Brief Explaination of
MultiVariate Number Generator (MultiVariRNG)

Pete Lealiiee Jr

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1 Introduction

The Multi-Variate Random Number Generator (MultiVariRNG) uses statistical theory to transform a random vector sample, $\bar{\mathbf{x}}$ -whose population has a mutually independent standard normal distribution, into a sample, $\bar{\mathbf{y}}$ -whose population has a distribution which is representative of an **input covariance matrix and mean vector**.

The methods used to accomplish this transformation are rooted from the following:

1. Cholesky Decomposition/Factorization
2. Definition of Standard Normal Distribution

2 Proofs

Note that the following proof is well known amongst those familiar with this transformation and this can be found in many textbooks; However, for those unfamiliar, it is proven below. Additionally, note that in the following proof, $\mathbf{C_i}$ is the covariance matrix which is representative of population $\bar{\mathbf{i}}$.

The proof begins with the initial assumption that the transformation to change the population's covariance takes the form

$$\bar{\mathbf{y}} = \mathbf{A}\bar{\mathbf{x}} \tag{1}$$

Recall the definition of the Covariance Matrix shown below, where $\bar{\boldsymbol{\mu}}_{\boldsymbol{y}}$ is the mean vector of the population $\bar{\mathbf{y}}$.

$$\mathbf{C_y} = E[(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_{\boldsymbol{y}})(\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_{\boldsymbol{y}})^T] \tag{2}$$

Expanding Eq. 1 using Eq. 2

$$\mathbf{C_y} = E[(\mathbf{A}\bar{\mathbf{x}} - \mathbf{A}\bar{\boldsymbol{\mu}}_{\boldsymbol{x}})(\mathbf{A}\bar{\mathbf{x}} - \mathbf{A}\bar{\boldsymbol{\mu}}_{\boldsymbol{x}})^T] \tag{3}$$

$$\mathbf{C_y} = E[\mathbf{A}(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}}_{\boldsymbol{x}})(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}}_{\boldsymbol{x}})^T\mathbf{A}^T] \tag{4}$$

$$\mathbf{C_y} = \mathbf{A}(E[(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}}_{\boldsymbol{x}})(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}}_{\boldsymbol{x}})^T])\mathbf{A}^T \tag{5}$$

$$\mathbf{C_y} = \mathbf{A}(\mathbf{C_x})\mathbf{A}^T \tag{6}$$

From this point, we define the population of the elements in the sample, $\bar{\mathbf{x}}$, to have a **Standard Normal Distribution** $\{1\}$ and be **Mutually Independent** $\{2\}$. These indicate the following about $\bar{\mathbf{x}}$:

1. **SND:** Elements have a Standard Deviation of One and Mean of Zero
2. **MI:** (Variance-)Covariance Matrix is Diagonal consisting of only variance terms in the form:

$$\mathbf{C_x} = \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sigma_k^2 \end{bmatrix} \tag{7}$$

Combining the indications stated above:

$$\mathbf{C_x} = \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sigma_k^2 \end{bmatrix} = \begin{bmatrix} 1^2 & 0 & \dots & 0 \\ 0 & 1^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1^2 \end{bmatrix} = \mathbf{I}_{k\times k} \tag{8}$$

Substituting Eq. 8 into Eq. 6

$$\mathbf{C_y} = \mathbf{A}(\mathbf{I})\mathbf{A}^T = \mathbf{A}\mathbf{A}^T \tag{9}$$

Solving for $\mathbf{A}$, assuming $\mathbf{C_y}$ is a hermitian/symmetric positive definite matrix, is easily done using **Cholesky decomposition**. Click the **link** for more information.

How do we use this information? Notice that if we want the distribution of a random variable to have a certain covariance, $\mathbf{C_i}$, then we can (Cholesky-)decompose the covariance matrix we want in order to get the lower triangular matrix, $\mathbf{A_i}$. Then we can transform a mutually-independent and standard-normal-distributed random variable, $\bar{\mathbf{x}}$, with $\mathbf{A_i}$ (Eq. 1) to get the transformed random variable $\bar{\mathbf{y}}$ which has covariance represented by $\mathbf{C_i}$.

Because both of the previous random variable populations will inherit the zero mean of the standard normal distribution, we can add an input mean vector to the transformed random variable, $\bar{\mathbf{y}}$, to gives us a final transformated random variable, $\bar{\mathbf{z}}$, with covariance $\mathbf{C_i}$ and mean $\bar{\boldsymbol{\mu}}_{\boldsymbol{i}}$:

$$\bar{\mathbf{z}} = \mathbf{A_i}\bar{\mathbf{x}} + \bar{\boldsymbol{\mu}}_{\boldsymbol{i}} \tag{10}$$

3 C++ Implementation

From the **Proofs** Section, notice that the only complicated things we need to find (or do) is:

1. Generate Mutually Independent Standard Normal Distributed Random Variables
2. Find a method to Cholesky Decompose an Applicable Matrix

Each of these is easily accomplished with C++ libraries as seen below:

1. The random numbers according to {1} is achieved in the code with `std::normaldistribution<type>` and its appropriate mechanisms as seen in the `src` code. Click the **link** for more information.
2. We can Cholesky decompose a matrix ({2}) with the `Eigen` library and the `Matrix` class member function `llt().matrixL()` and its appropriate mechanisms as seen in the `src` code. Click the **link** for general information.