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value given by the probe measurements is in fair agreement with the values determined from the cut-off curves. That the latter points seem somewhat too high may be due to tube dissymmetries which would cause the cut-off curve to have a greater slope, and cause the temperature determined therefrom to be too high.

It may be concluded that the shape of the cut-off curve can be satisfactorily explained on the basis of the random electron motion. The effect of this motion on other magnetron characteristics such as orbit shape, transit time, transit angle, tube noise, etc., appears not to have been considered in any publications up to the present time, but there can be little doubt that these are all affected in an important manner.

The Numerical Solution of Laplace's Equation

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This paper considers in detail numerical methods of solving Laplace's equation in an arbitrary two-dimensional region with given boundary values. The methods involve the solution of approximating difference equations by iterative procedures. Modifications of the standard Liebmann procedure are developed which lead to a great increase in the convenience and rapidity of obtaining such a numerical solution. These modifications involve the use of formulas which simultaneously improve a block of points in place of a single point; methods of operating on the differences of trial functions in place of the functions themselves; and also a method of extrapolating to the final solution of the difference equations. The theory underlying these procedures is considered in detail by a new method which involves the expansion of the error and difference functions in terms of eigenfunctions. This permits definite comparison of rates of convergence of various procedures. The techniques of handling practical problems are considered in detail.

§1. INTRODUCTION

THERE are many industrial and engineering problems in which solutions of Laplace's equation are needed. These solutions cannot often be obtained analytically because of the occurrence of irregular boundaries for which systems of separable coordinates are unknown. For example, our interest in this equation was aroused by the need for a convenient method of obtaining from the fringe photographs of photoelasticity the principal stresses in the interior of the specimen tested. The photographs give directly only the *difference* of the two principal stresses. On the boundary of the specimen the normal component of the stress vanishes, so here we have complete information. The *sum* of the principal stresses satisfies Laplace's equation; values of this sum in the interior of the specimen can be obtained from the fringe photographs by solving the two-dimensional Laplace equation, using the known boundary values.[1] A numerical method must be used to obtain this solution since the shape of the specimen is usually much too irregular to admit of analytical treatment. The same necessity for numerical solution of Laplace's equation arises in the calculation of electrostatic fields in regions enclosed by boundaries at known potentials; in the calculation of steady state temperature distributions in heat conduction problems; in the calculation of the shapes of films and membranes; and in certain problems of hydrodynamics and gravitation.

We shall restrict our discussion to the solution of the Laplace equation in a two-dimensional

[1] Details of the application to photoelasticity of the theory outlined in this paper will be found in Weller and Shortley, "Calculation of Internal Stresses in Photoelasticity," to be published soon in the *Journal of Applied Mechanics*. This reference should be consulted for an explicitly solved example.

region with given arbitrary boundary values, with some mention of the case where the normal derivative at the boundary is given instead of the boundary value. The methods we shall give can be readily extended to three dimensions, but the amount of labor involved in solving a three-dimensional problem will be tremendously greater than that required for a similar two-dimensional problem.

Any numerical method of handling a differential equation necessarily involves the replacement of the continuum of points on the boundary and in the interior of the region by a discrete set of points. The best method which has been previously given for handling Laplace's equation is an iterative procedure due to Liebmann.[2] In this, a square net is laid down over the region; approximate values are assigned to the function at the interior points—known values at the boundary points. One then traverses the net repeatedly, replacing the value at each interior point by the sum of the values at the four neighboring points, using the new values immediately in the improvement of the succeeding points. The function will then converge to a solution of the Laplace difference equation

$$[\omega(x, y+h)+\omega(x, y-h)+\omega(x+h, y)+\omega(x-h, y)-4\omega(x, y)]/h^2=0. \quad (1)$$

which will be an approximation to the solution of the Laplace differential equation—the smaller the interval h, the better the approximation.

[2] An examination of the literature of mathematical and applied physics indicates that numerical methods of solving Laplace's equation have received little attention. Liebmann's paper appears in Sitzungsber. der math.-phys. Klasse der Bayer. Akad., München, 1918, p. 385. The method is discussed by Phillips and Wiener, J. Math. and Phys. **2**, 105 (1923); Courant, Zeits. f. Angew. Math. u. Mech. **6**, 322 (1926); Frank von Mises, *Differential und Integralgleichungen der Mathematischen Physik* I, p. 735. In these references little attention is given to the application of the method to practical problems: none of the material contained in the present paper is discussed. A different successive approximation method for the solution of the difference equation is given by L. F. Richardson, Trans. Roy. Soc. **A210**, 307 (1910); Math. Gazette **12**, 415 (1925). This method is similar to the Liebmann procedure but does not use the new values, as they are obtained, in the improvement of succeeding points. This makes the convergence much slower (cf. section e., Appendix) unless considerable care is exercised in the assignment of certain arbitrary multiplying factors. The method does not have the general utility of the Liebmann, and at its best does not converge as rapidly as the modified methods we shall propose.

It has been shown quite generally by Richardson[2] that the difference between the solution of a differential equation and that of an approximating difference equation on a net of interval h is of the form

$$A(x, y)h^2+B(x, y)h^4+C(x, y)h^6+\cdots. \quad (2)$$

Since only even powers of h enter here, the difference solution approaches the differential solution rapidly as h approaches zero. If h is sufficiently small the error is proportional to h^2, and by making two solutions with different h, one can estimate the difference of each from the solution of the continuous problem.

In this paper we shall first investigate, by a new method, the rate of convergence of the iterative process to the final solution of the difference equation in the case of the Liebmann procedure. These considerations lead to the formulation of procedures of similar type (in particular the "nine-block" procedure) which converge much more rapidly than the Liebmann. They also lead to the development of an accurate method for extrapolating to the final solution of the difference equations. In this way we greatly reduce the amount of labor required in obtaining a numerical solution of Laplace's equation to any desired accuracy and render the task of calculating the solution entirely practical and convenient. The last section of this paper is devoted to a detailed discussion of practical techniques.

§2. Theory

In this section we shall for simplicity assume that we are working with a square net of interval h in a region with a rectangular boundary. The slight modifications of the theory needed in case of a more general net or an irregular boundary will be discussed in §3 under the heading of Technique. We shall first examine the rate of convergence of the Liebmann iteration procedure.

Denote the values of a function at the interior points of the lattice by ψ_i, where $i=1, 2, \cdots, n$. Denote the values at the boundary points by ψ_α, where $\alpha=n+1, n+2, \cdots, m$. For example, for the lattice of Fig. 1, $n=54$, $m=88$.

In the Liebmann procedure for solving the Laplace difference equation (1) with assigned boundary values ψ_α one starts by assuming any

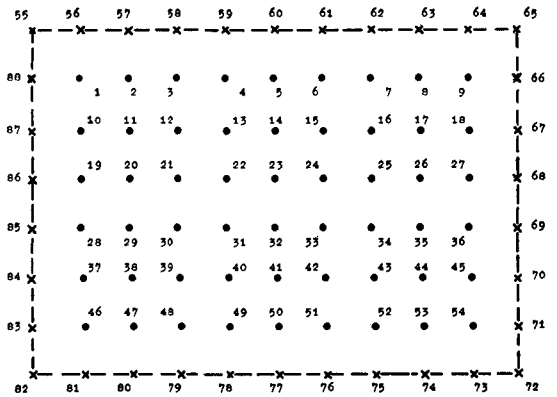

FIG. 1. A net with 54 interior points and 34 boundary points.

values ψ_i whatsoever at the interior points. Then one works over the lattice, e.g. in the order in which the points are numbered in Fig. 1, replacing each ψ_i by the mean of the four neighboring values to get an improved value $\bar{\psi}_i$. *The previously improved values are used in calculating this mean wherever they are available*, so that e.g. in Fig. 1,

$$\bar{\psi}_{13}=\tfrac{1}{4}(\bar{\psi}_{12}+\bar{\psi}_{4}+\psi_{14}+\psi_{22}). \tag{3}$$

The function $\bar{\psi}$ is then closer to the solution of the difference equation than was ψ.[3] Iteration of this process leads to values which are as close to the true solution as may be desired.

In this improvement,[4] $\bar{\psi}_i$ may be expressed as a linear combination of all the ψ_i and the ψ_α, since for example in (3) the $\bar{\psi}$'s on the right are themselves combinations of other $\bar{\psi}$'s and ψ's which in the last analysis reduce to combinations of ψ's alone. Hence we may write

$$\bar{\psi}_i=\sum_j T_{ij}\psi_j+\sum_\alpha U_{i\alpha}\psi_\alpha, \qquad \bar{\psi}_\alpha=\psi_\alpha, \tag{4}$$

where the T_{ij} and $U_{i\alpha}$ are numerical coefficients which depend on the size of the net and the order of improving the points, but not on the particular boundary values given nor the particular body values assumed.

Now let ω represent the true solution of the difference equation (1). The *trial function* ψ may then be expressed as the sum of the *true solution* ω and an *error* ϕ:

$$\psi_i=\omega_i+\phi_i, \qquad \psi_\alpha=\omega_\alpha+\phi_\alpha=\omega_\alpha. \tag{5}$$

Since the trial function must be chosen with the correct boundary values ω_α, the error ϕ will have vanishing boundary values: $\phi_\alpha=0$. Since the transformation (4) is *linear*, the function obtained when ψ is "improved" will be the sum of the functions obtained when ω and ϕ are improved: $\bar{\psi}=\bar{\omega}+\bar{\phi}$. But $\bar{\omega}=\omega$—*the improvement process is such that it does not change the true solution*—this is seen to be true when the improvement formula (3) operates on any solution of (1). Hence

$$\bar{\psi}=\omega+\bar{\phi}, \tag{6}$$

and in order to study the rate of convergence of an arbitrary function ψ with correct boundary values to the true solution ω on iteration of the improvement, we need only investigate the rate of convergence of an arbitrary "error" function ϕ with zero boundary values to zero as the same improvement formula is used repeatedly. The two rates of convergence are the same. Since $\phi_\alpha=0$, we have simply

$$\bar{\phi}_i=\sum_j T_{ij}\phi_j, \tag{7}$$

or briefly

$$\bar{\phi}=\boldsymbol{T}\phi. \tag{7'}$$

Now the possible functions ϕ are best classified in terms of the eigenfunctions (characteristic vectors) of the matrix $\boldsymbol{T}$. There will be a set of functions $\phi^{(1)}, \phi^{(2)}, \cdots, \phi^{(n)}$ such that

$$\boldsymbol{T}\phi^{(i)}=\lambda_i\phi^{(i)},$$

where λ_i is a constant. We show in the Appendix that *all of the $|\lambda|$'s will be less than unity*. This set of eigenfunctions may be considered as forming a complete set[5] in terms of which any function

[3] See the convergence proof in the Appendix, or in the references noted in footnote 2.

[4] The arguments of the next four paragraphs apply equally well to the other types of improvement procedures considered in the balance of the paper and in the Appendix.

[5] The arguments of the text at this point are not rigorously true and therefore the conclusions drawn from them need justification which is furnished in this footnote. The discussion in the text would be mathematically correct only if the matrix $\boldsymbol{T}$ were symmetric, which a calculation for a simple lattice shows to be not the case. For example for a square lattice of four interior points, improved in the order

1 2
3 4

ϕ (with zero boundary values) may be expanded:

$$\phi = a_1\phi^{(1)} + a_2\phi^{(2)} + \cdots + a_n\phi^{(n)}. \qquad (13)$$

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the matrix $\boldsymbol{T}$ is the following nonsymmetric array:

$$\begin{matrix} 0 & 8 & 8 & 0 \\ 0 & 2 & 2 & 8 \\ 0 & 2 & 2 & 8 \\ 0 & 1 & 1 & 4 \end{matrix} \qquad (8)$$

(Here the elements of the top row are T_{11}, T_{12}, T_{13}, T_{14}; etc.) A nonsymmetric matrix cannot in general be reduced to diagonal form; its roots will not necessarily be real, and it will not have a sufficient number of *eigenvectors* to permit expansion in terms of them. But it can be reduced to a unique stepmatrix with steps of the form

$$\begin{matrix} \lambda_q & 1 & 0 & \cdot & 0 & 0 \\ 0 & \lambda_q & 1 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \lambda_q & 1 \\ 0 & 0 & 0 & \cdot & 0 & \lambda_q \end{matrix} \qquad (9)$$

Associated with the root λ_q there will hence be a group of *invariant vectors* $\chi^{(q1)}$, $\chi^{(q2)}$, $\cdots$ satisfying the relations

$$\begin{aligned} \boldsymbol{T}\chi^{(q1)} &= \lambda_q\chi^{(q1)} \\ \boldsymbol{T}\chi^{(q2)} &= \lambda_q\chi^{(q2)} + \chi^{(q1)} \\ \boldsymbol{T}\chi^{(q3)} &= \lambda_q\chi^{(q3)} + \chi^{(q2)} \\ &\cdots\cdots\cdots; \end{aligned} \qquad (10)$$

of these only $\chi^{(q1)}$ is an eigenvector. The entire set of invariant vectors is complete and the expansion (13) must be made in terms of them. The canonical form of the matrix (8) is

$$\begin{matrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \qquad (11)$$

while the canonical form of the matrix for a six-point net improved in the order

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}$$

is

$$\begin{matrix} .3643 & 0 & 0 & 0 & 0 & 0 \\ 0 & .0625 & 0 & 0 & 0 & 0 \\ 0 & 0 & .0107 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad (12)$$

In spite, however, of the fact that $\boldsymbol{T}$ is not a symmetric matrix, it is convenient to speak in the text as if it were, because of the simplicity and familiarity of the simple eigenvalue theory of symmetric matrices. This is permissible because we are not led into any essential error thereby. It may be shown that repeated application of a stepmatrix composed of steps such as (9) to an arbitrary vector reduces finally to the *eigenvector* of the root of highest absolute value. Thus the conclusions of the text are correct. The rapidity of this convergence to the eigenvector of the highest root is however less, the greater the size of the step involving this particular root. But it seems likely from geometrical and asymptotic considerations that the largest roots are all real and belong to steps of order 1, and that complex roots and invariant vectors which are not eigenvectors occur only for the very small roots, where they are quickly wiped out by the iteration and do not cause trouble. These statements are borne out by the form of (11) and (12), in which the only invariant vectors which are not eigenvectors belong to the root 0. Hence all the text arguments may be said to be "practically" true.

For reference to the theory of nonsymmetric matrices see Bôcher, *Higher Algebra*, Chap. XXI, or Wedderburn, *Matrices*, Chap. III.

Then $\boldsymbol{T}\phi = \lambda_1 a_1\phi^{(1)} + \lambda_2 a_2\phi^{(2)} + \cdots,$

$$\boldsymbol{T}\boldsymbol{T}\phi = \boldsymbol{T}^2\phi = \lambda_1^2 a_1\phi^{(1)} + \lambda_2^2 a_2\phi^{(2)} + \cdots,$$

$$\cdots\cdots\cdots\cdots$$

$$\boldsymbol{T}^k\phi = \lambda_1^k a_1\phi^{(1)} + \lambda_2^k a_2\phi^{(2)} + \cdots.$$

So after k applications of $\boldsymbol{T}$, i.e., after k traverses, the coefficients of the various eigenfunctions in ϕ are decreased by λ^k. After a time only the function with the highest eigenvalue, say λ_1, will remain. The whole rate of convergence of ϕ to zero, and hence of ψ to ω will depend on the rate at which the $\phi^{(1)}$ part of ϕ is wiped out—i.e., on the size of λ_1.

These considerations furnish a practical method of finding λ_1 and $\phi^{(1)}$ for a given lattice and order of improvement. $\phi^{(1)}$ will clearly be "smooth," nodeless, and will have a maximum near the center of the region in question. Take a trial function of this general shape, iterate repeatedly, and soon all the terms involving $\phi^{(i)}$'s of lesser eigenvalue will have dropped out, leaving a function proportional to $\phi^{(1)}$, which on each traverse is reduced uniformly to λ_1 times itself.[6] In this way, the values of λ_1 for regions with square boundaries of various sizes were determined for the Liebmann procedure. The values are plotted in Fig. 2 in the form of a curve giving $-p^2 \log_e \lambda_1$ as a function of p, the number of intervals on a side of the square $[(p-1)^2 = n$,

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[6] The determination of this λ_1 value by iteration is facilitated by the following theorems, which apply to the improvement of functions of zero boundary value.

1. *If the nowhere negative function ϕ goes on improvement into a function $\bar{\phi}$ such that for all lattice points $\mu \leq \bar{\phi}_i/\phi_i \leq \nu$, then $\mu \leq \lambda_1 \leq \nu$.* This theorem follows if we show that the bounds on $\bar{\phi}/\phi$ imply the same bounds on $\boldsymbol{T}\bar{\phi}/\bar{\phi}$, and this follows by noting that in (7) all the T_{ij}'s are nonnegative (see Appendix for discussion of this property of T_{ij} in connection with the various types of improvement formulae).

2. When one has arrived at the stage where all the eigenfunctions have disappeared except the first and second, and if ϕ is everywhere positive, one will find that along the nodes of $\phi^{(2)}$, $\bar{\phi}_i/\phi_i = \lambda_1$ and is constant on iteration; in regions where $\phi^{(2)} > 0$, $\bar{\phi}_i/\phi_i < \lambda_1$ but increases and approaches λ_1 with iteration; in regions where $\phi^{(2)} < 0$, $\bar{\phi}_i/\phi_i > \lambda_1$ but decreases and approaches λ_1. To prove these statements consider a function of the form $\phi = \phi^{(1)} + \phi^{(2)}$. On iteration $\phi \rightarrow \bar{\phi} = \boldsymbol{T}\phi \rightarrow \boldsymbol{T}\bar{\phi}$. Investigation of the successive ratios, $\boldsymbol{T}\phi/\phi$ and $\boldsymbol{T}\bar{\phi}/\bar{\phi}$ gives the above results when we remember that $\lambda_2 < \lambda_1$. Hence *when one arrives at the point where the ratio $\bar{\phi}/\phi$ between successive functions is decreasing with iteration over roughly half the region, increasing over the other half, and standing constant along the line dividing these regions, the value of the ratio along this dividing line* is λ_1. By making use of this theorem one need not wait for $\phi^{(2)}$ to disappear but only for $\phi^{(3)}$, $\cdots$, to disappear in determining λ_1.

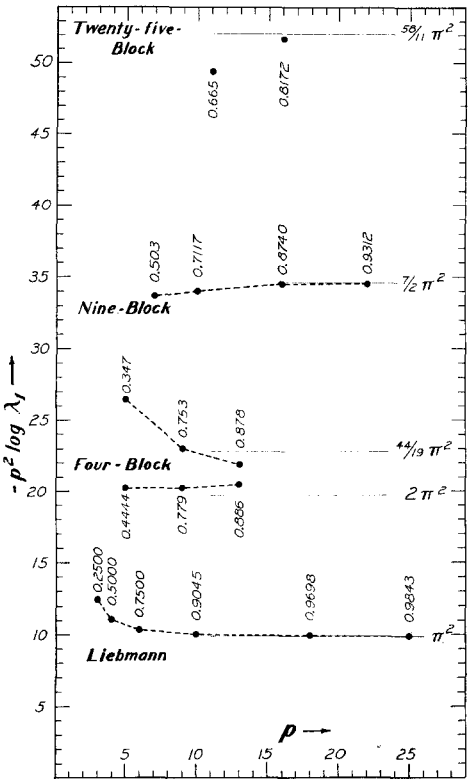


FIG. 2. Relative rates of convergence of the different procedures. The numbers opposite the points are the values of λ_1 for a square region with p intervals on a side.

the total number of interior points]. This form was chosen because asymptotically for large p, $-p^2 \log \lambda_1 \to \pi^2$, i.e. for large p

$$-\log \lambda_1 = 1-\lambda_1 = \pi^2/p^2. \qquad (14a)^7$$

[7] In the case of a rectangular region of p by q intervals, this asymptotic formula becomes

$$-\log \lambda_1 = 1-\lambda_1 = \tfrac{1}{2}\pi^2\left(\frac{1}{p^2}+\frac{1}{q^2}\right). \qquad (14b)$$

A lower eigenvalue $\lambda_{\mu,\nu}$, which belongs to the function with $\mu-1$ nodes in one direction and $\nu-1$ nodes in the other, has the same asymptotic value as that for the nodeless eigenfunction in a region of p/μ by q/ν intervals. Hence asymptotically

$$-\log \lambda_{\mu,\nu} = 1-\lambda_{\mu,\nu} = \tfrac{1}{2}\pi^2\left(\frac{\mu^2}{p^2}+\frac{\nu^2}{q^2}\right). \qquad (14c)$$

The derivation of these asymptotic relations is discussed in the Appendix. Although for small regions the form of $\phi^{(1)}$ changes considerably with a change in the order of improvement of the points, the value of λ_1 is to a good approximation independent of this order. The curve of Fig. 2 was obtained by the use of similar orders for the various size regions, but for all practical purposes it is valid for any order. No particular order seems to have appreciable advantage over any other. While we have no rigorous proof, we believe the asymptotic value to be independent of the order of improvement.

The number of traverses which must be made in order to reduce the coefficient of $\phi^{(1)}$ in (13) to a definite fraction ρ of its initial value is $(-\log \rho)/(-\log \lambda_1)$, so the *value of* $-\log \lambda_1$ *is a measure of the rapidity of convergence of the process.* Approximately, the rapidity of convergence varies inversely as the number of points in the square.

We can then visualize the improvement process as follows: after a number of traverses have been made, the error will be a pillow-shaped multiple of $\phi^{(1)}$. A traverse consists in going over this pillow bringing successive points down to the mean level of their four neighbors. In this way the height of the whole pillow is gradually reduced to zero. We then see that if we could reduce at one step a greater area of the pillow to the "mean" level of its neighbors, the pillow would disappear faster—we would be cutting it down by larger slices. This more rapid slicing would be accomplished if we divided the interior of the region up into square blocks each containing v^2 points and then used improvement formulas which would make a whole block at a time a harmonic function [in the sense of (1)] with boundary values determined by the values of the function at the $4v$ points just outside the block. Such improvement formulas can be obtained by solving the v^2 simultaneous equations which state that the value at each point in the block equals the mean of its four neighbors.

For example, we might divide the region of Fig. 1 up into blocks of nine points (nine-blocks) in the manner indicated in the drawing. We could then improve the six blocks successively in any order, for example the top row from left to right, then the bottom row from left to right. In this case, the improved values in the upper left-hand block would be calculated from the boundary values ψ_{56}, ψ_{57}, ψ_{58}, ψ_4, ψ_{13}, ψ_{22}, ψ_{30}, ψ_{29}, ψ_{28}, ψ_{86}, ψ_{87}, ψ_{88}. The simplest formulas which give harmonic improved values are

$$\bar{\psi}_2 = \tfrac{1}{112}(37\psi_{57}+11\psi_{56}+11\psi_{58}+11\psi_{88}+11\psi_4 +7\psi_{87}+7\psi_{13}+3\psi_{86}+3\psi_{22}+3\psi_{28}+3\psi_{30}+5\psi_{29});$$

$\bar{\psi}_{10}$, $\bar{\psi}_{12}$, $\bar{\psi}_{20}$ are given by corresponding formulas;

$$\bar{\psi}_1 = \tfrac{1}{4}(\psi_{56}+\psi_{88}+\bar{\psi}_2+\bar{\psi}_{10}); \qquad (15)$$

$\bar{\psi}_3$, $\bar{\psi}_{19}$, $\bar{\psi}_{21}$ are given by corresponding formulas;

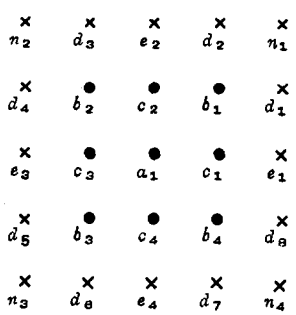
$$\bar{\psi}_{11} = \tfrac{1}{4}(\bar{\psi}_2+\bar{\psi}_{10}+\bar{\psi}_{12}+\bar{\psi}_{20}).$$

In the improvement of the next block, $\bar{\psi}_3$, $\bar{\psi}_{12}$, $\bar{\psi}_{21}$ would be used as three of the boundary values.

By improving by blocks in this way one obtains considerably lower λ values and hence convergence to the final solution after fewer traverses than for the straight Liebmann process. We show in the Appendix that the asymptotic values of $-p^2 \log \lambda$ for a v^2-block are v times the values given by (14), so the convergence is v times as fast as for the Liebmann. But the formulas used are so complex that it takes v times as much labor per traverse, so that little saving in work is made in this way.

Nine-block procedure

But it is possible to modify the improvement formulas[8] so that the improvement can be by nine-blocks with practically no more work per improvement than in the case of the Liebmann. In the nine-block



we require that the value

$$a_1 = \tfrac{1}{16}[d_1+d_2+d_3+d_4+d_5+d_6+d_7+d_8) + 2(e_1+e_2+e_3+e_4)]$$

as would be obtained by using (15); but we may now obtain the values b_i by requiring that they be the means of their four diagonal neighbors, i.e. that

$$b_1 = \tfrac{1}{4}(a_1+e_1+e_2+n_1), \quad \text{etc.},^9$$

and finally that the c points be the means of their immediate neighbors:

$$c_1 = \tfrac{1}{4}(a_1+b_1+e_1+b_4), \quad \text{etc.},$$

using in the last two formulas the improved a and b values.

Precisely the same reasoning which we have applied to the Liebmann procedure may be used in the study of the rate of convergence of this nine-block procedure; empirical values of $\phi^{(1)}$ and λ_1 for nets of various sizes may be similarly obtained, except that in this case the disappearance of all functions below $\phi^{(1)}$ will take place much more rapidly since all eigenvalues will be much smaller. The values of $-p^2 \log \lambda_1$ obtained in this way are plotted in Fig. 2—they are seen to be about 3.5 times the corresponding values for the Liebmann procedure. The asymptotic value of $-p^2 \log \lambda_1$ for this procedure is $7\frac{1}{2}\pi^2 = 34.5$ and other asymptotic eigenvalues for this procedure are given by writing $7\frac{1}{2}\pi^2$ in place of π^2 in (14) (see Appendix). These values are essentially independent of the order of improving the blocks.

Since the number of traverses necessary to accomplish a certain reduction in the coefficient of an eigenfunction in the error is inversely proportional to $-\log \lambda_1$, the number of times the Liebmann procedure must be iterated in order to accomplish reduction by the same factor as in one traverse by the nine-block procedure is just the ratio of the curves of Fig. 2. Thus we may conclude that in general *the nine-block procedure will cut the time required to solve a problem to given accuracy by a factor of almost* 3.5 *as compared to the straight Liebmann method.* This remains true when an extrapolation procedure to be discussed later is used to remove the first eigenfunction after the second has disappeared. The slightly greater inconvenience in improvement of the center-point of the nine-block is made up by a certain advantage in this extrapolation procedure.

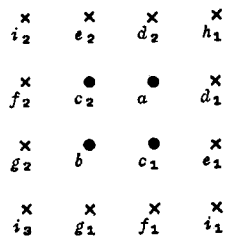
[8] This modification amounts to a change in the set of difference equations which are satisfied by the final solution, so that the final solution for this procedure will be slightly different from that for the Liebmann. That this does not in general impair the accuracy of approximation to the continuous solution is indicated by arguments in Section c of the Appendix.

[9] In case one of the n points is not available, the following formula, derived by the methods outlined in the Appendix, may be used for the corresponding b point:

$$b_1 = \tfrac{1}{8}(2a_1+e_1+2d_1+2d_2+e_2).$$

Four-block procedure

It is possible to use blocks of four in place of blocks of nine, by the following formulas:



$$a=\tfrac{1}{24}(7d_1+7d_2+2e_1+2e_2+2f_1+2f_2+g_1+g_2),$$
$$b=\tfrac{1}{4}(a+f_1+f_2+i_3),$$
$$c_1=\tfrac{1}{4}(a+b+e_1+f_1), \text{ etc.}$$

The formulas should be used in this order, using the improved values of a and b as they are obtained. The value of λ_1 varies according to which of the four points is taken as the point a to which to apply the long formula, in relation to the order of improvement. The upper curve of Fig. 2 is for the case in which the long formula is applied to the point of each four block which is nearest the boundary of the region, the lower curve when this formula is applied to the point nearest the center—improving the blocks row by row as one reads the words of a book. The two asymptotic values given are for this same order of improvement, the higher one, $4\tfrac{4}{19}\pi^2$, when the long formula is applied to one of the points on the left side of each block, the lower, $2\pi^2$, when this formula is applied to a point on the right side.

One sees that the four-block converges somewhat more than twice as fast as the straight Liebmann, and is a definite saving in time, in spite of the fact that the labor per improvement is even more than that for the nine-block.

Twenty-five-block procedure

In the direction of blocks of size larger than nine, one could use a block of 16; in this case it would be necessary to find all four central values directly from the boundary values by means of fairly cumbersome formulae. One can do a twenty-five-block almost as easily, calculating only four points by cumbersome formulae (see Fig. 3). One first obtains the values at the points a_1, a_2, a_3, a_4 by formulas similar to the following:

$$a_1=\tfrac{1}{528}(38d_1+38d_2+76e_1+76e_2+49f_1+49f_2+26g_1+26g_2 +11h_1+11h_2+11i_1+11i_2+18j_1+18j_2+17k_1+17k_2 +12l_1+12l_2+6m_1+6m_2).$$

Then the values at the points marked b are obtained from these and the boundary values by taking the mean of the four diagonal neighbors,[9] and finally the values at the points marked c are obtained from the means of the four closest neighbors. This procedure gives an asymptotic value $-p^2\log\lambda_1=5\tfrac{8}{11}\pi^2=52.0$, and hence considerably more rapid convergence than the nine-block. But it is probably quicker on the whole to introduce only nine-blocks since they are so much simpler and easier to improve.

One compensating factor in favor of the 25-block is that the central b and the four central c values need not be calculated until the iteration is completely finished since they do not enter into the improvement formula for any other points.

It is of course possible to handle still larger blocks by similar methods with even lower λ_1 values, but in these cases so many points would need very complicated improvement formulas that they would certainly not save time on the whole. The optimum block size seems to be nine, with perhaps some arguments in favor of 25.

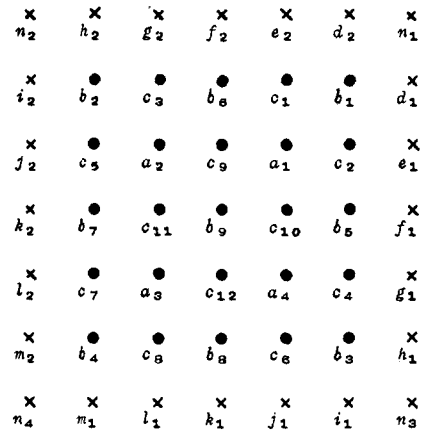


FIG. 3.

§3. TECHNIQUE

In this section we shall summarize the practical techniques which our experience has shown to be best adapted to the solution of the problem in hand.

Choice of net

The first thing of course is to lay over the region in question a lattice. It is usually most convenient to take a square lattice, although there are probably cases where a rectangular lattice is advisable, and cases where it is advisable to change the lattice spacing near sections of the boundary where the function representing the boundary values has exceptionally high or low curvature.[10] The net points at which the value of the function is to be defined are the interior points at the intersections of the bars of the lattice and the boundary points at the intersection of the lattice bars with the boundary of the region, as indicated in Fig. 4.

[10] See Appendix for improvement formulas for these cases.

It is advisable always to start by solving the difference equations with a very coarse net and then successively cut the net spacing in half. The previous solutions of the coarser nets are desirable because at each halving of the net constant, the convergence is about four times as slow (cf. Fig. 2). Hence it is economical of time to get as good a function as possible from a coarse net to use as the initial trial function for the finer net, so as to keep the error in this function as small as possible.

The net should be split up into as many nine-blocks as possible and the remainder into as many four-blocks as possible. Each nine-block or four-block must be surrounded by points at the regular net spacing. These 12 or 8 surrounding points will ordinarily be interior points of the net but may, if a lattice line coincides with a section of the boundary, be boundary points, as at the extreme right of Fig. 4. It seems advisable to observe these requirements rather than to go to the trouble of using formulas modified to take care of unusual distances of the points surrounding the blocks.[9] The more nine- and four-blocks one can use, the lower will be the eigenvalues and the faster the convergence. It occurs to one that it might be advantageous to use blocks which overlap, but we find that the extra labor involved with such blocks more than offsets the gain in rate of convergence.

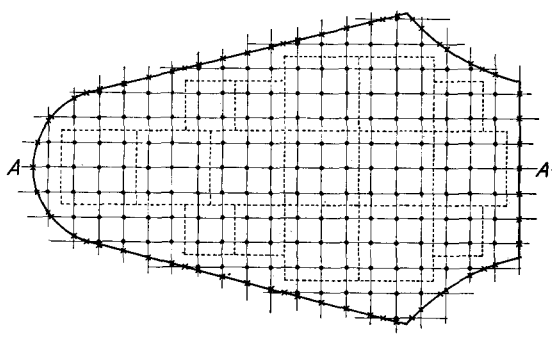


FIG. 4. Location of interior points (circles) and boundary points (crosses).

The reduction of the net spacing should be stopped when one has achieved the desired accuracy in the solution, which may of course not be greater than the accuracy with which the boundary values are known. The amount by which the solution of the difference equations differs from that of the differential equation may be judged from the changes in final solutions for successively finer nets [cf. (2), which however may be expected to apply to the block procedures only on the average since the mode of division of successive nets into blocks will be different].[11]

Choice of trial function

The values of the function at the boundary points are of course determined by the given function which determines the boundary values for the region. In case this function is given only in the shape of a table of values at a discrete set of points around the boundary, the values at the boundary net-points must be determined by interpolation in this table, which is usually done most conveniently by plotting the given boundary values in some convenient way and reading off the values at the net points.

The coarsest net used will usually converge so rapidly that one need not exercise particular care in guessing trial values for the interior points. When the net spacing is halved, the new trial function has the values of the final solution of the coarse net at the points which the two nets have in common and values interpolated from this solution at the remainder of the points. *This interpolation may be very conveniently done by formulas which guarantee that Laplace's equation will be approximately satisfied at these points:* If in Fig. 5 the black dots represent the points of the coarse net, the letters points at which interpolated values are necessary, one takes the value at a point marked a as the mean of the values at the four dots diagonally surrounding a, and then the value at a point marked b as the

[11] Some indication of the error in the difference solution may be obtained by comparing, at various points, the average of the four nearest neighbors and the average of the four diagonal neighbors. The error is certainly greater than the difference between these two averages. For the Liebmann procedure in a square net of $p \times p$ intervals, we can show that asymptotically the error is at most of the order of $0.06\ p^2$ times the difference between the value at a point and the average of its four diagonal neighbors. This is the case where the difference between the value at a point and the average of its four neighbors for the continuous solution is constant over the net. If the continuous solution is less regular the factor 0.06 is to be decreased. Such definite quantitative statements are not easy to make for other procedures.

mean of the values at the four nearest neighbors, two dots and two a points.

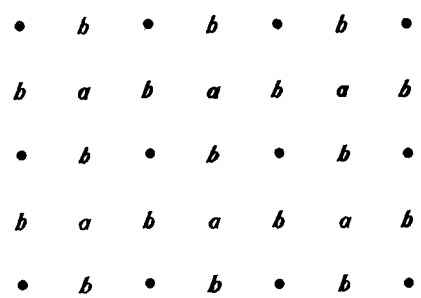
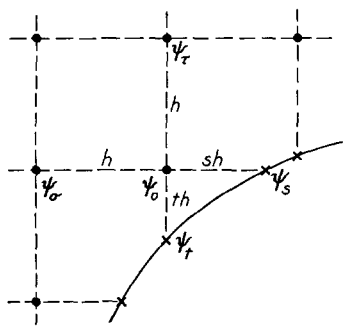


FIG. 5. Interpolation.

Improvement of points near boundary

An interior point which has boundary values for neighbors will ordinarily be surrounded by two interior points at the regular net spacing h



and two boundary points which are closer than h, say at distances sh and th. However it may be surrounded by three interior points, in which case either s or t may be taken as unity. The improved value of ψ_0 in this case is chosen so that Laplace's equation is approximately satisfied at the five points in question. The formula for this improved value, which is a special case of Eq. (23) of the Appendix, is

$$\psi_0=\frac{st}{(s+t)}\left\{\frac{1}{s(1+s)}\psi_s+\frac{1}{1+s}\psi_\sigma+\frac{1}{t(1+t)}\psi_t+\frac{1}{1+t}\psi_\tau\right\}. \quad (16)$$

The complete value of the coefficients of ψ_σ and ψ_τ and the whole terms in ψ_s and ψ_t if these are constant boundary values should be calculated once for all and written on the net. The value ψ_0 is then obtained very simply by multiplying ψ_σ and ψ_τ by known coefficients and adding a known constant.

Axes of symmetry

If there is an axis of symmetry such that the function values are known to be the same at corresponding points on opposite sides of this axis (AA might be such an axis in Fig. 4), it is only necessary to work with the net on one side of this axis. One imagines that each time a value is changed, the value at the corresponding point is simultaneously changed, and in improving values near or on the symmetry axis, one uses these imaginary values as needed. For example, if the left-hand row of points in Fig. 6 (p. 346) lie along a symmetry axis, and one is using the simple Liebmann formula for these points, the improvement formula for r is

$$\psi_r=\tfrac{1}{4}(\psi_s+2\psi_t+\psi_u).$$

One obtains somewhat more rapid convergence in the Liebmann case if the net is chosen so that the symmetry axis is along one of the rows of net points rather than between two rows, and in the block procedures, if the blocks straddle the symmetry axis rather than be located unsymmetrically.

In a similar fashion one can handle a straight section of the boundary where the boundary condition is that the normal derivative of ψ have the value zero, or even have given nonzero values.

Use of the difference function

In most cases the labor of improvement will be materially reduced if one obtains the difference between the first improved value and the original value, and then applies the improvement formula to this difference function. The difference function is easier to handle because it usually has many fewer significant figures than the original function, because it has zero boundary values, and because it permits of convenient extrapolation. In the notation of (5), (6), (7),

$$\psi=\omega+\phi:$$

the trial function equals the true solution plus

the error; when this is improved, one obtains

$$\bar{\psi}=\omega+T\phi.$$

The difference between $\bar{\psi}$ and ψ we call the difference function

$$\delta=\bar{\psi}-\psi=T\phi-\phi.$$

Since this has zero boundary values, if we apply the improvement process to the difference function we obtain successively the functions

$$\begin{aligned} T\delta &= T^2\phi - T\phi, \\ T^2\delta &= T^3\phi - T^2\phi, \\ &\cdots\cdots\cdots \\ T^{n-1}\delta &= T^n\phi - T^{n-1}\phi. \end{aligned} \tag{17}$$

If we add the sum of these functions to $\bar{\psi}$, we obtain

$$\bar{\psi}+T\delta+T^2\delta+\cdots+T^{n-1}\delta=\omega+T^n\phi, \tag{18}$$

which is the same function we would have obtained by improving ψ itself n times.

Extrapolation

The greatest advantage of using the difference function lies in the fact that it is possible at a certain point to make an accurate extrapolation to find the sum of the infinite series of terms of the type (17) which must be added to $\bar{\psi}$ to obtain the true solution ω. [If $n=\infty$ in (18),

$$\bar{\psi}+\sum_{i=1}^{\infty}T^i\delta=\omega,$$

since ϕ disappears after an infinite number of applications of T.] Since δ has vanishing boundary values, it may be expanded in the series, cf. (13),

$$\delta=b_1\phi^{(1)}+b_2\phi^{(2)}+\cdots+b_n\phi^{(n)}. \tag{19}$$

As T is applied successively to δ, the functions $\phi^{(2)}, \cdots, \phi^{(n)}$ disappear before $\phi^{(1)}$, and after a time $T^i\delta$ will become a fairly smooth, pillow-shaped, one-signed function decreasing uniformly by a factor λ_1 at each iteration. At this stage, one has

$$\begin{aligned} T^k\delta &= c\phi^{(1)}, \\ T^{k+1}\delta &= \lambda_1 c\phi^{(1)}, \\ T^{k+2}\delta &= \lambda_1^2 c\phi^{(1)}, \\ &\cdots\cdots\cdots \end{aligned}$$

so that

$$T^k\delta+T^{k+1}\delta+T^{k+2}\delta+\cdots$$
$$=c\phi^{(1)}(1+\lambda_1+\lambda_1^2+\cdots)=T^k\delta/(1-\lambda_1),$$

which furnishes the evaluation of the desired sum. The time saved by this extrapolation depends on the relative rate of convergence of the first and second eigenfunctions—after the second eigenfunction has been iterated out of δ, the first may be extrapolated out.[12]

Resumé

We conclude with a detailed resumé of the calculation for a given net, making use of the difference function and extrapolation.

Having chosen the trial function, the improvement formula to be used for each point, and the order of going over the net, *go over the trial function ψ once to obtain the function $\bar{\psi}$.*

Calculate the difference function $\delta=\bar{\psi}-\psi$. This will have zero boundary values. It need not be computed at points whose *unimproved* function values are not used in the improvement of any other point. In the nine-block procedure, this usually applies to four of the nine points in each block, e.g. to points 41, 42, 50, 51 in the bottom center block of Fig. 1 if the blocks are improved in the order of reading a book.

Improve the function δ repeatedly in the same order and with the same improvement formulas as used for ψ, to obtain successively $T\delta$, $T^2\delta$, $T^3\delta$, $\cdots$. When this function has become fairly smooth, one-signed,[13] and falling approximately

[12] For a large net in a boundary approximately square or circular, the extrapolation will reduce the number of necessary traverses by a factor of 2.5 since $\log\lambda_2/\log\lambda_1\approx 2.5$ in this case, cf. (14c). This factor is reduced as the region becomes more rectangular or elongated, dropping to about 1.3 for a rectangle 3 times as long as broad. But the factor is increased by the presence of axes of symmetry since in such cases the second eigenfunction will have a node along the axis of symmetry and the fact that all the successive approximation functions are automatically symmetrical means that the coefficient of this second eigenfunction in an expansion such as (19) is identically zero; in this case one will be ready for extrapolation as soon as the third eigenfunction has disappeared. For a region approximately square or circular with two crossing symmetry axes, extrapolation reduces the labor by a factor of about 5. For a rectangular region with one axis of symmetry across the short way, the factor is a maximum of 3.2, falling to 1.8 for a rectangle 3 times as long as broad.

[13] Cases sometimes occur where $T^k\delta$ is two-signed with an approximately stationary node and looks more like the second eigenfunction than the first. In this case it is profitable to extrapolate at this stage.

at a uniform rate, calculate the ratio $T^k\delta/T^{k-1}\delta$ at various points of the net. If the higher eigenfunctions were completely eliminated, this ratio would be constant and equal to λ_1. It does not pay to wait for this to become strictly true before extrapolating. Fig. 2 and the discussion of footnote 6 assist in finding an early approximation to the value of λ_1.

Using an approximate λ_1 determined in this way, *add together* $\bar{\psi}$, $T\delta$, $T^2\delta$, $\cdots$, $T^{k-1}\delta$, $T^k\delta/(1-\lambda_1)$ *to obtain a new trial function* ψ'. *Apply the improvement process to* ψ' *to obtain* $\bar{\psi}'$. (*ψ' need be determined only at those points at which its value is used in determining* $\bar{\psi}'$. In the case of the nine-block procedure this fact makes a considerable saving in the time of extrapolation.) The function ψ' should be very close to the final function ω, its departure from ω is measured by the difference between ψ' and $\bar{\psi}'$. If necessary, *calculate a new difference function* $\delta'=\bar{\psi}'-\psi'$ *and repeat the procedure.* The more constant one requires the ratio $T^k\delta/T^{k-1}\delta$ to be before extrapolating the better will be the function ψ', but it seems advisable to make a fairly early extrapolation, and then continue the improvement, for two reasons. First, the new δ' will in general be much smaller than $T^k\delta$, and hence it will be much easier to improve it than to continue improving $T^k\delta$. Second, there is a possibility that one may have made an error in obtaining $\bar{\psi}$ or one of the $T^i\delta$'s, and if this has occurred, it will show up in ψ', so it is advisable to make a fresh start before one does too much work, just as a check.

In order to obtain a certain number of decimal places in the final solution ω, it is necessary to carry more places in $\bar{\psi}$ and δ in order to avoid large rounding-off errors. It is necessary that a change of $1/(1-\lambda_1)$ in the last figure carried in δ represent a change of less than 1 in the place where the last significant figure of ω is to be located. It is desirable for ease of calculation of λ_1 that the number of figures in δ should be chosen more generously than this. For a net not exceeding a couple of hundred points, *it is convenient to carry two more figures in* $\bar{\psi}$ *and* δ *than one wants in the final solution.* One then need carry the process only to the point where the estimated error of extrapolation is less than 50 in the last decimal place to get an error of less than $\frac{1}{2}$ in the last place of ω.

With regard to the time and labor involved in the numerical solution of problems by these methods, no general statement can be made since so much depends upon the accuracy required. To engineering accuracy of one percent a reasonably complicated problem such as arises in photoelasticity can be solved in two or three hours, after the boundary values have been determined.[1] For problems requiring greater accuracy we feel that the method is sufficiently convenient to be entirely practical and deserving of wide application to physical problems.

We believe that methods similar to those discussed above can be applied to the solution of Poisson's equation $\Delta\psi=\rho(x, y)$. We are planning to consider the numerical solution of this equation in the near future.[14]

We are indebted to Professor L. H. Thomas for helpful discussions concerning approximation formulas.

[14] The heat conduction and vibration equation, $\Delta\psi=\lambda\psi$ and Schrödinger's equation $\Delta\psi+\rho(x,y)\psi=\lambda\psi$ are more complicated because an eigenvalue λ must be determined at the same time that ψ is determined. A generalization of the Liebmann procedure suitable for this type of equation has been considered by Kimball and Shortley, Phys. Rev. **45**, 815 (1934).