

# Rotational Motion of a Rigid Reference Frame

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This document describes the computations that are performed by the class `RigidReferenceFrame` and its subclasses to determine the rotational motion (rotation, angular velocity, and angular acceleration) of a rigid frame.

## Definitions

We consider in this document a rigid reference frame that is defined by two bodies  $B_1$  and  $B_2$  at positions  $\mathbf{q}_1$  and  $\mathbf{q}_2$ , respectively. A basis of the reference frame is defined by three vectors:

- the *fore* vector  $\mathbf{F}$  which is along the axis  $\mathbf{q}_2 - \mathbf{q}_1$ ;
- the *normal* vector  $\mathbf{N}$  which is orthogonal to  $\mathbf{F}$  and is such that the velocity of the frame,  $\dot{\mathbf{q}}_2 - \dot{\mathbf{q}}_1$ , is in the plane  $(\mathbf{F}, \mathbf{N})$ ;
- the *binormal* vector  $\mathbf{B}$  which is orthogonal to  $\mathbf{F}$  and  $\mathbf{N}$  such that  $(\mathbf{F}, \mathbf{N}, \mathbf{B})$  forms a direct trihedron.

There are obviously many possible choices for  $(\mathbf{F}, \mathbf{N}, \mathbf{B})$ . In practice, it is convenient to choose  $\mathbf{B}$  before  $\mathbf{N}$  so that the basis is defined exclusively using vector products:

$$\begin{cases} \mathbf{F} &= \mathbf{r} \\ \mathbf{B} &= \mathbf{r} \wedge \dot{\mathbf{r}} \\ \mathbf{N} &= \mathbf{B} \wedge \mathbf{F} \end{cases} \quad (1)$$

where we have defined  $\mathbf{r} := \mathbf{q}_2 - \mathbf{q}_1$ . Since we'll need to use  $\dot{\mathbf{r}}$  later, it is important to note here that  $\dot{\mathbf{r}} = \dot{\mathbf{q}}_2 - \dot{\mathbf{q}}_1$  where  $\dot{\mathbf{q}}_1$  is the acceleration exerted on  $B_1$  by the rest of the system (and similarly,  $\dot{\mathbf{q}}_2$  is the acceleration exerted on  $B_2$  by the rest of the system).

It is trivial to check that these definitions satisfy the properties above, and in particular that they determine a direct orthogonal basis. The corresponding orthonormal basis is:

$$\begin{cases} \mathbf{f} &= \frac{\mathbf{F}}{|\mathbf{F}|} \\ \mathbf{b} &= \frac{\mathbf{B}}{|\mathbf{B}|} \\ \mathbf{n} &= \frac{\mathbf{N}}{|\mathbf{N}|} \end{cases} \quad (2)$$

These vectors are sufficient to define the rotation of the reference frame at any point in time.

## Derivatives of normalized vectors

In what follows, we will need to compute the time derivatives of the elements of the trihedron  $(\mathbf{f}, \mathbf{n}, \mathbf{b})$ . To help with this we prove two formulæ that define the first and second derivatives of  $\mathbf{V}/|\mathbf{V}|$  based on that of  $\mathbf{V}$ .

The first derivative is:

$$\begin{aligned}
 \frac{d}{dt} \frac{\mathbf{v}}{|\mathbf{v}|} &= \frac{|\mathbf{v}|\dot{\mathbf{v}} - \frac{d|\mathbf{v}|}{dt}\mathbf{v}}{|\mathbf{v}|^2} \\
 &= \frac{|\mathbf{v}|\dot{\mathbf{v}} - \frac{(\mathbf{v} \cdot \dot{\mathbf{v}})}{|\mathbf{v}|}\mathbf{v}}{|\mathbf{v}|^2} \\
 &= \frac{\dot{\mathbf{v}}}{|\mathbf{v}|} - \mathbf{v} \frac{(\mathbf{v} \cdot \dot{\mathbf{v}})}{|\mathbf{v}|^3}
 \end{aligned} \tag{3}$$

The second derivative is somewhat more complicated:

$$\begin{aligned}
 \frac{d^2}{dt^2} \frac{\mathbf{v}}{|\mathbf{v}|} &= \frac{d}{dt} \left( \frac{|\mathbf{v}|^2 \dot{\mathbf{v}} - (\mathbf{v} \cdot \dot{\mathbf{v}})\mathbf{v}}{|\mathbf{v}|^3} \right) \\
 &= \frac{|\mathbf{v}|^3 \frac{d}{dt} (|\mathbf{v}|^2 \dot{\mathbf{v}} - (\mathbf{v} \cdot \dot{\mathbf{v}})\mathbf{v}) - 3|\mathbf{v}|(\mathbf{v} \cdot \dot{\mathbf{v}})(|\mathbf{v}|^2 \dot{\mathbf{v}} - (\mathbf{v} \cdot \dot{\mathbf{v}})\mathbf{v})}{|\mathbf{v}|^6} \\
 &= \frac{2(\mathbf{v} \cdot \dot{\mathbf{v}})\dot{\mathbf{v}} + |\mathbf{v}|^2 \ddot{\mathbf{v}} - (|\dot{\mathbf{v}}|^2 + (\mathbf{v} \cdot \ddot{\mathbf{v}}))\mathbf{v} - (\mathbf{v} \cdot \dot{\mathbf{v}})\dot{\mathbf{v}}}{|\mathbf{v}|^3} - 3 \frac{|\mathbf{v}|^3 (\mathbf{v} \cdot \dot{\mathbf{v}})\dot{\mathbf{v}} - |\mathbf{v}|(\mathbf{v} \cdot \dot{\mathbf{v}})^2 \mathbf{v}}{|\mathbf{v}|^6} \\
 &= \frac{\ddot{\mathbf{v}}}{|\mathbf{v}|} - 2\dot{\mathbf{v}} \frac{(\mathbf{v} \cdot \dot{\mathbf{v}})}{|\mathbf{v}|^3} - \mathbf{v} \frac{|\dot{\mathbf{v}}|^2 + (\mathbf{v} \cdot \ddot{\mathbf{v}})}{|\mathbf{v}|^3} + 3\mathbf{v} \frac{(\mathbf{v} \cdot \dot{\mathbf{v}})^2}{|\mathbf{v}|^5}
 \end{aligned} \tag{4}$$

## Angular velocity

To compute the angular velocity, we derive the vectors (1) and obtain:

$$\begin{cases} \dot{\mathbf{F}} &= \dot{\mathbf{r}} \\ \dot{\mathbf{B}} &= \mathbf{r} \wedge \dot{\mathbf{r}} \\ \dot{\mathbf{N}} &= \dot{\mathbf{B}} \wedge \mathbf{F} + \mathbf{B} \wedge \dot{\mathbf{F}} \end{cases} \tag{5}$$

Injecting these expressions in the derivative formula (3) makes it possible to compute the trihedron of the derivatives  $(\dot{\mathbf{f}}, \dot{\mathbf{n}}, \dot{\mathbf{b}})$  of (2). The angular velocity is then written:

$$\boldsymbol{\omega} = (\dot{\mathbf{n}} \cdot \mathbf{b})\mathbf{f} + (\dot{\mathbf{b}} \cdot \mathbf{f})\mathbf{n} + (\dot{\mathbf{f}} \cdot \mathbf{n})\mathbf{b}$$

## Angular acceleration

To compute the angular acceleration, we derive the vectors (5) and obtain:

$$\begin{cases} \ddot{\mathbf{F}} &= \ddot{\mathbf{r}} \\ \ddot{\mathbf{B}} &= \dot{\mathbf{r}} \wedge \dot{\mathbf{r}} + \mathbf{r} \wedge \mathbf{r}^{(3)} \\ \ddot{\mathbf{N}} &= \ddot{\mathbf{B}} \wedge \mathbf{F} + 2\dot{\mathbf{B}} \wedge \dot{\mathbf{F}} + \mathbf{B} \wedge \ddot{\mathbf{F}} \end{cases} \tag{6}$$

Injecting these expressions in the second derivative formula (4) makes it possible to compute the trihedron of the second derivatives  $(\ddot{\mathbf{f}}, \ddot{\mathbf{n}}, \ddot{\mathbf{b}})$  of (2). The angular acceleration is then written:

$$\begin{aligned}
 \dot{\boldsymbol{\omega}} &= (\ddot{\mathbf{n}} \cdot \mathbf{b})\mathbf{f} + (\dot{\mathbf{n}} \cdot \dot{\mathbf{b}})\mathbf{f} + (\dot{\mathbf{n}} \cdot \mathbf{b})\dot{\mathbf{f}} \\
 &\quad + (\ddot{\mathbf{b}} \cdot \mathbf{f})\mathbf{n} + (\dot{\mathbf{b}} \cdot \dot{\mathbf{f}})\mathbf{n} + (\dot{\mathbf{b}} \cdot \mathbf{f})\dot{\mathbf{n}} \\
 &\quad + (\ddot{\mathbf{f}} \cdot \mathbf{n})\mathbf{b} + (\dot{\mathbf{f}} \cdot \dot{\mathbf{n}})\mathbf{b} + (\dot{\mathbf{f}} \cdot \mathbf{n})\dot{\mathbf{b}}
 \end{aligned}$$

## Jerk

The alert reader will have noticed the presence of the jerk,  $\mathbf{r}^{(3)}$  in (6). This section explains how it is calculated.

Recall that  $\ddot{\mathbf{r}} = \ddot{\mathbf{q}}_2 - \ddot{\mathbf{q}}_1$  where  $\ddot{\mathbf{q}}_1$  is the acceleration exerted on  $B_1$  by the rest of the system. Define  $\mathbf{F}_1(\mathbf{q}, t)$  to be the vector field representing the acceleration exerted by all the bodies in the system *other than*  $B_1$  on the position  $\mathbf{q}$  at time  $t$ . It follows from the definition that  $\ddot{\mathbf{q}}_1(t) = \mathbf{F}_1(\mathbf{q}_1(t), t)$  where we have made the dependency on time explicit. The jerk is the total derivative with respect to  $t$ :

$$\mathbf{q}_1^{(3)}(t) = \left. \frac{d}{d\tau} \mathbf{F}_1(\mathbf{q}, \tau) \right|_{\mathbf{q}=\mathbf{q}_1(t), \tau=t}$$

Looking at the coordinate along the axis  $\mathbf{x}$  we can write the total derivative as follow:

$$\frac{d}{dt} \mathbf{F}_{1,x} = \frac{\partial \mathbf{F}_{1,x}}{\partial x} \frac{dx}{dt} + \frac{\partial \mathbf{F}_{1,x}}{\partial y} \frac{dy}{dt} + \frac{\partial \mathbf{F}_{1,x}}{\partial z} \frac{dz}{dt}$$

and similarly along the axes  $\mathbf{y}$  and  $\mathbf{z}$ . The partial derivatives are the first row of the Jacobian, so we obtain:

$$\frac{d \mathbf{F}_1}{dt} = \nabla \mathbf{F}_1 \cdot \frac{d \mathbf{q}}{dt}$$

which gives us the jerk as:

$$\mathbf{q}_1^{(3)}(t) = \nabla \mathbf{F}_1(\mathbf{q}_1(t), t) \cdot \dot{\mathbf{q}}_1(t)$$

## Application to the body surface reference frame

While the formulæ above were derived assuming a rigid reference frame defined by two bodies  $B_1$  and  $B_2$ , they remain valid for a reference frame defined by a single body  $B_1$  provided that we give a proper definition of the vector  $\mathbf{r}$ . We now consider how they apply to the *body surface* reference frame.

It is convenient to choose  $\mathbf{r}$  to be a unit vector rotating with the surface of the body; without loss of generality,  $\mathbf{r}$  may be written:

$$\mathbf{r} = \mathbf{x} \cos(\omega t + \varphi) + \mathbf{y} \sin(\omega t + \varphi)$$

where  $\mathbf{x}$  and  $\mathbf{y}$  form a direct orthonormal basis of the equatorial plane of the body. The time derivatives of  $\mathbf{r}$  are:

$$\begin{cases} \dot{\mathbf{r}} &= -\mathbf{x}\omega \sin(\omega t + \varphi) + \mathbf{y}\omega \cos(\omega t + \varphi) \\ \ddot{\mathbf{r}} &= -\mathbf{x}\omega^2 \cos(\omega t + \varphi) - \mathbf{y}\omega^2 \sin(\omega t + \varphi) = -\omega^2 \mathbf{r} \\ \mathbf{r}^{(3)} &= \mathbf{x}\omega^3 \sin(\omega t + \varphi) - \mathbf{y}\omega^3 \cos(\omega t + \varphi) = -\omega^2 \dot{\mathbf{r}} \end{cases}$$

Using (1) it's easy to see that:

$$\begin{cases} \mathbf{F} &= \mathbf{r} \\ \mathbf{B} &= \omega \mathbf{z} \\ \mathbf{N} &= \dot{\mathbf{r}} \end{cases}$$

where  $\mathbf{z}$  is along the rotation axis such that  $(\mathbf{x}, \mathbf{y}, \mathbf{z})$  form a direct trihedron. The orthonormal vectors follow immediately, noting that  $|\mathbf{r}| = 1$ :

$$\begin{cases} \mathbf{f} &= \mathbf{r} \\ \mathbf{b} &= \mathbf{z} \\ \mathbf{n} &= \frac{\dot{\mathbf{r}}}{\omega} \end{cases}$$

The first derivatives of the orthonormal trihedron can be computed directly:

$$\begin{cases} \dot{\mathbf{f}} &= \dot{\mathbf{r}} \\ \dot{\mathbf{b}} &= 0 \\ \dot{\mathbf{n}} &= \frac{\ddot{\mathbf{r}}}{\omega} = -\omega \mathbf{r} \end{cases}$$

which yields, not surprisingly,  $\boldsymbol{\omega} = \omega \mathbf{z}$ . The second derivatives are equally straightforward:

$$\begin{cases} \ddot{\mathbf{f}} &= \ddot{\mathbf{r}} = -\mathbf{r} \\ \dot{\mathbf{b}} &= 0 \\ \dot{\mathbf{n}} &= -\omega \dot{\mathbf{r}} \end{cases}$$

and yield, as expected,  $\dot{\boldsymbol{\omega}} = 0$ .