1 Lec6:

1.1 Def: converging sequences

(x_n) converges to L if given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - L| < \epsilon \quad \forall m \geq N$.

n

1.2 Example:

Show that

$$\lim_{m\to\infty} \frac{3m+2}{m+1} = 3$$

using the definition of convergence.

We need to show that given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|\frac{3m+2}{m+1} - 3| < \epsilon, \forall n \geq N$

We compute

$$|\frac{3m+2}{m+1} - 3| = |\frac{3m+2-(3m+3)}{m+1}| = |-\frac{1}{m-1}| = \frac{1}{m+1} < \frac{1}{m}$$

Given $\epsilon > 0$, by the archimeadia property, $\exists N \in \mathbb{N}$, $\frac{1}{N} < \epsilon$ Then, $\forall m \geq N$, we have

$$|\frac{3m+2}{m+1} - 3| < \frac{1}{m} \leq \frac{1}{N} < \epsilon$$

1.3 Thoerem(Uniqueness of Limits):

Suppose that $L_1, L_2 \in \mathbb{R}$ and (x_n) is a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = L_1$, and $\lim_{n\to\infty} x_n = L_2$. Then $L_1 = L_2$

Proof: observe that if $b \in \mathbb{R}, b \geq 0$ such that $b < \epsilon \forall \epsilon > 0$, then b = 0 by the Archimediam property.

So to show that $L_1 = L_2$, it suffices to prove that $|L_1 - L_2| < \epsilon \ \forall \epsilon > 0$.

For $m \in \mathbb{N}$, we have

$$|L_1 - L_2| = |L_1 - x_n + x_n - L_2| \leq |L_1 - x_n| + |x_n - L_2| \qquad (*)$$

Let $\epsilon > 0$ be given, since $\lim_{m\to\infty} x_n = L_1$, then $\exists N_1 \in \mathbb{N}$ such that

$$|x_m - L_1| < \epsilon/2, \forall m \geq N_1$$

.

Similarly, we can show that since $\lim_{m\to\infty} x_n = L_2$, then $\exists N_2 \in \mathbb{N}$ such that

$$|x_m - L_2| < \epsilon/2, \forall m \geq N_2$$

. Set $N = max\{N_1, N_2\}$, then from (*) we have

$$|L_1 - L_2| < \epsilon/2 * 2 = \epsilon$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$|L_1 - L_2| = 0 \Rightarrow L_1 = L_2$$

1.4 Def 2.5:

We say that a sequqnce (x$_n$ in $\mathbb{R}$) is bounded if there is $M \in \mathbb{R}$ such that

$$|x_n| \leq M, \forall m \in \mathbb{N}$$

that is, x$_n \in [-M, M], \forall m \in \mathbb{N}$

1.5 Theorem 2.6

Every convergent sequency (x$_n$) is bounded.

Proof: We need to find $M \in \mathbb{R}$ such that $|x_n| \leq M, \forall m \in \mathbb{N}$. Let $L \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = L$. We have $\forall n \in \mathbb{N}$

$$|x_n| = |x_n - L + L| \leq |x_n - L| + |L|$$

Take $\epsilon = 1$, then there exists N such that

$$|x_n - L| < 1, \forall m \geq N$$

Set

$$M = Max\{1 + |L|, |x_1|, |x_2|, ..|x_{N-1}|\}$$

Then for $m < N$, we have $|x_n| \leq M$, also, for $m \geq N$, we have

$$|x_n| \leq |x_n - L + L| < 1 + |L| \leq M$$

. Hence, (x_n) is bounded.

1.6 Example

Consider the sequence $((S_m)_{m=1}^{\infty})$, where

$$S_m = 1 + 1/2 + 1/3 + ... + 1/m$$

Show that (S_m) is unbounded and conclude that S_n diverges

It suffices to show that the sequence S_m that have m $= 2^k, \forall k \in \mathbb{N}$, this sequence is unbounded.

We have

$$S_2 = 1 + 1/2, k = 1$$

$$S_4 = 1 + 1/2 + 1/3 + 1/4 > 1 + 1/2 + 1/4 + 1/4 = 1 + 2 * 1/2, k = 2$$

The idea is to show by induction that $S_{2^k} = 1+1/2+...+1/2^k > 1+k*1/2$