An agent based order book simulation of an asset price.

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Abstract

In the present work we give an overview of an algorithm for simulating a limit order book for an asset based on a simulation of a collection of descision making autonomous agents.

1 Introduction

Agent based modeling an an approach to modeling complex systems composed of a multitude of interacting elements by simulating the behavior of each element of the system (an agent). It has it's roots in physical simulations of systems with large number of constituents in cases when the approximations of mean field theory cannot be applied. Since then, the approach has been applied heavily to the simulation of economical systems, as they are the result of the interaction of many humans, each solving an optimization problem. It differs from top-down simulation methods that try to establish some global behavior of the system and simulate it without taking into consideration the inner workings of the system. While in general, an agent based model is more computationally expensive to simulate, it's interesting not only from the perspective of simulating the behavior of the system, but also in examining the behavior of the each agent in the system and the effects if small changes of agent behavior on the global behavior. In many cases, the interaction of a large number of even very simple agents can lead to complex behavior on the system level.

This is what we try to reproduce in the present work. We simulate a population of traders that make decisions to place a buy or sell order or not to act based on stochastic external data. These orders are tracked in an order book and fulfilled if there is a matching order of the opposite type. The agents do not actually solve an optimization problem, neither do they keep track of their fulfilled orders. In this sense, they act as very simple signal processors that translate a time series of external data into a time series of buy and sell orders. We keep track of track of the price of the asset in two ways: the price of the last fulfilled sell order and the price of the current open sell order. We try to reproduce some stylized fact about real world asset prices by simulating this population of traders: mainly heavy-tailed returns. We also examine the resulting time series graphically to asses the appearance of any patterns that are characteristic to real world security prices.

The article is structured as follows: in section 2 we give a definition of an order book that we will use in our simulation.

2 Limit order book

First we define a buy (sell) limit order as an order to buy (sell) an asset at or below (above) a certain price. A limit order book is a collection of such limit orders. In such a collection the highest price of a buy order (the bid price) must always be lower than then lowest price of a sell order (the ask price). This is due to the fact that any buy order at a price higher than the ask price can be fulfilled immediately and the same applies to new sell orders. If this is not the case due to errors in bookkeeping, the offending order is said to be 'crossing the book'. The difference between the bid and the ask price is called the spread.

In our implementation, we keep the limit order book in two separate heaps - for the sell and for the buy orders. A heap is a fitting data structure for an order book because it allows for fast retrieval of the largest or smallest element in the heap. When a new buy (sell) order comes in, we check it against the lowest (highest) price in the sell (buy) heap. If the new order price is higher (lower), we fulfill the new order and pop the lowest (highest) order from the heap.

3 Agent based order book simulation

Our model consists of a population of n agents $A = \{a_i\}_{i=1}^{n}$ that interact with the limit order book by placing buy and sell orders in it at discrete time steps, which we index with $t \in \mathbb{Z}$. At each time step the agent a_i decides whether to place an order and the type of the order.

We define an order as a 2-tuple (o, p), where $p \in \mathbb{R}$ is the price of the order and

$$o \in \{-1, 1\}$$

is the type of the order: if $o = 1$ we have a buy order, and if $o = -1$ - a sell order.

The decision of an agent of whether or not to place an order at time t and the type of the order can be represented by defining an agent as two functions $a_i = (o_i, p_i)$, where:

$$\begin{aligned} o_i &: \mathbb{Z} \to \{-1, 0, 1\} \\ & t \mapsto o_{it} \\ p_i &: \tilde{\mathbb{Z}} \to \mathbb{R} \\ & t \mapsto p_{it}, \end{aligned}$$

where we have defined $\tilde{\mathbb{Z}} \subset \mathbb{Z}$ as the subset $\{t : o_{it} \neq 0\}$.

o_i is the *order function* of the agent. If $o_{it} = 0$, the agent a_i chooses not to act at time t. If $o_{it} = 1$ (resp. $o_{it} = -1$), the agent a_i chooses to place a buy (resp. sell) order at time t.

p_i is the *price function* of the agent. It is only defined for values of (i, t) for which and agent decides to place an order and it's value p_{it} is the price of the limit order of the agent a_i at time t.

In this way, we have defined an agent in the most general way possible. Each agent can take input of any form at time t and apply any internal decision making process on this input. Whatever this inner workings of the agents might be, their input in the form (o_{it}, p_{it}) can be processed by the limit order book as described in the previous section. In the next section we will describe a simple noise following agent that generated limit orders based on a white noise process as input

4 Noise agent model

The definition of our agents is an extension of the definition of a noise agent in [1]. There the authors examine the interaction of a three types of agents - a noise agent, a fundamental agent and a technical agent. For our simple model we implement only the noise agent.

A noise agent is one that makes decisions on their trading strategy based only of external noise. They don't track their fulfilled buy and sell orders and don't make any effort to make profits. In this sense they are noise agents - in a system containing agents that try to optimize their profits, the noise agents are the one generating noise in the system.

In [1], the agents interact by outputting only an order type $o_{it} \in \{-1, 1\}$ at each time step, the sum of which over i gives the excess demand $D(t)$ at time t. This then affects the market price P_t of the asset by the relation:

$$R_t = \arctan \frac{D(t)}{n\lambda}, \tag{1}$$

where r_t is the return of the asset:

$$R_t = \frac{P_t}{P_{t-1}} - 1 \tag{2}$$

and λ is the market depth. In this framework, the agents place either a buy or sell market order at each time step. The parameter λ controls how much of the market is represented by the agents in the model and thus the impact of the agents on the price.

We extend this model by simulating an order book. In our model the agents represent the whole market and they place limit orders instead of market orders. Of course a limit order can in effect be the same as a market, if it's a buy (resp. sell) order at price higher (lower) than the current ask (bid) price. Due to the added complexity that an order book structure brings to the model, as a first iteration we only simulate a population of noise agents.

The noise agent is defined as follows: Let there be a Gaussian white noise process:

$$I_t \sim \mathcal{N}(0, \sigma_I^2); \quad I_{t_1} \perp\!\!\!\perp I_{t_2}, \forall t_1 \neq t_2. \tag{3}$$

This process represents an external signal of information about the asset we are simulating. It might be any news that might influence the perception of an agent about the value of the asset.

Each of the noise agents a_i has three parameters (c_i, r_i, η_i). These have the following meaning:

- $c_i \in \mathbb{R}$ is the *personal sentiment* of the agent towards the asset. The value of this parameter will be added to I_t at each time step to determine the *personal view* $I_t + c_i$ of the agent towards the asset at this time.

- $r_i \in \mathbb{R}$ is the *reaction parameter* of the agent. It is used to determine the price at which an agent will place an order if they decide to do so. If $r_i >= 0$, the agent will place market orders as limit orders that can fulfilled immediately. If $r_i < 0$, the agent will place limit orders, with larger in magnitude values of r_i leading to order at prices further apart from the current market price.

- $\eta_i \in [0, 1]$ is the *order threshold decay* parameter of the agent. A large value for this parameter will make the agent place orders further apart on average.

Whether an agent a_i places an order at time t is tracked by an internal threshold τ_{it}. If the absolute value of the personal view of the agent is larger than on equal to this threshold, the agent places an order, the type of which depends on the sign of the personal view. This decision making process is encoded in the order function of the agent:

$$o_{it} = \begin{cases} 1, & \tau_{it} \geq I_t + c_i \\ 0, & -\tau_{it} < I_t + x_i < \tau_{it} \\ -1, & I_t + c_i \geq \tau_{it} \end{cases} \tag{4}$$

Initially, the thresholds of all agents are set to 0, meaning that at the first time step all agents place an order. After an agent places an order, their threshold is set to a value of two times the variance σ_I^2 of the information process I_t. If the agent did not place an order at the previous time step, their threshold declines proportionally to the inverse of their order period T_i:

$$\tau_{it} = \begin{cases} 2\sigma_I^2, & o_{i(t-1)} \neq 0 \\ \eta_i \tau_{i(t-1)}, & o_{i(t-1)} = 0 \end{cases} \tag{5}$$

If an agent a_i decides to place an order at time t, the price of the order is determined by the price function of the agent:

$$p_{it} = P_{t-1} + r_i(I_t + c_i). \tag{6}$$

We can see from this formula the meaning of the reaction parameter r_i. The type of the order depends on the sign of the personal sentiment $I_t + c_i$. If it is positive, the agent places a buy order. If $r_i >= 0$, the price of the buy order will be larger than or equal to the current market price P_{t-1}. Such an order can be fulfilled immediately. If $r_i < 0$, the agent will place a limit buy order at a price lower than the current market price. This order will be logged into the order book and will wait for a corresponding sell order to be fulfilled. The same logic applies in the case of a sell order.

5 Implementation

An algorithm for simulating a population of the above defined agents is implemented in the GitHub repository `https://github.com/PetarChernev/agent-based-modeling`. A run of the simulation can be run by executing the `main.py` python file after installing the dependencies. An example of a run can be seen in the figures below.

In this file we first define the distributions of agent parameters, define each agent by drawing samples of those distributions and then simulate the market for a set number of time steps. Each time step consists of calling the decision making function for each agent consecutively and processing their order through the order book. The price P_t is to to the price of the last fulfilled order for each time step. If no orders were fulfilled in a time step, the price is set to it's previous value.

We are interested in seeing the effects of different distributions of parameters on the behavior of the system. We also try to reproduce some stylized facts about real world asset prices by choosing specific distributions of parameters. In the next section we lay out the most interesting results after running the simulation many times for different parameters.

6 Choosing the parameters

In this section we describe different configurations of agent parameters, our reasoning for choosing them and the results of the simulation with those parameters. For quick iteration over the different options for the parameters, we examine the results of the simulation graphically, by plotting the price time series and the number of buy and sell orders for each period, as well as a histogram of the distribution of returns.

1. We first initialize a population of $n = 100$ identical traders with parameters:

$$c_i = 0, r_i = 1, \eta_i = 0.8, \forall i = 1, \dots, n \tag{7}$$

   We simulate a $T = 1000$ time steps. The variance of the information noise process is set to $\sigma_I^2 = 1$. The results are show in Fig. 1. The price seems correlated to the number of unfulfilled buy orders and anticorrelated to the number of unfulfilled sell orders, which is what we expect. It would seem that we are successful in modeling the supply and demand for the asset through our order book model and their effect on the price of the asset.

   As we can see, both buy and sell orders change with steps. This is due to the fact that all agents make the same decisions at each time step and so either there are no new orders placed, or there are exactly 100 new buy or sell orders. Due to this homogeneity of the system, there are a lot of time steps (78% of all time steps) where the price doesn't change. We don't even bother to examine the distribution of returns for this system and move on to trying to improve the result by introducing some variability in the agents.

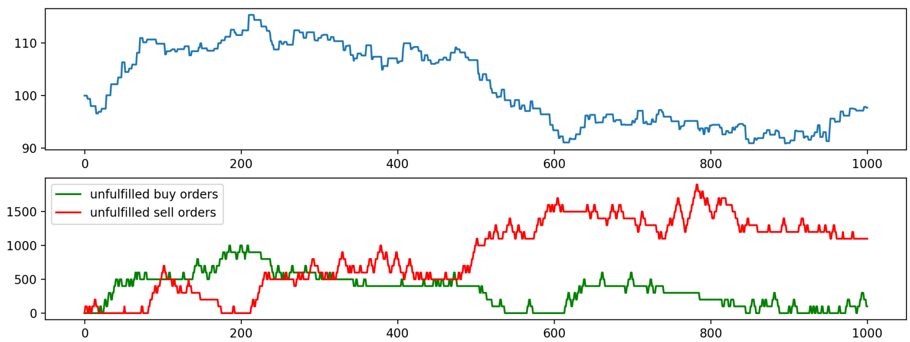

Figure 1: Strike price and number of unfulfilled buy and sell orders for 100 identical noise agents with parameters $c_i = 0, r_i = 1, \eta_i = 0.8$.

2. As a next step, we make the personal biases c_i of each agent i.i.d normally distributed random variables:

$$c_i \sim \mathcal{N}(0, \sigma_c^2), c_i \perp\!\!\!\perp c_j, \forall i \neq j \tag{8}$$

   We choose $\sigma_c^2 = 0.5$. The personal biases have half the variance of the information noise process $\sigma_I^2 = 1$, so the system should still be governed mainly by the white noise, but there should be some variability in the decisions made by each agent. The results of the simulation are shown in Fig. 2.

   As we can see, there is an increase in volatility in both the price and the numbers of unfulfilled orders. This is due to the variability of the personal sentiment of the traders c_i. Each trader makes a decision whether to buy or sell based on the value of the sum of the information noise and their personal sentiment $I_t + c_i$. This allows for a situation where some traders interpret the news at time t positively and place buy orders, while other view them negatively and place sell orders.

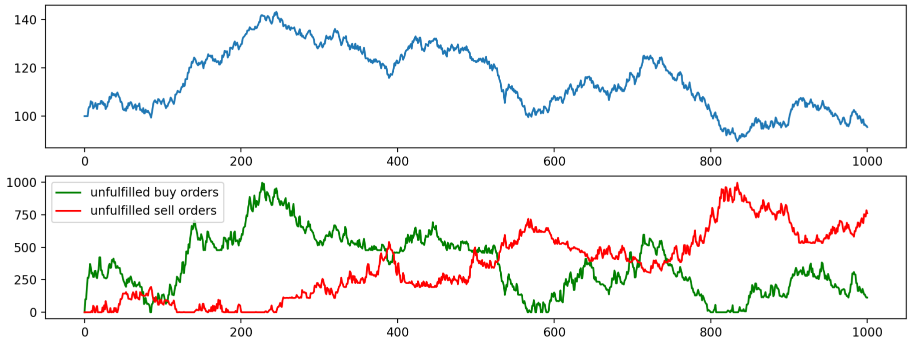

Figure 2: Strike price and number of unfulfilled buy and sell orders for 100 identical noise agents with parameters $c_i \sim \mathcal{N}(0, 0.5), r_i = 1, \eta_i = 0.8$.

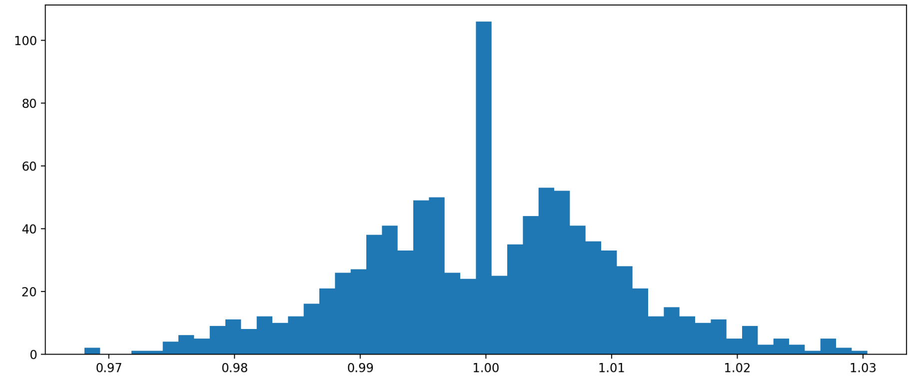

Figure 3: Histogram of the distribution of returns for 100 identical noise agents with parameters $c_i \sim \mathcal{N}(0, 0.5), r_i = 1, \eta_i = 0.8$..

We can see graphically that the prices are correlated with the number of unfulfilled buy orders and inversely correlated with the number of sell orders. Indeed, the correlation coefficients for this run are:

$$\mathrm{cor}(P_t, b_t) = 0.939,$$
$$\mathrm{cor}(P_t, s_t) = -0.683.$$

The correlation coefficient between the numbers of buy orders is:

$$\mathrm{cor}(s_t, b_t) = -0.684.$$

In Fig. 3, we can see a histogram of the distribution of returns for this model. This distribution takes an unexpected bimodal form with an additional concentration of returns at 0. In fact, a large number (8.2%) of the returns, which are exactly 0. There is a drop in the number of returns for values close to zero and then an increase as the absolute value of the returns grows, until a stationary point is reached and the distribution takes a normal looking shape. While a deeper analysis is needed to fully describe this behavior, an initial explanation for it is due to the way that agents decide the price at which to place orders. An agent places an order at price

$$\begin{aligned} p_{it} &= s_{t-1} + r_i(I_t + c_i) \\ &= s_{t-1} + I_t + c_i \end{aligned}$$

only if the absolute value of the news plus their personal sentiment $I_t + c_i$ is larger than their threshold τ_{it}. This threshold decreases with time and is reset to σ_I^2 each time the trader places an order. This makes placing orders at small values of $I_t + c_i$ less likely, as is requires that the absolute value of $I_t + c_i$ was not higher than the threshold for all previous time steps since the last order. This might explain the drop in probability for very small changes in the price of the asset.

This leads us to examine the reaction parameter r_i of the traders. By introducing variability there, we can make the traders place their orders at different prices levels that are determined not only by the value of $I_t + c_i$.

3. As discussed in the previous point, we want to fix the problem of very low probability of very small returns by introducing some variability in the way that traders choose the price level of orders. We make their reaction parameter r_i a i.i.d normally distributed variable:

$$r_i \sim \mathcal{N}(\mu_r, \sigma_r^2), r_i \perp\!\!\!\perp r_j, \forall i \neq j \tag{9}$$

By varying the reaction parameter and allowing negative values for it, we allow the agents to also place limit orders. We are interested in seeing the effect of this new agent behavior on the system. With this in mind, we choose $\mu_r = 1, \sigma_r^2 = 1$. On average 15.9% of the agents will be placing limit orders. A run of the simulation for those parameters can be seen in Fig. 4. In Fig. 5 we see the distribution of returns exhibits similar bimodal behavior as when the reactions were constant, this time without the spike at $R_t = 0$.

We can try completely decoupling the decision of whether to place an order or not from the price level of the order by modifying the price function of our agent to only use their reaction parameter:

$$p_{it} = P_{t-1} + o_{it} r_i. \tag{10}$$

In this way, the information noise and personal sentiment $I_t + c_i$ only play a role in the order placing decision o_{it}, which influences the price level of the order only by it's sign. We expect the returns have a distribution closer to the distribution of r_i.

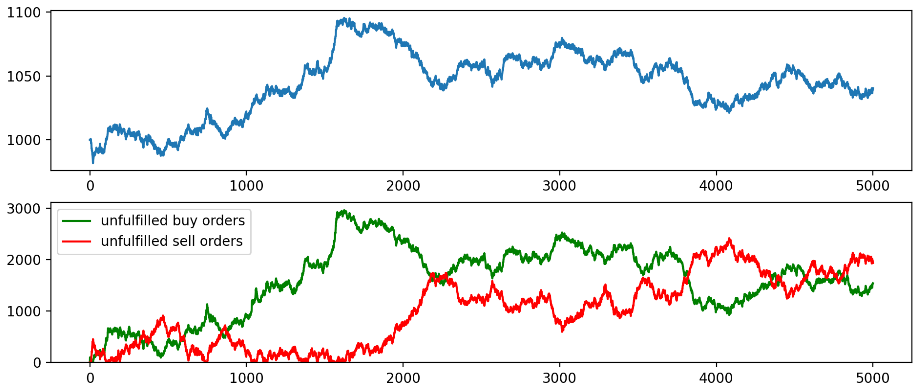

Figure 4: Histogram of the distribution of returns for 100 identical noise agents with parameters $c_i \sim \mathcal{N}(0, 0.5), r_i = 1, \eta_i = 0.8$..

On Fig. 6 we can see a histogram of the returns for the above model for $T = 5000$ time steps. We see no improvement in the bimodality of the returns with this change, so we revert it.

While there are some models with built in bimodality that are able to fit stock return data in under certain conditions [2], the behavior of our model is too extreme to be considered as a good fit for any realistic asset price.

4. We have tried modifying the behavior of our agents in several ways and have seen little improvement toward a more desirable distribution of returns. This leads us to look for a solution in other parts of the system. As discussed in section 5, at each time step, we simulate the decisions of all agents consecutively. The price that we log for each time step is the last strike price of a fulfilled order at this time step or the last price from the previous time step if no order were fulfilled. It is possible that this is the cause of the bimodality of returns. Agents place their orders one after the other and some of these orders are fulfilled. When an order is fulfilled, it is matched with the order of the opposite type in the order book that has the closest price to the current price out of all the orders of this type in the order book. This means that the agents that are simulated later in the time step in general experience a higher spread. Since we log only the last strike price, this could lead to lower returns being less likely.

   To check if this proposition has some merit, we modify our model to simulate one agent per time step. This means that the price will remain unchanged if and agent doesn't place an order or places a limit order that is logged in the order book. On the other hand, any fulfilled order will be reflected in the price time series. We have to increase the value of the simulation time, as now each time step represent the decision of only a single agent.

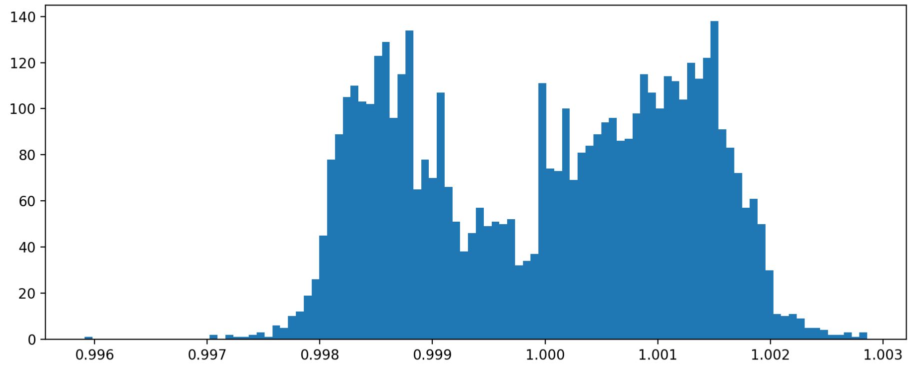

Figure 5: Histogram of the distribution of returns for 100 identical noise agents with parameters $c_i \sim \mathcal{N}(0, 0.5), r_i = 1, \eta_i = 0.8$..

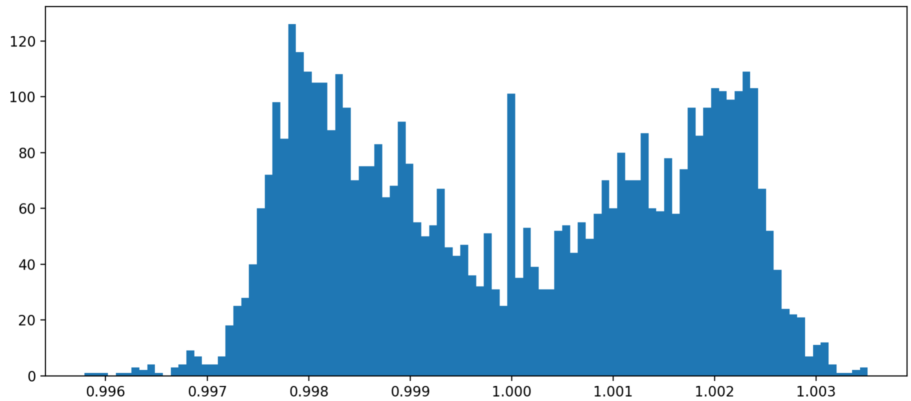

Figure 6: Histogram of the distribution of returns for 100 identical noise agents with parameters $c_i \sim \mathcal{N}(0, 0.5), r_i = 1, \eta_i = 0.8$..

For a run of the above described model for $T = 2.10^4$ time steps, $78,56\%$ of the time steps do not have a change of price, which makes the histogram of the distribution of returns very heavily concentrated at 1. We further modify the model to only log a new price when an order is fulfilled. A histogram of the distribution of non-1 returns for a run of the model can be seen in Fig. 7. This distribution looks more promising, even thought there are some values if higher probability then we would expect if the returns followed some well defining probability distribution and that are unlikely to be explained purely by random variation with a sample size of 77589 returns. Moreover, on Fig. 8, we can see the price history of this run of the model. We see the price varying above what looks like an exponentially decaying lower bound. This behavior is very undesirable and seeing how have only changed how we log the prices, not the underlying model itself, raises some worrying questions about the overall validity of the model.

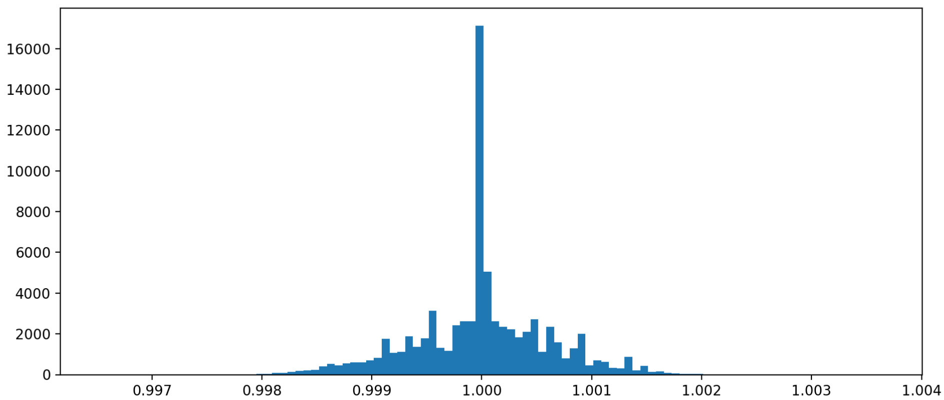

Figure 7: Histogram of the distribution of returns for 100 identical noise agents with parameters $c_i \sim \mathcal{N}(0, 0.5), r_i = 1, \eta_i = 0.8$..

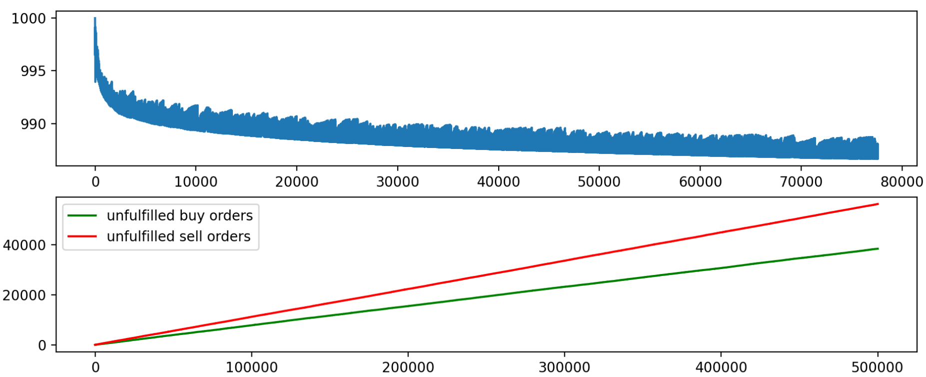

Figure 8: Histogram of the distribution of returns for 100 identical noise agents with parameters $c_i \sim \mathcal{N}(0, 0.5), r_i = 1, \eta_i = 0.8$..

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