On the rotating-pulsating reference frame

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Notation

Time Unless otherwise mentioned, all variables implicitly depend on time, a one-dimensional real oriented inner product space T. All equations are for those functions evaluated at time t. Unless otherwise mentioned, all predicates involving variables dependent on time are implicitly quantified for all times t.

For example, we may write $f \in \mathbb{R}$ for $f : T \to \mathbb{R}$.

Derivative The notation $\frac{\mathrm{d}\,f}{\mathrm{d}\,x}$ represents the evaluation at x of the differential $\mathrm{d}\,f$, where the expression f is taken as a function of x. Within f, any occurrences of x represent a free variable, rather than a function of time. The derivative with respect to time $\frac{\mathrm{d}\,f}{\mathrm{d}\,t}$ may be written $\dot{f}$.

For example, we may write

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\mathrm{d}\,\mathrm{e}^x}{\mathrm{d}\,x} = \frac{\mathrm{d}}{\mathrm{d}\,t}\mathrm{e}^x = \mathrm{e}^x\dot{x},$$

which is implicitly

$$\frac{\mathrm{d}}{\mathrm{d}\,\tau}\left(\frac{\mathrm{d}\,\mathrm{e}^\xi}{\mathrm{d}\,\xi}\bigg|_{\xi=x(\tau)}\right)\bigg|_{\tau=t} = \frac{\mathrm{d}\,\mathrm{e}^{x(\tau)}}{\mathrm{d}\,\tau}\bigg|_{\tau=t} = \mathrm{e}^{x(t)}\dot{x}(t).$$

This also applies for multidimensional x; in particular we may write

$$\frac{\mathrm{d}\,f(\boldsymbol{x})}{\mathrm{d}\,\boldsymbol{x}}^\top \text{ for } (\boldsymbol{\nabla} f(t,\,\cdot\,))(t, \boldsymbol{x}(t)).$$

This notation facilitates changes of variables, which are the main focus of this document.

Note that d here is always the differential, not the exterior derivative; thus for vector spaces V and W and $f : V \to W$, we have $\mathrm{d}\,f : V \to V^* \otimes W$, and $\mathrm{d}^2\,f : V \to V^* \otimes V^* \otimes W$, rather than $\mathrm{d}^2 = 0$.

As it is somewhat impractical to construct a notation which makes pullbacks of two-forms natural, and as we do not perform changes of variables on curls, we eschew the exterior derivative entirely, and merely get rid of orientation-dependent identifications by writing, for $\boldsymbol{v}$ and $\boldsymbol{w}$ implicitly dependent on $\boldsymbol{q}$,

$$(\mathbf{rot}_{\boldsymbol{q}}\,\boldsymbol{v})\boldsymbol{w} \text{ for } (\boldsymbol{\nabla} \times \boldsymbol{v}(t,\,\cdot\,))(\boldsymbol{q}(t)) \times \boldsymbol{w}(t, \boldsymbol{q}(t)).$$

Reference frames Script capital letters denote reference frames. For all frames $\mathcal{F}$, $Q^\mathcal{F}$ is the space of displacements from the origin in frame $\mathcal{F}$ (representing positions in space), a three-dimensional real inner product space associated with $\mathcal{F}$. Variables $\boldsymbol{q}^\mathcal{F}$ or $\boldsymbol{q}_i^\mathcal{F}$ have values in $Q^\mathcal{F}$.

Reference frames are defined in relation to each other by invertible transformations; thus if $\mathcal{G}$ is defined by $\boldsymbol{q}^\mathcal{G} \coloneqq \boldsymbol{g}(\boldsymbol{q}^\mathcal{F})$, a function f that depends on $\boldsymbol{q}^\mathcal{F}$ can be taken as a function that depends on $\boldsymbol{q}^\mathcal{G}$, and differentiated accordingly.

For example, for f implicitly dependent on $\boldsymbol{q}^\mathcal{F}$ as well as t, we may write

$$\frac{\mathrm{d}\,f}{\mathrm{d}\,\boldsymbol{q}^\mathcal{G}} = \frac{\mathrm{d}\,f}{\mathrm{d}\,\boldsymbol{q}^\mathcal{F}}\frac{\mathrm{d}\,\boldsymbol{q}^\mathcal{F}}{\mathrm{d}\,\boldsymbol{q}^\mathcal{G}},$$

which is implicitly

$$\mathrm{d}(f(t,\,\cdot\,) \circ \boldsymbol{g}(t,\,\cdot\,)^{-1})(\boldsymbol{q}^\mathcal{G}(t)) = \mathrm{d}(f(t,\,\cdot\,))(\boldsymbol{g}(t, \boldsymbol{q}^\mathcal{G}(t)))\,\mathrm{d}\,\boldsymbol{g}(t,\,\cdot\,)^{-1}.$$

1 Geometric potential

Let $\boldsymbol{q}^{\mathcal{F}}$ be a field of free-falling trajectories such that $\dot{\boldsymbol{q}}^{\mathcal{F}} = 0$ at time t; the field $\ddot{\boldsymbol{q}}^{\mathcal{F}}$ is the field of *geometric accelerations at rest*.

In all reference frames considered, the geometric accelerations at rest have a constant curl throughout space, thus, at time t,

$$\mathbf{rot}_{q^{\mathcal{F}}}\, \ddot{\boldsymbol{q}}^{\mathcal{F}} = \boldsymbol{A}^{\mathcal{F}}$$

for some $\boldsymbol{A}^{\mathcal{F}}$ which does not depend on $\boldsymbol{q}^{\mathcal{F}}$.

The *geometric potential* $V^{\mathcal{F}}$ of a frame $\mathcal{F}$ is defined on $Q^{\mathcal{F}}$ at time t by the equation

$$\ddot{\boldsymbol{q}}^{\mathcal{F}} = -\frac{\mathrm{d}\,V^{\mathcal{F}}}{\mathrm{d}\,\boldsymbol{q}^{\mathcal{F}}}^{\top} + \frac{1}{2}\boldsymbol{A}^{\mathcal{F}}\boldsymbol{q}^{\mathcal{F}}. \tag{1.1}$$

The geometric potential $V^{\mathcal{F}}$ implictly depends on $\boldsymbol{q}^{\mathcal{F}}$ as well as t.

The acceleration $\ddot{\boldsymbol{q}}^{\mathcal{F}}$ of a free-falling trajectory $\boldsymbol{q}^{\mathcal{F}}$ is the *geometric acceleration*, which, for a given frame, depends on time, position, and velocity. The gradient of the geometric potential is the *rotation-free geometric acceleration at rest*; it depends on time and position. If the geometric acceleration at rest is irrotational, it is equal to the rotation-free geometric acceleration at rest.

Note that unless the geometric acceleration at rest is irrotational, the geometric potential depends on the choice of the origin of $\mathcal{F}$.

2 Inertial frame

Let $\mathcal{I}$ be an inertial frame. Then for all free-falling trajectories $\boldsymbol{q}^{\mathcal{I}}$,

$$\ddot{\boldsymbol{q}}^{\mathcal{I}} = -\frac{\mathrm{d}\,V^{\mathcal{I}}}{\mathrm{d}\,\boldsymbol{q}^{\mathcal{I}}}^{\top}. \tag{2.1}$$

3 Rotating frame

Consider the rotating reference frame $\mathcal{R}$ defined by

$$\boldsymbol{q}^{\mathcal{R}} \coloneqq \boldsymbol{R}(\boldsymbol{q}^{\mathcal{I}} - \boldsymbol{q}_0^{\mathcal{I}}). \tag{3.1}$$

Velocities in $\mathcal{R}$ are related to velocities in $\mathcal{I}$ as follows:

$$\dot{\boldsymbol{q}}^{\mathcal{R}} = -\boldsymbol{R}\dot{\boldsymbol{q}}_0^{\mathcal{I}} + \boldsymbol{\omega}\boldsymbol{q}^{\mathcal{R}} + \boldsymbol{R}\dot{\boldsymbol{q}}^{\mathcal{I}}, \tag{3.2}$$

where

$$\dot{\boldsymbol{R}} = \boldsymbol{\omega}\boldsymbol{R}.$$

Accelerations in $\mathcal{R}$ are related to accelerations in $\mathcal{I}$ as follows:

$$\begin{aligned}\ddot{\boldsymbol{q}}^{\mathcal{R}} &= -\boldsymbol{\omega}\boldsymbol{R}\dot{\boldsymbol{q}}_0^{\mathcal{I}} - \boldsymbol{R}\ddot{\boldsymbol{q}}_0^{\mathcal{I}} + \dot{\boldsymbol{\omega}}\boldsymbol{q}^{\mathcal{R}} + \boldsymbol{\omega}\dot{\boldsymbol{q}}^{\mathcal{R}} + \boldsymbol{\omega}\boldsymbol{R}\dot{\boldsymbol{q}}^{\mathcal{I}} + \boldsymbol{R}\ddot{\boldsymbol{q}}^{\mathcal{I}} \\ &= -\boldsymbol{\omega}\boldsymbol{R}\dot{\boldsymbol{q}}_0^{\mathcal{I}} - \boldsymbol{R}\ddot{\boldsymbol{q}}_0^{\mathcal{I}} + \dot{\boldsymbol{\omega}}\boldsymbol{q}^{\mathcal{R}} + \boldsymbol{\omega}\dot{\boldsymbol{q}}^{\mathcal{R}} + \boldsymbol{\omega}(\dot{\boldsymbol{q}}^{\mathcal{R}} - \boldsymbol{\omega}\boldsymbol{q}^{\mathcal{R}} + \boldsymbol{R}\dot{\boldsymbol{q}}_0^{\mathcal{I}}) + \boldsymbol{R}\ddot{\boldsymbol{q}}^{\mathcal{I}} \\ &= -\boldsymbol{R}\ddot{\boldsymbol{q}}_0^{\mathcal{I}} + \underbrace{\dot{\boldsymbol{\omega}}\boldsymbol{q}^{\mathcal{R}}}_{\text{Euler}} + \underbrace{2\boldsymbol{\omega}\dot{\boldsymbol{q}}^{\mathcal{R}}}_{\text{Coriolis}} \underbrace{-\boldsymbol{\omega}^2\boldsymbol{q}^{\mathcal{R}}}_{\text{centrifugal}} + \boldsymbol{R}\ddot{\boldsymbol{q}}^{\mathcal{I}}.\end{aligned} \tag{3.3}$$

Observe that for a free-falling trajectory,

$$\boldsymbol{R}\ddot{\boldsymbol{q}}^{\mathcal{I}} = -\boldsymbol{R}\frac{\mathrm{d}\,V^{\mathcal{I}}}{\mathrm{d}\,\boldsymbol{q}^{\mathcal{I}}}^{\top} = -\left(\frac{\mathrm{d}\,V^{\mathcal{I}}}{\mathrm{d}\,\boldsymbol{q}^{\mathcal{I}}}\boldsymbol{R}^{-1}\right)^{\top} = -\left(\frac{\mathrm{d}\,V}{\mathrm{d}\,\boldsymbol{q}^{\mathcal{I}}}\frac{\mathrm{d}\,\boldsymbol{q}^{\mathcal{I}}}{\mathrm{d}\,\boldsymbol{q}^{\mathcal{R}}}\right)^{\top} = -\frac{\mathrm{d}\,V^{\mathcal{I}}}{\mathrm{d}\,\boldsymbol{q}^{\mathcal{R}}}^{\top}.$$

At rest in $\mathcal{R}$, *i.e.*, for $\dot{\boldsymbol{q}}^{\mathcal{R}} = 0$, we have

$$\begin{aligned}\ddot{\boldsymbol{q}}^{\mathcal{R}} &= -\boldsymbol{R}\ddot{\boldsymbol{q}}_0^{\mathcal{I}} + \dot{\boldsymbol{\omega}}\boldsymbol{q}^{\mathcal{R}} - \boldsymbol{\omega}^2\boldsymbol{q}^{\mathcal{R}} + \boldsymbol{R}\ddot{\boldsymbol{q}}^{\mathcal{I}} \\ &= \dot{\boldsymbol{\omega}}\boldsymbol{q}^{\mathcal{R}} - \left(\frac{\mathrm{d}}{\mathrm{d}\,\boldsymbol{q}^{\mathcal{R}}}\left(\boldsymbol{q}^{\mathcal{R}} \cdot \boldsymbol{R}\ddot{\boldsymbol{q}}_0^{\mathcal{I}} - \frac{(\boldsymbol{q}^{\mathcal{R}})^{\top}\boldsymbol{\omega}^{\top}\boldsymbol{\omega}\boldsymbol{q}^{\mathcal{R}}}{2} + V^{\mathcal{I}}\right)\right)^{\top},\end{aligned} \tag{3.4}$$

so that the geometric potential is

$$V^{\mathcal{R}} = \boldsymbol{q}^{\mathcal{R}} \cdot \boldsymbol{R}\ddot{\boldsymbol{q}}_0^{\mathcal{I}} - \frac{\boldsymbol{\omega}\boldsymbol{q}^{\mathcal{R}} \cdot \boldsymbol{\omega}\boldsymbol{q}^{\mathcal{R}}}{2} + V^{\mathcal{I}}. \tag{3.5}$$

4 Rotating-pulsating frame

The rotating-pulsating reference frame $\mathcal{P}$ is defined by

$$\boldsymbol{q}^{\mathcal{P}} \coloneqq \frac{\boldsymbol{q}^{\mathcal{R}}}{r}. \tag{4.1}$$

For velocities in $\mathcal{P}$, we have

$$\dot{\boldsymbol{q}}^{\mathcal{P}} = -\frac{\dot{r}}{r^2}\boldsymbol{q}^{\mathcal{R}} + \frac{1}{r}\dot{\boldsymbol{q}}^{\mathcal{R}} = -\frac{\dot{r}}{r}\boldsymbol{q}^{\mathcal{P}} + \frac{1}{r}\dot{\boldsymbol{q}}^{\mathcal{R}}. \tag{4.2}$$

For accelerations in $\mathcal{P}$,

$$\begin{aligned}
\ddot{\boldsymbol{q}}^{\mathcal{P}} &= \frac{\dot{r}^2}{r^2}\boldsymbol{q}^{\mathcal{P}} - \frac{\ddot{r}}{r}\boldsymbol{q}^{\mathcal{P}} - \omega\dot{\boldsymbol{q}}^{\mathcal{P}} - \frac{\dot{r}}{r^2}\dot{\boldsymbol{q}}^{\mathcal{R}} + \frac{1}{r}\ddot{\boldsymbol{q}}^{\mathcal{R}} \\
&= \frac{\dot{r}^2}{r^2}\boldsymbol{q}^{\mathcal{P}} - \frac{\ddot{r}}{r}\boldsymbol{q}^{\mathcal{P}} - \frac{\dot{r}}{r}\dot{\boldsymbol{q}}^{\mathcal{P}} - \frac{\dot{r}}{r}\left(\dot{\boldsymbol{q}}^{\mathcal{P}} + \frac{\dot{r}}{r}\boldsymbol{q}^{\mathcal{P}}\right) + \frac{1}{r}\ddot{\boldsymbol{q}}^{\mathcal{R}} \\
&= -\frac{\ddot{r}}{r}\boldsymbol{q}^{\mathcal{P}} - 2\frac{\dot{r}}{r}\dot{\boldsymbol{q}}^{\mathcal{P}} + \frac{1}{r}\ddot{\boldsymbol{q}}^{\mathcal{R}} \qquad (4.3) \\
&= -\frac{\ddot{r}}{r}\boldsymbol{q}^{\mathcal{P}} - 2\frac{\dot{r}}{r}\dot{\boldsymbol{q}}^{\mathcal{P}} + \frac{1}{r}\left(\dot{\boldsymbol{\omega}}\boldsymbol{q}^{\mathcal{R}} + 2\boldsymbol{\omega}\dot{\boldsymbol{q}}^{\mathcal{R}} - \frac{\mathrm{d}\,V^{\mathcal{R}}}{\mathrm{d}\,\boldsymbol{q}^{\mathcal{R}}}^{\top}\right) \\
&= -\frac{\ddot{r}}{r}\boldsymbol{q}^{\mathcal{P}} - 2\frac{\dot{r}}{r}\dot{\boldsymbol{q}}^{\mathcal{P}} + \dot{\boldsymbol{\omega}}\boldsymbol{q}^{\mathcal{P}} + 2\boldsymbol{\omega}\left(\dot{\boldsymbol{q}}^{\mathcal{P}} + \frac{\dot{r}}{r}\boldsymbol{q}^{\mathcal{P}}\right) - \frac{1}{r^2}\frac{\mathrm{d}\,V^{\mathcal{R}}}{\mathrm{d}\,\boldsymbol{q}^{\mathcal{P}}}^{\top} \\
&= \left(2\boldsymbol{\omega} - 2\frac{\dot{r}}{r}\mathbb{1}\right)\dot{\boldsymbol{q}}^{\mathcal{P}} + \left(2\frac{\dot{r}}{r}\boldsymbol{\omega} + \dot{\boldsymbol{\omega}}\right)\boldsymbol{q}^{\mathcal{P}} - \frac{\ddot{r}}{r}\boldsymbol{q}^{\mathcal{P}} - \frac{1}{r^2}\frac{\mathrm{d}\,V^{\mathcal{R}}}{\mathrm{d}\,\boldsymbol{q}^{\mathcal{P}}}^{\top} \\
&= \left(2\boldsymbol{\omega} - 2\frac{\dot{r}}{r}\mathbb{1}\right)\dot{\boldsymbol{q}}^{\mathcal{P}} + \underbrace{\left(2\frac{\dot{r}}{r}\boldsymbol{\omega} + \dot{\boldsymbol{\omega}}\right)}_{A^{\mathcal{P}}}\boldsymbol{q}^{\mathcal{P}} - \left(\frac{\mathrm{d}}{\mathrm{d}\,\boldsymbol{q}^{\mathcal{P}}}\underbrace{\left(\frac{\ddot{r}\boldsymbol{q}^{\mathcal{P}}\cdot\boldsymbol{q}^{\mathcal{P}}}{2r} + \frac{V^{\mathcal{R}}}{r^2}\right)}_{V^{\mathcal{P}}}\right)^{\top}. \qquad (4.4)
\end{aligned}$$

5 Rotating-pulsating frame of the Kepler problem

Consider a system consisting of two point masses with time-independent gravitational parameters μ_1 and μ_2, subject to Newtonian gravity. A test mass is then subject to the potential

$$\begin{aligned}
V^{\mathcal{I}} &= -\frac{\mu_1}{|\boldsymbol{q}_1^{\mathcal{I}} - \boldsymbol{q}^{\mathcal{I}}|} - \frac{\mu_2}{|\boldsymbol{q}_2^{\mathcal{I}} - \boldsymbol{q}^{\mathcal{I}}|} \\
&= -\frac{\mu_1}{|\boldsymbol{q}_1^{\mathcal{R}} - \boldsymbol{q}^{\mathcal{R}}|} - \frac{\mu_2}{|\boldsymbol{q}_2^{\mathcal{R}} - \boldsymbol{q}^{\mathcal{R}}|} \\
&= -\frac{1}{r}\underbrace{\left(\frac{\mu_1}{|\boldsymbol{q}_1^{\mathcal{P}} - \boldsymbol{q}^{\mathcal{P}}|} + \frac{\mu_2}{|\boldsymbol{q}_2^{\mathcal{P}} - \boldsymbol{q}^{\mathcal{P}}|}\right)}_{=:V'}.
\end{aligned}$$

Let $\boldsymbol{q}_0^{\mathcal{I}}$ be the barycentre,

$$\boldsymbol{q}_0^{\mathcal{I}} \coloneqq \frac{\mu_1\boldsymbol{q}_1^{\mathcal{I}} + \mu_2\boldsymbol{q}_2^{\mathcal{I}}}{\mu_1 + \mu_2}.$$

We have $\ddot{\boldsymbol{q}}_0^{\mathcal{I}} = \boldsymbol{0}$. Let $\boldsymbol{R}$ be such that $\boldsymbol{q}_1^{\mathcal{R}}$ and $\boldsymbol{q}_2^{\mathcal{R}}$ both lie on the x-axis, with $\boldsymbol{q}_1^{\mathcal{R}}$ on the negative side, and such that $\boldsymbol{\omega}$ is in the positive $x \wedge y$ direction.

Let $\boldsymbol{r} \coloneqq \boldsymbol{q}_1^{\mathcal{I}} - \boldsymbol{q}_2^{\mathcal{I}}$, and $r \coloneqq |\boldsymbol{r}|$, so that

$$\boldsymbol{q}_1^{\mathcal{R}} = \begin{pmatrix} -\frac{\mu_2}{\mu_1+\mu_2}r \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{q}_2^{\mathcal{R}} = \begin{pmatrix} \frac{\mu_1}{\mu_1+\mu_2}r \\ 0 \\ 0 \end{pmatrix},$$

and, in the pulsating frame,

$$\boldsymbol{q}_1^{\mathcal{P}} = \begin{pmatrix} -\frac{\mu_2}{\mu_1+\mu_2} \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{q}_2^{\mathcal{P}} = \begin{pmatrix} \frac{\mu_1}{\mu_1+\mu_2} \\ 0 \\ 0 \end{pmatrix}.$$

If the eccentricity vanishes, $\dot{\boldsymbol{\omega}} = \mathbf{0}$, so that the Euler force vanishes, and the geometric acceleration at rest in $\mathcal{R}$ is irrotational. Further, r is constant, thus so are $\boldsymbol{q}_1^{\mathcal{R}}$ and $\boldsymbol{q}_2^{\mathcal{R}}$, and therefore the geometric potential $V^{\mathcal{R}}$ is constant. The critical points of $V^{\mathcal{R}}$ are thus fixed; they are the *Lagrange points.* However, when the eccentricity does not vanish, the Euler force appears, and the critical points of the geometric potential are not fixed.

Observe that ωr^2 is the areal velocity of the Kepler problem, so that

$$\frac{\mathrm{d}\,\omega r^2}{\mathrm{d}\,t} = \dot{\omega} r^2 + 2\omega\dot{r}r = 0, \text{ and therefore } \dot{\omega} + 2\frac{\dot{r}}{r}\omega = 0.$$

Since the rotational axis is invariant in the Kepler problem,

$$\boldsymbol{A}^{\mathcal{P}} = \dot{\boldsymbol{\omega}} + 2\frac{\dot{r}}{r}\boldsymbol{\omega} = \mathbf{0},$$

i.e., the geometric acceleration at rest is irrotational: the pulsation of the reference frame eliminates the Euler force.

Further, observe that, since $\ddot{\boldsymbol{r}} = -\frac{\mu_1+\mu_2}{r^2}\hat{\boldsymbol{r}}$, we have

$$\ddot{r} - r\omega^2 = \ddot{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}} = -\frac{\mu_1 + \mu_2}{r^2}.$$

Consider now the geometric potential

$$\begin{aligned} V^{\mathcal{P}} &= \frac{\ddot{r}\boldsymbol{q}^{\mathcal{P}} \cdot \boldsymbol{q}^{\mathcal{P}}}{2r} + \frac{V^{\mathcal{R}}}{r^2} \\ &= \frac{\ddot{r}\boldsymbol{q}^{\mathcal{P}} \cdot \boldsymbol{q}^{\mathcal{P}}}{2r} - \frac{\boldsymbol{\omega}\boldsymbol{q}^{\mathcal{R}} \cdot \boldsymbol{\omega}\boldsymbol{q}^{\mathcal{R}}}{2r^2} + \frac{V^{\mathcal{I}}}{r^2} \\ &= \frac{\ddot{r}\boldsymbol{q}^{\mathcal{P}} \cdot \boldsymbol{q}^{\mathcal{P}}}{2r^2} - \frac{\boldsymbol{\omega}\boldsymbol{q}^{\mathcal{P}} \cdot \boldsymbol{\omega}\boldsymbol{q}^{\mathcal{P}}}{2} + \frac{V'}{r^3}. \end{aligned}$$

For $\boldsymbol{q}^P$ in the xy plane,

$$\begin{aligned} V^{\mathcal{P}} &= \left(\frac{\ddot{r}}{r} - \omega^2\right)\frac{\left(q^{\mathcal{P}}\right)^2}{2} + \frac{V'}{r^3} \\ &= \frac{1}{r^3}\left(-\frac{(\mu_1 + \mu_2)\left(q^{\mathcal{P}}\right)^2}{2} + V'\right). \end{aligned}$$

Thus, in the xy plane, the geometric potential, while not constant, varies only by multiplication by a position-independent scalar; in particular, its critical points are fixed.