On an Article by Celledoni et al.

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2019-11-13

This document provides clarifications, corrections, and accuracy improvements to the formulæ presented in [CFSZ07]. It follows the notation and conventions of that paper. Note that the preprint [CFSZ07] differs in some of the formulæ from the final publication [CFSZ08], and that we follow the former because the latter introduced errors.

Preamble

We remind the reader of the derivation formulæ for the Jacobian elliptic functions ([OLBC10], section 22.13(i)):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{sn} u &= \operatorname{cn} u \operatorname{dn} u \\ \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{cn} u &= -\operatorname{sn} u \operatorname{dn} u \\ \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{dn} u &= -k^2 \operatorname{sn} u \operatorname{cn} u \end{cases}$$

and for the hyperbolic functions ([OLBC10], section 4.34):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{th} u &= \operatorname{sech}^2 u \\ \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{sech} u &= -\operatorname{sech} u \operatorname{th} u \end{cases}$$

The equations of motion

We start by writing equation (1) of [CFSZ07] in coordinates. The coordinates of $\boldsymbol{m}$ and $\boldsymbol{I}$ are defined by:

$$\boldsymbol{m} \coloneqq \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

and:

$$\boldsymbol{I} \coloneqq \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

with $I_1 \leq I_2 \leq I_3$.

Euler's equation $\dot{\boldsymbol{m}} = [\boldsymbol{m}, \boldsymbol{\omega}]$ can be written in coordinates in the principal axes frame:

$$\dot{\boldsymbol{m}} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \times \begin{pmatrix} m_1/I_1 \\ m_2/I_2 \\ m_3/I_3 \end{pmatrix}$$

thus:

$$\begin{cases} \dot{m}_1 &= m_2 m_3 \left(1/I_3 - 1/I_2\right) \\ \dot{m}_2 &= m_3 m_1 \left(1/I_1 - 1/I_3\right) \\ \dot{m}_3 &= m_1 m_2 \left(1/I_2 - 1/I_1\right) \end{cases} \tag{1}$$

Solution of Euler's equation

The solution of Euler's equation has three cases depending on the initial value of $\boldsymbol{m}$ (more precisely, on the sign of $\Delta_2 = m_1^2 \frac{I_{12}}{I_1} + m_3^2 \frac{I_{32}}{I_3}$, see discussion below). Figure 1 illustrates the possible evolutions of $\boldsymbol{m}$. The sphere is the surface $|\boldsymbol{m}| = G$, which is an invariant of motion. The planes are the surfaces $\Delta_2 = 0$ and separate different modes of the motion. The blue curve is called case (i) in [CFSZ07]: $\boldsymbol{m}$ follows a periodic curve, and when that curve is close to the m_1 axis we have a classical case of precession. The red curve is case (ii), and again the motion of $\boldsymbol{m}$ is periodic and exhibits precession when the curve remains close to the m_3 axis. The green curve is case (iii): $\boldsymbol{m}$ takes an infinite amount of time to reach the point $(0, G, 0)$; furthermore, the motion is unstable as any perturbation moves it either to the blue or the red region where $\boldsymbol{m}$ oscillates between points close to $(0, G, 0)$ and $(0, -G, 0)$; this is the Джанибеков effect.

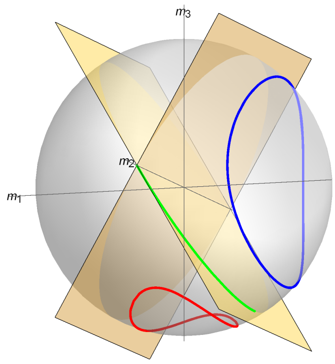

Figure 1. Possible trajectories of $\boldsymbol{m}$: the blue and red curves are cases (i) and (ii), respectively, and correspond to motion with precession. The green curve is the (unstable) case (iii) and any perturbation demonstrates the Джанибеков effect.

The solutions may also be visualized by intersecting the sphere $|\boldsymbol{m}| = G$ with ellipsoids defined by the value of the kinetic energy T, which is also a constant of motion. Since $T = \frac{G^2 - \Delta_2}{2I_2}$, different values of T determine the same modes as above.

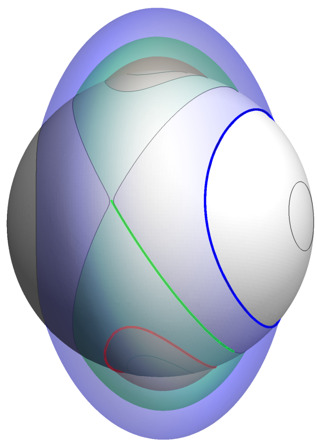

Figure 2. Possible trajectories of $\boldsymbol{m}$: the sphere is identical to that of Figure 1. The ellipsoids are surfaces of equal kinetic energy and intersect the sphere on the blue, red, and green curves depending on the value of T .

In the rest of this section, we describe our notation and derive (corrected) formulæ for the three cases described above.

Notation

[CFSZ07] uses a dimensionless formulation where $|\boldsymbol{m}| = 1$, and absolute values for I_{jh} and Δ_j. We prefer to use a dimensionful formulation where $|\boldsymbol{m}| = G$, and to avoid absolute values. Thus we define:

$$I_{jh} \coloneqq I_j - I_h \qquad \Delta_j \coloneqq G^2 - 2TI_j \qquad B_{jh} \coloneqq \sqrt{\pm\frac{I_j\Delta_h}{I_{jh}}}$$

$$k \coloneqq \sqrt{-\frac{\Delta_1 I_{32}}{\Delta_3 I_{21}}} \qquad \lambda_1 \coloneqq \sqrt{\frac{\Delta_1 I_{32}}{I_1 I_2 I_3}} \qquad \lambda_3 \coloneqq \sqrt{\frac{\Delta_3 I_{12}}{I_1 I_2 I_3}}$$

With these definitions, $I_{jh} \geq 0$ if and only if $j \geq h$, and we will prove later that $\Delta_1 \geq 0$, $\Delta_3 \leq 0$, and that Δ_2 can have either sign. The sign under the radical in the definition of B_{jh} is $+$ if $h = 1$ and $j \geq h$, or $h = 3$ and $j < h$; it is $-$ otherwise (note that we never use B_{j2} in the analysis below). At this point it is also useful to observe that:

$$B_{31}^2 + B_{13}^2 = \frac{\Delta_1 I_3 - \Delta_3 I_1}{I_{31}} = G^2$$

Physically, I_{jh} has the dimension of an inertial momentum L^2M. G has the dimension of an angular momentum $L^2MT^{-1}A$. Δ_j has the same dimension as G^2. B_{jh} has the same dimension as $\sqrt{\Delta_h}$, i.e., the same dimension as G. λ_1 and λ_3 have the same dimension as the quotient $\frac{G}{I_j}$, i.e., $T^{-1}A$ which is appropriate for their usage.

Case (i)

Case (i) of the solution of Euler's equation in section 2.2 of [CFSZ07] is:

$$\boldsymbol{m}_t = \begin{pmatrix} \sigma B_{13}\,\mathrm{dn}(\lambda t - \nu, k) \\ -B_{21}\,\mathrm{sn}(\lambda t - \nu, k) \\ B_{31}\,\mathrm{cn}(\lambda t - \nu, k) \end{pmatrix}$$

If we derive this expression with respect to t, inject in into (1), and eliminate the elliptic functions we obtain:

$$\begin{cases} -\sigma\lambda k^2 B_{13} & = -B_{21}B_{31}\left(1/I_3 - 1/I_2\right) \\ -\lambda B_{21} & = \sigma B_{13}B_{31}\left(1/I_1 - 1/I_3\right) \\ -\lambda B_{31} & = -\sigma B_{13}B_{21}\left(1/I_2 - 1/I_1\right) \end{cases} \tag{2}$$

The last equation of (2) yields the following value for λ:

$$\begin{aligned} \lambda = \sigma\frac{B_{13}B_{21}}{B_{31}}\frac{I_1 - I_2}{I_1 I_2} &= \sigma\sqrt{\frac{I_1\Delta_3}{I_{13}}\frac{I_2\Delta_1}{I_{21}}\frac{I_{31}}{I_3\Delta_1}}\frac{I_1 - I_2}{I_1 I_2} \\ = \sigma\sqrt{\frac{\Delta_3}{I_{21}I_1I_2I_3}}(I_1 - I_2) &= -\sigma\sqrt{\frac{\Delta_3 I_{21}}{I_1 I_2 I_3}} = -\sigma\lambda_3 \end{aligned}$$

The sign change when moving $I_1 - I_2$ under the radical is necessary because $I_1 - I_2 < 0$.

It is straightforward to check that this value of λ also satisfies the other equations of (2). Note that it differs in sign from the one given by [CFSZ07]: the sign error is visible in that it does not yield the proper precession direction.

Case (ii)

Case (ii) of the solution of Euler's equation in section 2.2 of [CFSZ07] is:

$$\boldsymbol{m}_t = \begin{pmatrix} B_{13}\,\mathrm{cn}(\lambda t - \nu, k^{-1}) \\ -B_{23}\,\mathrm{sn}(\lambda t - \nu, k^{-1}) \\ \sigma B_{31}\,\mathrm{dn}(\lambda t - \nu, k^{-1}) \end{pmatrix}$$

Just as we did above, we derive this expression with respect to t, inject in into (1), and eliminate the elliptic functions:

$$\begin{cases} -\lambda B_{13} & = -\sigma B_{23} B_{31} \big(1/I_3 - 1/I_2\big) \\ -\lambda B_{23} & = \sigma B_{13} B_{31} \big(1/I_1 - 1/I_3\big) \\ -\sigma \lambda k^{-2} B_{31} & = -B_{13} B_{23} \big(1/I_2 - 1/I_1\big) \end{cases} \tag{3}$$

The first equation of (3) yields the following value for λ:

$$\begin{aligned} \lambda &= \sigma \frac{B_{23} B_{31}}{B_{13}} \frac{I_2 - I_3}{I_2 I_3} = \sigma \sqrt{\frac{I_2 \Delta_3}{I_{23}} \frac{I_3 \Delta_1}{I_{31}} \frac{I_{13}}{I_1 \Delta_3}} \frac{I_2 - I_3}{I_2 I_3} \\ &= \sigma \sqrt{\frac{\Delta_1}{I_{23} I_1 I_2 I_3}} (I_2 - I_3) = -\sigma \sqrt{\frac{\Delta_1 I_{23}}{I_1 I_2 I_3}} = -\sigma \lambda_1 \end{aligned}$$

Again, note the change of sign due to the fact that $I_2 - I_3 < 0$. And again, the same value of λ can be shown to satisfy the other equations of (3).

Case (iii)

Case (iii) of the solution of Euler's equation in section 2.2 of [CFSZ07] is clearly incorrect as it implies that m_1 and m_3 always have the same sign, whereas it is straightforward to choose initial conditions where they do not (because the separatrix is made of two planes, see Figure 1). Instead, we introduce an extra parameter $\sigma'' = \pm 1$ and posit a solution of the form:

$$\boldsymbol{m}_t = \begin{pmatrix} \sigma' B_{13} \operatorname{sech}(\lambda t - \nu) \\ G \operatorname{th}(\lambda t - \nu) \\ \sigma'' B_{31} \operatorname{sech}(\lambda t - \nu) \end{pmatrix}$$

Deriving this expression and injecting it into (1) yields:

$$\begin{cases} -\sigma' \lambda B_{13} & = \sigma'' G B_{31} \big(1/I_3 - 1/I_2\big) \\ \lambda G & = \sigma' \sigma'' B_{13} B_{31} \big(1/I_1 - 1/I_3\big) \\ -\sigma'' \lambda B_{31} & = \sigma' G B_{13} \big(1/I_2 - 1/I_1\big) \end{cases} \tag{4}$$

The second equation of (4) gives the following value for λ:

$$\lambda = \sigma' \sigma'' \frac{B_{13} B_{31}}{G} \frac{I_3 - I_1}{I_1 I_3} = \sigma' \sigma'' \frac{1}{G} \sqrt{\frac{I_1 \Delta_3}{I_{13}} \frac{I_3 \Delta_1}{I_{31}}} \frac{I_3 - I_1}{I_1 I_3} = \sigma' \sigma'' \frac{1}{G} \sqrt{-\frac{\Delta_1 \Delta_3}{I_1 I_3}}$$

In this case it is a bit less obvious that the other equations yield the same value of λ. We detail the derivation for the first equation, using the fact that ${\sigma'}^2 = 1$:

$$\begin{aligned} \lambda &= -\sigma' \sigma'' G \frac{B_{31}}{B_{13}} \frac{I_2 - I_3}{I_2 I_3} = -\sigma' \sigma'' G \sqrt{\frac{I_3 \Delta_1}{I_{31}} \frac{I_{13}}{I_1 \Delta_3}} \frac{I_2 - I_3}{I_2 I_3} \\ &= -\sigma' \sigma'' G \sqrt{-\frac{\Delta_1}{I_1 I_3 \Delta_3}} \frac{I_2 - I_3}{I_2} = \sigma' \sigma'' G \sqrt{-\frac{\Delta_1}{I_1 I_3 \Delta_3}} \left(\frac{I_3}{I_2} - 1 \right) \end{aligned}$$

Now note that in case (iii) we have $2 T I_2 = G^2$ thus $1/I_2 = \frac{2T}{G^2}$. λ can be rewritten as:

$$\lambda = \sigma' \sigma'' G \sqrt{-\frac{\Delta_1}{I_1 I_3 \Delta_3}} \left(\frac{2 T I_3}{G^2} - 1 \right) = \sigma' \sigma'' \frac{1}{G} \sqrt{-\frac{\Delta_1 \Delta_3}{I_1 I_3}}$$

where we have used the fact that $2 T I_3 - G^2 = -\Delta_3 = 2T(I_3 - I_2) > 0$.

We then define:

$$\lambda_2 \coloneqq \frac{1}{G} \sqrt{-\frac{\Delta_1 \Delta_3}{I_1 I_3}}$$

It is easy to see that λ_2 is the common value of λ_1 and λ_3 in case (iii), that σ' and σ'' are free parameters and that:

$$\lambda = \sigma' \sigma'' \lambda_2$$

Note that λ_2 has the same dimension as the quotient $\frac{\Delta_j}{G I_j}$, which has the same dimension as $\frac{G}{I_j}$, namely, $T^{-1}A$.

Phase and initial value

The phase ν and the free parameters σ, σ' and σ'' are determined from the initial value $\boldsymbol{m}_0$ by setting $t = 0$.

Case (i)

We have:

$$\boldsymbol{m}_0 = \begin{pmatrix} \sigma B_{13} \,\mathrm{dn}(-\nu, k) \\ -B_{21} \,\mathrm{sn}(-\nu, k) \\ B_{31} \,\mathrm{cn}(-\nu, k) \end{pmatrix}$$

First, we set σ to be the sign of m_{01}. Then, forming the quotient of the last two coordinates we find:

$$\frac{m_{02}}{m_{03}} = \frac{B_{21}}{B_{31}} \,\mathrm{tg}(\mathrm{am}(\nu, k))$$

thus:

$$\mathrm{arctg}\left(\frac{m_{02}}{m_{03}} \frac{B_{31}}{B_{21}}\right) = \mathrm{am}(\nu, k)$$

and finally we obtain ν as:

$$\nu = F\left(\mathrm{arctg}\left(\frac{m_{02}}{m_{03}} \frac{B_{31}}{B_{21}}\right), k\right)$$

Case (ii)

Starting from:

$$\boldsymbol{m}_0 = \begin{pmatrix} B_{13} \,\mathrm{cn}(-\nu, k^{-1}) \\ -B_{23} \,\mathrm{sn}(-\nu, k^{-1}) \\ \sigma B_{31} \,\mathrm{dn}(-\nu, k^{-1}) \end{pmatrix}$$

we set σ to be the sign of m_{03} and form the quotient of the first two coordinates. We obtain:

$$\frac{m_{02}}{m_{01}} = \frac{B_{23}}{B_{13}} \,\mathrm{tg}(\mathrm{am}(\nu, k^{-1}))$$

and for ν:

$$\nu = F\left(\mathrm{arctg}\left(\frac{m_{02}}{m_{01}} \frac{B_{13}}{B_{23}}\right), k^{-1}\right)$$

Case (iii)

The initial value $\boldsymbol{m}_0$ is:

$$\boldsymbol{m}_0 = \begin{pmatrix} \sigma' B_{13} \,\mathrm{sech}(-\nu) \\ G \,\mathrm{th}(-\nu) \\ \sigma'' B_{31} \,\mathrm{sech}(-\nu) \end{pmatrix}$$

σ' and σ'' are set to be the signs of m_{01} and m_{03}, respectively. The second coordinate immediately gives:

$$\nu = -\,\mathrm{argth}\left(\frac{m_{02}}{G}\right)$$

Implementation considerations

Some of the formulæ given by [CFSZ07] do not lend themselves to an easy implementation or lead to numerical inaccuracies. We describe in this section the modifications we make to these formulæ in our implementation.

The quantity Δ_j

We notice that the computation of Δ_j may entail cancellations, so we go back to the definition of $|\boldsymbol{m}|$ and of the kinetic energy:

$$\begin{cases} G^2 &= m_1^2 + m_2^2 + m_3^2 \\ 2T &= \dfrac{m_1^2}{I_1} + \dfrac{m_2^2}{I_2} + \dfrac{m_3^2}{I_3} \end{cases}$$

When, for instance, $j = 2$, this yields:

$$\begin{aligned} \Delta_2 &= m_1^2\left(1 - \frac{I_2}{I_1}\right) + m_3^2\left(1 - \frac{I_2}{I_3}\right) \\ &= m_1^2\frac{I_{12}}{I_1} + m_3^2\frac{I_{32}}{I_3} \end{aligned}$$

and similarly:

$$\begin{cases} \Delta_1 &= m_2^2\dfrac{I_{21}}{I_2} + m_3^2\dfrac{I_{31}}{I_3} \\ \Delta_3 &= m_1^2\dfrac{I_{13}}{I_1} + m_2^2\dfrac{I_{23}}{I_2} \end{cases}$$

It is easy to see that Δ_1 and Δ_3 are the sums of terms of the same sign, so they can be computed without cancellations. Furthermore, $\Delta_1 \geq 0$ and $\Delta_3 \leq 0$. Δ_2 can have either sign, which correspond exactly to cases (i) ($\Delta_2 < 0$), (ii) ($\Delta_2 > 0$) and (iii) ($\Delta_2 = 0$).

The elliptic modulus

For the computation of the elliptic functions and integrals [CFSZ07] gives the value of the elliptic modulus k but we need the value of the complementary parameter $m_c = 1 - m$ (see [OLBC10], section 19.1.2 for an overview of the notation). In case (i) we have:

$$m_c = 1 - k^2 = 1 + \frac{\Delta_1 I_{32}}{\Delta_3 I_{21}}$$

where we have used $\Delta_3 \leq 0$. This can be rewritten as follows:

$$\begin{aligned} m_c &= \frac{\Delta_3 I_{21} + \Delta_1 I_{32}}{\Delta_3 I_{21}} = \frac{(G^2 - 2TI_3)(I_2 - I_1) + (G^2 - 2TI_1)(I_3 - I_2)}{\Delta_3 I_{21}} \\ &= \frac{G^2(I_3 - I_1) + 2TI_2(I_1 - I_3)}{\Delta_3 I_{21}} = \frac{\Delta_2 I_{31}}{\Delta_3 I_{21}} \end{aligned}$$

Similarly, in case (ii):

$$m_c = 1 - k^{-2} = 1 + \frac{\Delta_3 I_{21}}{\Delta_1 I_{32}} = \frac{\Delta_1 I_{32} + \Delta_3 I_{21}}{\Delta_1 I_{32}} = \frac{\Delta_2 I_{31}}{\Delta_1 I_{32}}$$

In both cases we have $m_c \geq 0$.

References

[CFSZ07] E. Celledoni, F. Fassò, N. Säfström, and A. Zanna. “The exact computation of the free rigid body motion and its use in splitting methods”. Preprint. Oct. 2007.

[CFSZ08] E. Celledoni, F. Fassò, N. Säfström, and A. Zanna. “The exact computation of the free rigid body motion and its use in splitting methods”. In: *SIAM J. Scientific Computing* 30 (May 2008), pp. 2084–2112.

[OLBC10] F. Olver, D. Lozier, R. Boisvert, and C. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.