Data Structures and Algorithms

Lecture 3: Sorting

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Last time

Searching and complexity

- Big-O notation
- Binary search vs linear search

Plan for today:

- **Sorting** algorithms
- Selection sort and merge sort

Asymptotic analysis

Principle 0: measure number of basic operations as function of input size

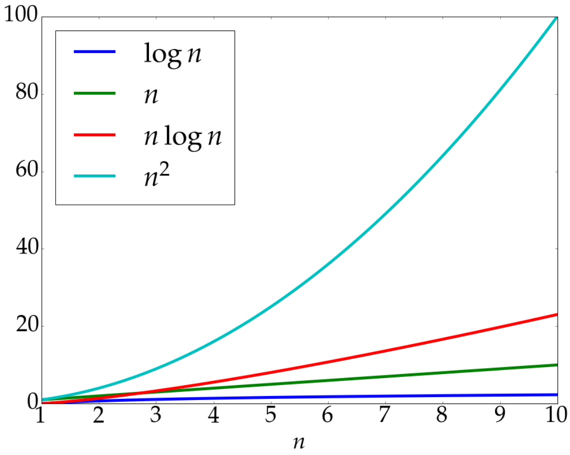
Principle 1: focus on worst-case analysis

Principle 2: ignore constant factors and lower-order terms

Principle 3: only care about large inputs

Formal way to describe this approach:

- Big-O notation: upper bound on worst-case running time



Big O: for **large enough inputs**, an $O(n)$ algorithm will be slower than $O(\log(n))$

Basic operations

Operations that a **computer can perform "quickly"** (constant time $O(1)$ for any input)

- Arithmetic operations (eg `x*y`) (for not too big numbers)
- Comparisons (eg `x > 0`)
- Assign a variable (eg `x = 2`), read/write memory

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What if the data structure is more complicated?

- For example, if `L` is a `list`: `L.append(), L[5]` ?
- Are these basic constant-time operations?
- Wait for Lecture 4... assume for now that list operations are constant time

Complexity classes

Fast algorithm: worst-case running time grows slowly with input size

- $O(1)$: constant running time — basic operations
- $O(\log n)$: logarithmic running time — binary search
- $O(n)$: linear running time — linear search
- $O(n \log n)$: log-linear running time — ??
- $O(n^c)$: polynomial running time — ??
- $O(c^n)$: exponential running time — ??

Sorting algorithms

So if we have an unsorted list, should we sort it first?

- Suppose complexity $O(\mathit{sort}(n))$
- Is it less work to sort and then do binary search than to do linear search?
- Is $\mathit{sort}(n) + \log(n) < n$?
- No...

But what if we need to search repeatedly, say k times?

- Is $\mathit{sort}(n) + k\log(n) < kn$?
- Depends on k...

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In words: Find smallest item and move to front (swap with first unsorted item). Repeat with remaining unsorted items.

Selection sort algorithm

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  - Swap its position with the first unsorted element

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Python:

```
def selection_sort(L):
    M = L[:] # make a copy to preserve original list
    n = len(M)
    for index in range(n):
        min_index = find_min_index(M, index) # index with smallest element
        M[index], M[min_index] = M[min_index], M[index] # swap positions
    return M
```

Let’s assume the function is implemented. What is its complexity?

Selection sort complexity

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- Each pass: search for the smallest element in $O(n)$
- Total $O(n^2)$

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Can we do better?

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Can we do better?

- Yes! **Merge sort** is $O(n \log n)$
- But you can't do any better than that...

How to merge two sorted lists?

x = | 24 | 32 | 56 |

y = | 19 | 57 | 61 |

z = | |

How to merge two sorted lists?

x =	24	32	56	i1 = 0
y =	19	57	61	i2 = 0
z =				

How to merge two sorted lists?

x =	24	32	56	i1 = 0
y =	19	57	61	i2 = 1
z =	19			

How to merge two sorted lists?

x = | 24 | 32 | 56 | i1 = 1

y = | 19 | 57 | 61 | i2 = 1

z = | 19 | 24 |

How to merge two sorted lists?

x =	24	32	56	i1 = 2
y =	19	57	61	i2 = 1
z =	19	24	32	

How to merge two sorted lists?

x = | 24 | 32 | 56 | i1 = 3

y = | 19 | 57 | 61 | i2 = 1

z = | 19 | 24 | 32 | 56 |

How to merge two sorted lists?

x = | 24 | 32 | 56 | i1 = 3

24	32	56

y = i2 = 2

19	57	61

z =

19	24	32	56	57

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24	32	56

i1 = 3

y =

19	57	61

i2 = 3

z =

19	24	32	56	57	61

How to merge two sorted lists?

x = | 24 | 32 | 56 | i1 = 3

y = | 19 | 57 | 61 | i2 = 3

z = | 19 | 24 | 32 | 56 | 57 | 61 |

What is the complexity of this operation?

- Lengths of lists are n_1 and n_2
- Comparisons $O(\max\{n_1, n_2\})$
- Two lists of lengths n_1 and n_2: $O(n_1 + n_2)$ copy operations (need to copy each item)

Sidebar: recursion

The factorial of n is the product of integers $1, ..., n$.

- As a function: $\text{fact}(n) = n * (n-1) * (n-2) * \cdots * 2 * 1$
- By convention, $\text{fact}(0) = 1$

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```python
def fact(n):
    result = 1
    for i in range(1, n+1):
        result = result * i
    return result
print(fact(4))
```

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But we can also write the factorial as follows:

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$$\text{fact}(n) = n * \text{fact}(n-1), \text{ for } n > 0$$

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Factorial can be expressed as a smaller version of itself:

```python
def fact_rec(n):
    if n == 0:
        return 1
    else:
        return n*fact_rec(n-1)
print(fact_rec(4))
```

This is called **recursion**

- Function calls itself
- Can make some problems easier to define -> merge sort!

Merge sort idea

Divide and conquer:

- Identify smallest possible "base case" subproblems that are easy to solve
- Divide large problem and solve smaller subproblems
- Find a way to combine subproblem solutions to solve larger problems

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Merge sort:

- Base case: if list length $n < 2$, the list is sorted
- Divide: if list length $n \geq 2$, split into two lists and merge sort each

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Merge sort:

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- Divide: if list length $n \geq 2$, split into two lists and merge sort each
- Combine (merge) the results of the two smaller merge sorts

Merge sort

Dividing

56	24	99	32	9	61	57	79

Merging

Merge sort

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Merge sort

Dividing

56	24	99	32	9	61	57	79

56	24	99	32

9	61	57	79

56	24

99	32

56

24

Merging

Merge sort

Dividing

56	24	99	32	9	61	57	79

56	24	99	32

9	61	57	79

56	24

99	32

56

24

Merging

24	56

Merge sort

Dividing

56	24	99	32	9	61	57	79

56	24	99	32

9	61	57	79

56	24

99	32

56

24

99

32

Merging

24	56

Merge sort

Dividing

56	24	99	32	9	61	57	79

56	24	99	32

9	61	57	79

56	24

99	32

56

24

99

32

Merging

24	56

32	99

Merge sort

Dividing

| 56 | 24 | 99 | 32 | 9 | 61 | 57 | 79 |

| 56 | 24 | 99 | 32 | | 9 | 61 | 57 | 79 |

| 56 | 24 | | 99 | 32 |

| 56 | | 24 | | 99 | | 32 |

Merging

| 24 | 56 | | 32 | 99 |

| 24 | 32 | 56 | 99 |

Merge sort

Dividing

| 56 | 24 | 99 | 32 | 9 | 61 | 57 | 79 |

| 56 | 24 | 99 | 32 | | 9 | 61 | 57 | 79 |

| 56 | 24 | | 99 | 32 | | 9 | 61 | | 57 | 79 |

| 56 | | 24 | | 99 | | 32 |

Merging

| 24 | 56 | | 32 | 99 |

| 24 | 32 | 56 | 99 |

Merge sort

Dividing

56	24	99	32	9	61	57	79

56	24	99	32

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56 24 99 32 9 61

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24	56

32	99

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Merge sort

Dividing

[56 | 24 | 99 | 32 | 9 | 61 | 57 | 79]

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Merging

[24 | 56] [32 | 99] [9 | 61]

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Dividing

| 56 | 24 | 99 | 32 | 9 | 61 | 57 | 79 |

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Merging

| 24 | 56 | | 32 | 99 | | 9 | 61 |

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Dividing

56	24	99	32	9	61	57	79

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[56 | 24] [99 | 32] [9 | 61] [57 | 79]

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Merging

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24	32	56	99

Merge sort

Dividing

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99	32

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24	56

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Merge sort

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- Does need some more space due to copying lists

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- $O(\log n)$: logarithmic running time — binary search
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- Exploit the fact that lists tend to be partly sorted already

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Review

Sorting is a canonical computer science problem

- We've looked at two (of many) algorithms
- Selection sort involves repeatedly finding minimum element – intuitive but slow
- Merge sort is blazingly fast and has a neat recursive structure

Workshop after the break

- Implement sorting
- More looping and function practice

Workshop

Workshop zip file on the Hub

- HTML instructions
- At some point, you'll need the `.py`-file with skeleton code (open in Spyder)