Rotational Motion of a Rigid Reference Frame

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This document describes the computations that are performed by the class `RigidReferenceFrame` and its subclasses to determine the rotational motion (rotation, angular velocity, and angular accelation) of a rigid frame.

Definitions

We consider in this document a rigid reference frame defined by two bodies B_1 and B_2 at positions $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, respectively. We construct a basis of the reference frame using three vectors having the following properties:

- the *fore* vector $\boldsymbol{F}$ which is along the axis $\boldsymbol{q}_2 - \boldsymbol{q}_1$;
- the *normal* vector $\boldsymbol{N}$ which is orthogonal to $\boldsymbol{F}$ and is such that the velocity $\dot{\boldsymbol{q}}_2 - \dot{\boldsymbol{q}}_1$ is in the plane $(\boldsymbol{F}, \boldsymbol{N})$;
- the *binormal* vector $\boldsymbol{B}$ which is orthogonal to $\boldsymbol{F}$ and $\boldsymbol{N}$ such that $(\boldsymbol{F}, \boldsymbol{N}, \boldsymbol{B})$ forms a direct trihedron.

There are obviously many possible choices for $(\boldsymbol{F}, \boldsymbol{N}, \boldsymbol{B})$. In practice, it is convenient to choose $\boldsymbol{B}$ before $\boldsymbol{N}$ so that the basis is computed exclusively using vector products:

$$\begin{cases} \boldsymbol{F} &= \boldsymbol{r} \\ \boldsymbol{B} &= \boldsymbol{r} \wedge \dot{\boldsymbol{r}} \\ \boldsymbol{N} &= \boldsymbol{B} \wedge \boldsymbol{F} \end{cases} \tag{1}$$

where we have defined $\boldsymbol{r} \coloneqq \boldsymbol{q}_2 - \boldsymbol{q}_1$. Since we'll need to use $\ddot{\boldsymbol{r}}$ later, it is important to note here that $\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{q}}_2 - \ddot{\boldsymbol{q}}_1$ where $\ddot{\boldsymbol{q}}_1$ is the acceleration exerted on B_1 by the rest of the system (and similarly, $\ddot{\boldsymbol{q}}_2$ is the acceleration exerted on B_2 by the rest of the system).

It is trivial to check that these definitions satisfy the properties above, and in particular that they determine a direct orthogonal basis. The corresponding orthonormal basis is:

$$\begin{cases} \boldsymbol{f} &= \dfrac{\boldsymbol{F}}{|\boldsymbol{F}|} \\ \boldsymbol{b} &= \dfrac{\boldsymbol{B}}{|\boldsymbol{B}|} \\ \boldsymbol{n} &= \dfrac{\boldsymbol{N}}{|\boldsymbol{N}|} \end{cases} \tag{2}$$

These vectors are sufficient to define the rotation of the reference frame at any point in time.

Derivatives of normalized vectors

In what follows, we will need to compute the time derivatives of the elements of the trihedron $(\boldsymbol{f}, \boldsymbol{n}, \boldsymbol{b})$. To help with this we prove two formulæ that define the first and second derivatives of $\boldsymbol{V}/|\boldsymbol{V}|$ based on that of $\boldsymbol{V}$.

The first derivative is:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\boldsymbol{V}}{|\boldsymbol{V}|} &= \frac{|\boldsymbol{V}|\dot{\boldsymbol{V}} - \frac{\mathrm{d}\,|\boldsymbol{V}|}{\mathrm{d}\,t}\boldsymbol{V}}{|\boldsymbol{V}|^2} \\ &= \frac{|\boldsymbol{V}|\dot{\boldsymbol{V}} - \frac{(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})}{|\boldsymbol{V}|}\boldsymbol{V}}{|\boldsymbol{V}|^2} \\ &= \frac{\dot{\boldsymbol{V}}}{|\boldsymbol{V}|} - \boldsymbol{V}\frac{(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})}{|\boldsymbol{V}|^3}\end{aligned} \tag{3}$$

The second derivative is somewhat more complicated:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}\frac{\boldsymbol{V}}{|\boldsymbol{V}|} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{|\boldsymbol{V}|^2\dot{\boldsymbol{V}} - (\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\boldsymbol{V}}{|\boldsymbol{V}|^3}\right) \\ &= \frac{|\boldsymbol{V}|^3\frac{\mathrm{d}}{\mathrm{d}\,t}\left(|\boldsymbol{V}|^2\dot{\boldsymbol{V}} - (\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\boldsymbol{V}\right) - 3|\boldsymbol{V}|(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\left(|\boldsymbol{V}|^2\dot{\boldsymbol{V}} - (\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\boldsymbol{V}\right)}{|\boldsymbol{V}|^6} \\ &= \frac{2(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\dot{\boldsymbol{V}} + |\boldsymbol{V}|^2\ddot{\boldsymbol{V}} - \left(|\dot{\boldsymbol{V}}|^2 + (\boldsymbol{V}\cdot\ddot{\boldsymbol{V}})\right)\boldsymbol{V} - (\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\dot{\boldsymbol{V}}}{|\boldsymbol{V}|^3} - 3\frac{|\boldsymbol{V}|^3(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\dot{\boldsymbol{V}} - |\boldsymbol{V}|(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})^2\boldsymbol{V}}{|\boldsymbol{V}|^6} \\ &= \frac{\ddot{\boldsymbol{V}}}{|\boldsymbol{V}|} - 2\dot{\boldsymbol{V}}\frac{(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})}{|\boldsymbol{V}|^3} - \boldsymbol{V}\frac{|\dot{\boldsymbol{V}}|^2 + (\boldsymbol{V}\cdot\ddot{\boldsymbol{V}})}{|\boldsymbol{V}|^3} + 3\boldsymbol{V}\frac{(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})^2}{|\boldsymbol{V}|^5}\end{aligned} \tag{4}$$

Angular velocity

To compute the angular velocity, we start by deriving the vectors (1) and obtain:

$$\begin{cases}\dot{\boldsymbol{F}} &= \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{B}} &= \boldsymbol{r}\wedge\ddot{\boldsymbol{r}} \\ \dot{\boldsymbol{N}} &= \dot{\boldsymbol{B}}\wedge\boldsymbol{F} + \boldsymbol{B}\wedge\dot{\boldsymbol{F}}\end{cases} \tag{5}$$

Injecting these expressions in the derivative formula (3) makes it possible to compute the trihedron of the derivatives $(\dot{\boldsymbol{f}}, \dot{\boldsymbol{n}}, \dot{\boldsymbol{b}})$ of (2). The angular velocity is then written:

$$\boldsymbol{\omega} = (\dot{\boldsymbol{n}}\cdot\boldsymbol{b})\boldsymbol{f} + (\dot{\boldsymbol{b}}\cdot\boldsymbol{f})\boldsymbol{n} + (\dot{\boldsymbol{f}}\cdot\boldsymbol{n})\boldsymbol{b}$$

Angular acceleration

To compute the angular acceleration, we start by deriving the vectors (5) and obtain:

$$\begin{cases}\ddot{\boldsymbol{F}} &= \ddot{\boldsymbol{r}} \\ \ddot{\boldsymbol{B}} &= \dot{\boldsymbol{r}}\wedge\ddot{\boldsymbol{r}} + \boldsymbol{r}\wedge\boldsymbol{r}^{(3)} \\ \ddot{\boldsymbol{N}} &= \ddot{\boldsymbol{B}}\wedge\boldsymbol{F} + 2\dot{\boldsymbol{B}}\wedge\dot{\boldsymbol{F}} + \boldsymbol{B}\wedge\ddot{\boldsymbol{F}}\end{cases} \tag{6}$$

Injecting these expressions in the second derivative formula (4) makes it possible to compute the trihedron of the second derivatives $(\ddot{\boldsymbol{f}}, \ddot{\boldsymbol{n}}, \ddot{\boldsymbol{b}})$ of (2). The angular acceleration is then written:

$$\begin{aligned}\dot{\boldsymbol{\omega}} = (\ddot{\boldsymbol{n}}\cdot\boldsymbol{b})\boldsymbol{f} + (\dot{\boldsymbol{n}}\cdot\dot{\boldsymbol{b}})\boldsymbol{f} + (\dot{\boldsymbol{n}}\cdot\boldsymbol{b})\dot{\boldsymbol{f}} \\ + (\ddot{\boldsymbol{b}}\cdot\boldsymbol{f})\boldsymbol{n} + (\dot{\boldsymbol{b}}\cdot\dot{\boldsymbol{f}})\boldsymbol{n} + (\dot{\boldsymbol{b}}\cdot\boldsymbol{f})\dot{\boldsymbol{n}} \\ + (\ddot{\boldsymbol{f}}\cdot\boldsymbol{n})\boldsymbol{b} + (\dot{\boldsymbol{f}}\cdot\dot{\boldsymbol{n}})\boldsymbol{b} + (\dot{\boldsymbol{f}}\cdot\boldsymbol{n})\dot{\boldsymbol{b}}\end{aligned}$$

Jerk

The alert reader will have noticed the presence of the jerk, $\boldsymbol{r}^{(3)}$ in (6). This section explains how it is calculated.

Consider a system of n massive bodies $B_k, k \in [1, n]$ located at positions $\boldsymbol{q_k}(t)$ at time t. The acceleration field at point $\boldsymbol{q}$ is:

$$\boldsymbol{\Gamma}(\boldsymbol{q}, \boldsymbol{q}_k(t)) = \sum_{k=1}^{n} \frac{\mu_k(\boldsymbol{q}_k - \boldsymbol{q})}{|\boldsymbol{q}_k - \boldsymbol{q}|^3}$$

The jerk vector field is the total derivative of this field with respect to time:

$$\frac{\mathrm{d}\,\boldsymbol{\Gamma}}{\mathrm{d}\,t} = \frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{q}} \frac{\mathrm{d}\,\boldsymbol{q}}{\mathrm{d}\,t} + \sum_{k=1}^{n} \frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{q}_k} \frac{\mathrm{d}\,\boldsymbol{q}_k}{\mathrm{d}\,t} = \frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} + \sum_{k=1}^{n} \frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{q}_k} \dot{\boldsymbol{q}}_k$$

To compute it, let's first focus on the partial derivatives with respect to $\boldsymbol{q}_k$:

$$\begin{aligned}
\frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{q}_k} &= \frac{\partial}{\partial \boldsymbol{q}_k} \frac{\mu_k(\boldsymbol{q}_k - \boldsymbol{q})}{|\boldsymbol{q}_k - \boldsymbol{q}|^3} \\
&= \mu_k \left(\frac{\frac{\partial}{\partial \boldsymbol{q}_k}(\boldsymbol{q}_k - \boldsymbol{q})}{|\boldsymbol{q}_k - \boldsymbol{q}|^3} - 3 \frac{(\boldsymbol{q}_k - \boldsymbol{q}) \otimes \frac{\partial}{\partial \boldsymbol{q}_k} |\boldsymbol{q}_k - \boldsymbol{q}|}{|\boldsymbol{q}_k - \boldsymbol{q}|^4} \right) \\
&= \mu_k \left(\frac{\mathbb{1}}{|\boldsymbol{q}_k - \boldsymbol{q}|^3} - 3 \frac{(\boldsymbol{q}_k - \boldsymbol{q}) \otimes \frac{\mathbb{1} \cdot (\boldsymbol{q}_k - \boldsymbol{q})}{|\boldsymbol{q}_k - \boldsymbol{q}|}}{|\boldsymbol{q}_k - \boldsymbol{q}|^4} \right) \\
&= \mu_k \left(\frac{\mathbb{1}}{|\boldsymbol{q}_k - \boldsymbol{q}|^3} - 3 \frac{(\boldsymbol{q}_k - \boldsymbol{q}) \otimes (\boldsymbol{q}_k - \boldsymbol{q})}{|\boldsymbol{q}_k - \boldsymbol{q}|^5} \right)
\end{aligned}$$

Note that this expression is a symmetric bilinear form, as expected since the gravitational acceleration derives from a potential, and therefore its Jacobian (the Hessian of the potential) is symmetric.

The partial derivative with respect to $\boldsymbol{q}$ is then easy to compute by noting that each term in $\frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{q}}$ has the same form as the partial derivative above, with a sign change due to deriving with respect to $\boldsymbol{q}$ instead of $\boldsymbol{q}_k$:

$$\frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{q}} = \sum_{k=1}^{n} \mu_k \left(-\frac{\mathbb{1}}{|\boldsymbol{q}_k - \boldsymbol{q}|^3} + 3 \frac{(\boldsymbol{q}_k - \boldsymbol{q}) \otimes (\boldsymbol{q}_k - \boldsymbol{q})}{|\boldsymbol{q}_k - \boldsymbol{q}|^5} \right)$$

Putting all these calculations together we obtain the jerk exercised by the bodies B_k on point $\boldsymbol{q}$ at time t:

$$\frac{\mathrm{d}\,\boldsymbol{\Gamma}}{\mathrm{d}\,t} = \sum_{k=1}^{n} \mu_k \left(\frac{\mathbb{1}}{|\boldsymbol{q}_k - \boldsymbol{q}|^3} - 3 \frac{(\boldsymbol{q}_k - \boldsymbol{q}) \otimes (\boldsymbol{q}_k - \boldsymbol{q})}{|\boldsymbol{q}_k - \boldsymbol{q}|^5} \right) (\dot{\boldsymbol{q}}_k - \dot{\boldsymbol{q}}) \tag{7}$$

The computation of $\dddot{\boldsymbol{r}} = \dddot{\boldsymbol{q}}_2 - \dddot{\boldsymbol{q}}_1$ is then straightforward. $\dddot{\boldsymbol{q}}_1$ is obtained by evaluating (7) at $\boldsymbol{q} = \boldsymbol{q}_1$ using a system from which B_1 is excluded. Similarly, $\dddot{\boldsymbol{q}}_2$ is obtained by evaluation (7) at $\boldsymbol{q} = \boldsymbol{q}_2$ using a system from which B_2 is excluded.

Application to the body surface reference frame

While the formulæ above were derived assuming a rigid reference frame defined by two bodies B_1 and B_2, they remain valid for a reference frame defined by a single body B_1 provided that we give a proper definition of the vector $\boldsymbol{r}$. We now consider how they apply to the *body surface* reference frame.

It is convenient to choose $\boldsymbol{r}$ to be a unit vector rotating with the surface of the body; without loss of generality, $\boldsymbol{r}$ may be written:

$$\boldsymbol{r} = \boldsymbol{x} \cos(\omega t + \varphi) + \boldsymbol{y} \sin(\omega t + \varphi)$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ form a direct orthonormal basis of the equatorial plane of the body. The time derivatives of $\boldsymbol{r}$ are:

$$\begin{cases}
\dot{\boldsymbol{r}} &= -\boldsymbol{x}\omega \sin(\omega t + \varphi) + \boldsymbol{y}\omega \cos(\omega t + \varphi) \\
\ddot{\boldsymbol{r}} &= -\boldsymbol{x}\omega^2 \cos(\omega t + \varphi) - \boldsymbol{y}\omega^2 \sin(\omega t + \varphi) = -\omega^2 \boldsymbol{r} \\
\boldsymbol{r}^{(3)} &= \boldsymbol{x}\omega^3 \sin(\omega t + \varphi) - \boldsymbol{y}\omega^3 \cos(\omega t + \varphi) = -\omega^2 \dot{\boldsymbol{r}}
\end{cases}$$

Using (1) it's easy to see that:

$$\begin{cases} \boldsymbol{F} &= \boldsymbol{r} \\ \boldsymbol{B} &= \omega \boldsymbol{z} \\ \boldsymbol{N} &= \dot{\boldsymbol{r}} \end{cases}$$

where $\boldsymbol{z}$ is along the rotation axis such that $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ form a direct trihedron. The orthonormal vectors follow immediately, noting that $|\boldsymbol{r}| = 1$:

$$\begin{cases} \boldsymbol{f} &= \boldsymbol{r} \\ \boldsymbol{b} &= \boldsymbol{z} \\ \boldsymbol{n} &= \dfrac{\dot{\boldsymbol{r}}}{\omega} \end{cases}$$

The first derivatives of the orthonormal trihedron can be computed directly:

$$\begin{cases} \dot{\boldsymbol{f}} &= \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{b}} &= 0 \\ \dot{\boldsymbol{n}} &= \dfrac{\ddot{\boldsymbol{r}}}{\omega} = -\omega \boldsymbol{r} \end{cases}$$

which yields, not surprisingly, $\boldsymbol{\omega} = \omega \boldsymbol{z}$. The second derivatives are equally straightforward:

$$\begin{cases} \ddot{\boldsymbol{f}} &= \ddot{\boldsymbol{r}} = -\boldsymbol{r} \\ \dot{\boldsymbol{b}} &= 0 \\ \dot{\boldsymbol{n}} &= -\omega \dot{\boldsymbol{r}} \end{cases}$$

and yield, as expected, $\dot{\boldsymbol{\omega}} = 0$.