On a Formula by Cohen, Hubbard and Oesterwinter

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2021-03-13

This document proves and generalizes a formula given in [**CohenHubbardOesterwinter1973**] to compute the velocity of a body in the context of 12th-order numerical integration of n-body systems.

Statement

[**CohenHubbardOesterwinter1973**] states that

> the formula for a velocity component [is] of the form:
>
> $$\dot{x}_{n+1} = \frac{x_n - x_{n-1} + h^2 \sum_{i=0}^{12} \beta_i \ddot{x}_{n-i}}{h} \tag{1}$$

They then proceed to give explicit values for the (rational) coefficients β_i without explaining how they are computed. This makes it impossible to use this formula for integrators of a different order.

Finite Differences

[**Fornberg1987**] gives finite difference formulæ for any order and for kernels of arbitrary size. Given a sufficient regular function $f(x)$ and a family $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_N)$ of points (which may not be equidistant), the m-th derivative at any point x_0 may be approximated to order $n - m + 1$ as:

$$f^{(m)}(x_0) \cong \sum_{\nu=0}^{n} \delta^m_{n,\nu}(\boldsymbol{\alpha}) f(\alpha_\nu)$$

where the coefficients $\delta^m_{n,\nu}$ are dependent on $\boldsymbol{\alpha}$ but independent of f.

If the $\boldsymbol{\alpha}$ are equally spaced with step h (i.e., $\alpha_\nu = x_0 + \nu h$) then we can write a forward formula as follows:

$$f^{(m)}(x_0) = \sum_{\nu=0}^{n} \delta^m_{n,\nu}(x_0, h) f(x_0 + \nu h) + \mathcal{O}(h^{n-m+1})$$

In this case equation (3.8) from [**Fornberg1987**] may be rewritten as follows, restoring x_0:

$$\begin{aligned}\delta^m_{n,\nu} &= \frac{1}{\alpha_n - \alpha_\nu}((\alpha_n - x_0)\delta^m_{n-1,\nu} - m\delta^{m-1}_{n-1,\nu}) \\ &= \frac{1}{(n-\nu)h}(nh\delta^m_{n-1,\nu} - m\delta^{m-1}_{n-1,\nu})\end{aligned}$$

It is convenient to extract the powers of h from the coefficients:

$$\delta^m_{n,\nu}(x_0, h) = \frac{\lambda^m_{n,\nu}}{h^m}$$

where $\lambda^m_{n,\nu}$ is independent from x_0 and h. We can check that equation (3.8) is still verified by multiplying both sides by h^m:

$$\lambda^m_{n,\nu} = \frac{1}{n-\nu}(n\lambda^m_{n-1,\nu} - m\lambda^{m-1}_{n-1,\nu})$$

Derivation

In this section we derive equation (1) and obtain an explicit expression for the coefficients appearing in the sum.

We consider a sufficient regular function $f(x)$ and write its Taylor series near x_0:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &= f(x_0) + (x - x_0) f'(x_0) + \sum_{n=2}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \end{aligned}$$

Conclusion

We have demonstrated how [**Fukushima2018**] uses different techniques from the ones detailed in [**Fukushima2011a**] in order to handle the logarithmic singularities of the B and D complete integrals of the second kind: while [**Fukushima2011a**] divides the leading logarithmic term $\log \frac{m_c}{16}$, [**Fukushima2018**] divides the complete logarithmic term $\log q(m_c)$.