Assignment I
Stratified Media Spectral's Green Functions

EE4620 Spectral Domain Methods in Electromagnetics

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Abstract

In this assignment, three types of stratified medias are investigated and the Spectral Green's Functions in these stratified medias are evaluated.

I. Spectral Green's Functions

The Spectral Green's Function give the electric and magnetic far-field, as a result of the elementary electric and magnetic current density. The Spectral Green's Function relating an elementary electric current density to electric far-field is defined by

$$\bar{\mathbf{G}}^{\mathbf{EJ}} = \begin{bmatrix} -\frac{v_{TM}k_x^2+v_{TE}k_y^2}{k_\rho^2} & \frac{(v_{TE}-v_{TE})k_xk_y}{k_\rho^2} \\ \frac{(v_{TE}-v_{TE})k_xk_y}{k_\rho^2} & -\frac{v_{TE}k_x^2+v_{TM}k_y^2}{k_\rho^2} \\ \zeta\frac{k_x}{k}i_{TM} & \zeta\frac{k_y}{k}i_{TM} \end{bmatrix}, \tag{1}$$

where v_{TM}, v_{TE}, and i_{TM} are the transverse magnetic (TM) voltage, transverse electric (TE) voltage, and TM current respectively, and ζ is the impedance of the medium. The Spectral Green's Function relating an elementary magnetic current density to electric far-field is defined by

$$\bar{\mathbf{G}}^{\mathbf{EM}} = \begin{bmatrix} \frac{(v_{TM}-v_{TE})k_xk_y}{k_\rho^2} & -\frac{v_{TM}k_x^2+v_{TE}k_y^2}{k_\rho^2} \\ \frac{(v_{TM}k_y^2+v_{TE})k_x^2}{k_\rho^2} & \frac{(v_{TE}-v_{TM})k_xk_y}{k_\rho^2} \\ Z^{TM}\frac{k_y}{k_z}i_{TM} & Z^{TM}\frac{k_x}{k_z}i_{TM} \end{bmatrix}, \tag{2}$$

where Z_{TM} is the TM component impedance.

The TE and TM components of the impedance are defined respectively by

$$Z^{TM} = \zeta\frac{k_z}{k}, \tag{3a}$$

$$Z^{TE} = \zeta\frac{k}{k_z}. \tag{3b}$$

II. Grounded Dielectric

In the grounded dielectric structure, an elementary electric current source in the x-direction is located on top of dielectric with thickness h and shorted by a perfect electric conductor (PEC). Consequently, there are forward and backward propagating waves in the dielectric, while there is only forward propagating wave in free space.

For a coordinate system with origin at the interface between the PEC and dielectric[1], and dielectric-free space interface located at $z = h$, the waves in dielectric and free space are defined as

$$v_d(z) = V_d^+e^{-jk_{z,d}z} + V_d^-e^{jk_{z,d}z}, \tag{4a}$$

$$v_0(z) = V_0^+e^{-jk_{z,0}z}, \tag{4b}$$

where the wave in the dielectric is the superposition of the forward and backward propagating waves, $k_{z,d}$ is the z-component of the wave propagation vector in the dielectric, V_d^+ and V_d^- is the amplitude of the forward and backward propagating wave in the dielectric, $k_{z,0}$ is the z-component of the wave propagating in free space, and V_0^+ is the amplitude of the forward propagating wave in free space.

The solutions to the waves propagating in free space and dielectric must satisfy the boundary conditions across the two interfaces, dielectric-PEC and free space-dielectric.

[1]It is useful to define the origin of the coordinate system at the source's plane, consequently, a derivation of the stratified media solutions for origin at the free space-dielectric interface, and PEC located at $z = -h$ is provided in Appendix.A

A. Dielectric-PEC Interface

At the dielectric-PEC interface, $z = 0$, the voltage must be equal to zero

$$v(z=0) = 0 \text{ V}; \tag{5}$$

therefore, the wave propagating in the dielectric at $z = 0$ is given by

$$\begin{aligned} v_d(z=0) &= V_d^+ e^{-jk_{z,d}z} + V_d^- e^{jk_{z,d}z} \mid_{z=0} \\ &= V_d^+ + V_d^- = 0, \end{aligned} \tag{6}$$

from which follows that $V_d^+ = -V_d^-$. Thus, the wave propagating in the dielectric is

$$\begin{aligned} v_d(z) &= V_d^+ e^{-jk_{z,d}z} - V_d^+ e^{jk_{z,d}z} \\ &= V_d^+ (e^{-jk_{z,d}z} - e^{jk_{z,d}z}) \\ &= -2jV_d^+ \frac{e^{jk_{z,d}z} - e^{-jk_{z,d}z}}{2j} \\ &= -2jV_d^+ \sin(k_{z,d}z), \end{aligned} \tag{7}$$

where Euler's formula for $\sin$ is applied.

B. Free Space-Dielectric Interface

At the free space-dielectric interface, $z = h$, the voltage must be continuous across the interface and is defined by

$$v(z=h) = (Z_a || Z_{sd}) \cdot 1 \text{ A} = Z_a || Z_{sd}, \tag{8}$$

where Z_{sd} is the impedance seen from the interface towards the shorted dielectric.

The wave in free space at $z = h$ is

$$v_0(z=h) = V_0^+ e^{-jk_{z,0}z} \mid_{z=h} = V_0^+ e^{-jk_{z,0}h} = Z_0 || Z_{sd}, \tag{9}$$

from which follows that the amplitude of the free space propagating wave is $V_0^+ = (Z_0||Z_{sd})e^{jk_{z,0}h}$. Thus, the wave propagating in free space is

$$v_0(z) = (Z_0||Z_{sd}) e^{jk_{z,0}h} e^{-jk_{z,0}h}. \tag{10}$$

The wave in the dielectric at $z = h$ is

$$\begin{aligned} v_d(z=h) &= V_d^+ e^{-jk_{z,d}z} + V_d^- e^{jk_{z,d}z} \mid_{z=h} \\ &= -2jV_d^+ \frac{e^{jk_{z,d}h - e^{-jk_{z,d}h}}}{2j} \\ &= -2jV_d^+ \sin(k_{z,d}h) = Z_0||Z_{sd}; \end{aligned} \tag{11}$$

therefore the amplitude of the forward propagating wave in the dielectric is

$$V_d^+ = -\frac{(Z_0||Z_{sd})}{2j} \frac{1}{\sin(k_{z,d}h)}. \tag{12}$$

Substituting Eq.12 in Eq.7 gives the wave propagating in the dielectric

$$v_d(z) = (Z_0||Z_{sd}) \frac{\sin(k_{z,d}z)}{\sin(k_{z,d}h)} \tag{13}$$

C. Shorted Dielectric Impedance

The input impedance of the shorted dielectric as seen from the elementary current source (from the free space-dielectric interface) is

$$Z_{sd} = Z_d \frac{Z_{PEC} + jZ_d \tan(k_{z,d}h)}{Z_d + jZ_{PEC} \tan(k_{z,d}h)} \mid_{Z_{PEC}=0} = jZ_d \tan(k_{z,d}h), \tag{14}$$

where Z_d is the wave impedance in the dielectric.

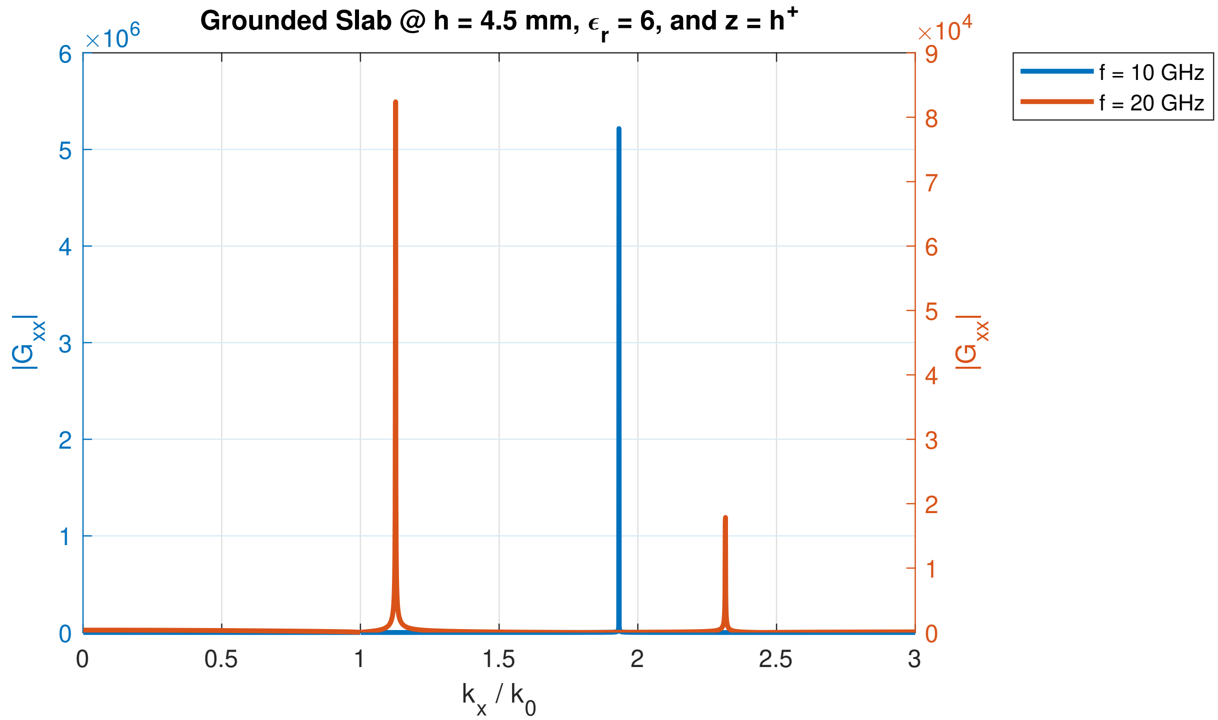

Fig. 1. Spectral Green's Functions G_{xx}^{EJ}-component for a stratified media consisting of a grounded dielectric as a function of k_x/k_0 at $f = 10$ GHz and $f = 20$ GHz for $k_x = [0 \;\; 3k_0]$ and $k_y = 0$, evaluated at $z = h^+$.

D. Wave Solutions

The free space and the dielectric wave solutions in Eq.10 and Eq.13 must be projected to its TE and TM components defined by

$$v_d^{TE}(z) = (Z_0^{TE}||Z_{sd}^{TE})\frac{\sin(k_{z,d}z)}{\sin(k_{z,d}h)}, \tag{15a}$$

$$v_d^{TM}(z) = (Z_0^{TM}||Z_{sd}^{TM})\frac{\sin(k_{z,d}z)}{\sin(k_{z,d}h)}, \tag{15b}$$

$$v_0^{TE}(z) = (Z_0^{TE}||Z_{sd}^{TE})e^{jk_{z,0}h}e^{-jk_{z,0}z}, \tag{15c}$$

$$v_0^{TM}(z) = (Z_0^{TM}||Z_{sd}^{TM})e^{jk_{z,0}h}e^{-jk_{z,0}z}. \tag{15d}$$

E. Spectral Green's Function Solution

The elementary electric current source is oriented in the x-direction, consequently, only the G_{xx}^{EJ}, G_{yx}^{EJ}, and G_{zx}^{EJ} components of the Spectral Green's Function contribute to the electric field. The G_{xx}^{EJ} component for the grounded dielectric stratified media is plotted as a function of k_x/k_0 at $f = 10$ GHz and $f = 20$ GHz in Fig.1 for $k_x = [0 \;\; 3k_0]$ and $k_y = 0$, evaluated at $z = h^+$. At $f = 10$ GHz, one there is one peak for an x-component of the wave vector between $k_x = [0 \;\; 3k_0]$. While, at $f = 20$ GHz, there are two peaks for an x-component of the wave vector between $k_x = [0 \;\; 3k_0]$.

III. Superstrate

In the superstrate structure, a PEC at the origin is followed by free space with thickness h and then a dielectric with thickness h_s, on top of the dielectric another free space medium is located. An elementary magnetic current source in the x-direction is located on top of the PEC. Consequently, there are forward and backward propagating waves in the first free space medium and the dielectric, and only forward propagating wave in the second free space medium (on top of the dielectric).

For a coordinate system with origin at the elementary magnetic current source, first free space-dielectric interface located at $z = h$, and second interface located at $z = h + h_s$, the waves in the first free space, dielectric, and second free space mediums are defined as

$$v_{0,1}(z) = V_{0,1}^{+}e^{-jk_{z.0}z} + V_{0,1}^{-}e^{jk_{z,0}z}, \tag{16a}$$

$$v_d(z) = V_d^{+}e^{-jk_{z,d}z} + V_d^{-}e^{jk_{z,d}z}, \tag{16b}$$

$$v_{0,2}(z) = V_{0,2}^{+} e^{-jk_{z,0}z}, \tag{16c}$$

where $V_{0,1}^{+}$ and $V_{0,1}^{-}$ are the forward and backward propagating waves' amplitudes in the first free space medium, and $V_{0.2}^{+}$ and $V_{0,2}^{+}$ in the second free space medium.

The solution to the waves must satisfy the boundary conditions across the three interfaces, PEC-free space, free space-dielectric, and dielectric-free space.[2]

A. Wave Solutions

The wave solutions in the first free space medium, dielectric, and second free space medium are projected to TE and TM components and defined by

$$v_{0,1}^{TE}(z) = \frac{e^{j2k_{z,0}h}}{\Gamma_1^{TE} + e^{j2k_{z,0}h}} e^{-jk_{z,0}z}(1 + \Gamma_1^{TE} e^{-j2k_{z,0}h} e^{j2k_{z,0}z}), \tag{17a}$$

$$v_{0,1}^{TM}(z) = \frac{e^{j2k_{z,0}h}}{\Gamma_1^{TM} + e^{j2k_{z,0}h}} e^{-jk_{z,0}z}(1 + \Gamma_1^{TM} e^{-j2k_{z,0}h} e^{j2k_{z,0}z}), \tag{17b}$$

$$v_{d}^{TE}(z) = \frac{e^{jk_{z,0}h} e^{jk_{z,d}h}(1 + \Gamma_1^{TE})}{(\Gamma_1^{TE} + e^{j2k_{z,0}h})(1 + \Gamma_2^{TE} e^{-j2k_{z,d}h_s})} (e^{-jk_{z,d}z} + e^{jk_{z,d}z}\Gamma_2^{TE} e^{-j2k_{z,d}(h+h_s)}), \tag{17c}$$

$$v_{d}^{TM}(z) = \frac{e^{jk_{z,0}h} e^{jk_{z,d}h}(1 + \Gamma_1^{TM})}{(\Gamma_1^{TM} + e^{j2k_{z,0}h})(1 + \Gamma_2^{TM} e^{-j2k_{z,d}h_s})} (e^{-jk_{z,d}z} + e^{jk_{z,d}z}\Gamma_2^{TM} e^{-j2k_{z,d}(h+h_s)}), \tag{17d}$$

$$v_{0,2}^{TE}(z) = \frac{e^{jk_{z,0}h} e^{-jk_{z,d}h_s} e^{jk_{z,0}(h+h_s)}(1 + \Gamma_1^{TE})(1 + \Gamma_2^{TE})}{(\Gamma_1^{TE} + e^{j2k_{z,0}h})(1 + \Gamma_2^{TE} e^{-j2k_{z,d}h_s})} e^{-jk_{z,0}z}, \tag{17e}$$

$$v_{0,2}^{TM}(z) = \frac{e^{jk_{z,0}h} e^{-jk_{z,d}h_s} e^{jk_{z,0}(h+h_s)}(1 + \Gamma_1^{TM})(1 + \Gamma_2^{TE})}{(\Gamma_1^{TM} + e^{j2k_{z,0}h})(1 + \Gamma_2^{TM} e^{-j2k_{z,d}h_s})} e^{-jk_{z,0}z}, \tag{17f}$$

where Γ_1 and Γ_2 are the reflection coefficients as seen from the first free space medium towards the dielectric and dielectric towards the second free space medium respectively.

B. Reflection Coefficients

The reflection coefficients are defined by

$$\Gamma_1 = \frac{Z_{df} - Z_0}{Z_{df} + Z_0}, \tag{18a}$$

$$\Gamma_2 = \frac{Z_0 - Z_d}{Z_0 + Z_d}, \tag{18b}$$

where Z_{df} is the impedance seen from the first free space medium towards the dielectric, and Z_d is the characteristic impedance of the dielectric. Furthermore, the reflection coefficients must be projected to their TE and TM components.

C. Impedance Seen From First Free Space Medium Towards Dielectric

The input impedance of the dielectric and second free space medium as seen from the first free space medium is

$$Z_{df} = Z_d \frac{Z_0 + jZ_d \tan(k_d h_s)}{Z_d + jZ_0 \tan(k_d h_s)}, \tag{19}$$

where h_s is the length of the dielectric.

D. Spectral Green's Function Solution

The elementary magnetic source is oriented in the x-direction, consequently, only the G_{xx}^{EM}, G_{yx}^{EM}, and G_{zx}^{EM} components of the Spectral Green's Function contribute to the electric field. The G_{yx}^{EM} component for the superstrate stratified media is plotted as a function of k_y/k_0 at $f = 8$ GHz, $f = 8.5$ GHz, $f = 9$ GHz, $f = 9.5$ GHz, and $f = 10$ GHz in Fig.2 for $k_y = [0 \;\; k_0]$ and $k_x = 0$, evaluated at $z = h + h_s^{+}$. At larger frequencies, the wave experience larger y-components at $k_y < 0.5k_0$ with local peak slightly before $k_y = k_0$. At smaller frequencies, the y-components are small at $k_y < 0.5k_0$ compared to larger frequencies. However, all frequencies experience local peak slightly before $k_y = k_0$.

[2]Derivations of the solutions for the waves in the three mediums are not shown.

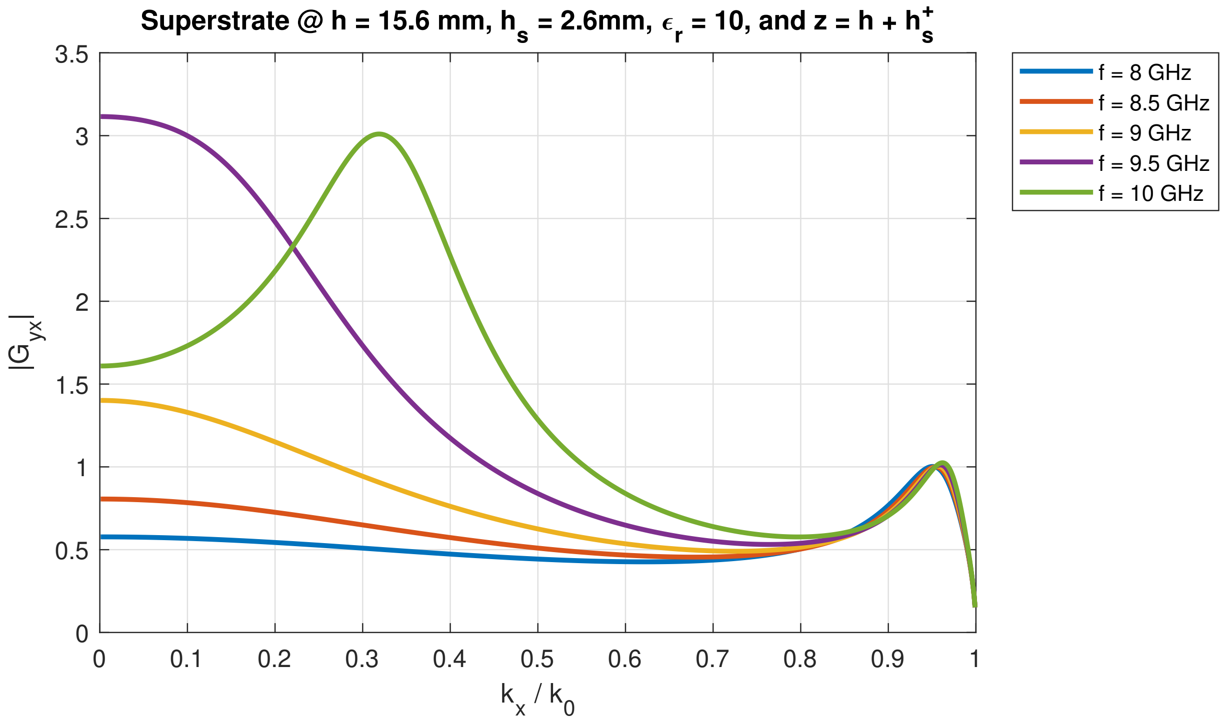

Fig. 2. Spectral Green's Functions G_{yx}^{EM}-component for a stratified media consisting of a superstrate as a function of k_y/k_0 at $f = 8$ GHz, $f = 8.5$ GHz, $f = 9$ GHz, $f = 9.5$ GHz, and $f = 10$ GHz for $k_y = [0 \;\; k_0]$ and $k_x = 0$, evaluated at $z = h + h_s^+$.

IV. Semi-Infinite Superstrate

In the semi-infinite superstrate structure, a PEC at the origin is followed by free space with thickness h and then a semi-infinite dielectric. An elementary magnetic current source in the y-direction is located on top of the PEC. Consequently, there are forward and backward propagating waves in the free space medium, and only forward propagating wave in the dielectric.

For a coordinate system with origin at the elementary magnetic current source, and free space-dielectric interface located at $z = h$, the waves in the free space and dielectric are defined as

$$v_0(z) = V_0^+ e^{-jk_{z,0}z} + V_0^- e^{jk_{z,0}z}, \tag{20a}$$

$$v_d(z) = V_d^+ e^{-jk_{z,0}z}. \tag{20b}$$

The solution to the waves must satsify the boundary conditions across the two interfaces, PEC-free space and free space-dielectric.

A. Wave Solutions

The wave solutions in free space and dielectric medium are projected to TE and TM components and defined by

$$v_0^{TE}(z) = \frac{e^{-jk_{z,0}z}}{1 + \Gamma^{TE} e^{-j2k_{z,0}h}} (1 + \Gamma^{TE} e^{-j2k_{z,0}h} e^{j2k_{z,0}z}), \tag{21a}$$

$$v_0^{TM}(z) = \frac{e^{-jk_{z,0}z}}{1 + \Gamma^{TM} e^{-j2k_{z,0}h}} (1 + \Gamma^{TM} e^{-j2k_{z,0}h} e^{j2k_{z,0}z}), \tag{21b}$$

$$v_d^{TE}(z) = \frac{e^{-jk_{z,0}h} e^{jk_{z,d}h}}{1 + \Gamma^{TE} e^{-j2k_{z,0}h}} (1 + \Gamma^{TE}) e^{-jk_{z,d}z}, \tag{21c}$$

$$v_d^{TM}(z) = \frac{e^{-jk_{z,0}h} e^{jk_{z,d}h}}{1 + \Gamma^{TM} e^{-j2k_{z,0}h}} (1 + \Gamma^{TM}) e^{-jk_{z,d}z}, \tag{21d}$$

where Γ is the reflection coefficient as seen from free space towards the dielectric medium.

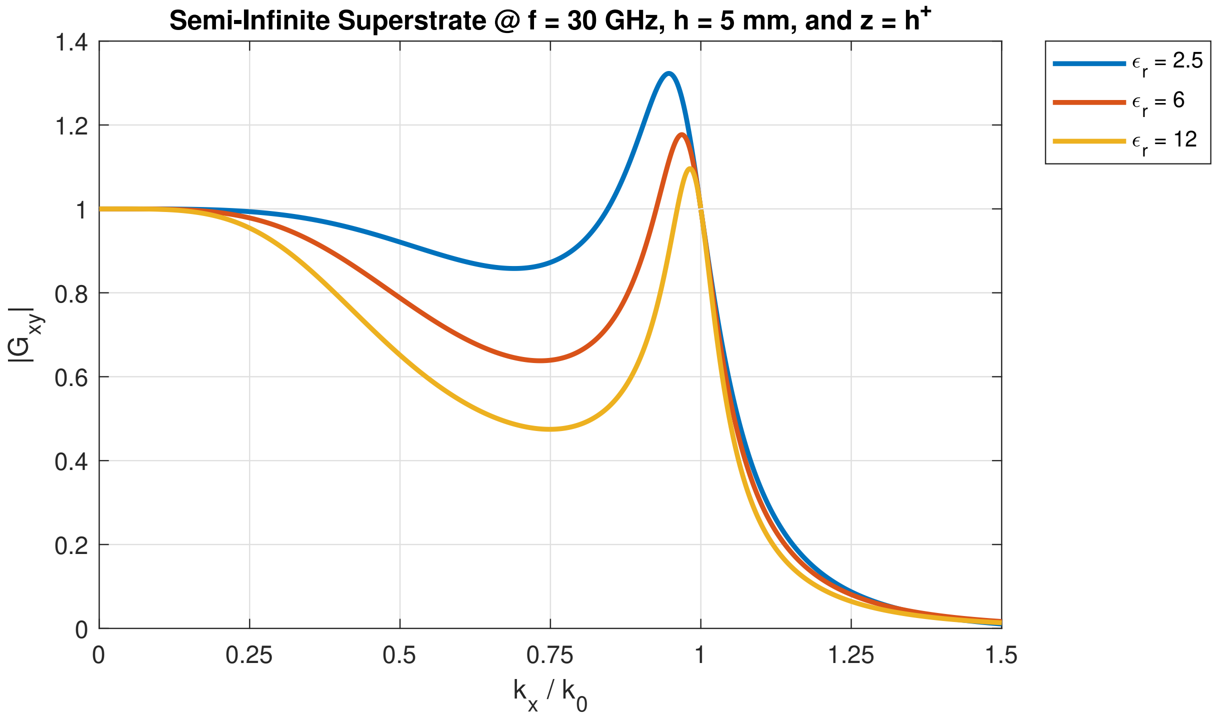

Fig. 3. Spectral Green's Functions G_{xy}^{EM}-component for a stratified media consisting of a semi-infinite superstrate as a function of k_x/k_0 at $f = 30$ GHz and $\epsilon_r = 2.5$, $\epsilon_r = 6$, and $\epsilon_r = 12$ for $k_x = [0 \;\; 2k_0]$ and $k_y = 0$, evaluated at $z = h^+$.

B. Reflection Coefficients

The reflection coefficient is defined by

$$\Gamma = \frac{Z_d - Z_0}{Z_d + Z_0}, \tag{22}$$

where it must be projected to its TE and TM components.

C. Spectral Green's Function Soluitions

The elementary magnetic source is oriented in the y-direction, consequently, only the G_{xy}^{EM}, G_{yy}^{EM}, and G_{zy}^{EM} components of the Spectral Green's Function contribute to the electric field. The G_{xy}^{EM} component for the semi-infinite superstrate stratified media is plotted as a function of k_x/k_0 at $f = 30$ GHz for $\epsilon_r = 2.5$, $\epsilon = 6$, and $\epsilon = 12$ in Fig.3 for $k_x = [0 \;\; 2k_0]$ and $k_y = 0$, evaluated at $z = h^+$. The G_{xy}^{EM} component is dominant at $k_x < k_0$ with a peak slightly before $k_x = k_0$. At low k_x (close to zero), the component is equal to one, with a slight deep before the peak. The deep decreases while the peak slightly increases for lower relative permittivity. At $k_x > k_0$, the component decreases exponentially. Therefore, this stratified media is a good structure for lens antennas, because the exponential decrease at $k_x > k_0$ results in less power trapped in the lens dielectric.

Appendix A
Grounded Dielectric With Origin at Free Space-Dielectric Interface

The forward and backward propagating waves in the dielectric are reversed than the ones defined in Section.II

$$v_d(z) = V_d^+ e^{jk_{z,d}z} + V_d^- e^{-jk_{z,d}z}, \tag{23a}$$

$$v_0(z) = V_0^+ e^{-jk_{z,0}z}. \tag{23b}$$

At the dielectric-PEC interface, $z = -h$, the wave in the dielectric is

$$V(z = -h) = V_d^+ e^{-jk_{z,d}h} + V_d^- e^{jk_{z,d}h} = 0, \tag{24}$$

therefore the forward and backward propagating waves' amplitudes are related by $V_d^+ e^{-2jk_{z,d}h} = -V_d^-$; consequently

$$\begin{aligned} v_d(z) &= V_d^+ e^{jk_{z,d}z} - V_d^+ e^{-2jk_{z,d}h} e^{jk_{z,d}z} \\ &= V_d^+ e^{-jk_{z,d}h} (e^{jk_{z,d}h} e^{jk_{z,d}z} - e^{-jk_{z,d}h} e^{-jk_{z,d}z}) \\ &= 2jV_d^+ e^{-jk_{z,d}h} \sin(k_{z,d}(z+h)), \end{aligned} \tag{25}$$

which satisfies the boundary condition at $z = -h$, for $v_d(z = -h) = 0$. At the free space-dielectric interface, $z = 0$, the wave in free space is

$$v_0(z = 0) = V_a^+ = Z_0 || Z_{sd}; \tag{26}$$

hence, the wave propagating in free space is

$$v_0(z) = (Z_0 || Z_{sd}) e^{-jk_{z,0}z}. \tag{27}$$

While the dielectric propagating wave is

$$v(z = 0) = V_d^+ + V_d^- = V_d^+ (1 - e^{-2jk_{z,d}h}) = (Z_0 || Z_{sd}), \tag{28}$$

and the amplitude of the forward propagating wave in the dielectric is

$$V_d^+ = \frac{Z_0 || Z_{sd}}{2je^{-jk_{z,d}h}} \frac{1}{\sin(k_{z,d}h)}. \tag{29}$$

Therefore the dielectric propagating wave is

$$v_d(z) = (Z_0 || Z_{sd}) \frac{\sin(k_{z,d}(h+z))}{\sin(k_{z,d}h)}. \tag{30}$$

The impedance seen from the free space-dielectric interface towards the PEC is

$$Z_{sd} = -jZ_d \tan(k_{z,d}h). \tag{31}$$

Projecting the waves to their TE and TM components results in

$$v_d^{TE}(z) = (Z_0^{TE} || Z_{sd}^{TE}) \frac{\sin(k_{z,d}(h+z))}{\sin(k_{z,d}h)}, \tag{32a}$$

$$v_d^{TM}(z) = (Z_0^{TM} || Z_{sd}^{TM}) \frac{\sin(k_{z,d}(h+z))}{\sin(k_{z,d}h)}, \tag{32b}$$

$$v_0^{TE}(z) = (Z_0^{TE} || Z_{sd}^{TE}) e^{-jk_{z,0}z}, \tag{32c}$$

$$v_0^{TM}(z) = (Z_0^{TM} || Z_{sd}^{TM}) e^{-jk_{z,0}z}. \tag{32d}$$

Appendix B
Grounded Dielectric MatLab Code

The spectral domains techniques library is uploaded to spectral-techniques-library and the scripts used in this assignment to spectral-techniques-assignment-I. To use the scripts either place the library in the parent directory where the assignment directory is located or edit the library path.