) has proved that the solution for the stress depends on two composite elastic constants (known as Dundurs constants) provided conditions (i), (ii), (iii) are satisfied. For bimaterials that consist of two dissimilar monoclinic materials with symmetry plane at $x_3=0$, Ting (1995a) has proved that the solution for the stress depends on two composite elastic constants α and β in addition to p_1 and p_2 in the two materials. They reduce to Dundurs constants in the isotropic limit.

5.11 A. N. Stroh (1926–1962)

Not all results presented in this chapter are due to Stroh. Nevertheless his 1958 and 1962 papers laid the foundations of a new formalism for two-dimensional deformations of anisotropic elastic materials. His influence reaches almost every chapter we will present in the rest of the book. A fact seldom mentioned in the literature is that Stroh, in his 1958 paper on dislocations and cracks, also presented an alternate formalism in terms of the reduced elastic compliances $s'_{\alpha\beta}$. Unknown to him at that time the alternate formalism recovered some of Lekhnitskii's results. In particular, he reproduced Lekhnitskii's sextic equation in his equation (74). He was the first person to obtain explicit expressions in terms of $s'_{\alpha\beta}$ of the matrices $\mathbf{N}_3$ of (5.5-5) and $\mathbf{N}_3^{(-1)}$ to be introduced in (6.2-4) for general anisotropic elastic materials, and the inverse of the tensor $\mathbf{L}$ of $(5.5\text{-}17)_3$ for monoclinic materials with the symmetry plane at $x_3=0$. These will be studied in Chapter 6. His 1962 paper on steady state problems to be discussed in Chapter 12 has provided great impetus to the study of surface waves in anisotropic elastic materials. In but a few paragraphs he showed that the condition for determining the speed of a subsonic surface wave is far less restrictive than originally speculated by Synge (1956). It is amazing that he accomplished so much in those two papers at the time when the formalism was still in its infancy.

The reader perhaps wants to know more about the master who has inspired so many researchers in the field. The following biography and the list of publications of Stroh are based on information provided by Professor P. Chadwick of the University of East Anglia and by the M.I.T. Museum Archives through Professor Ali S. Argon. Professor Argon was also instrumental in securing a photo of Stroh from the M.I.T. Museum with the assistance of Ms Kara Schneiderman.

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Biography of A. N. Stroh

Professor A. N. Stroh was born in Queenstown, South Africa on April 4, 1926. He attended Queens College (1934-42) and Rhodes University College (1943-47) where he was a lecturer in physics from 1947 to 1950. He received B.Sc. (1945) in mathematics and physics with distinction and M.Sc. (1947) in physics with distinction from the University of South Africa. Stroh moved to England in 1951 and did research at H. H. Wills Physical Laboratory at the University of Bristol. He was initially supervised by J. D. Eshelby and later, when Eshelby left Bristol in 1952, by N. F. (later Sir Nevill) Mott. Stroh received his Ph. D. from Bristol in 1953. In 1954-55 he was on a D.S.I.R. Senior Research Award at the Cavendish Laboratory in Cambridge. He then moved to University of Sheffield as a lecturer in Physics. During this period he became a Fellow of the Institute of Physics. Stroh came to Massachusetts Institute of Technology in 1958 as a Sloan Foreign Post-doctoral Research Fellow, and became an Associate Professor of Mechanical Engineering in 1959. In 1961-62 he was on leave from his department to do research in the Insulation Laboratory. On a rainy afternoon of Friday, September 21, 1962, on his way to a new post at the Boeing Scientific Research Laboratories in Seattle, Stroh was killed when his car skidded and went over an embankment on a highway near Steamboat Springs, Colorado. He was survived by his mother, Mrs. Iris M. Stroh of Queenstown, South Africa.

Stroh was never married. He was active in mountaineering, and was a member of the Appalachian Mountain Club. He had a brush with death in the summer of 1961 while a member of a mountain climbing team in British Columbia.

Publications of A. N. Stroh

1. J. D. Eshelby and A. N. Stroh, "Dislocations in thin plates," *Phil. Mag.* **42**, 1401-1405, 1951.
2. F. C. Frank and A. N. Stroh, "On the theory of kinking," *Proc. Phys. Soc.* B**65**, 811-821 1952.
3. A. N. Stroh, "The mean shear stress in an array of dislocations and latent hardening," *Proc. Phys. Soc.* B**66**, 2-6, 1953.
4. A. N. Stroh, "A theoretical calculation of the stored energy in a work-hardening material," *Proc. Roy. Soc. Lond.* A**218**, 391-400, 1953.
5. A. N. Stroh, "Constrictions and jogs in extended dislocations," *Proc. Phys. Soc.* B**67**, 427-436, 1954.
6. A. N. Stroh, "The formation of cracks as a result of plastic flow," *Proc. Roy. Soc. Lond.* A**223**, 404-414, 1954.

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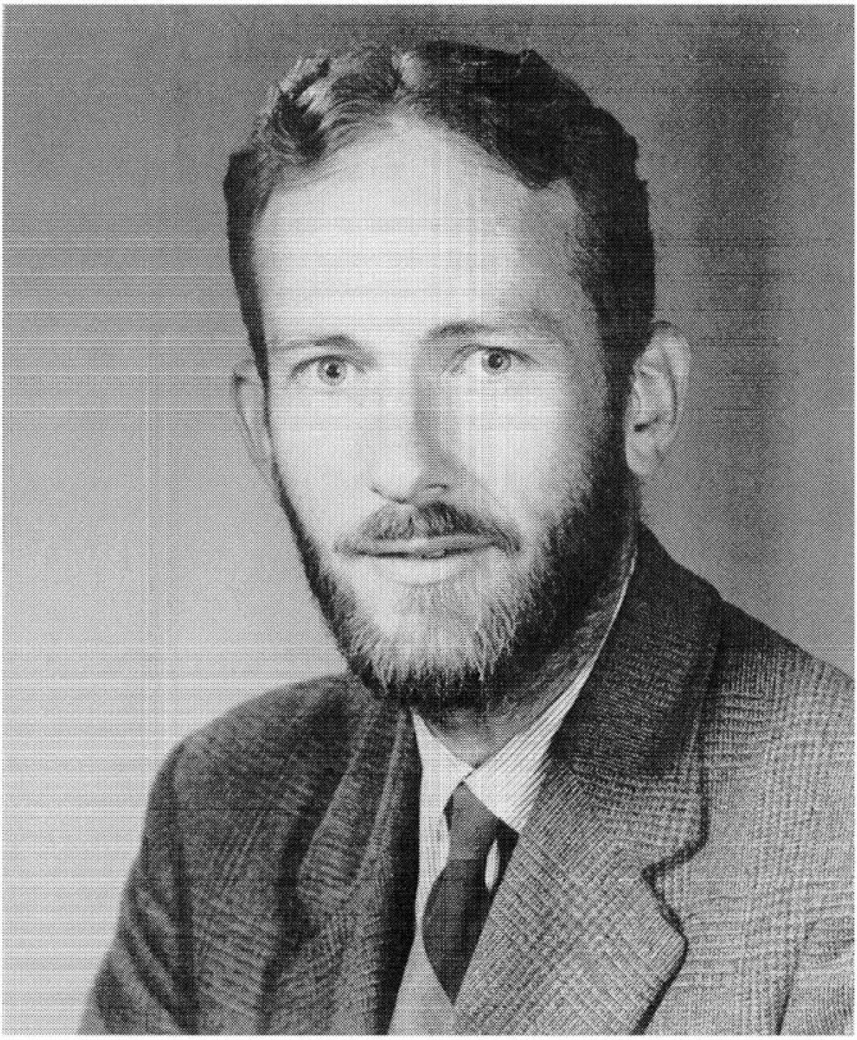

Alan Neil Stroh (1926–1962)

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7. A. N. Stroh, "The formation of cracks in plastic flow. II," *Proc. Roy. Soc. Lond.* A**232**, 548-560, 1955.
8. A. N. Stroh, "Brittle fracture and yielding," *Phil. Mag.* **46**, 968-972, 1955.
9. A. N. Stroh, "The strength of Lomer–Cottrell sessile dislocations," *Phil. Mag.* **1**, 489-502, 1956.
10. A. N. Stroh, "The existence of microcracks after cold-work," *Phil. Mag.* **2**, 1-4, 1957.
11. B. A. Bilby, L. R. T. Gardner, and A. N. Stroh, "Continuous distributions of dislocations and the theory of plasticity," *Proc. 9th Int. Congress Appl. Mech.*, VIII, 35-44, 1957.
12. A. N. Stroh, "A theory of the fracture of metals," *Adv. Physics* **6**, 418-465, 1957.
13. A. N. Stroh, "The cleavage of metal single crystals," *Phil. Mag.* **3**, 597-606, 1958.
14. A. N. Stroh, "Dislocations and cracks in anisotropic elasticity," *Phil. Mag.* **3**, 625-646, 1958.
15. A. N. Stroh, "Crack nucleation in body-centered cubic metals," in *Fracture*, ed. by B. L. Averbach, D. K. Felbeck, G. T. Hahn, and D. A. Thomas, The Technology Press of MIT and Wiley, New York, 117-122, 1959.
16. A. N. Stroh, "A simple model of a propagating crack," *J. Mech. Phys. Solids* **8**, 119-122, 1960.
17. A. N. Stroh, "The deformation of a tilt boundary under applied forces," *Acta Metallurgica* **9**, 315-319, 1961.
18. A. N. Stroh, "Steady state problems in anisotropic elasticity," *J. Math. Phys.* **41**, 77-103, 1962.
19. A. N. Stroh, "Force on a moving dislocation," *Phys. Rev.* **128**, 55-61, 1962.

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