

# On a Formula by Cohen, Hubbard and Oesterwinter

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2021-03-13

This document proves and generalizes a formula given in [CHO73] to compute the velocity of a body in the context of numerical integration of  $n$ -body systems.

## Statement

[CHO73, p. 20] states that

the formula for a velocity component [is] of the form:

$$\dot{x}_{n+1} = \frac{1}{h} \left( x_n - x_{n-1} + h^2 \sum_{i=0}^{12} \beta_i \ddot{x}_{n-i} \right) \quad (1)$$

They then proceed to tabulate explicit values for the (rational) coefficients  $\beta_i$  without explaining how they are computed. This makes it impossible to use this formula for integrators of a different order or to construct similar formulæ for slightly different purposes.

## Finite difference formulæ

In this section we prove a backward difference formula that makes it possible to compute an approximation of any derivative of a function on an equally-spaced grid at any desired order. We then derive a corollary based on the second derivative  $f''(x)$  that is useful for the following sections.

**Lemma.** *Given two integers  $m \leq n$ , there exists a family of rational numbers  $\lambda_{n,v}^m$ , with  $0 \leq v \leq n$ , such that for any sufficiently regular function  $f$ :*

$$f^{(m)}(x_0) = \frac{(-1)^m}{h^m} \sum_{v=0}^n \lambda_{n,v}^m f(x_0 - vh) + \mathcal{O}(h^{n-m+1}) \quad (2)$$

**Proof.** [For88] gives finite difference formulæ for any order and for kernels of arbitrary size. Given a family  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_N)$  of points (which may not be equidistant), the  $m$ -th derivative at any point  $x_0$  may be approximated to order  $n - m + 1$  as:

$$f^{(m)}(x_0) \cong \sum_{v=0}^n \delta_{n,v}^m(\alpha) f(\alpha_v)$$

where  $m \leq n$  and the coefficients  $\delta_{n,v}^m$  are dependent on  $\alpha$  but independent of  $f$ .

If the  $\alpha$  are equally spaced with step  $-h$  (i.e.,  $\alpha_v = x_0 - vh$ ) then we can write a backward formula as follows:

$$f^{(m)}(x_0) = \sum_{v=0}^n \delta_{n,v}^m(\alpha) f(x_0 - vh) + \mathcal{O}(h^{n-m+1}) \quad (3)$$

In this case equation (3.8) from [For88] may be rewritten as follows, restoring  $x_0$ :

$$\begin{aligned}\delta_{n,v}^m &= \frac{1}{\alpha_n - \alpha_v} ((\alpha_n - x_0)\delta_{n-1,v}^m - m\delta_{n-1,v}^{m-1}) \\ &= \frac{1}{(v-n)h} (-nh\delta_{n-1,v}^m - m\delta_{n-1,v}^{m-1})\end{aligned}\quad (4)$$

It is possible to extract the powers of  $h$  from the coefficients by defining, for  $v < n$ :

$$\lambda_{n,v}^m := \delta_{n,v}^m(\alpha)(-1)^m h^m$$

Substituting into equation (4) we obtain:

$$\lambda_{n,v}^m = \frac{1}{n-v} (n\lambda_{n-1,v}^m - m\lambda_{n-1,v}^{m-1})$$

and we see that  $\lambda_{n,v}^m$  is independent from  $x_0$  and  $h$  and therefore that  $\delta_{n,v}^m$  is independent from  $x_0$ . Note that the definition of  $\lambda_{n,v}^m$  extends immediately to  $n = v$ , that is to  $\lambda_{n,n}^m$ , using equation (3.10) of [For88].

Substituting:

$$\delta_{n,v}^m(\alpha) = (-1)^m \frac{\lambda_{n,v}^m}{h^m}$$

into equation (3) yields equation (2), thus proving the lemma.  $\square$

**Corollary.** *Applying the lemma to the second derivative  $f''(x)$  gives the following result when  $m \geq 2$ :*

$$f^{(m)}(x_0) = \frac{(-1)^m}{h^{m-2}} \sum_{v=0}^{n-2} \lambda_{n-2,v}^{m-2} f''(x_0 - vh) + \mathcal{O}(h^{n-m+1}) \quad (5)$$

$\square$

## A formula for symmetric linear multistep integrators

In this section we derive a formula suitable for computing the velocity of a body knowing its positions and accelerations at preceding times. This is the formula we use after integration using a symmetry linear multistep integrator.

**Proposition.** *There exists a family of rational numbers  $\eta_{n,v}$ , with  $0 \leq v \leq n-2$ , such that for any sufficiently regular function  $f$ :*

$$f'(x_0) = \frac{1}{h} \left( f(x_0) - f(x_0 - h) + h^2 \sum_{v=0}^{n-2} \eta_{n,v} f''(x_0 - vh) \right) + \mathcal{O}(h^n) \quad (6)$$

**Proof.** We start by writing the Taylor series of  $f(x_0 - h)$  for  $h$  close to 0, extracting the leading terms:

$$\begin{aligned}f(x_0 - h) &= \sum_{m=0}^{\infty} \frac{f^{(m)}(x_0)}{m!} (-1)^m h^m \\ &= f(x_0) - hf'(x_0) + \sum_{m=2}^{\infty} \frac{f^{(m)}(x_0)}{m!} (-1)^m h^m \\ &= f(x_0) - hf'(x_0) + \sum_{m=2}^n \frac{f^{(m)}(x_0)}{m!} (-1)^m h^m + \mathcal{O}(h^{n+1})\end{aligned}\quad (7)$$

Injecting the expression from corollary (5) into this series we find:

$$f(x_0 - h) = f(x_0) - hf'(x_0) + h^2 \sum_{m=2}^n \sum_{v=0}^{n-2} \frac{\lambda_{n-2,v}^{m-2}}{m!} f''(x_0 - vh) + \mathcal{O}(h^{n+1})$$

The summations are independent and can be exchanged:

$$f(x_0 - h) = f(x_0) - hf'(x_0) + h^2 \sum_{v=0}^{n-2} \left( \sum_{m=2}^n \frac{\lambda_{n-2,v}^{m-2}}{m!} \right) f''(x_0 - vh) + \mathcal{O}(h^{n+1})$$

If we define:

$$\eta_{n,v} := \sum_{m=2}^n \frac{\lambda_{n-2,v}^{m-2}}{m!}$$

the equation (6) follows immediately.  $\square$

## The Cohen-Hubbard-Osterwinder formula

In this section we derive the Cohen-Hubbard-Osterwinder formula, equation (1).

**Proposition.** *There exists a family of rational numbers  $\beta_{n,v}$ , with  $0 \leq v \leq n-2$ , such that for any sufficiently regular function  $f$ :*

$$f'(x_0 + h) = \frac{1}{h}(f(x_0) - f(x_0 - h)) + h \sum_{v=0}^{n-2} \beta_{n,v} f''(x_0 - vh) + \mathcal{O}(h^n)$$

**Proof.** We start by writing the Taylor series of  $f'(x_0 + h)$  for  $h$  close to 0:

$$\begin{aligned} f'(x_0 + h) &= f'(x_0) + \sum_{m=2}^{\infty} \frac{h^{m-1}}{(m-1)!} f^{(m)}(x_0) \\ &= f'(x_0) + \sum_{m=2}^n \frac{h^{m-1}}{(m-1)!} f^{(m)}(x_0) + \mathcal{O}(h^n) \end{aligned}$$

and we replace  $f'(x_0)$  with its value from equation (7):

$$f'(x_0 + h) = \frac{1}{h}(f(x_0) - f(x_0 - h)) + \sum_{m=2}^n \frac{h^{m-1}}{(m-1)!} \left( 1 + \frac{(-1)^m}{m} \right) f^{(m)}(x_0) + \mathcal{O}(h^n)$$

As we did above, we replace  $f^{(m)}(x_0)$  using its expression from corollary (5) to obtain:

$$f'(x_0 + h) = \frac{1}{h}(f(x_0) - f(x_0 - h)) + h \sum_{m=2}^n \left( \frac{1}{(m-1)!} \left( (-1)^m + \frac{1}{m} \right) \sum_{v=0}^{n-2} \lambda_{n-2,v}^{m-2} f''(x_0 - vh) \right) + \mathcal{O}(h^n)$$

If we exchange the summations and define:

$$\beta_{n,v} = \sum_{m=2}^n \frac{\lambda_{n-2,v}^{m-2}}{(m-1)!} \left( (-1)^m + \frac{1}{m} \right)$$

we obtain the desired result, which is a reformulation of equation (1) where  $\beta_i$  in [CHO73] is our  $\beta_{14,i}$ .  $\square$

## References

- [CHO73] C. J. Cohen, E. C. Hubbard and C. Oesterwinter. *Astronomical Papers Prepared for the Use of the American Ephemeris and Nautical Almanac – Elements of the Outer Planets for One Million Years*. Vol. XXII. I. United States Government Printing Office, 1973.
- [For88] B. Fornberg. “Generation of Finite Difference Formulas on Arbitrarily Spaced Grids”. In: *Mathematics of Computation* 51.184 (Oct. 1988), pp. 699–706.  
DOI: 10.1090/S0025-5718-1988-0935077-0.