In [3]:

```
# getting the library that has some of the functions needed for simulations
import numpy as np
import matplotlib.pyplot as plt
from numba import jit
import time
from numba import njit, prange
import pandas as pd

%matplotlib inline
```

In [4]:

```
@jit(nopython=True)
def cdf_inv_fr(u, gamma):
  return ((pow(-np.log(u) , -gamma)))

@jit(nopython=True)
def cdf_inv_H(u , gamma, beta):
  return (pow(-np.log(u)/(pow(beta , -1/gamma)-1) , -gamma) )

@jit(nopython=True)
def armax(beta , gamma, n):
  x = np.zeros(n)
  #r = np.random.RandomState(random_state)
  u = np.random.uniform(0,1,1)[0]
  x0 = cdf_inv_fr(u,gamma)
  xi_lag = x0
  x[0] = x0
  #print(x0)
  t = 1
  for i in range(n-1):
    #r2 = np.random.RandomState(random_state + i)
    u = np.random.uniform(0,1,1)[0]
    zi = cdf_inv_H(u,gamma,beta)
    xi = beta*max(xi_lag , zi)
    xi_lag = xi
    x[t] = xi
    t = t + 1
    #print(zi)
  return x
```

In [5]:

```
@jit(nopython=True)
def theta_n_k(X, n, k_=1):
  sum = 0
  k = int(k_)

  X_k = np.partition(X, n-k-1)[n-k-1]
  #if k <= 1:
  # return 1

  #X_k = max(X[n-k:n])

  for j in range(n-1):
    # k-th top order equals n-k low order
    if X[j] <= X_k and X[j+1] > X_k:
      sum += 1

  if k == 0:
    return 1

  #if sum/k >= 1:
  #  return 1
  return sum/k


@jit(nopython=True)
def theta_GJ_k(X, n, k, delta):
  #n = X.shape[0]

  numerator = (delta*delta + 1) * theta_n_k(X,n, int(np.floor(delta*k)) + 1) - delta*(theta_n_k
```

```python
    denominator = (1 - delta)**2

    if numerator < 0:
      return 0

    #return (numerator/denominator if numerator/denominator <= 1 else 1)
    return numerator/denominator
```

In [6]:
```python
@jit(nopython = True)
def np_apply_along_axis(func1d, axis, arr):
  assert arr.ndim == 2
  assert axis in [0, 1]
  if axis == 0:
    result = np.empty(arr.shape[1])
    for i in range(len(result)):
      result[i] = func1d(arr[:, i])
  else:
    result = np.empty(arr.shape[0])
    for i in range(len(result)):
      result[i] = func1d(arr[i, :])
  return result
```

In [7]:
```python
@jit(nopython=True)
def simulate_mean_mse(n, theta, runs = 15, replicates = 10):
  all_values_mean = np.zeros(n)
  all_values_mse = np.zeros(n)

  #print("Theta_N", end = "")
  for run in range(runs):
    #print("\nrun = ", run + 1, "/", runs ,"...", sep = "", end = " ")
    a = np.zeros((replicates, n))
    for i in range(replicates):
      a[i] = armax(1-theta, 1, n)

    path = np.zeros((replicates, n))
    path2 = np.zeros((replicates, n))

    k_range = list(range(n))

    for k in k_range:
      #if k%100 == 0:
      #  print(int(k/100) + 1, end = " ")
      for j in range(replicates):
        path[j][k] = theta_n_k(a[j], n, k)
        path2[j][k] = (path[j][k] - theta)**2

    #one_run_mean = np.mean(path, axis = 0)

    all_values_mean += np_apply_along_axis(np.mean, 0, path)
    all_values_mse += np_apply_along_axis(np.mean, 0, path2)

  return all_values_mean/runs, all_values_mse/runs
```

In [8]:
```python
@jit(nopython=True)
def simulate_mean_mse_GJ(n, theta, delta = 0.25, runs = 15, replicates = 10):
  all_values_mean = np.zeros(n)
  all_values_mse = np.zeros(n)

  #print("Theta_N", end = "")
  for run in range(runs):
    #print("\nrun = ", run + 1, "/", runs ,"...", sep = "", end = " ")
    a = np.zeros((replicates, n))
    for i in range(replicates):
      a[i] = armax(1-theta, 1, n)

    path = np.zeros((replicates, n))
    path2 = np.zeros((replicates, n))
```

```python
    k_range = list(range(n))

    for k in k_range:
      #if k%100 == 0:
      #  print(int(k/100) + 1, end = " ")
      for j in range(replicates):
        path[j][k] = theta_GJ_k(a[j], n, k, delta)
        path2[j][k] = (path[j][k] - theta)**2

    #one_run_mean = np.mean(path, axis = 0)

  all_values_mean += np_apply_along_axis(np.mean, 0, path)
  all_values_mse += np_apply_along_axis(np.mean, 0, path2)

  return all_values_mean/runs, all_values_mse/runs
```

In [9]:

```python
@njit
def subsampling_GJ(X, n, theta, T = 2, delta = 0.25):
  r = int(np.floor(n/T))

  theta_Vi = np.zeros((T, r))

  #X = armax(1- theta, 1, n)

  for i in range(T):
    V = X[i:(r-1)*T+i+1:T]

    for j in range(1, r):
      theta_Vi[i][j] = theta_GJ_k(V, V.shape[0], j, delta)

  theta_sub = -1 * np.ones(n)

  for j in range(1, r):
    #temp = 1 - np.power(abs(1 - theta_Vi[:,j]), 1/T).mean(axis = 0)
    theta_sub[(j - 1)*T + 1] = 1 - ((np.power(1 - theta_Vi[:,j], 1/T)).mean())
    theta_sub[np.isnan(theta_sub)] = 0

  for k in range(1, n):
    if theta_sub[k] == -1:
      theta_sub[k] = theta_sub[k-1]

  return theta_sub
```

In [10]:

```python
@njit
def simulate_mean_mse_GJ_sub(n, T = 2, theta = 0.2, delta = 0.25, replicates = 10, runs = 15):
  all_values_mean = np.zeros(n)
  all_values_mse = np.zeros(n)

  for run in range(runs):
    #print("\nrun = ", run + 1, "/", runs ,"...", sep = "", end = " ")
    a = np.zeros((replicates, n))
    for i in range(replicates):
      a[i] = armax(1-theta, 1, n)

    path = np.zeros((replicates, n))
    path2 = np.zeros((replicates, n))

    k_range = list(range(n))


    for j in range(replicates):
      path[j] = subsampling_GJ(a[j], n, theta, T, delta)
      path2[j] = np.power((path[j] - theta), 2)

    #one_run_mean = np.mean(path, axis = 0)

    all_values_mean += np_apply_along_axis(np.mean, 0, path)
    all_values_mse += np_apply_along_axis(np.mean, 0, path2)
```

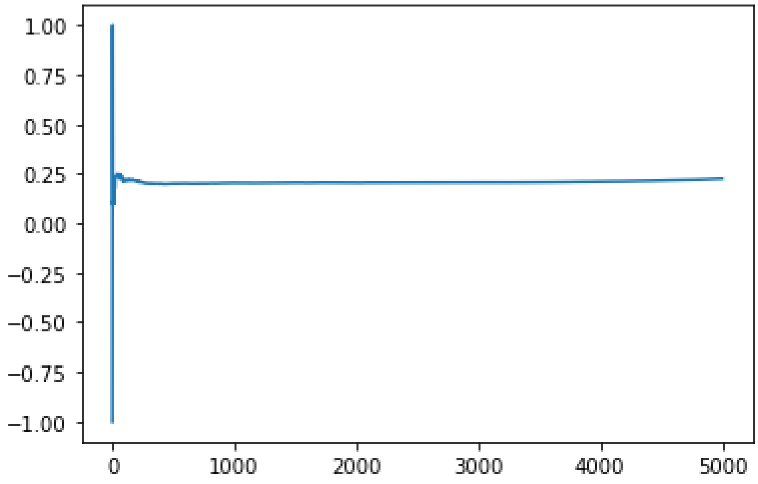
```
    return all_values_mean/runs, all_values_mse/runs
```

In [10]:
```
%%time
e, m = simulate_mean_mse_GJ_sub(5000)
```

```
Wall time: 2min 26s
```

In [11]:
```
plt.plot(e)
```

Out[11]:
```
[<matplotlib.lines.Line2D at 0x23153234288>]
```

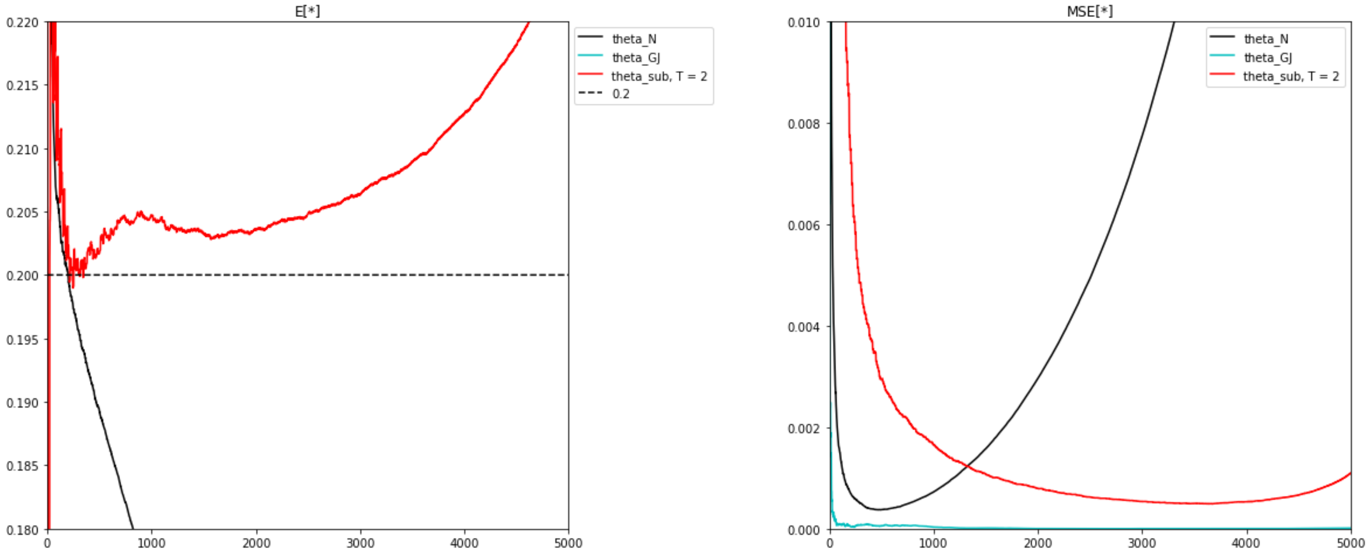


In [11]:
```
%%time
runs = 100
replicates = 10
n = 5000
theta = 0.2
delta = 0.25
e_1000_5, mse_1000_5 = simulate_mean_mse(n, theta, runs = runs, replicates = replicates)
e_1000_5_1, mse_1000_5_1 = simulate_mean_mse_GJ(n, theta, delta, runs = runs, replicates = repl
e_1000_5_2, mse_1000_5_2 = simulate_mean_mse_GJ_sub(n, 2, theta, delta, replicates, runs)
```

```
Wall time: 1h 52s
```

In [17]:
```
# plotting
alph = 0.5
straight_line = theta*np.ones(n)
k_range = np.arange(n)
fig, axs = plt.subplots(1, 2, figsize = (20,8))

fig.suptitle("Figure 10: Effect of subsampling in E[*] and MSE[*] for a Fretchet sequence (gamm
axs[0].set_ylim([0.18, 0.22])
axs[0].set_xlim([0,n])
axs[0].set_title("E[*]")
axs[0].plot(k_range, e_1000_5, 'k-', label = 'theta_N')
axs[0].plot(k_range, e_1000_5_1, 'c-', label = 'theta_GJ')
axs[0].plot(k_range, e_1000_5_2, 'r-', label = 'theta_sub, T = 2')
axs[0].plot(k_range, straight_line, 'k--', label = str(theta))
axs[0].legend(bbox_to_anchor=(1, 1))


axs[1].set_ylim([0, 0.01])
axs[1].set_xlim([0,n])
axs[1].set_title("MSE[*]")
axs[1].plot(k_range, mse_1000_5, 'k-', label = 'theta_N')
axs[1].plot(k_range, mse_1000_5_1, 'c-', label = 'theta_GJ')
axs[1].plot(k_range, mse_1000_5_2, 'r-', label = 'theta_sub, T = 2')

axs[1].legend(bbox_to_anchor=(1, 1))

fig.subplots_adjust(wspace=0.5)
```

Figure 10: Effect of subsampling in E[*] and MSE[*] for a Fretchet sequence (gamma = 1) and theta = 0.2 with subsampling



In [13]:
```
e_1000_5_1
```

Out[13]:
```
array([0.01      , 0.01      , 0.01177778, ..., 0.00229393, 0.00229455,
       0.00229527])
```

In [ ]: