

Rotational Motion of a Rigid Reference Frame

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This document describes the computations that are performed by the class `RigidReferenceFrame` and its subclasses to determine the rotational motion (rotation, angular velocity, and angular acceleration) of a rigid frame.

Definitions

We are considering in this section a rigid reference frame that is defined by two bodies B_1 and B_2 at positions q_1 and q_2 , respectively. A basis of the reference frame is defined by three vectors:

- the *fore* vector \mathbf{F} which is along the axis $q_2 - q_1$;
- the *normal* vector \mathbf{N} which is orthogonal to \mathbf{F} and is such that the velocity of the frame, $\dot{q}_2 - \dot{q}_1$ is in the plane (\mathbf{F}, \mathbf{N}) ;
- the *binormal* vector \mathbf{B} which is orthogonal to \mathbf{F} and \mathbf{N} such that $(\mathbf{F}, \mathbf{N}, \mathbf{B})$ forms a direct trihedron.

There are obviously many possible choices for $(\mathbf{F}, \mathbf{N}, \mathbf{B})$. In practice, it is convenient to choose \mathbf{B} before \mathbf{N} so that the basis is defined exclusively using vector products:

$$\begin{cases} \mathbf{F} &= \mathbf{r} \\ \mathbf{B} &= \mathbf{r} \wedge \dot{\mathbf{r}} \\ \mathbf{N} &= \mathbf{B} \wedge \mathbf{F} \end{cases} \quad (1)$$

where we have defined $\mathbf{r} := q_2 - q_1$. It is trivial to check that these definitions satisfy the properties above, and in particular that they determine a direct orthogonal basis. The corresponding orthonormal basis is:

$$\begin{cases} \mathbf{f} &= \frac{\mathbf{F}}{|\mathbf{F}|} \\ \mathbf{b} &= \frac{\mathbf{B}}{|\mathbf{B}|} \\ \mathbf{n} &= \frac{\mathbf{N}}{|\mathbf{N}|} \end{cases} \quad (2)$$

Derivatives of normalized vectors

In what follows, we will need to derive the elements of the trihedron $(\mathbf{f}, \mathbf{n}, \mathbf{b})$. To help with this we prove two formulæ that define the first and second derivative of $\mathbf{V}/|\mathbf{V}|$ based on that of \mathbf{V} .

The first derivative is:

$$\begin{aligned} \frac{d}{dt} \frac{\mathbf{V}}{|\mathbf{V}|} &= \frac{|\mathbf{V}| \dot{\mathbf{V}} - \frac{d|\mathbf{V}|}{dt} \mathbf{V}}{|\mathbf{V}|^2} \\ &= \frac{|\mathbf{V}| \dot{\mathbf{V}} - \frac{(\mathbf{V} \cdot \dot{\mathbf{V}})}{|\mathbf{V}|} \mathbf{V}}{|\mathbf{V}|^2} \\ &= \frac{|\mathbf{V}|^2 \dot{\mathbf{V}} - (\mathbf{V} \cdot \dot{\mathbf{V}}) \mathbf{V}}{|\mathbf{V}|^3} \end{aligned} \quad (3)$$

The second derivative is somewhat more complicated:

$$\begin{aligned}
 \frac{d^2}{dt^2} \frac{\mathbf{v}}{|\mathbf{v}|} &= \frac{d}{dt} \left(\frac{|\mathbf{v}|^2 \dot{\mathbf{v}} - (\mathbf{v} \cdot \dot{\mathbf{v}}) \mathbf{v}}{|\mathbf{v}|^3} \right) \\
 &= \frac{|\mathbf{v}|^3 \frac{d}{dt} (|\mathbf{v}|^2 \dot{\mathbf{v}} - (\mathbf{v} \cdot \dot{\mathbf{v}}) \mathbf{v}) - 3|\mathbf{v}|(\mathbf{v} \cdot \dot{\mathbf{v}})(|\mathbf{v}|^2 \dot{\mathbf{v}} - (\mathbf{v} \cdot \dot{\mathbf{v}}) \mathbf{v})}{|\mathbf{v}|^6} \\
 &= \frac{2(\mathbf{v} \cdot \dot{\mathbf{v}}) \dot{\mathbf{v}} + |\mathbf{v}|^2 \ddot{\mathbf{v}} - (|\dot{\mathbf{v}}|^2 + (\mathbf{v} \cdot \ddot{\mathbf{v}})) \mathbf{v} - (\mathbf{v} \cdot \dot{\mathbf{v}}) \dot{\mathbf{v}}}{|\mathbf{v}|^3} - 3 \frac{|\mathbf{v}|^3 (\mathbf{v} \cdot \dot{\mathbf{v}}) \dot{\mathbf{v}} - |\mathbf{v}| (\mathbf{v} \cdot \dot{\mathbf{v}})^2 \mathbf{v}}{|\mathbf{v}|^6} \\
 &= \frac{\ddot{\mathbf{v}}}{|\mathbf{v}|} - 2 \dot{\mathbf{v}} \frac{(\mathbf{v} \cdot \dot{\mathbf{v}})}{|\mathbf{v}|^3} - \mathbf{v} \frac{|\dot{\mathbf{v}}|^2 + (\mathbf{v} \cdot \ddot{\mathbf{v}})}{|\mathbf{v}|^3} + 3 \mathbf{v} \frac{(\mathbf{v} \cdot \dot{\mathbf{v}})^2}{|\mathbf{v}|^5} \quad (4)
 \end{aligned}$$

First derivative and rotation

Deriving the definitions (1) we obtain:

$$\begin{cases} \dot{\mathbf{F}} &= \dot{\mathbf{r}} \\ \dot{\mathbf{B}} &= \mathbf{r} \wedge \ddot{\mathbf{r}} \\ \dot{\mathbf{N}} &= \dot{\mathbf{B}} \wedge \mathbf{F} + \mathbf{B} \wedge \dot{\mathbf{F}} \end{cases}$$