In [2]:

```
# getting the library that has some of the functions needed for simulations
import numpy as np
import matplotlib.pyplot as plt
from numba import jit
import time
from numba import njit, prange
import pandas as pd

%matplotlib inline
```

Functions defined earlier

In [3]:

```
@jit(nopython=True)
def cdf_inv_fr(u, gamma):
  return ((pow(-np.log(u) , -gamma)))

@jit(nopython=True)
def cdf_inv_H(u , gamma, beta):
  return (pow(-np.log(u)/(pow(beta , -1/gamma)-1) , -gamma) )

@jit(nopython=True)
def armax(beta , gamma, n):
  x = np.zeros(n)
  #r = np.random.RandomState(random_state)
  u = np.random.uniform(0,1,1)[0]
  x0 = cdf_inv_fr(u,gamma)
  xi_lag = x0
  x[0] = x0
  #print(x0)
  t = 1
  for i in range(n-1):
    #r2 = np.random.RandomState(random_state + i)
    u = np.random.uniform(0,1,1)[0]
    zi = cdf_inv_H(u,gamma,beta)
    xi = beta*max(xi_lag , zi)
    xi_lag = xi
    x[t] = xi
    t = t + 1
    #print(zi)
  return x
```

In [4]:

```
@jit(nopython=True)
def theta_n_k(X, n, k_=1):
  sum = 0
  k = int(k_)

  X_k = np.partition(X, n-k-1)[n-k-1]
  #if k <= 1:
  # return 1

  #X_k = max(X[n-k:n])

  for j in range(n-1):
    # k-th top order equals n-k low order
    if X[j] <= X_k and X[j+1] > X_k:
      sum += 1

  if k == 0:
    return 1

  #if sum/k >= 1:
```

```
    #  return 1
    return sum/k


@jit(nopython=True)
def theta_GJ_k(X, n, k, delta):
    #n = X.shape[0]

    numerator = (delta*delta + 1) * theta_n_k(X,n, int(np.floor(delta*k)) + 1) - delta*(th
eta_n_k(X, n, int(np.floor(delta*delta*k)) + 1) + theta_n_k(X, n, k))
    denominator = (1 - delta)**2

    if numerator < 0:
        return 0

    #return (numerator/denominator if numerator/denominator <= 1 else 1)
    return numerator/denominator
```

In [5]:

```
@jit(nopython = True)
def np_apply_along_axis(func1d, axis, arr):
    assert arr.ndim == 2
    assert axis in [0, 1]
    if axis == 0:
        result = np.empty(arr.shape[1])
        for i in range(len(result)):
            result[i] = func1d(arr[:, i])
    else:
        result = np.empty(arr.shape[0])
        for i in range(len(result)):
            result[i] = func1d(arr[i, :])
    return result
```

In [6]:

```
@jit(nopython=True)
def simulate_mean_mse(n, theta, runs = 15, replicates = 10):
    all_values_mean = np.zeros(n)
    all_values_mse = np.zeros(n)

    #print("Theta_N", end = "")
    for run in range(runs):
        #print("\nrun = ", run + 1, "/", runs ,"...", sep = "", end = " ")
        a = np.zeros((replicates, n))
        for i in range(replicates):
            a[i] = armax(1-theta, 1, n)

        path = np.zeros((replicates, n))
        path2 = np.zeros((replicates, n))

        k_range = list(range(n))

        for k in k_range:
            #if k%100 == 0:
            #  print(int(k/100) + 1, end = " ")
            for j in range(replicates):
                path[j][k] = theta_n_k(a[j], n, k)
                path2[j][k] = (path[j][k] - theta)**2

        #one_run_mean = np.mean(path, axis = 0)

        all_values_mean += np_apply_along_axis(np.mean, 0, path)
        all_values_mse += np_apply_along_axis(np.mean, 0, path2)

    return all_values_mean/runs, all_values_mse/runs
```

In [7]:

```
@jit(nopython=True)
def simulate_mean_mse_GJ(n, theta, delta = 0.25, runs = 15, replicates = 10):
```

```
  all_values_mean = np.zeros(n)
  all_values_mse = np.zeros(n)

  #print("Theta_N", end = "")
  for run in range(runs):
    #print("\nrun = ", run + 1, "/", runs ,"...", sep = "", end = " ")
    a = np.zeros((replicates, n))
    for i in range(replicates):
      a[i] = armax(1-theta, 1, n)

    path = np.zeros((replicates, n))
    path2 = np.zeros((replicates, n))

    k_range = list(range(n))

    for k in k_range:
      #if k%100 == 0:
      #  print(int(k/100) + 1, end = " ")
      for j in range(replicates):
        path[j][k] = theta_GJ_k(a[j], n, k, delta)
        path2[j][k] = (path[j][k] - theta)**2

    #one_run_mean = np.mean(path, axis = 0)

  all_values_mean += np_apply_along_axis(np.mean, 0, path)
  all_values_mse += np_apply_along_axis(np.mean, 0, path2)

  return all_values_mean/runs, all_values_mse/runs
```

More information about functions in the file for fig 11

In [8]:

```
@njit
def subsampling_GJ(X, n, theta, T = 2, delta = 0.25):
  r = int(np.floor(n/T))

  theta_Vi = np.zeros((T, r))

  #X = armax(1- theta, 1, n)

  for i in range(T):
    V = X[i:(r-1)*T+i+1:T]

    for j in range(1, r):
      theta_Vi[i][j] = theta_GJ_k(V, V.shape[0], j, delta)

  theta_sub = -1 * np.ones(n)

  for j in range(1, r):
    #temp = 1 - np.power(abs(1 - theta_Vi[:,j]), 1/T).mean(axis = 0)
    theta_sub[(j - 1)*T + 1] = 1 - ((np.power(1 - theta_Vi[:,j], 1/T)).mean())
    theta_sub[np.isnan(theta_sub)] = 0

  for k in range(1, n):
    if theta_sub[k] == -1:
      theta_sub[k] = theta_sub[k-1]

  return theta_sub
```

In [9]:

```
@njit
def simulate_mean_mse_GJ_sub(n, T = 2, theta = 0.2, delta = 0.25, replicates = 10, runs
= 15):
  all_values_mean = np.zeros(n)
  all_values_mse = np.zeros(n)

  for run in range(runs):
    #print("\nrun = ", run + 1, "/", runs ,"...", sep = "", end = " ")
```

```
    a = np.zeros((replicates, n))
    for i in range(replicates):
      a[i] = armax(1-theta, 1, n)

    path = np.zeros((replicates, n))
    path2 = np.zeros((replicates, n))

    k_range = list(range(n))


    for j in range(replicates):
      path[j] = subsampling_GJ(a[j], n, theta, T, delta)
      path2[j] = np.power((path[j] - theta), 2)

    #one_run_mean = np.mean(path, axis = 0)

    all_values_mean += np_apply_along_axis(np.mean, 0, path)
    all_values_mse += np_apply_along_axis(np.mean, 0, path2)

  return all_values_mean/runs, all_values_mse/runs
```

In [10]:

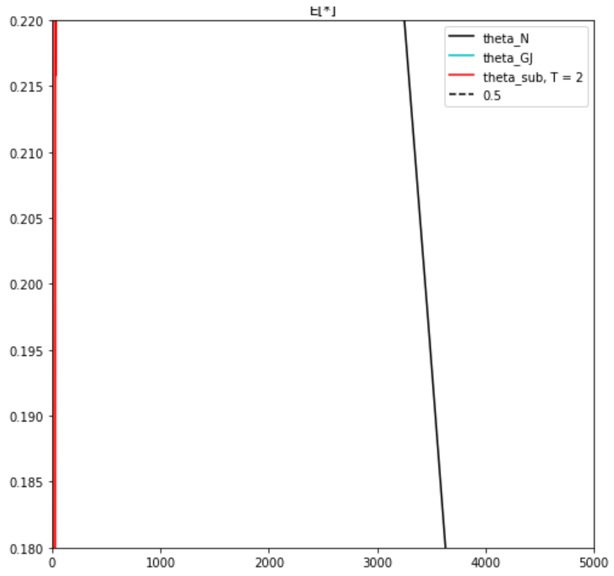
```
%%time
# calculating
runs = 100
replicates = 10
n = 5000
theta = 0.5
delta = 0.25
e_1000_5, mse_1000_5 = simulate_mean_mse(n, theta, runs = runs, replicates = replicates)
e_1000_5_1, mse_1000_5_1 = simulate_mean_mse_GJ(n, theta, delta, runs = runs, replicates
= replicates)
e_1000_5_2, mse_1000_5_2 = simulate_mean_mse_GJ_sub(n, 2, theta, delta, replicates, runs
)
```

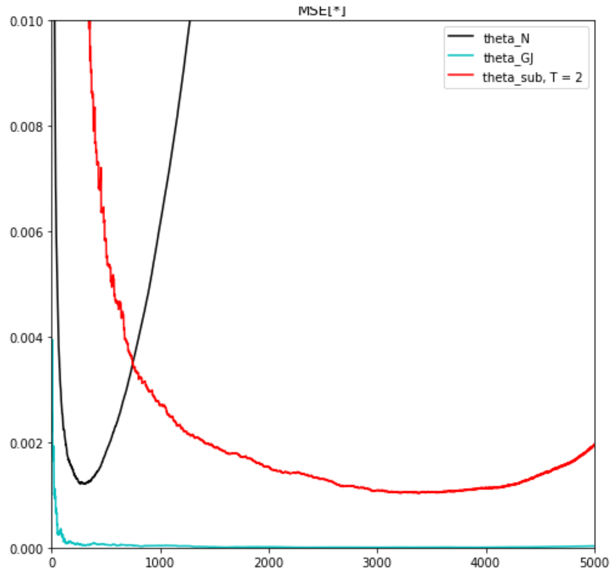
```
CPU times: user 35min 55s, sys: 2.71 s, total: 35min 57s
Wall time: 35min 56s
```

In [11]:

```
# plotting
alph = 0.5
straight_line = theta*np.ones(n)
k_range = np.arange(n)
fig, axs = plt.subplots(1, 2, figsize = (20,8))

fig.suptitle("Figure 10: Effect of subsampling in E[*] and MSE[*] for a Fretchet sequence
(gamma = 1) and theta = " + str(theta) + " with subsampling")
axs[0].set_ylim([0.18, 0.22])
axs[0].set_xlim([0,n])
axs[0].set_title("E[*]")
axs[0].plot(k_range, e_1000_5, 'k-', label = 'theta_N')
axs[0].plot(k_range, e_1000_5_1, 'c-', label = 'theta_GJ')
axs[0].plot(k_range, e_1000_5_2, 'r-', label = 'theta_sub, T = 2')
axs[0].plot(k_range, straight_line, 'k--', label = str(theta))
axs[0].legend(bbox_to_anchor=(1, 1))


axs[1].set_ylim([0, 0.01])
axs[1].set_xlim([0,n])
axs[1].set_title("MSE[*]")
axs[1].plot(k_range, mse_1000_5, 'k-', label = 'theta_N')
axs[1].plot(k_range, mse_1000_5_1, 'c-', label = 'theta_GJ')
axs[1].plot(k_range, mse_1000_5_2, 'r-', label = 'theta_sub, T = 2')

axs[1].legend(bbox_to_anchor=(1, 1))

fig.subplots_adjust(wspace=0.5)
```

Figure 10: Effect of subsampling in E[*] and MSE[*] for a Fretchet sequence (gamma = 1) and theta = 0.5 with subsampling


E[*]
theta_N
theta_GJ
theta_sub, T = 2
0.5
0.220
0.215
0.210
0.205
0.200
0.195
0.190
0.185
0.180
0
1000
2000
3000
4000
5000


MSE[*]
theta_N
theta_GJ
theta_sub, T = 2
0.010
0.008
0.006
0.004
0.002
0.000
0
1000
2000
3000
4000
5000

In [11]: