Assignment V Connected Slot Array Derivations

EE4620 Spectral Domain Methods in Electromagnetics

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Abstract

In this assignment, the longitudinal spectral *HM* Green's function, spectral slot voltage, spatial slot voltage, and active input impedance of a single infinite slot and periodic connected slot array are derived.

I. Single Infinite Slot

For a single infinite slot, the steps to derive the longitudinal spectral *HM* Green's function $D^{HM}(k_x)$, spectrum slot voltage $V(k_x)$, spatial slot voltage $v(x)$, and active input impedance Z_{in} are as follows

1) define the Magnetic Field Integral Equation (MFIE);
2) separate the magnetic current density $m_x(x,y)$ in longitudinal and transverse components and define the longitudinal spatial *hm* Green's function $d^{hm}(x)$;
3) substitute the Inverse Fourier Transform (IFT) of $d^{hm}(x)$ in the MFIE;
4) derive the longitudinal spectral *HM* Green's function $D^{HM}(k_x)$;
5) derive the spectral slot voltage $V(k_x)$;
6) derive the spatial slot voltage $v(x)$ by taking the IFT of $V(k_x)$;
7) derive the active input impedance Z_{in} using the spatial slot voltage $v(x)$.

A. Magnetic Field Integral Equation

The MFIE is defined by applying the Equivalence and image theorems on the slot, while assuming the slot is narrow. It is already defined as

$$-4m_x(x,y) * g_{xx}^{hm}(x,y) = j_{inc,y}(x,y), \tag{1}$$

where $j_{inc,y}(x,y)$ is the impressed current at the slot's feed with orientation along the y-direction and is given by

$$j_{inc,y}(x,y) = I_0 \frac{rect_{\delta_s}(x,y)}{\delta_s}. \tag{2}$$

The impressed current is constant and defined over the feed's width δ_s and slot's width w_s. Moreover, the orientation of the magnetic current density $m_x(x,y)$ on the slot is in the slot's axis, x-direction.

B. Define the Longitudinal Spatial **hm** Green's Function

The MFIE in Eq.1 is rewritten as

$$-4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m_x(x',y') g_{xx}^{hm}(x-x',y-y') dx'dy' = I_0 \frac{rect_{\delta_s}(x,y)}{\delta_s}; \tag{3}$$

the MFIE is evaluated at the slot's axis, i.e. $y=0$,

$$-4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m_x(x',y') g_{xx}^{hm}(x-x',-y') dx'dy' = I_0 \frac{rect_{\delta_s}(x)}{\delta_s}. \tag{4}$$

Assuming narrow slot, $m_x(x,y)$ can be separated in longitudinal and transverse components. The separation of variables of $m_x(x,y)$ leads to

$$m_x(x,y) = v(x)m_t(y), \tag{5}$$

where $v(x)$ and $m_t(y)$ are the longitudinal and transverse components of the magnetic current. Therefore, the MFIE becomes

$$-4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v(x')m_t(y') g_{xx}^{hm}(x-x',-y') dx'dy' = I_0 \frac{rect_{\delta_s}(x)}{\delta_s}, \tag{6a}$$

$$-4\int_{-\infty}^{\infty} v(x') \int_{-\infty}^{\infty} m_t(y') g_{xx}^{hm}(x-x',-y') dy'dx' = I_0 \frac{rect_{\delta_s}(x)}{\delta_s}. \tag{6b}$$

From Eq.6b, the longitudinal spatial *hm* Green's function $d^{hm}(x)$ is recognized as

$$d^{hm}(x) = \int_{-\infty}^{\infty} m_t(y) g_{xx}^{hm}(x, -y) dy; \tag{7}$$

therefore, the MFIE becomes

$$-4 \int_{-\infty}^{\infty} v(x') d(x - x') dx' = I_0 \frac{rect_{\delta_s}(x)}{\delta_s}. \tag{8}$$

*C. Substitute the Inverse Fourier Transform of the Longitudinal Spatial **hm** Green's Function in the MFIE*

The IFT of $d^{hm}(x)$ is defined by

$$d^{hm}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} D^{HM}(k_x) e^{-jk_x x} dk_x, \tag{9}$$

therefore, when substituted in the MFIE, the left hand-side (LHS) of Eq.8 becomes

$$-4\frac{1}{2\pi} \int_{-\infty}^{\infty} v(x') \int_{-\infty}^{\infty} D^{HM}(k_x) e^{-jk_x(x-x')} dk_x dx' = -4\frac{1}{2\pi} \int_{-\infty}^{\infty} D^{HM}(k_x) \int_{-\infty}^{\infty} v(x') e^{jk_x x'} dx' e^{-jk_x x} dk_x, \tag{10}$$

The Fourier Transform (FT) of $v(x)$ is recognized as

$$V(k_x) = \int_{-\infty}^{\infty} v(x) e^{jk_x x} dx; \tag{11}$$

therefore the MFIE becomes

$$-4\frac{1}{2\pi} \int_{-\infty}^{\infty} D^{HM}(k_x) \int_{-\infty}^{\infty} v(x') e^{jk_x x'} dx' e^{-jk_x x} dk_x = I_0 \frac{rect_{\delta_s}(x)}{\delta_s}, \tag{12a}$$

$$-4\frac{1}{2\pi} \int_{-\infty}^{\infty} D^{HM}(k_x) V(k_x) e^{-jk_x x} dk_x = I_0 \frac{rect_{\delta_s}(x)}{\delta_s}. \tag{12b}$$

*D. Derive the Longitudinal Spectral **HM** Green's Function*

Expressing the spatial Green's function $g_{xx}^{hm}(x, y)$ as the IFT of the spectral Green's function $G_{XX}^{HM}(k_x, k_y)$, in the definition of $d^{hm}(x)$, gives

$$\begin{aligned} d^{hm}(x) &= \int_{-\infty}^{\infty} m_t(y') g_{xx}^{hm}(x, -y') dy' \\ &= \int_{-\infty}^{\infty} m_t(y') \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{XX}^{HM}(k_x, k_y) e^{-jk_x x} e^{jk_y y'} dk_x dk_y dy' \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{XX}^{HM}(k_x, k_y) \int_{-\infty}^{\infty} m_t(y') e^{jk_y y'} dy' e^{-jk_x x} dk_x dk_y. \end{aligned} \tag{13}$$

Assuming $m_t(y)$ is edge-singular, then, its FT is zeroth-order Bessel function of the first kind and defined by

$$\int_{-\infty}^{\infty} m_t(y') e^{jk_y y'} dy' = J_0(\frac{k_y w_s}{2}), \tag{14}$$

therefore $d^{hm}(x)$ becomes

$$d^{hm}(x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{XX}^{HM}(k_x, k_y) J_0(\frac{k_y w_s}{2}) e^{-jk_x x} dk_x dk_y. \tag{15}$$

From the definition of the IFT of $d^{hm}(x)$ in Eq.9, $D^{HM}(k_x)$ is derived as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} D^{HM}(k_x) e^{-jk_x x} dk_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{XX}^{HM}(k_x, k_y) J_0(\frac{k_y w_s}{2}) dk_y e^{-jk_x x} dk_x, \tag{16a}$$

$$D^{HM}(k_x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{XX}^{HM}(k_x, k_y) J_0(\frac{k_y w_s}{2}) dk_y. \tag{16b}$$

For stratified media, $G_{XX}^{HM}(k_x, k_y)$ is

$$G_{XX}^{HM}(k_x, k_y) = -\frac{i_{TE} k_x^2 + i_{TM} k_y^2}{k_\rho^2}; \tag{17}$$

therefore, $D^{HM}(k_x)$ becomes

$$D^{HM}(k_x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{XX}^{HM}(k_x,k_y)J_0(\frac{k_y w_s}{2})dk_y, \tag{18a}$$

$$D^{HM}(k_x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{i_{TE}k_x^2 + i_{TM}k_y^2}{k_\rho^2}J_0(\frac{k_y w_s}{2})dk_y. \tag{18b}$$

E. Derive the Spectral Slot Voltage

The right hand-side (RHS) of the MFIE, in Eq.12b, is expressed in terms of its IFT as

$$I_0\frac{rect_{\delta_s}(x)}{\delta_s} = I_0\frac{1}{2\pi}\int_{-\infty}^{\infty} sinc(\frac{k_x\delta_s}{2})e^{-jk_x x}dk_x; \tag{19}$$

therefore, the MFIE in Eq.12b becomes

$$-4\frac{1}{2\pi}\int_{-\infty}^{\infty} D^{HM}(k_x)V(k_x)e^{-jk_x x}dk_x = I_0\frac{1}{2\pi}\int_{-\infty}^{\infty} sinc(\frac{k_x\delta_s}{2})e^{-jk_x x}dk_x, \tag{20}$$

which is valid for every x, hence, the integrands are equal and the equation is simplified to

$$-4D^{HM}(k_x)V(k_x) = I_0 sinc(\frac{k_x\delta_s}{2}). \tag{21}$$

Therefore, $V(k_x)$ is

$$V(k_x) = -I_0\frac{sinc(\frac{k_x\delta_s}{2})}{4D^{HM}(k_x)}. \tag{22}$$

F. Derive the Spatial Slot Voltage

The spatial slot voltage $v(x)$ is derived by taking the IFT of $V(k_x)$

$$v(x) = -I_0\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{sinc(\frac{k_x\delta_s}{2})}{4D^{HM}(k_x)}e^{-jk_x x}dk_x. \tag{23}$$

G. Derive the Active Input Impedance

The active input impedance is equal to the ratio between the total voltage and current on the feed's interface, V_0 and I_0 respectively, defined by

$$Z_{in} = \frac{V_0}{I_0}. \tag{24}$$

The total voltage on the feed's interface is

$$V_0 = \frac{1}{\delta_s}\int_{-\delta_s/2}^{\delta_s/2} v(x)dx; \tag{25}$$

therefore, the input impedance becomes

$$\begin{aligned} Z_{in} &= \frac{1}{I_0}\frac{1}{\delta_s}\int_{-\delta_s/2}^{\delta_s/2} v(x)dx \\ &= -\frac{1}{I_0}\frac{1}{\delta_s}\int_{-\delta_s/2}^{\delta_s/2} I_0\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{sinc(\frac{k_x\delta_s}{2})}{4D^{HM}(k_x)}e^{-jk_x x}dk_x dx \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{sinc(\frac{k_x\delta_s}{2})}{4D^{HM}(k_x)}\frac{1}{\delta_s}\int_{-\delta_s/2}^{\delta_s/2} e^{-jk_x x}dx dk_x, \end{aligned} \tag{26}$$

where the IFT of the rectangular function is recognized as

$$\frac{1}{\delta_s}\int_{-\delta_s/2}^{\delta_s/2} e^{-jk_x x}dx = sinc(\frac{k_x\delta_s}{2}). \tag{27}$$

The input impedance is

$$Z_{in} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{sinc^2(\frac{k_x\delta_s}{2})}{4D^{HM}(k_x)}dk_x. \tag{28}$$

II. Periodic Connected Slot Array

For a periodic connected slot array, the derivations of the periodic longitudinal spectral Green's function $D_{\infty}^{HM}(k_x)$, $V(k_x)$, $v(x)$, and $Z_a, slot$ follow similar procedures to the infinite slot derivations and are

1) define the Magnetic Field Integral Equation (MFIE);
2) separate the magnetic current density $m_x(x,y)$ in longitudinal and transverse components and define the periodic longitudinal spatial *hm* Green's function $d_{\infty}^{hm}(x)$;
3) substitute the Inverse Fourier Transform (IFT) of $d^{hm}(x)$ in the MFIE;
4) derive the periodic longitudinal spectral *HM* Green's function $D_{\infty}^{HM}(k_x)$;
5) derive the spectral slot voltage $V(k_x)$;
6) derive the spatial slot voltage $v(x)$ by taking the Discrete Inverse Fourier Transform (DIFT) of $V(k_x)$;
7) derive the active input impedance $Z_{a,slot}$ using the spatial slot voltage $v(x)$.

However, for a periodic connected slot array the feeds are infinite, spaced in the x and y-directions by d_x and d_y respectively, and $V(k_x)$ is discrete.

A. Magnetic Field Integral Equation

Similarly to the single infinite slot, the MFIE is

$$-4m_x(x,y) * g_{xx}^{hm}(x,y) = j_{inc,y}(x,y), \tag{29}$$

where $j_{inc,y}(x,y)$ is the sum of the infinite delta-gap sources on the periodic slot array, in contrast to the single source in the single infinite slot, and is given by

$$j_{inc,y}(x,y) = \sum_{n_x,n_y} I_0 \frac{rect_{\delta_s}(x-n_xd_x, y-n_yd_y)}{\delta_s} e^{-jk_{x0}n_xd_x} e^{-jk_{y0}n_yd_y}. \tag{30}$$

The array is divided into infinite $d_x \times d_y$ cells with centers at the middle of each respective (n_x, n_y) cell's delta-gap source.

B. Define the Periodic Longitudinal Spatial **hm** Green's Function

The MFIE in Eq.29 is rewritten as

$$-4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m_x(x',y')g_{xx}^{hm}(x-x',y-y')dx'dy' = \sum_{n_x,n_y} I_0 \frac{rect_{\delta_s}(x-n_xd_x, y-n_yd_y)}{\delta_s} e^{-jk_{x0}n_xd_x} e^{-jk_{y0}n_yd_y}, \tag{31}$$

the MFIE is evaluated at the array's axis, i.e. $y=0$ and $n_y=0$,

$$-4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m_x(x',y')g_{xx}^{hm}(x-x',-y')dx'dy' = \sum_{n_x=-\infty}^{\infty} I_0 \frac{rect_{\delta_s}(x-n_xd_x)}{\delta_s} e^{-jk_{x0}n_xd_x}. \tag{32}$$

The total magnetic field is the sum of the field contributions of each cell, hence, the integration domain is decomposed as

$$-4\sum_{n_x,n_y}\int_{n_xd_x-d_x/2}^{n_xd_x+d_x/2}\int_{n_yd_y-d_y/2}^{n_yd_y+d_y/2} m_x(x',y')g_{xx}^{hm}(x-x',-y')dy'dx' = \sum_{n_x=-\infty}^{\infty} I_0 \frac{rect_{\delta_s}(x-n_xd_x)}{\delta_s} e^{-jk_{x0}n_xd_x}, \tag{33}$$

where the magnetic current and spatial Green's function convolution is carried over each cell and summed.

Similarly to the single infinite slot, $m_x(x,y)$ can be separated in longitudinal and transverse components, however, in this case there is a periodicity in the magnetic current components. The separation of variables of $m_x(x,y)$ leads to

$$m_x(x,y) = v(x)m_t(y), \tag{34a}$$

$$v(x+n_xd_x) = v(x)e^{-jk_{x0}n_xd_x}, \tag{34b}$$

$$m_t(y+n_yd_y) = m_t(y)e^{-jk_{y0}n_yd_y}. \tag{34c}$$

Therefore, the MFIE becomes

$$-4\sum_{n_x,n_y}\int_{n_xd_x-d_x/2}^{n_xd_x+d_x/2}\int_{n_yd_y-d_y/2}^{n_yd_y+d_y/2} v(x')m_t(y')g_{xx}^{hm}(x-x',-y')dy'dx' = \sum_{n_x=-\infty}^{\infty} I_0 \frac{rect_{\delta_s}(x-n_xd_x)}{\delta_s} e^{-jk_{x0}n_xd_x}, \tag{35}$$

To simplify the integration limits and make the integrals independent of the cell's indecies (n_x, n_y), a change of variables is applied defined by

$$x'' = x' - n_xd_x, \tag{36a}$$

$$y'' = y' - n_yd_y; \tag{36b}$$

therefore, the LHS of the MFIE in Eq.35 becomes

$$\begin{aligned}
&-4\sum_{n_x,n_y}\int_{n_xd_x-d_x/2}^{n_xd_x+d_x/2}\int_{n_yd_y-d_y/2}^{n_yd_y+d_y/2} v(x')m_t(y')g_{xx}^{hm}(x-x',-y')dy'dx' \\
&\qquad = -4\sum_{n_x,n_y}\int_{-d_x/2}^{d_x/2}\int_{-d_y/2}^{d_y/2} v(x''+n_xd_x)m_t(y''+n_yd_y)g_{xx}^{hm}(x-x''-n_xd_x,-y''-n_yd_y)dy''dx''
\end{aligned} \tag{37}$$

Substituting Eq.34b and Eq.34c in Eq.37 leads to

$$\begin{aligned}
&-4\sum_{n_x,n_y}\int_{-d_x/2}^{d_x/2}\int_{-d_y/2}^{d_y/2} v(x''+n_xd_x)m_t(y''+n_yd_y)g_{xx}^{hm}(x-x''-n_xd_x,-y''-n_yd_y)dy''dx'' \\
&= -4\sum_{n_x,n_y}\int_{-d_x/2}^{d_x/2}\int_{-d_y/2}^{d_y/2} v(x'')e^{-jk_{x0}n_xd_x}m_t(y'')e^{-jk_{y0}n_yd_y}g_{xx}^{hm}(x-x''-n_xd_x,-y''-n_yd_y)dy''dx'' \\
&= -4\sum_{n_x=-\infty}^{\infty}\int_{-d_x/2}^{d_x/2} v(x'')e^{-jk_{x0}n_xd_x}\sum_{n_y=-\infty}^{\infty}\int_{-d_y/2}^{d_y/2} m_t(y'')e^{-jk_{y0}n_yd_y}g_{xx}^{hm}(x-x''-n_xd_x,-y''-n_yd_y)dy''dx''.
\end{aligned} \tag{38}$$

From Eq.38, the periodic longitudinal spatial *hm* Green's function $d_\infty^{hm}(x)$ is recognized as

$$d_\infty^{hm}(x) = \sum_{n_y=-\infty}^{\infty}\int_{-d_y/2}^{d_y/2} m_t(y)g_{xx}^{hm}(x,-y-n_yd_y)e^{-jk_{y0}n_yd_y}dy; \tag{39}$$

therefore, the MFIE becomes

$$-4\sum_{n_x=-\infty}^{\infty}\int_{-d_x/2}^{d_x/2} v(x'')d_\infty^{hm}(x-x''-n_xd_x)e^{-jk_{x0}n_xd_x}dx'' = \sum_{n_x=-\infty}^{\infty} I_0\frac{rect_{\delta_s}(x-n_xd_x)}{\delta_s}e^{-jk_{x0}n_xd_x}. \tag{40}$$

*C. Substitute the Inverse Fourier Transform of the Periodic Longitudinal Spatial **hm** Green's Function in the MFIE*

The IFT of $d_\infty^{hm}(x)$ is defined by

$$d_\infty^{hm}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} D_\infty^{HM}(k_x)e^{-jk_xx}dk_x, \tag{41}$$

therefore, when substituted in the MFIE, the LHS of Eq.40 becomes

$$\begin{aligned}
&-4\sum_{n_x=-\infty}^{\infty}\int_{-d_x/2}^{d_x/2} v(x'')\frac{1}{2\pi}\int_{-\infty}^{\infty} D_\infty^{HM}(k_x)e^{-jk_x(x-x''-n_xd_x)}dk_xe^{-jk_{x0}n_xd_x}dx'' \\
&\qquad = -4\frac{1}{2\pi}\int_{-\infty}^{\infty} D_\infty^{HM}(k_x)\int_{-d_x/2}^{d_x/2} v(x'')e^{jk_xx''}dx''\sum_{n_x=-\infty}^{\infty} e^{jk_xn_xd_x}e^{-jk_{x0}n_xd_x}e^{-jk_xx}dk_x \\
&\qquad = -4\frac{1}{2\pi}\int_{-\infty}^{\infty} D_\infty^{HM}(k_x)\int_{-d_x/2}^{d_x/2} v(x'')e^{jk_xx''}dx''\sum_{n_x=-\infty}^{\infty} e^{j(k_x-k_{x0})n_xd_x}e^{-jk_xx}dk_x.
\end{aligned} \tag{42}$$

The FT of $v(x)$ is recognized as

$$V(k_x) = \int_{-d_x/2}^{d_x/2} v(x)e^{jk_xx}dx; \tag{43}$$

therefore the MFIE becomes

$$-4\frac{1}{2\pi}\int_{-\infty}^{\infty} D_\infty^{HM}(k_x)V(k_x)\sum_{n_x=-\infty}^{\infty} e^{j(k_x-k_{x0})n_xd_x}e^{-jk_xx}dk_x = \sum_{n_x=-\infty}^{\infty} I_0\frac{rect_{\delta_s}(x-n_xd_x)}{\delta_s}e^{-jk_{x0}n_xd_x}. \tag{44}$$

Applying the Floquet theorem defined by

$$\sum_{n_x=-\infty}^{\infty} e^{j(k_x-k_{x0})n_xd_x} = \frac{2\pi}{d_x}\sum_{m_x=-\infty}^{\infty}\delta(k_x-k_{xm}), \tag{45a}$$

$$k_{xm} = k_{x0} - \frac{2\pi m_x}{d_x}. \tag{45b}$$

on the LHS side of the MFIE gives

$$\begin{aligned}
&-4\frac{1}{2\pi}\int_{-\infty}^{\infty} D_{\infty}^{HM}(k_x)V(k_x)\sum_{n_x=-\infty}^{\infty} e^{j(k_x-k_{x0})n_x d_x}e^{-jk_x x}dk_x \\
&= -4\frac{1}{2\pi}\int_{-\infty}^{\infty} D_{\infty}^{HM}(k_x)V(k_x)\frac{2\pi}{d_x}\sum_{m_x=-\infty}^{\infty}\delta(k_x-k_{xm})e^{-jk_x x}dk_x \\
&= -4\frac{1}{d_x}\sum_{m_x=-\infty}^{\infty}\int_{-\infty}^{\infty} D_{\infty}^{HM}(k_x)V(k_x)e^{-jk_x x}\delta(k_x-k_{xm})dk_x \\
&= -4\frac{1}{d_x}\sum_{m_x=-\infty}^{\infty} D_{\infty}^{HM}(k_{xm})V(k_{xm})e^{-jk_{xm}x},
\end{aligned} \tag{46}$$

where the definition of the delta function δ is applied. Therefore, the MFIE becomes

$$-4\frac{1}{d_x}\sum_{m_x=-\infty}^{\infty} D_{\infty}^{HM}(k_{xm})V(k_{xm})e^{-jk_{xm}x} = \sum_{n_x=-\infty}^{\infty} I_0\frac{rect_{\delta_s}(x-n_x d_x)}{\delta_s}e^{-jk_{x0}n_x d_x}. \tag{47}$$

*D. Derive the Periodic Longitudinal Spectral **HM** Green's Function*

Expressing the spatial Green's function $g_{xx}^{hm}(x,y)$ as the IFT of the spectral Green's function $G_{XX}^{HM}(k_x,k_y)$ in the definition of $d_{\infty}^{hm}(x)$, gives

$$\begin{aligned}
d_{\infty}^{hm}(x) &= \sum_{n_y=-\infty}^{\infty}\int_{-d_y/2}^{d_y/2} m_t(y'')g_{xx}^{hm}(x,-y''-n_y d_y)e^{-jk_{y0}n_y d_y}dy'' \\
&= \sum_{n_y=-\infty}^{\infty}\int_{-d_y/2}^{d_y/2} m_t(y'')\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{XX}^{HM}(k_x,k_y)e^{-jk_x x}e^{-jk_y(-y''-n_y d_y)}dk_x dk_y e^{-jk_{y0}n_y d_y}dy'' \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-d_y/2}^{d_y/2} m_t(y'')e^{jk_y y''}dy'' G_{XX}^{HM}(k_x,k_y)e^{-jk_x x}\sum_{n_y=-\infty}^{\infty} e^{j(k_y-k_{y0})n_y d_y}dk_x dk_y \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-d_y/2}^{d_y/2} m_t(y'')e^{jk_y y''}dy'' G_{XX}^{HM}(k_x,k_y)e^{-jk_x x}\frac{2\pi}{d_y}\sum_{m_y=-\infty}^{\infty}\delta(k_y-k_{ym})dk_x dk_y \\
&= \frac{1}{d_y}\frac{1}{2\pi}\sum_{m_y=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-d_y/2}^{d_y/2} m_t(y'')e^{jk_y y''}dy'' G_{XX}^{HM}(k_x,k_y)e^{-jk_x x}\delta(k_y-k_{ym})dk_x dk_y,
\end{aligned} \tag{48}$$

where the Floquet theorem is applied, and has Floquet modes defined by

$$k_{ym} = k_{y0} - \frac{2\pi m_y}{d_y}. \tag{49}$$

Assuming $m_t(y)$ is edge-singular, then, its FT is zeroth-order Bessel function of first kind, similarly to the single infinite slot, therefore $d_{\infty}^{hm}(x)$ becomes

$$\begin{aligned}
d_{\infty}^{hm}(x) &= \frac{1}{d_y}\frac{1}{2\pi}\sum_{m_y=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_0(\frac{k_y w_s}{2})G_{XX}^{HM}(k_x,k_y)e^{-jk_x x}\delta(k_y-k_{ym})dk_y dk_x \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{d_y}\sum_{m_y=-\infty}^{\infty} G_{XX}^{HM}(k_x,k_{ym})J_0(\frac{k_{ym}w_s}{2})e^{-jk_x x}dk_x.
\end{aligned} \tag{50}$$

From the definition of the IFT of $d_{\infty}^{hm}(x)$ in Eq.41, $D_{\infty}^{HM}(k_x)$ is derived as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} D_{\infty}^{HM}(k_x)e^{-jk_x x}dk_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{d_y}\sum_{m_y=-\infty}^{\infty} G_{XX}^{HM}(k_x,k_{ym})J_0(\frac{k_{ym}w_s}{2})e^{-jk_x x}dk_x, \tag{51a}$$

$$D_{\infty}^{HM}(k_x) = \frac{1}{d_y}\sum_{m_y=-\infty}^{\infty} G_{XX}^{HM}(k_x,k_{ym})J_0(\frac{k_{ym}w_s}{2}). \tag{51b}$$

E. Derive the Spectral Slot Voltage

The IFT on RHS of the MFIE, in Eq.47, is taken and the Floquet theorem applied, thus, the RHS becomes

$$\sum_{n_x=-\infty}^{\infty} I_0 \frac{rect_{\delta_s}(x-n_x d_x)}{\delta_s} e^{-jk_{x0}n_x d_x} = I_0 \frac{1}{d_x} \sum_{m_x=-\infty}^{\infty} sinc(\frac{k_{xm}\delta_s}{2}) e^{-jk_{xm}x}; \tag{52}$$

therefore, the MFIE in Eq.47 becomes

$$-4\frac{1}{d_x} \sum_{m_x=-\infty}^{\infty} D_{\infty}^{HM}(k_{xm}) V(k_{xm}) e^{-jk_{xm}x} = I_0 \frac{1}{d_x} \sum_{m_x=-\infty}^{\infty} sinc(\frac{k_{xm}\delta_s}{2}) e^{-jk_{xm}x}, \tag{53}$$

which is valid for every x, hence, the summating terms are equals and the equation is simplified to

$$-4D_{\infty}^{HM}(k_{xm}) V(k_{xm}) = I_0 sinc(\frac{k_{xm}\delta_s}{2}). \tag{54}$$

Therefore, $V(k_x)$ is

$$\boxed{V(k_{xm}) = -I_0 \frac{sinc(\frac{k_{xm}\delta_s}{2})}{4D_{\infty}^{HM}(k_{xm})}.} \tag{55}$$

F. Derive the Spatial Slot Voltage

The spatial slot voltage $v(x)$ is derived by taking the DIFT of $V(k_x)$

$$\boxed{v(x) = -I_0 \frac{1}{d_x} \sum_{m_x=-\infty}^{\infty} \frac{sinc(\frac{k_{xm}\delta_s}{2})}{4D_{\infty}^{HM}(k_{xm})} e^{-jk_{xm}x}.} \tag{56}$$

G. Derive the Active Input Impedance

Similarly to the single infinite slot, the active input impedance is

$$\begin{aligned} Z_{a,slot} &= \frac{1}{I_0}\frac{1}{\delta_s} \int_{-\delta_s/2}^{\delta_s/2} v(x)dx \\ &= \frac{1}{I_0}\frac{1}{\delta_s} \int_{-\delta_s/2}^{\delta_s/2} -I_0 \frac{1}{d_x} \sum_{m_x=-\infty}^{\infty} \frac{sinc(\frac{k_{xm}\delta_s}{2})}{4D_{\infty}^{HM}(k_{xm})} e^{-jk_{xm}x} dx \\ &= -\frac{1}{d_x} \sum_{m_x=-\infty}^{\infty} \frac{sinc(\frac{k_{xm}\delta_s}{2})}{4D_{\infty}^{HM}(k_{xm})} \frac{1}{\delta_s} \int_{-\delta_s/2}^{\delta_s/2} e^{-jk_{xm}x} dx, \end{aligned} \tag{57}$$

where the IFT of the rectangular function is recognized. Therefore, $Z_{a,slot}$ is

$$\boxed{Z_{a,slot} = -\frac{1}{d_x} \sum_{m_x=-\infty}^{\infty} \frac{sinc^2(\frac{k_{xm}\delta_s}{2})}{4D_{\infty}^{HM}(k_{xm})}.} \tag{58}$$