Rotational Motion of a Rigid Reference Frame

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This document describes the computations that are performed by the class `RigidReferenceFrame` and its subclasses to determine the rotational motion (rotation, angular velocity, and angular accelation) of a rigid frame.

Definitions

We are considering in this section a rigid reference frame that is defined by two bodies B_1 and B_2 at positions q_1 and q_2, respectively. A basis of the reference frame is defined by three vectors:

- the *fore* vector $\boldsymbol{F}$ which is along the axis $q_2 - q_1$;
- the *normal* vector $\boldsymbol{N}$ which is orthogonal to $\boldsymbol{F}$ and is such that the velocity of the frame, $\dot{q_2} - \dot{q_1}$ is in the plane $(\boldsymbol{F}, \boldsymbol{N})$;
- the *binormal* vector $\boldsymbol{B}$ which is orthogonal to $\boldsymbol{F}$ and $\boldsymbol{N}$ such that $(\boldsymbol{F}, \boldsymbol{N}, \boldsymbol{B})$ forms a direct trihedron.

There are obviously many possible choices for $(\boldsymbol{F}, \boldsymbol{N}, \boldsymbol{B})$. In practice, it is convenient to choose $\boldsymbol{B}$ before $\boldsymbol{N}$ so that the basis is defined exclusively using vector products:

$$\begin{cases} \boldsymbol{F} & = \boldsymbol{r} \\ \boldsymbol{B} & = \boldsymbol{r} \wedge \dot{\boldsymbol{r}} \\ \boldsymbol{N} & = \boldsymbol{B} \wedge \boldsymbol{F} \end{cases} \tag{1}$$

where we have defined $\boldsymbol{r} \coloneqq q_2 - q_1$. Is trivial to check that these definitions satisfy the properties above, and in particular that they determine a direct orthogonal basis. The corresponding orthonormal basis is:

$$\begin{cases} \boldsymbol{f} & = \dfrac{\boldsymbol{F}}{|\boldsymbol{F}|} \\ \boldsymbol{b} & = \dfrac{\boldsymbol{B}}{|\boldsymbol{B}|} \\ \boldsymbol{n} & = \dfrac{\boldsymbol{N}}{|\boldsymbol{N}|} \end{cases} \tag{2}$$

Derivatives of normalized vectors

In what follows, we will need to derive the elements of the trihedron $(\boldsymbol{f}, \boldsymbol{n}, \boldsymbol{b})$. To help with this we prove two formulæ that define the first and second derivative of $\boldsymbol{V}/|\boldsymbol{V}|$ based on that of $\boldsymbol{V}$.

The first derivative is:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\boldsymbol{V}}{|\boldsymbol{V}|} &= \frac{|\boldsymbol{V}|\dot{\boldsymbol{V}} - \frac{\mathrm{d}\,|\boldsymbol{V}|}{\mathrm{d}\,t}\boldsymbol{V}}{|\boldsymbol{V}|^2} \\ &= \frac{|\boldsymbol{V}|\dot{\boldsymbol{V}} - \frac{(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})}{|\boldsymbol{V}|}\boldsymbol{V}}{|\boldsymbol{V}|^2} \\ &= \frac{|\boldsymbol{V}|^2\dot{\boldsymbol{V}} - (\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\boldsymbol{V}}{|\boldsymbol{V}|^3} \end{aligned}$$

The second derivative is somewhat more complicated:

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}\frac{\boldsymbol{V}}{|\boldsymbol{V}|} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{|\boldsymbol{V}|^2\dot{\boldsymbol{V}} - \left(\boldsymbol{V}\cdot\dot{\boldsymbol{V}}\right)\boldsymbol{V}}{|\boldsymbol{V}|^3}\right) \\
&= \frac{|\boldsymbol{V}|^3\frac{\mathrm{d}}{\mathrm{d}\,t}\left(|\boldsymbol{V}|^2\dot{\boldsymbol{V}} - \left(\boldsymbol{V}\cdot\dot{\boldsymbol{V}}\right)\boldsymbol{V}\right) - 3|\boldsymbol{V}|\left(\boldsymbol{V}\cdot\dot{\boldsymbol{V}}\right)\left(|\boldsymbol{V}|^2\dot{\boldsymbol{V}} - \left(\boldsymbol{V}\cdot\dot{\boldsymbol{V}}\right)\boldsymbol{V}\right)}{|\boldsymbol{V}|^6} \\
&= \frac{}{|\boldsymbol{V}|^6} - 3\frac{|\boldsymbol{V}|^3\left(\boldsymbol{V}\cdot\dot{\boldsymbol{V}}\right)\dot{\boldsymbol{V}} - |\boldsymbol{V}|\left(\boldsymbol{V}\cdot\dot{\boldsymbol{V}}\right)^2\boldsymbol{V}}{|\boldsymbol{V}|^6}
\end{aligned}
$$

First derivative and rotation

Deriving the definitions (1) we obtain:

$$
\begin{cases}
\dot{\boldsymbol{F}} &= \dot{\boldsymbol{r}} \\
\dot{\boldsymbol{B}} &= \boldsymbol{r}\wedge\ddot{\boldsymbol{r}} \\
\dot{\boldsymbol{N}} &= \dot{\boldsymbol{B}}\wedge\boldsymbol{F} + \boldsymbol{B}\wedge\dot{\boldsymbol{F}}
\end{cases}
$$