Hiding Computations in Projection

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This document describes the computations that are performed by the method `VisibleSegments` of class `Perspective` to determine the parts of a segment that are hidden by a sphere when seen from a pinhole camera.

Figure 1 illustrates an example of the problem. The points drawn in black define the problem: A and B are the extremities of the segment; C is the centre of the sphere; R is the radius of the sphere; and K is the location of the camera. The points drawn in red will be computed as part of the resolution. Of particular interest is the plane KAB, in which we willl do much of the analysis below. The figure shows the circle formed by the intersection of this plane with the sphere, as well as H, the center of that circle and the projection of C on KAB.

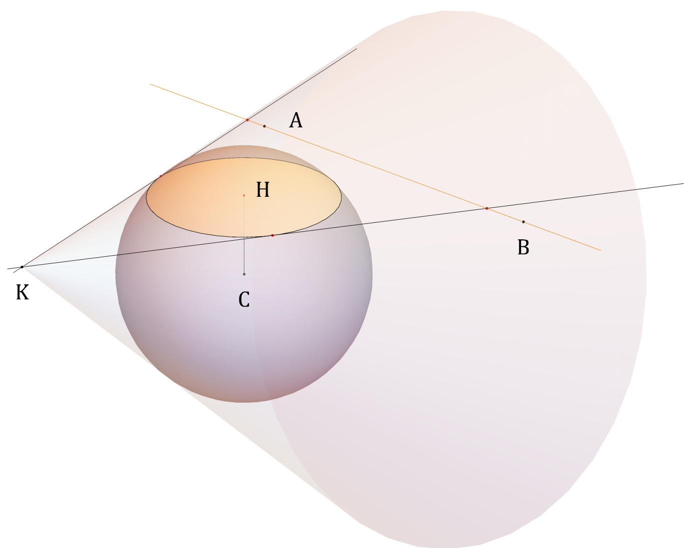

Figure 1. A 3-dimensional example. In this case, the only segment visible from K is the part of AB between B and the red point immediately to its left.

Camera inside the sphere

We start our analysis by eliminating a case that would cause anomalies in the analysis below. If

$$\overrightarrow{\mathrm{KC}} \cdot \overrightarrow{\mathrm{KC}} < R^2$$

then the camera is inside the sphere and the segment is hidden irrespective of its position.

Sphere and segment in distinct half-spaces

Consider the plane containing K and orthogonal to KC; it separates the entire space into two half-spaces. If the segment AB is entirely within the half-space that does not contain C then the segment is not hidden (remember that K is not inside the sphere). This is the case if the following inequalities are both true:

$$\overrightarrow{KA} \cdot \overrightarrow{KC} < 0$$
$$\overrightarrow{KB} \cdot \overrightarrow{KC} < 0.$$

Projection of C on KAB

For simplicity we will do the rest of our analysis in the plane KAB and we will often use $(\overrightarrow{KA}, \overrightarrow{KB})$ as a basis of that plane. Let H be the orthogonal projection of C on KAB. and define α, β to be its coordinates in $(\overrightarrow{KA}, \overrightarrow{KB})$

$$\overrightarrow{KH} = \alpha\overrightarrow{KA} + \beta\overrightarrow{KB}.$$

Note that $\overrightarrow{KH} = \overrightarrow{KC} + \overrightarrow{CH}$. By definition, $\overrightarrow{CH}$ is orthogonal to both $\overrightarrow{KA}$ and $\overrightarrow{KB}$:

$$\overrightarrow{KA} \cdot \overrightarrow{CH} = 0$$
$$\overrightarrow{KB} \cdot \overrightarrow{CH} = 0.$$

Decomposing $\overrightarrow{CH}$ we obtain:

$$\overrightarrow{KA} \cdot \overrightarrow{KH} = \overrightarrow{KA} \cdot \overrightarrow{KC}$$
$$\overrightarrow{KB} \cdot \overrightarrow{KH} = \overrightarrow{KB} \cdot \overrightarrow{KC}.$$

Expanding $\overrightarrow{KH}$ on the basis $(\overrightarrow{KA}, \overrightarrow{KB})$ gives a linear system of two equations with two unknowns:

$$\alpha\overrightarrow{KA} \cdot \overrightarrow{KA} + \beta\overrightarrow{KA} \cdot \overrightarrow{KB} = \overrightarrow{KA} \cdot \overrightarrow{KC}$$
$$\alpha\overrightarrow{KA} \cdot \overrightarrow{KB} + \alpha\overrightarrow{KB} \cdot \overrightarrow{KB} = \overrightarrow{KB} \cdot \overrightarrow{KC}.$$

The determinant of this system is

$$D = (\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KB}) - (\overrightarrow{KA} \cdot \overrightarrow{KB})^2,$$

which is non-zero if and only if $A \neq B$. The solutions are thus:

$$\alpha = \frac{(\overrightarrow{KB} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KC}) - (\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KB} \cdot \overrightarrow{KC})}{D}$$
$$\beta = \frac{(\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KC}) - (\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KC})}{D}.$$

Once $\overrightarrow{KH}$ is determined we can compute $\overrightarrow{CH} = \overrightarrow{KH} - \overrightarrow{KC}$. If $\overrightarrow{CH} \cdot \overrightarrow{CH} \geq R^2$, the sphere is either tangent to the plane KAB or doesn't intersect it. In these cases, there is no hiding.

If the sphere intersects KAB, then the intersection is a circle whose radius we denote by r in the rest of this analysis. The radius is such that $r^2 = R^2 - \overrightarrow{CH} \cdot \overrightarrow{CH}$.

An optimization where the circle is outside the wedge KAB

A useful optimization at this stage is to determine if the circle intersects the wedge KAB. If it does not, then there is no hiding and the segment AB is entirely visible. This can happen because the circle is away from the wedge on the side of KA or because it is away from the wedge on the side of KB. Figure 2 illustrates the latter case: note that the circle must not intersect KB, so the distance between H and KB must be at least r; the figure shows the case where the circle is tangent to KB at N, which is the one that interests us in this section.

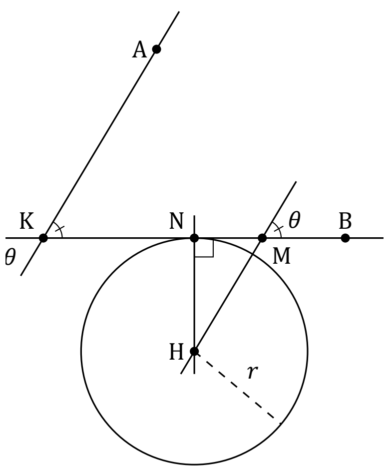

Figure 2. Circle lying outside the wedge KAB on the side of KB.

Observe that, if θ is the angle between $\overrightarrow{KA}$ and $\overrightarrow{KB}$, we have

$$\overrightarrow{KA} \cdot \overrightarrow{KB} = |\overrightarrow{KA}||\overrightarrow{KB}| \cos\theta$$

Let M be the point where H projects on KB parallel to KA. Since α and β are the coordinates of $\overrightarrow{KH}$ in the basis $(\overrightarrow{KA}, \overrightarrow{KB})$, we have

$$\overrightarrow{HM} = \alpha\overrightarrow{KA}, \tag{1}$$

and elementary trigonometry in the triangle HNM yields

$$\overrightarrow{HM} \cdot \overrightarrow{HM} = \frac{r^2}{\sin^2\theta} = \frac{r^2}{1-\cos^2\theta}. \tag{2}$$

Eliminating $\overrightarrow{HM}$ between equations (1) and (2) we obtain the following conditions on α for H to be outside the wedge KAB on the side of KB:

$$\alpha \leq 0$$

$$\alpha^2 \geq r^2 \frac{\overrightarrow{KB} \cdot \overrightarrow{KB}}{(\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KB}) - (\overrightarrow{KA} \cdot \overrightarrow{KB})^2}.$$

It's straightforward to prove similar conditions on β for the circle to be outside the wedge KAB on the side of KA.

Construction of the cone

Remember that the circle of centre H and radius r is the intersection of the sphere with the plane KAB. As shown on figure 1, hiding is determined by the cone of apex K tangent to the sphere. This cone intersects KAB in two straight lines that go through K and are tangent to the circle at points P and P′. In this section we are going to determine the coordinates of P and P′ in the basis $(\overrightarrow{KA}, \overrightarrow{KB})$. Figure 3 illustrates the construction of these points.

P is characterized by the two equations:

$$\overrightarrow{PH} \cdot \overrightarrow{PH} = r^2 \tag{3}$$

$$\overrightarrow{PH} \cdot \overrightarrow{KP} = 0. \tag{4}$$

Equation (4) may be rewritten as

$$\overrightarrow{PH} \cdot (\overrightarrow{KH} + \overrightarrow{HP}) = 0,$$

which yields, when combined with equation (3)

$$\overrightarrow{PH} \cdot \overrightarrow{KH} = r^2. \tag{5}$$

Define now γ and δ to be the coordinates of $\overrightarrow{PH}$ in $(\overrightarrow{KA}, \overrightarrow{KB})$

$$\overrightarrow{PH} = \gamma\overrightarrow{KA} + \delta\overrightarrow{KB}.$$

Equation (5) is linear in the coordinates of P and can be rewritten as

$$\gamma\overrightarrow{KA} \cdot \overrightarrow{KH} + \delta\overrightarrow{KB} \cdot \overrightarrow{KH} = r^2.$$

Note that $\overrightarrow{KA} \cdot \overrightarrow{KH}$ and $\overrightarrow{KB} \cdot \overrightarrow{KH}$ cannot both be 0 unless K is on AB, so we can either express γ as a function of δ or vice-versa. If we do the former we obtain

$$\gamma = \frac{r^2 - \delta\overrightarrow{KB} \cdot \overrightarrow{KH}}{\overrightarrow{KA} \cdot \overrightarrow{KH}}.$$

Equation (3) is quadratic in the coordinates of P and can be rewritten as

$$(\gamma\overrightarrow{KA} + \delta\overrightarrow{KB})^2 = r^2.$$

Pluging the value of γ above we get

$$\delta^2((\overrightarrow{KB} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KH})^2 + 2(\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KH})(\overrightarrow{KB} \cdot \overrightarrow{KH}) + \overrightarrow{KA} \cdot \overrightarrow{KA}(\overrightarrow{KB} \cdot \overrightarrow{KH})^2) +$$
$$2\delta r^2((\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KH}) - (\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KH})) +$$
$$r^2(r^2(\overrightarrow{KA} \cdot \overrightarrow{KA}) - (\overrightarrow{KA} \cdot \overrightarrow{KH})^2) = 0.$$

This equation always has two solutions because the sphere intersects KAB.

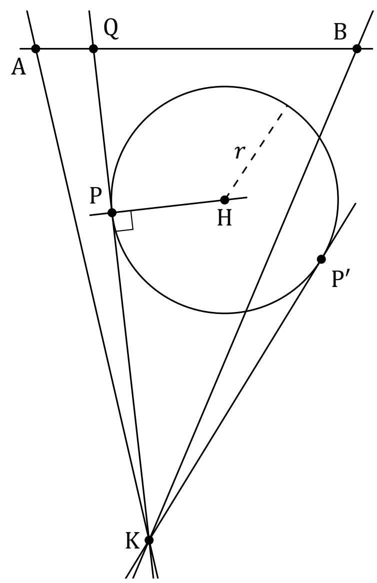

Figure 3. Construction of P and P′ and of the cone.

Intersection of the cone and the line AB

Having determined the location of points P and P′ we need to find the points Q and Q′ where the lines KP and KP′, respectively, intersect the line AB. Since Q is on AB there is a λ such that

$$\overrightarrow{AQ} = \lambda\overrightarrow{AB},$$

or equivalently

$$\overrightarrow{KQ} - \overrightarrow{KA} = \lambda\overrightarrow{AB}. \tag{6}$$

We can take the scalar product of equation (6) with $\overrightarrow{PH}$, and, noting that $\overrightarrow{KQ}$ is orthogonal to $\overrightarrow{PH}$ we obtain

$$-\overrightarrow{KA}\cdot\overrightarrow{PH} = \lambda\overrightarrow{AB}\cdot\overrightarrow{PH}, \text{ or, } \lambda = -\frac{\overrightarrow{KA}\cdot\overrightarrow{PH}}{\overrightarrow{AB}\cdot\overrightarrow{PH}}.$$

Obviously there is a similar equation with P′ which may be used to compute λ', the position of Q′ on AB.

The point Q is in the segment AB if and only if $0 \le \lambda \le 1$. This does not tell us, however, where Q is located with respect to the cone, to the sphere, or for that matter if it is in front or behind the camera. This is what we need to determine next.

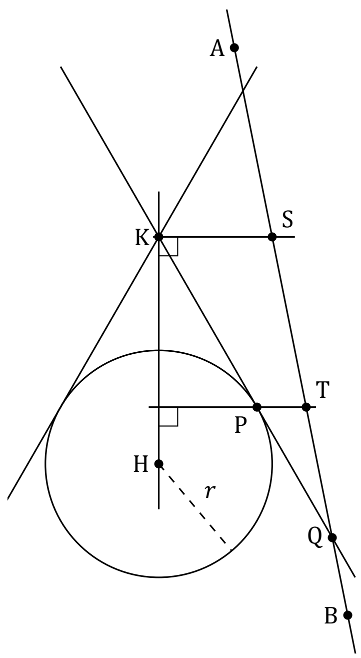

Figure 4. Definition of S and T.

Location of Q and Q′

Having computed the values of λ we need to determine where Q is located with respect to the cone, the sphere and the camera. To do this we are going to define two new points, S and T, as shown on figure 4. S is the intersection of AB with the line orthogonal to KH at K. T is the intersection of AB with the line orthogonal to KH at P. We are going to find the positions of S and T on AB, and, by comparing then with λ, figure out the location of Q with respect to S and T.

We locate S on AB as follows

$$\overrightarrow{KS} = \overrightarrow{KA} + \sigma\overrightarrow{AB}.$$

Taking the scalar product of this equation with $\overrightarrow{KH}$, and noting that $\overrightarrow{KS}$ is orthogonal to $\overrightarrow{KH}$ we obtain

$$\sigma = -\frac{\overrightarrow{KA}\cdot\overrightarrow{KH}}{\overrightarrow{AB}\cdot\overrightarrow{KH}}$$

Similarly, we locate T on AB as

$$\overrightarrow{KT} = \overrightarrow{KA} + \tau\overrightarrow{AB}.$$

Since $\overrightarrow{KT} = \overrightarrow{KH} + \overrightarrow{HP} + \overrightarrow{PT}$ this can be written

$$\overrightarrow{PT} = \overrightarrow{KA} + \tau\overrightarrow{AB} - \overrightarrow{KH} + \overrightarrow{PH}$$

Again, taking the scalar product of this equation with $\overrightarrow{KH}$, and noting that $\overrightarrow{KT}$ is orthogonal to $\overrightarrow{KH}$ we obtain

$$\tau = \frac{\overrightarrow{KH}\cdot\overrightarrow{KH} - \overrightarrow{PH}\cdot\overrightarrow{KH} - \overrightarrow{KA}\cdot\overrightarrow{KH}}{\overrightarrow{AB}\cdot\overrightarrow{KH}}$$

We can now determine if Q (or alternatively, λ) is an "interesting" intersection, i.e., one that intersects the cone behind the sphere when seen from the camera. That's because the location of S let us separate what is behind the camera from what is in front of the camera, and the location of T determines where the cone starts (points in front of T cannot be in the cone, although they can be in the sphere).

First, assume that A and B are in the same order as S and T on AB. Then we have $\sigma \le \tau$ and the intersection Q is farther than T (as seen from the camera) if and only if $\tau < \lambda$. Q is located on the cone and it is an interesting intersection (and it participates in hiding) if it falls within the segment AB. This is the situation illustrated in figure 4.

Conversely, if A and B are in the reverse order as S and T on AB we have $\tau \le \sigma$ and the intersection is farther than T if and only if $\lambda < \tau$.

The same analysis must be performed for λ' corresponding to Q′.

Point at infinity

There is another special case that requires some care: if the line AB is in "hyperbolic" position, i.e., intersects both halves of the cone, then one of the values λ, λ' is smaller than σ and the other is greater than σ (this is the situation shown in figure 4). Exactly one of λ, λ' will be retained by the preceding analysis. To account for the fact that the points farther than Q (such as B in figure 4) are inside the cone, we need to add to our analysis an extra value of λ equal to an infinity with the sign of $\tau - \sigma$.

Intersection with the circle

To complete the analysis we need to compute the intersection Q of the circle with the line AB. Q is on the circle, thus

$$\overrightarrow{\mathrm{HQ}} \cdot \overrightarrow{\mathrm{HQ}} = r^2; \tag{7}$$

it is also on the line AB and therefore there is a μ such that

$$\overrightarrow{\mathrm{KQ}} = \overrightarrow{\mathrm{KA}} + \mu\overrightarrow{\mathrm{AB}}.$$

We can rewrite $\overrightarrow{\mathrm{HQ}}$ as follows

$$\overrightarrow{\mathrm{HQ}} = \overrightarrow{\mathrm{KQ}} - \overrightarrow{\mathrm{KH}} = \overrightarrow{\mathrm{KA}} + \mu\overrightarrow{\mathrm{AB}} - \overrightarrow{\mathrm{KH}} = \overrightarrow{\mathrm{HA}} + \mu\overrightarrow{\mathrm{AB}}.$$

Inserting this value in equation (7) we obtain

$$r^2 = (\overrightarrow{\mathrm{HA}} + \mu\overrightarrow{\mathrm{AB}})^2,$$

meaning that μ is a solution of

$$\mu\overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{AB}} + 2\mu\overrightarrow{\mathrm{HA}} \cdot \overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{HA}} \cdot \overrightarrow{\mathrm{HA}} - R^2 = 0.$$

Depending on the location of AB with respect to the circle, there can be 0, 1, or 2 intersections.

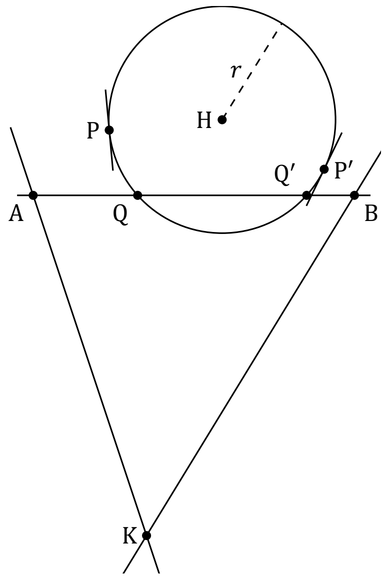

Figure 5. Intersection with the sphere.

Concluding the analysis

If we take the union of the values of λ and μ and order them, it is straightforward to find the visible segments by remembering that $0 \leq \lambda, \mu \leq 1$ for the points that are in the segment AB.