On an Article by Celledoni et al.

Pascal Leroy (phl)

2020-06-03

This document provides clarifications, corrections, and accuracy improvements to the formulæ presented in [CFSZ07]. It follows the notation and conventions of that paper. Note that the preprint [CFSZ07] differs in some of the formulæ from the final publication [CFSZ08], but we generally follow the former.

Preamble

We remind the reader of the derivation formulæ for the Jacobian elliptic functions ([OLBC10], section 22.13(i)):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{sn} u & = \operatorname{cn} u \operatorname{dn} u \\ \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{cn} u & = -\operatorname{sn} u \operatorname{dn} u \\ \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{dn} u & = -k^2 \operatorname{sn} u \operatorname{cn} u \end{cases}$$

and for the hyperbolic functions ([OLBC10], section 4.34):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{th} u & = \operatorname{sech}^2 u \\ \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{sech} u & = -\operatorname{sech} u \operatorname{th} u \end{cases}$$

The equations of motion

We start by writing equation (1) of [CFSZ07] in coordinates. The coordinates of $\boldsymbol{m}$ and $\boldsymbol{I}$ are defined by:

$$\boldsymbol{m} \coloneqq \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

and:

$$\boldsymbol{I} \coloneqq \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

with $I_1 \leq I_2 \leq I_3$.

Euler's equation $\dot{\boldsymbol{m}} = [\boldsymbol{m}, \boldsymbol{\omega}]$ can be written in coordinates in the principal axes frame:

$$\dot{\boldsymbol{m}} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \times \begin{pmatrix} m_1/I_1 \\ m_2/I_2 \\ m_3/I_3 \end{pmatrix}$$

thus:

$$\begin{cases} \dot{m}_1 & = m_2 m_3 (1/I_3 - 1/I_2) \\ \dot{m}_2 & = m_3 m_1 (1/I_1 - 1/I_3) \\ \dot{m}_3 & = m_1 m_2 (1/I_2 - 1/I_1) \end{cases} \tag{1}$$

Solution of Euler's equation

The solution of Euler's equation has three cases depending on the initial value of $\boldsymbol{m}$ (more precisely, on the sign of $\Delta_2 = m_1^2 \frac{I_{12}}{I_1} + m_3^2 \frac{I_{32}}{I_3}$, see discussion below). Figure 1 illustrates the possible evolutions of $\boldsymbol{m}$. The sphere is the surface $|\boldsymbol{m}| = G$, which is an invariant of motion. The planes are the surfaces $\Delta_2 = 0$ and separate different modes of the motion. The blue curve is called case (i) in [CFSZ07]: $\boldsymbol{m}$ follows a periodic curve, and when that curve is close to the m_1 axis we have a classical case of precession. The red curve is case (ii), and again the motion of $\boldsymbol{m}$ is periodic and exhibits precession when the curve remains close to the m_3 axis. The green curve is case (iii): $\boldsymbol{m}$ takes an infinite amount of time to reach the point $(0, G, 0)$; furthermore, the motion is unstable as any perturbation moves it either to the blue or the red region where $\boldsymbol{m}$ oscillates between points close to $(0, G, 0)$ and $(0, -G, 0)$; this is the Джанибеков effect.

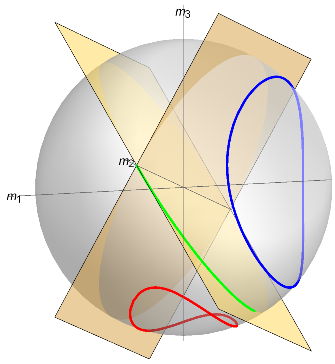

Figure 1. Possible trajectories of $\boldsymbol{m}$: the blue and red curves are cases (i) and (ii), respectively, and correspond to motion with precession. The green curve is the (unstable) case (iii) and any perturbation demonstrates the Джанибеков effect.

The solutions may also be visualized by intersecting the sphere $|\boldsymbol{m}| = G$ with ellipsoids defined by the value of the kinetic energy T, which is also a constant of motion. Since $T = \frac{G^2 - \Delta_2}{2 I_2 \,\mathrm{rad}^2}$, different values of T determine the same modes as above.

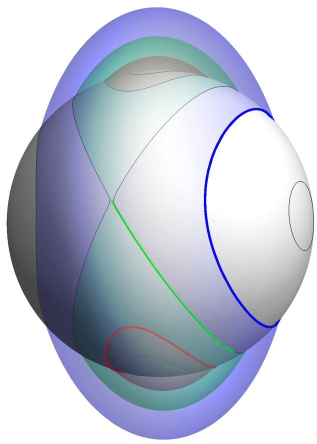

Figure 2. Possible trajectories of $\boldsymbol{m}$: the sphere is identical to that of Figure 1. The ellipsoids are surfaces of equal kinetic energy and intersect the sphere on the blue, red, and green curves depending on the value of T .

In the rest of this section, we describe our notation and derive (corrected) formulæ for the three cases described above.

Notation

[CFSZ07] uses a dimensionless formulation where $|\boldsymbol{m}| = 1$, and absolute values for I_{jh} and Δ_j. We prefer to use a dimensionful formulation where $|\boldsymbol{m}| = G$, and to avoid absolute values. Thus we define:

$$I_{jh} \coloneqq I_j - I_h \qquad \Delta_j \coloneqq G^2 - 2TI_j \,\mathrm{rad}^2 \qquad B_{jh} \coloneqq \sqrt{\pm\frac{I_j\Delta_h}{I_{jh}}}$$

$$k \coloneqq \sqrt{-\frac{\Delta_1 I_{32}}{\Delta_3 I_{21}}} \qquad \lambda_1 \coloneqq \sqrt{\frac{\Delta_1 I_{32}}{I_1 I_2 I_3}} \qquad \lambda_3 \coloneqq \sqrt{\frac{\Delta_3 I_{12}}{I_1 I_2 I_3}}$$

With these definitions, $I_{jh} \geq 0$ if and only if $j \geq h$, and we will prove later that $\Delta_1 \geq 0$, $\Delta_3 \leq 0$, and that Δ_2 can have either sign. The sign under the radical in the definition of B_{jh} is $+$ if $h = 1$ and $j \geq h$, or $h = 3$ and $j < h$; it is $-$ otherwise (note that we never use B_{j2} in the analysis below). At this point it is also useful to observe that:

$$B_{31}^2 + B_{13}^2 = \frac{\Delta_1 I_3 - \Delta_3 I_1}{I_{31}} = G^2$$

Physically, I_{jh} has the dimension of a moment of inertia $[L^2M]$. G has the dimension of an angular momentum $[L^2MT^{-1}A]$. T has the dimension of an energy $[L^2MT^{-2}]$. Δ_j has the same dimension as G^2. B_{jh} has the same dimension as $\sqrt{\Delta_h}$, i.e., the same dimension as G. λ_1 and λ_3 have the same dimension as the quotient $\frac{G}{I_j}$, i.e., $[T^{-1}A]$ which is appropriate for their usage.

Case (i)

Case (i) of the solution of Euler's equation in section 2.2 of [CFSZ07] is:

$$\boldsymbol{m}_t = \begin{pmatrix} \sigma B_{13}\,\mathrm{dn}(\lambda t - \nu, k) \\ -B_{21}\,\mathrm{sn}(\lambda t - \nu, k) \\ B_{31}\,\mathrm{cn}(\lambda t - \nu, k) \end{pmatrix}$$

If we derive this expression with respect to t, inject in into (1), and eliminate the elliptic functions we obtain:

$$\begin{cases} -\sigma\lambda k^2 B_{13} &= -B_{21}B_{31}(1/I_3 - 1/I_2) \\ -\lambda B_{21} &= \sigma B_{13}B_{31}(1/I_1 - 1/I_3) \\ -\lambda B_{31} &= -\sigma B_{13}B_{21}(1/I_2 - 1/I_1) \end{cases} \tag{2}$$

The last equation of (2) yields the following value for λ:

$$\lambda = \sigma\frac{B_{13}B_{21}}{B_{31}}\frac{I_1 - I_2}{I_1 I_2} = \sigma\sqrt{\frac{I_1\Delta_3}{I_{13}}\frac{I_2\Delta_1}{I_{21}}\frac{I_{31}}{I_3\Delta_1}}\frac{I_1 - I_2}{I_1 I_2}$$
$$= \sigma\sqrt{\frac{-\Delta_3}{I_{21}I_1I_2I_3}}(I_1 - I_2) = -\sigma\sqrt{\frac{-\Delta_3 I_{21}}{I_1 I_2 I_3}} = -\sigma\lambda_3$$

The sign change when moving $I_1 - I_2$ under the radical is necessary because $I_1 - I_2 < 0$.

It is straightforward to check that this value of λ also satisfies the other equations of (2). Note that it differs in sign from the one given by [CFSZ07]: the sign error is visible in that it does not yield the proper precession direction.

Case (ii)

Case (ii) of the solution of Euler's equation in section 2.2 of [CFSZ07] is:

$$\boldsymbol{m}_t = \begin{pmatrix} B_{13}\,\mathrm{cn}(\lambda t - \nu, k^{-1}) \\ -B_{23}\,\mathrm{sn}(\lambda t - \nu, k^{-1}) \\ \sigma B_{31}\,\mathrm{dn}(\lambda t - \nu, k^{-1}) \end{pmatrix}$$

Just as we did above, we derive this expression with respect to t, inject in into (1), and eliminate the elliptic functions:

$$\begin{cases} -\lambda B_{13} & = -\sigma B_{23} B_{31}(1/I_3 - 1/I_2) \\ -\lambda B_{23} & = \sigma B_{13} B_{31}(1/I_1 - 1/I_3) \\ -\sigma\lambda k^{-2} B_{31} & = -B_{13} B_{23}(1/I_2 - 1/I_1) \end{cases} \tag{3}$$

The first equation of (3) yields the following value for λ:

$$\begin{aligned} \lambda &= \sigma \frac{B_{23} B_{31}}{B_{13}} \frac{I_2 - I_3}{I_2 I_3} = \sigma \sqrt{\frac{I_2 \Delta_3}{I_{23}} \frac{I_3 \Delta_1}{I_{31}} \frac{I_{13}}{I_1 \Delta_3}} \frac{I_2 - I_3}{I_2 I_3} \\ &= \sigma \sqrt{\frac{-\Delta_1}{I_{23} I_1 I_2 I_3}} (I_2 - I_3) = -\sigma \sqrt{\frac{-\Delta_1 I_{23}}{I_1 I_2 I_3}} = -\sigma \lambda_1 \end{aligned}$$

Again, note the change of sign due to the fact that $I_2 - I_3 < 0$. And again, the same value of λ can be shown to satisfy the other equations of (3).

Case (iii)

Case (iii) of the solution of Euler's equation in section 2.2 of [CFSZ07] is clearly incorrect as it implies that m_1 and m_3 always have the same sign, whereas it is straightforward to choose initial conditions where they do not (because the separatrix is made of two planes, see Figure 1). Instead, we introduce an extra parameter $\sigma'' = \pm 1$ and posit a solution of the form:

$$\boldsymbol{m}_t = \begin{pmatrix} \sigma' B_{13} \operatorname{sech}(\lambda t - \nu) \\ G \operatorname{th}(\lambda t - \nu) \\ \sigma'' B_{31} \operatorname{sech}(\lambda t - \nu) \end{pmatrix}$$

Deriving this expression and injecting it into (1) yields:

$$\begin{cases} -\sigma' \lambda B_{13} & = \sigma'' G B_{31}(1/I_3 - 1/I_2) \\ \lambda G & = \sigma' \sigma'' B_{13} B_{31}(1/I_1 - 1/I_3) \\ -\sigma'' \lambda B_{31} & = \sigma' G B_{13}(1/I_2 - 1/I_1) \end{cases} \tag{4}$$

The second equation of (4) gives the following value for λ:

$$\lambda = \sigma' \sigma'' \frac{B_{13} B_{31}}{G} \frac{I_3 - I_1}{I_1 I_3} = \sigma' \sigma'' \frac{1}{G} \sqrt{\frac{I_1 \Delta_3}{I_{13}} \frac{I_3 \Delta_1}{I_{31}}} \frac{I_3 - I_1}{I_1 I_3} = \sigma' \sigma'' \frac{1}{G} \sqrt{-\frac{\Delta_1 \Delta_3}{I_1 I_3}}$$

In this case it is a bit less obvious that the other equations yield the same value of λ. We detail the derivation for the first equation, using the fact that ${\sigma'}^2 = 1$:

$$\begin{aligned} \lambda &= -\sigma' \sigma'' G \frac{B_{31}}{B_{13}} \frac{I_2 - I_3}{I_2 I_3} = -\sigma' \sigma'' G \sqrt{\frac{I_3 \Delta_1}{I_{31}} \frac{I_{13}}{I_1 \Delta_3}} \frac{I_2 - I_3}{I_2 I_3} \\ &= -\sigma' \sigma'' G \sqrt{-\frac{\Delta_1}{I_1 I_3 \Delta_3}} \frac{I_2 - I_3}{I_2} = \sigma' \sigma'' G \sqrt{-\frac{\Delta_1}{I_1 I_3 \Delta_3}} \left(\frac{I_3}{I_2} - 1 \right) \end{aligned}$$

Now note that in case (iii) we have $2T I_2 \operatorname{rad}^2 = G^2$ thus $1/I_2 = 2T \operatorname{rad}^2 / G^2$. λ can be rewritten as:

$$\lambda = \sigma' \sigma'' G \sqrt{-\frac{\Delta_1}{I_1 I_3 \Delta_3}} \left(\frac{2T I_3 \operatorname{rad}^2}{G^2} - 1 \right) = \sigma' \sigma'' \frac{1}{G} \sqrt{-\frac{\Delta_1 \Delta_3}{I_1 I_3}}$$

where we have used the fact that $2T I_3 \operatorname{rad}^2 - G^2 = -\Delta_3 = 2T \operatorname{rad}^2 (I_3 - I_2) \geq 0$.

We then define:

$$\lambda_2 \coloneqq \frac{1}{G} \sqrt{-\frac{\Delta_1 \Delta_3}{I_1 I_3}}$$

It is easy to see that λ_2 is the common value of λ_1 and λ_3 in case (iii), that σ' and σ'' are free parameters and that:

$$\lambda = \sigma'\sigma''\lambda_2$$

Note that λ_2 has the same dimension as the quotient $\frac{\Delta_j}{GI_j}$, which has the same dimension as $\frac{G}{I_j}$, namely, $[T^{-1}A]$.

Phase and initial value

The phase ν and the free parameters σ, σ' and σ'' are determined from the initial value $\boldsymbol{m}_0$ by setting $t = 0$.

Case (i)

We have:

$$\boldsymbol{m}_0 = \begin{pmatrix} \sigma B_{13}\,\mathrm{dn}(-\nu, k) \\ -B_{21}\,\mathrm{sn}(-\nu, k) \\ B_{31}\,\mathrm{cn}(-\nu, k) \end{pmatrix}$$

First, we set σ to be the sign of m_{01}. Then, forming the quotient of the last two coordinates we find:

$$\frac{m_{02}}{m_{03}} = \frac{B_{21}}{B_{31}}\,\mathrm{tg}(\mathrm{am}(\nu, k))$$

This equation defines ν modulo $2K(k)$ because $\mathrm{am}(\nu + 2K(k), k) = \mathrm{am}(\nu, k) + \pi$ ([OLBC10], equation 22.16.2). It comes:

$$\mathrm{arctg}\left(\frac{m_{02}}{m_{03}}\frac{B_{31}}{B_{21}}\right) = \mathrm{am}(\nu, k)$$

and finally we obtain ν as:

$$\nu = F\left(\mathrm{arctg}\left(\frac{m_{02}}{m_{03}}\frac{B_{31}}{B_{21}}\right), k\right)$$

Any determination of the arc tangent works, because $F(\pi + \phi, k) = 2K(k) + F(\phi, k)$ ([OLBC10], equation 19.2.10). In pratice we use the `atan2` function.

Case (ii)

Starting from:

$$\boldsymbol{m}_0 = \begin{pmatrix} B_{13}\,\mathrm{cn}(-\nu, k^{-1}) \\ -B_{23}\,\mathrm{sn}(-\nu, k^{-1}) \\ \sigma B_{31}\,\mathrm{dn}(-\nu, k^{-1}) \end{pmatrix}$$

we set σ to be the sign of m_{03} and form the quotient of the first two coordinates. We obtain:

$$\frac{m_{02}}{m_{01}} = \frac{B_{23}}{B_{13}}\,\mathrm{tg}(\mathrm{am}(\nu, k^{-1}))$$

and for ν:

$$\nu = F\left(\mathrm{arctg}\left(\frac{m_{02}}{m_{01}}\frac{B_{13}}{B_{23}}\right), k^{-1}\right)$$

The same comments as above apply regarding the computation of the arc tangent.

Case (iii)

The initial value $\boldsymbol{m}_0$ is:

$$\boldsymbol{m}_0 = \begin{pmatrix} \sigma' B_{13}\,\mathrm{sech}(-\nu) \\ G\,\mathrm{th}(-\nu) \\ \sigma'' B_{31}\,\mathrm{sech}(-\nu) \end{pmatrix}$$

σ' and σ'' are set to be the signs of m_{01} and m_{03}, respectively. The second coordinate immediately gives:

$$\nu = -\,\mathrm{argth}\left(\frac{m_{02}}{G}\right)$$

Implementation considerations

Some of the formulæ given by [CFSZ07] do not lend themselves to an easy implementation or lead to numerical inaccuracies. We describe in this section the modifications we make to these formulæ in our implementation.

The quantity Δ_j

We notice that the computation of Δ_j as written in [CFSZ07] entails cancellations, so we go back to the definition of $|\boldsymbol{m}|$ and of the kinetic energy:

$$\begin{cases} G^2 & = m_1^2 + m_2^2 + m_3^2 \\ 2T\,\mathrm{rad}^2 & = \dfrac{m_1^2}{I_1} + \dfrac{m_2^2}{I_2} + \dfrac{m_3^2}{I_3} \end{cases}$$

When, for instance, $j = 2$, this yields:

$$\begin{aligned} \Delta_2 &= m_1^2\left(1 - \frac{I_2}{I_1}\right) + m_3^2\left(1 - \frac{I_2}{I_3}\right) \\ &= m_1^2\frac{I_{12}}{I_1} + m_3^2\frac{I_{32}}{I_3} \end{aligned}$$

and similarly:

$$\begin{cases} \Delta_1 & = m_2^2\dfrac{I_{21}}{I_2} + m_3^2\dfrac{I_{31}}{I_3} \\ \Delta_3 & = m_1^2\dfrac{I_{13}}{I_1} + m_2^2\dfrac{I_{23}}{I_2} \end{cases}$$

It is easy to see that Δ_1 and Δ_3 are the sums of terms of the same sign, so they can be computed without cancellations. Furthermore, $\Delta_1 \geq 0$ and $\Delta_3 \leq 0$. Δ_2 can have either sign, which correspond exactly to cases (i) ($\Delta_2 < 0$), (ii) ($\Delta_2 > 0$) and (iii) ($\Delta_2 = 0$).

The elliptic modulus

For the computation of the elliptic functions and integrals [CFSZ07] gives the value of the elliptic modulus k but we need the value of the complementary parameter $m_c = 1 - m$ (see [OLBC10], section 19.1.2 for an overview of the notation). In case (i) we have:

$$m_c = 1 - k^2 = 1 + \frac{\Delta_1 I_{32}}{\Delta_3 I_{21}}$$

This can be rewritten as follows:

$$\begin{aligned} m_c &= \frac{\Delta_3 I_{21} + \Delta_1 I_{32}}{\Delta_3 I_{21}} = \frac{(G^2 - 2TI_3)(I_2 - I_1) + (G^2 - 2TI_1)(I_3 - I_2)}{\Delta_3 I_{21}} \\ &= \frac{G^2(I_3 - I_1) + 2TI_2(I_1 - I_3)}{\Delta_3 I_{21}} = \frac{\Delta_2 I_{31}}{\Delta_3 I_{21}} \end{aligned}$$

Similarly, in case (ii):

$$m_c = 1 - k^{-2} = 1 + \frac{\Delta_3 I_{21}}{\Delta_1 I_{32}} = \frac{\Delta_1 I_{32} + \Delta_3 I_{21}}{\Delta_1 I_{32}} = \frac{\Delta_2 I_{31}}{\Delta_1 I_{32}}$$

In both cases we have $m_c \geq 0$.

Integration of the rotation matrix

In order to be compatible with our geometrical libraries, our notation differs from that of [CFSZ07].

Notation

[CFSZ07] describe the physical space as a three-dimensional vector space where vectors like $\boldsymbol{M}$ live. In this vector space, they pick orthonormal bases like $\{\boldsymbol{E}_1^b, \boldsymbol{E}_2^b, \boldsymbol{E}_3^b\}$ which they identify with the canonical basis of $\mathbb{R}^3$ to obtain a coordinate representation $\boldsymbol{m}$ of $\boldsymbol{M}$.

They then define active rotations in the physical (vector) space. For instance they explain that $\mathcal{P}_t$ takes $\boldsymbol{M}$ to $\boldsymbol{E}_3^b$ and transforms the basis $\mathcal{B}_t$ into the basis $\mathcal{B}^b$. By contrast, our libraries operate on coordinate representations, not abstract vectors, and implement passive rotations where the physical space is represented by multiple copies of $\mathbb{R}^3$ with different coordinate systems. Therefore, we view $\mathcal{P}_t$ as transforming $\boldsymbol{M}$ with coordinates $\boldsymbol{m}$ in the coordinate system $\mathcal{B}^b$ of the body into $\boldsymbol{M}$ with coordinates $\boldsymbol{e}_3$ in the coordinate system $\mathcal{B}_t$. Confusingly, [CFSZ07] appear to use passive rotations when they write matrices, so their P_t has semantics similar to that of our $\mathcal{P}_t$.

In what follows (and in our code) we try to use the same symbols as [CFSZ07] with the understanding that our rotations, written in script font, are passive, and that the entities denoted by $\mathcal{B}$ are coordinate systems in multiple copies of $\mathbb{R}^3$, not bases in a single vector space.

[CFSZ07] decompose the attitude rotation $\mathcal{Q}_t$ of the body as follows:

$$\mathcal{Q}_t : \mathcal{B}^b \xrightarrow{\mathcal{P}_t} \mathcal{B}_t \xrightarrow{\mathcal{Y}_t} \mathcal{B}' \xrightarrow{\mathcal{R}} \mathcal{B}^s$$

where $\mathcal{P}_t$ maps $\boldsymbol{m}$ onto $\boldsymbol{e}_3^b$, $\mathcal{Y}_t$ is a rotation of angle $\psi(t)$ around $\boldsymbol{m}$, and $\mathcal{R}$ maps $\boldsymbol{e}_3^s$ onto $\boldsymbol{m}$, where they assume that $\mathcal{Q}_{t_0} = \mathbb{1}$. This yields the following decomposition for $\mathcal{R}$:

$$\mathcal{R} : \mathcal{B}' \xrightarrow{\mathcal{Y}_{t_0}^{-1}=\mathbb{1}} \mathcal{B}_t \xrightarrow{\mathcal{P}_{t_0}^{-1}} \mathcal{B}^b$$

This is not sufficient for our purpose, however, because in practical situations $\mathcal{Q}_{t_0}$ cannot be chosen. Therefore we decompose $\mathcal{R}$ as follows:

$$\mathcal{R} : \mathcal{B}' \xrightarrow{\mathcal{Y}_{t_0}^{-1}=\mathbb{1}} \mathcal{B}_t \xrightarrow{\mathcal{P}_{t_0}^{-1}} \mathcal{B}^b \xrightarrow{\mathcal{Q}_{t_0}} \mathcal{B}^s$$

[CFSZ07] derive the following expression for $\dot{\psi}(t)$ (which they write slightly differently):

$$\begin{aligned}
\dot{\psi}(t) &= \frac{2T\,\mathrm{rad}^2}{G} + \frac{\Delta_2}{GI_2}\left(\frac{1}{1+\frac{I_{12}I_{23}G^2}{I_2^2\Delta_1\Delta_3}m_2^2}\right) \\
&= \frac{2T\,\mathrm{rad}^2}{G} + \frac{\Delta_2}{GI_2}\left(\frac{1}{1-\frac{G^2}{B_{21}^2B_{23}^2}m_2^2}\right)
\end{aligned}$$

Case (i)

In case (i) we have $m_2 = -B_{21}\,\mathrm{sn}(\lambda t - \nu, k)$ and the above expression becomes:

$$\dot{\psi}(t) = \frac{2T\,\mathrm{rad}^2}{G} + \frac{\Delta_2}{GI_2}\left(\frac{1}{1-\frac{G^2}{B_{23}^2}\,\mathrm{sn}^2(\lambda t - \nu, k)}\right)$$

This expression can be integrated using formula 110.04 of [BF54] with $\alpha = G/B_{23}$ to yield:

$$\psi(t) = \frac{2T\,\mathrm{rad}^2}{G}t + \frac{\Delta_2}{\lambda GI_2}\,\Pi\left(\mathrm{am}(\lambda t - \nu, k), \frac{G^2}{B_{23}^2}, k\right)$$

Note that this differs from the formula given by [CFSZ07] in the value of n.

Case (ii)

In case (ii) we have $m_2 = -B_{23}\,\mathrm{sn}(\lambda t - \nu, k^{-1})$ and a computation similar to the one above gives:

$$\dot{\psi}(t) = \frac{2T\,\mathrm{rad}^2}{G} + \frac{\Delta_2}{GI_2}\left(\frac{1}{1 - \frac{G^2}{B_{21}^2}\,\mathrm{sn}^2(\lambda t - \nu, k^{-1})}\right)$$

and:

$$\psi(t) = \frac{2T\,\mathrm{rad}^2}{G} t + \frac{\Delta_2}{\lambda G I_2}\,\Pi\left(\mathrm{am}(\lambda t - \nu, k^{-1}), \frac{G^2}{B_{21}^2}, k^{-1}\right)$$

Case (iii)

In case (iii) we have $m_2 = G\,\mathrm{th}(\lambda t - \nu)$ and:

$$\dot{\psi}(t) = \frac{2T\,\mathrm{rad}^2}{G} + \frac{\Delta_2}{GI_2}\left(\frac{1}{1 - \frac{G^4}{B_{21}^2 B_{23}^2}\,\mathrm{th}^2(\lambda t - \nu)}\right)$$

which can be integrated using:

$$\int \frac{1}{1 - n\,\mathrm{th}^2(\lambda t - \nu)}\,\mathrm{d}\,t = \frac{t}{1-n} - \frac{\sqrt{n}}{(1-n)\lambda}\,\mathrm{argth}\big(\sqrt{n}\,\mathrm{th}(\lambda t - \nu)\big)$$

Implementation considerations

The approach in this section relies on the intermediate basis $\{\boldsymbol{m}, \dot{\boldsymbol{m}}, [\boldsymbol{m}, \dot{\boldsymbol{m}}]\}$. Unfortunately, it is not suitable for a practical implementation because it has an essential singularity when $\dot{\boldsymbol{m}} = 0$: while the condition $\dot{\boldsymbol{m}} = 0$ is physically a constant of motion, $\dot{\boldsymbol{m}}$ itself is not. This means that any inaccuracies in numerical computations of $\dot{\boldsymbol{m}}$ (cancellations, underflow) may cause it to switch from 0 to non-0 and back.

When $\dot{\boldsymbol{m}} = 0$ at $t = 0$, the motion may be computed at all times assuming a constant $\boldsymbol{m}$, so the singularity may be eliminated. But when $\dot{\boldsymbol{m}} \neq 0$ at $t = 0$ and it later becomes 0, there is no way to find a basis at t that continuously corresponds to the one at $t = 0$.

While it might be possible to deal with the neighbourhoods of the stable zeros (m_1 and m_3) by handling the relevant regions with distinct formulæ (e.g., the low-order precession approximations), this is infeasible for m_2 where any neighbourhood of the singularity propagates all the way around the separatrix. The reader is invited to consult figure 1.

In conclusion, we tried to use this approach but had to abandon it because of the impossibility of handling the singularity.

Integration of the quaternion

[CFSZ07] use a rotation $\mathcal{P}_t$ to map $\boldsymbol{m}$ onto $\boldsymbol{e}_3$ and obtain the following quaternionic representation for that rotation:

$$\begin{cases} p_1 & = \dfrac{p_3 m_1 + p_0 m_2}{G + m_3} \\ p_2 & = \dfrac{p_3 m_2 - p_0 m_1}{G + m_3} \\ p_0^2 + p_3^2 & = \dfrac{G + m_3}{2G} \end{cases}$$

and the angle $\psi(t)$ for the rotation $\mathcal{Y}_t$ around $\boldsymbol{e}_3$:

$$\dot{\psi}(t) = \frac{2T\,\mathrm{rad}^2 + Gm_3/I_3}{G + m_3} + 4G\frac{p_3\dot{p}_0 - p_0\dot{p}_3}{G + m_3}$$

Solving these equations they obtain:

$$\begin{cases} p_0 &= \sqrt{\dfrac{1+m_3/G}{2}} \\ p_1 &= \dfrac{m_2}{\sqrt{2G(G+m_3)}} \\ p_2 &= \dfrac{-m_1}{\sqrt{2G(G+m_3)}} \\ p_3 &= 0 \end{cases}$$

and:

$$\psi(t) = \frac{G}{I_3}t + \frac{GI_{31}}{\lambda I_1 I_3}\left(\Pi\left(\mathrm{am}(\lambda t - \nu, k), -\left(\frac{B_{31}}{B_{13}}\right)^2, k\right) + f(t)\right)$$

They note that these formulæ are only applicable if $m_3 \neq -G$ but go on applying them to case (i) where $m_3 = B_{31}\,\mathrm{cn}(\lambda t - \nu, k)$ can be negative.

One should note at this point that [CFSZ07] make an error when copying the definition of $f_1(u)$ from formula 361.54 of [BF54]. This error is corrected in [CFSZ08], but the sign of $f(s)$ in the definition of $\psi(t)$ is still incorrect and should read:

$$f(s) \coloneqq -B_{31}\frac{B_{13}}{B_{21}}\,\mathrm{arctg}\left(\frac{B_{21}}{B_{13}}\,\mathrm{sd}(\lambda s - \nu, k)\right)$$

An alternative quaternionic solution

[CFSZ07] do not explain how they handle cases (ii) and (iii). It is intriguing to note, though, that the formulæ above work well for case (ii) where $m_3 = \sigma B_{31}\,\mathrm{dn}(\lambda t - \nu, k^{-1})$. The reason is that, without loss of generality, we can apply to the principal axes of the body a rotation $\mathcal{S}$ that maps m_3 onto $-m_3$ (there are many possible choices for $\mathcal{S}$, but for simplicity we pick one that flips the sign of either m_1 or m_2). With this rotation, the multiplier σ disappears from m_3 and we have $m_3 \geq 0$ for all times, which ensures that the denominator of the quaternionic coordinates $G + m_3$ can be safely computed. The key insight here is that the coordinate where the Jacobi funtion dn appears does not change sign, and is therefore a better choice for rotating $\boldsymbol{m}$ to $\boldsymbol{e_i}$.

Observing that, in case (i), $m_1 = \sigma B_{13}\,\mathrm{dn}(\lambda t - \nu, k)$, we will construct a rotation $\mathcal{S}$ to make m_1 positive and then a rotation $\mathcal{P}_t$ that maps $\boldsymbol{m}$ onto $\boldsymbol{e}_1$. Similarly, in case (iii) the function sech appears in the expressions of m_1 and m_3 and is always positive, so we will construct $\mathcal{S}$ to make both m_1 and m_3 positive and choose $\mathcal{P}_t$ to map $\boldsymbol{m}$ onto $\boldsymbol{e}_1$ or $\boldsymbol{e}_3$. In the rest of this section we detail the calculations used to compute $\mathcal{S}$, $\mathcal{P}_t$ and $\psi(t)$.

With the introduction of $\mathcal{S}$, we are effectively introducing a new base $\mathcal{B}^p$ for the "preferred" principal axes of the body, and the rotation diagrams above are modified as follows for $\mathcal{Q}_t$:

$$\mathcal{Q}_t : \mathcal{B}^b \xrightarrow{\mathcal{S}} \mathcal{B}^p \xrightarrow{\mathcal{P}_t} \mathcal{B}_t \xrightarrow{\mathcal{Y}_t} \mathcal{B}' \xrightarrow{\mathcal{R}} \mathcal{B}^s$$

And for $\mathcal{R}$:

$$\mathcal{R} : \mathcal{B}' \xrightarrow{\mathcal{Y}_{t_0}^{-1} = \mathbb{1}} \mathcal{B}_t \xrightarrow{\mathcal{P}_{t_0}^{-1}} \mathcal{B}^p \xrightarrow{\mathcal{S}^{-1}} \mathcal{B}^b \xrightarrow{\mathcal{Q}_{t_0}} \mathcal{B}^s$$

With this choice of $\mathcal{S}$, the free parameters σ, σ', and σ'' appearing in the three cases of the resolution of Euler's equation are all 1.

Integrals

We start by computing two integrals that are useful to obtain $\psi(t)$. They are valid for $0 \leq a < 1$. As far as we can tell the following integral, which is useful for cases

(i) and (ii), is missing from [BF54]:

$$\int \frac{1}{1 + a\,\mathrm{dn}(u,k)}\,\mathrm{d}u = \frac{1}{1-a^2}\left[\Pi\left(\mathrm{am}(u,k), \frac{a^2k^2}{a^2-1}, k\right) - a\sqrt{\frac{1-a^2}{a^2(k^2-1)+1}}\,\mathrm{arctg}\left(\sqrt{\frac{a^2(k^2-1)+1}{1-a^2}}\,\mathrm{sc}(u,k)\right)\right] \tag{5}$$

Also, the following integral is useful for case (iii):

$$\int \frac{1}{1 + a\,\mathrm{sech}(u)}\,\mathrm{d}u = u + \frac{2a}{\sqrt{1-a^2}}\,\mathrm{arctg}\left(\frac{a-1}{\sqrt{1-a^2}}\,\mathrm{th}\left(\frac{u}{2}\right)\right) \tag{6}$$

Case (i)

In case (i), we define $\mathcal{P}_t$ to map $\boldsymbol{m}$ onto $\boldsymbol{e}_1$. A computation similar to that in [CFSZ07] yields that rotation in quaternionic form:

$$\begin{cases} p_2 & = \dfrac{p_1 m_2 + p_0 m_3}{G + m_1} \\ p_3 & = \dfrac{p_1 m_3 - p_0 m_2}{G + m_1} \\ p_0^2 + p_1^2 & = \dfrac{G + m_1}{2G} \end{cases}$$

and, for the angle $\psi(t)$ of the rotation $\mathcal{Y}_t$ around $\boldsymbol{e}_1$:

$$\dot{\psi}(t) = \frac{2T\,\mathrm{rad}^2 + Gm_1/I_1}{G + m_1} + 4G\frac{p_1\dot{p}_0 - p_0\dot{p}_1}{G + m_1}$$

We then write $p_0 = c_0\sqrt{1 + m_1/G}$ and $p_1 = c_1\sqrt{1 + m_1/G}$ and pick $c_0 = 1/\sqrt{2}$ and $c_1 = 0$. The quaternion simplifies to:

$$\begin{cases} p_0 & = \sqrt{\dfrac{1 + m_1/G}{2}} \\ p_1 & = 0 \\ p_2 & = \dfrac{m_3}{\sqrt{2G(G + m_1)}} \\ p_3 & = \dfrac{-m_2}{\sqrt{2G(G + m_1)}} \end{cases}$$

and the angle to:

$$\dot{\psi}(t) = \frac{2T\,\mathrm{rad}^2 + Gm_1/I_1}{G + m_1} = \frac{G^2 - \Delta_1 + Gm_1}{I_1(G + m_1)} = \frac{G}{I_1} - \frac{\Delta_1/I_1}{G + m_1} = \frac{G}{I_1} - \frac{\Delta_1}{GI_1}\frac{1}{1 + m_1/G}$$

Using equation (5) with $a = B_{13}/G$ and simplifying the various coefficients we obtain:

$$\psi(t) = \frac{G}{I_1}t + \frac{GI_{13}}{\lambda I_1 I_3}\,\Pi\left(\mathrm{am}(\lambda t - \nu, k), \frac{I_1 I_{32}}{I_3 I_{12}}, k\right) - \mathrm{arctg}\left(\sqrt{\frac{I_2 I_{31}}{I_3 I_{21}}}\,\mathrm{sc}(\lambda t - \nu, k)\right)$$

Case (ii)

In case (ii) $\mathcal{P}_t$ maps $\boldsymbol{m}$ onto $\boldsymbol{e}_3$ and we follow the computation given in [CFSZ07]. We repeat their results here in dimensionful form. The quaternion is:

$$\begin{cases} p_0 &= \sqrt{\dfrac{1+m_3/G}{2}} \\ p_1 &= \dfrac{m_2}{\sqrt{2G(G+m_3)}} \\ p_2 &= \dfrac{-m_1}{\sqrt{2G(G+m_3)}} \\ p_3 &= 0 \end{cases}$$

and the angle:

$$\dot{\psi}(t) = \frac{G}{I_3} - \frac{\Delta_3}{GI_3}\frac{1}{1+m_3/G}$$

Using equation (5) with $a = B_{31}/G$ and simplifying the various coefficients we obtain:

$$\psi(t) = \frac{G}{I_3}t + \frac{GI_{31}}{\lambda I_1 I_3}\,\Pi\left(\operatorname{am}(\lambda t - \nu, k^{-1}), \frac{I_3 I_{21}}{I_1 I_{23}}, k^{-1}\right) + \operatorname{arctg}\left(\sqrt{\frac{I_2 I_{31}}{I_1 I_{32}}}\,\operatorname{sc}(\lambda t - \nu, k^{-1})\right)$$

Case (iii)

In case (iii) we start by defining $\mathcal{S}$ so as to make both m_1 and m_3 positive. This is always possible, perhaps by flipping m_2. We then choose $\mathcal{P}_t$ to map $\boldsymbol{m}$ onto $\boldsymbol{e}_1$ or $\boldsymbol{e}_3$. Which one we pick is explained below.

Assume that we map $\boldsymbol{m}$ onto $\boldsymbol{e}_1$. Then using equation (6) with $a = B_{13}/G$ and simplifying the coefficients we obtain:

$$\begin{cases} \psi(t) &= \left(\dfrac{G}{I_1} - \dfrac{\Delta_1}{GI_1}\right)t - \dfrac{2B_{13}\Delta_1}{\lambda G B_{31} I_1}\operatorname{arctg}\left(\dfrac{B_{13}-G}{B_{31}}\operatorname{th}\left(\dfrac{\lambda t - \nu}{2}\right)\right) \\ &= \dfrac{G}{I_2}t - 2\operatorname{arctg}\left(\dfrac{B_{13}-G}{B_{31}}\operatorname{th}\left(\dfrac{\lambda t - \nu}{2}\right)\right) \end{cases}$$

Because $B_{13} = G$ when $B_{31} = 0$, this formula is only usable if $B_{31} \neq 0$. For safety, we rotate onto $\boldsymbol{e}_1$ if and only if $B_{13} < B_{31}$.

Conversely when we map $\boldsymbol{m}$ onto $\boldsymbol{e}_3$, we have $a = B_{31}/G$ and:

$$\psi(t) = \frac{G}{I_2}t + 2\operatorname{arctg}\left(\frac{B_{31}-G}{B_{13}}\operatorname{th}\left(\frac{\lambda t - \nu}{2}\right)\right)$$

We use this formula when $B_{13} \geq B_{31}$ so we are sure that B_{13} is non-0.

References

[BF54] P. F. Byrd and M. D. Friedman. *Handbook of elliptic integrals for engineers and physicists*. Springer-Verlag, 1954.

[CFSZ07] E. Celledoni, F. Fassò, N. Säfström and A. Zanna. 'The exact computation of the free rigid body motion and its use in splitting methods'. Preprint at `https://www.math.unipd.it/~fasso/research/papers/norway.pdf`. Oct. 2007.

[CFSZ08] E. Celledoni, F. Fassò, N. Säfström and A. Zanna. 'The exact computation of the free rigid body motion and its use in splitting methods'. In: *SIAM J. Scientific Computing* 30 (May 2008), pp. 2084–2112.

[OLBC10] F. Olver, D. Lozier, R. Boisvert and C. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.