Data Structures and Algorithms

Lecture 7: Weighted graphs

Heikki Peura

h.peura@imperial.ac.uk

Previously

Graphs:

- Representing a network as a graph
- Breadth-first search

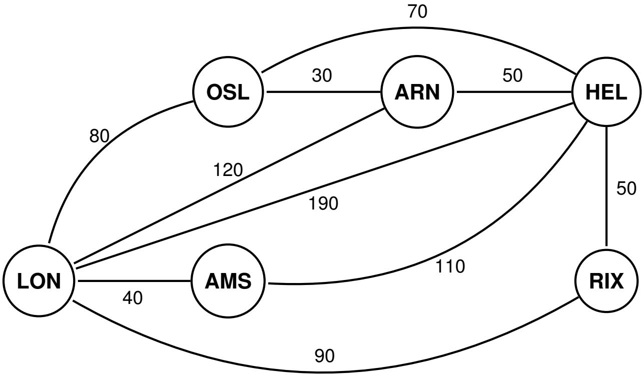
Plan for today:

- Weighted graphs
- Dijkstra's shortest-path algorithm

Shortest paths

Suppose you're travelling around Europe

- Know flights and their prices
- Cheapest price?



What is the shortest path from LON to HEL using BFS?

Why not just use BFS?

BFS already allows us to find the shortest paths?

- Assuming that all edges have length 1...

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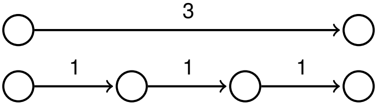
Write each edge as a series of length one edges?



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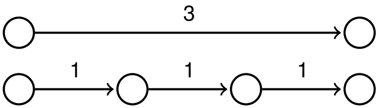

- Graph becomes **impractical**...

Need a new algorithm: **Dijkstra's shortest-paths algorithm**

We need a new algorithm

Dijkstra's algorithm: shortest paths from node s

1. Initially mark s "explored"
2. Find "costs" of all edges from explored nodes to unexplored nodes (details to follow)
3. Pick the "cheapest" edge, mark its end node explored
4. Repeat until every node in the graph has been explored
5. Check the final paths (details to follow)

Dijkstra example

1. Initially mark *s* “explored”
2. Find “costs” of all edges **from explored** nodes **to unexplored** nodes **as cost of explored node plus edge cost**
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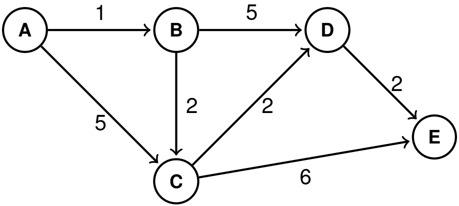

Edges from explored to unexplored

Nodes explored:

Costs of explored nodes

A	B	C	D	E
∞	∞	∞	∞	∞

Dijkstra example

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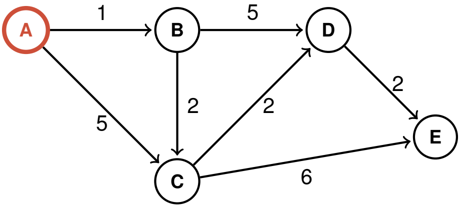

Edges from explored to unexplored

Nodes explored: A

Costs of explored nodes

A	B	C	D	E
0	∞	∞	∞	∞

Dijkstra example

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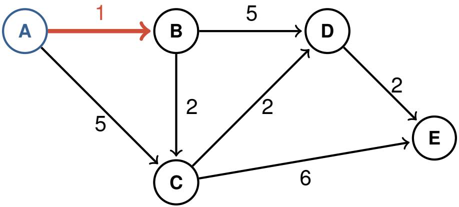

Edges from explored to unexplored

AB
1

Nodes explored: A

Costs of explored nodes

A	B	C	D	E
0	∞	∞	∞	∞

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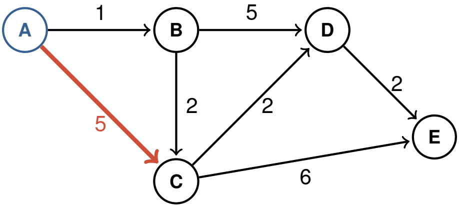

Edges from explored to unexplored

AB	AC
1	5

Nodes explored: A

Costs of explored nodes

A	B	C	D	E
0	∞	∞	∞	∞

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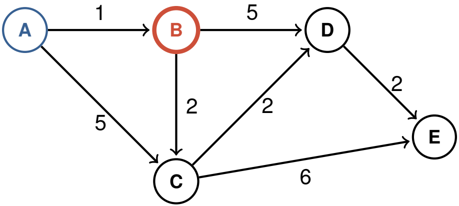

Edges from explored to unexplored

AC
5

Nodes explored: A, B

Costs of explored nodes

A	B	C	D	E
0	1	∞	∞	∞

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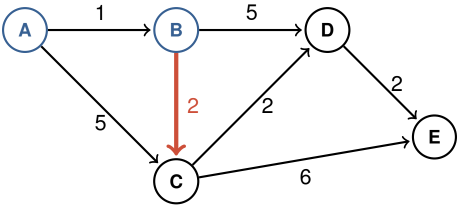

Edges from explored to unexplored

AC	BC
5	3

Nodes explored: A, B

Costs of explored nodes

A	B	C	D	E
0	1	∞	∞	∞

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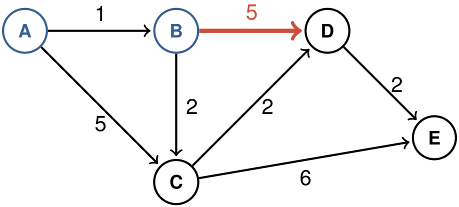

Edges from explored to unexplored

AC	BC	BD
5	3	6

Nodes explored: A, B

Costs of explored nodes

A	B	C	D	E
0	1	∞	∞	∞

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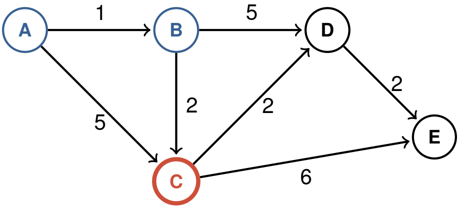

Edges from explored to unexplored

BD
6

Nodes explored: A, B, C

Costs of explored nodes

A	B	C	D	E
0	1	3	∞	∞

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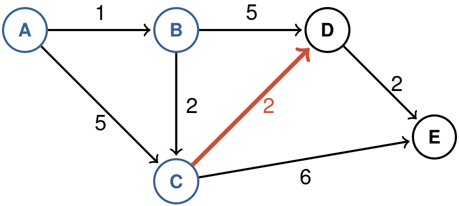

Edges from explored to unexplored

BD	CD
6	5

Nodes explored: A, B, C

Costs of explored nodes

A	B	C	D	E
0	1	3	∞	∞

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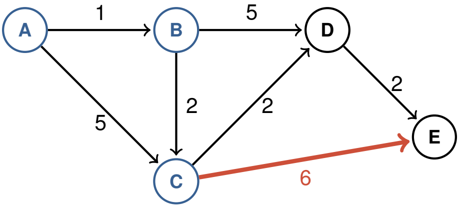

Edges from explored to unexplored

BD	CD	CE
6	5	9

Nodes explored: A, B, C, D

Costs of explored nodes

A	B	C	D	E
0	1	3	∞	∞

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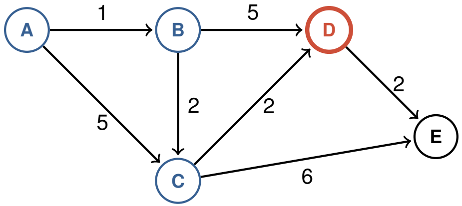

Edges from explored to unexplored

CE
9

Nodes explored: A, B, C, D

Costs of explored nodes

A	B	C	D	E
0	1	3	5	∞

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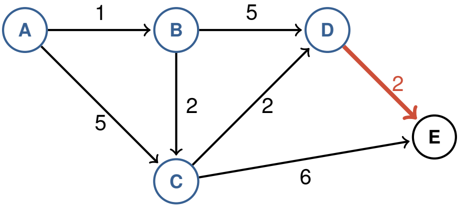

Edges from explored to unexplored

CE	DE
9	7

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Costs of explored nodes

A	B	C	D	E
0	1	3	5	∞

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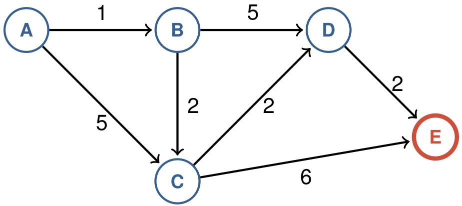

Edges from explored to unexplored

CE	DE
9	7

Nodes explored: A, B, C, D, E

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0	1	3	5	7

The gist

- We start with no information: “infinite” costs (zero for start node)

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- Second iteration: the lowest-cost edge from explored to unexplored must be shortest distance to the that node...

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- We then know the shortest distance to the two explored nodes
- Second iteration: the lowest-cost edge from explored to unexplored must be shortest distance to the that node...
  - Any other path would need to take a costlier route
- Whenever we add a “cheapest” node to “explored” nodes, we have the shortest distance to that node

Thinking about data structures

What data structure would you use for:

- The nodes we have explored?
- The distances to explored nodes?

More formally

Input: (directed) graph $G = (V, E)$

- V: set of n vertices, E: set of m edges
- Each edge e has length (cost) $c_e \geq 0$
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Assume:

- That such paths exist (connected graph)
- $c_e \geq 0$ (important!!)

Dijkstra's algorithm

For a graph $G = (V, E)$: V: n vertices, E: m edges, edge (v, w) has length c_{vw}

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Dijkstra's running time?

Input: graph $G = (V, E)$ of m edges, n vertices

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- **Total complexity** $O(mn)$

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- **Iterations of while loop**: $n - 1$
- **Work per iteration**: $O(m)$
- **Total complexity** $O(mn)$
- **Can we do better?**

How could we improve the algorithm?

Are we doing too much work?

Dijkstra example redux

Main loop: While $X \neq V$:

- Go through all edges (v, w) starting in X and ending in $V - X$
- **Pick the edge that minimizes $A[v] + c_{vw}$, call it (v^*, w^*)**
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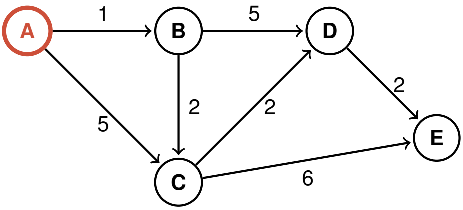

$A = [0, \infty, \infty, \infty, \infty]$
$X = \{A\}$
$D = [(B, \infty), (C, \infty), (D, \infty), (E, \infty)]$ (store Dijkstra scores for unexplored)

Dijkstra example redux

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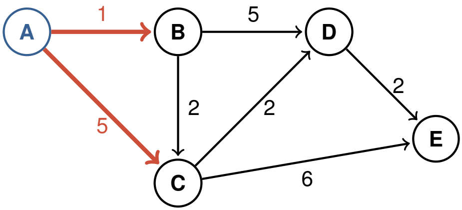

$A = [0, \infty, \infty, \infty, \infty]$
$X = \{A\}$
$D = [(B, 1), (C, 5), (D, \infty), (E, \infty)]$ (store Dijkstra scores for unexplored)

Dijkstra example redux

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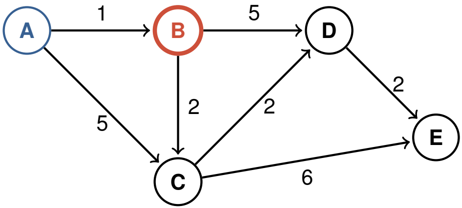

$A = [0, 1, \infty, \infty, \infty]$
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Dijkstra example redux

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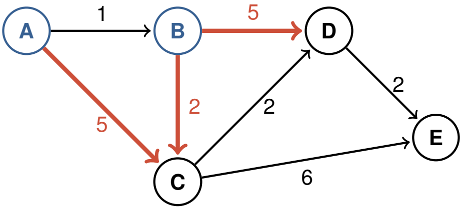

$A = [0, 1, \infty, \infty, \infty]$
$X = \{A, B\}$
$D = [(C, 3), (D, 6), (E, \infty)]$ (store Dijkstra scores for unexplored)

Dijkstra example redux

Main loop: While $X \neq V$:

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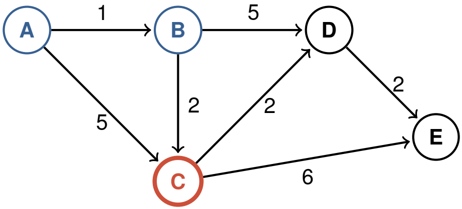

$A = [0, 1, 3, \infty, \infty]$
$X = \{A, B, C\}$
$D = [(D, 6), (E, \infty)]$ (store Dijkstra scores for unexplored)

Dijkstra example redux

Main loop: While $X \neq V$:

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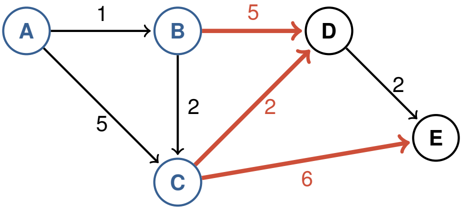
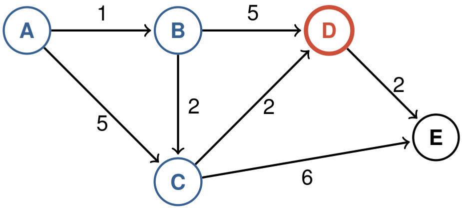


$A = [0, 1, 3, \infty, \infty]$
$X = \{A, B, C\}$
$D = [(D, 5), (E, 9)]$ (store Dijkstra scores for unexplored)

Dijkstra example redux

Main loop: While $X \neq V$:

- Go through all edges (v, w) starting in X and ending in $V - X$
- **Pick the edge that minimizes $A[v] + c_{vw}$, call it (v^*, w^*)**
- Add vertex w^* to X and set $A[w^*] = A[v^*] + c_{v^* w^*}$



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$X = \{A, B, C, D\}$
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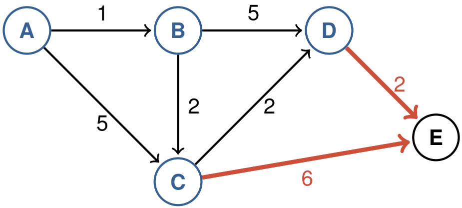

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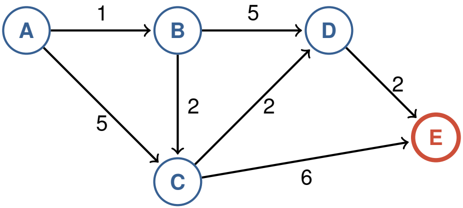

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What do we want from D?

We store Dijkstra scores for unprocessed nodes in D

- In each iteration, we need to find the minimum score
- We also recalculate Dijkstra scores for edges starting from each node that we process (replace a Dikstra score or remove the old score and add a new score)

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If only there was a data structure that performed these operations in $O(\log n)$ time...

- Then Dijkstra would run in $O((m + n) \log n)$ time

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Implementation is a great exercise for you to try

- An optional problem asks you to use a built-in data structure to speed up Dijkstra
- Conceptually, a heap is a type of a **tree**, where the highest-priority item is on top, and links to lower-priority items in branches below.
- **Difficulty**: keeping the order when adding items with different priorities

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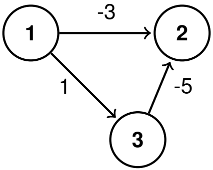

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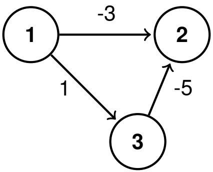

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With negative edge weights, **we need another algorithm**:

- **Bellman-Ford** — for you to explore

Review

Shortest paths

- Dijkstra's algorithm
- Data structure selection matters!
- In Dijkstra's case: heap

Workshop after the break

- Try Dijkstra in Python
- Recover shortest paths too...
- Shortest paths on the Tube

Workshop

Workshop zip file on the Hub

- HTML instructions
- At some point, you'll need the `.py`-file with skeleton code (open in Spyder)