

# On the rotating-pulsating reference frame

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## Notation

**Time** Unless otherwise mentioned, all variables implicitly depend on time, a one-dimensional real oriented inner product space  $T$ . All equations are for those functions evaluated at time  $t$ . Unless otherwise mentioned, all predicates involving variables dependent on time are implicitly quantified for all times  $t$ .

For example, we may write  $f \in \mathbb{R}$  for  $f : T \rightarrow \mathbb{R}$ .

**Derivative** The notation  $\frac{d f}{d x}$  represents the evaluation at  $x$  of the differential  $d f$ , where the expression  $f$  is taken as a function of  $x$ . Within  $f$ , any occurrences of  $x$  represent a free variable, rather than a function of time. The derivative with respect to time  $\frac{d f}{d t}$  may be written  $\dot{f}$ .

For example, we may write

$$\frac{d}{d t} \frac{d e^x}{d x} = \frac{d}{d t} e^x = e^x \dot{x},$$

which is implicitly

$$\left. \frac{d}{d \tau} \left( \frac{d e^\xi}{d \xi} \right) \right|_{\xi=x(\tau)} \Big|_{\tau=t} = \left. \frac{d e^{x(\tau)}}{d \tau} \right|_{\tau=t} = e^{x(t)} \dot{x}(t).$$

This also applies for multidimensional  $x$ ; in particular we may write

$$\frac{d f(\mathbf{x})}{d \mathbf{x}}^\top \text{ for } (\nabla f(t, \cdot))(t, \mathbf{x}(t)).$$

This notation facilitates changes of variables, which are the main focus of this document.

Note that  $d$  here is always the differential, not the exterior derivative; thus for vector spaces  $V$  and  $W$  and  $f : V \rightarrow W$ , we have  $d f : V \rightarrow V^* \otimes W$ , and  $d^2 f : V \rightarrow V^* \otimes V^* \otimes W$ , rather than  $d^2 = 0$ .

As it is somewhat impractical to construct a notation which makes pullbacks of two-forms natural, and as we do not perform changes of variables on curls, we eschew the exterior derivative entirely, and merely get rid of orientation-dependent identifications by writing, for  $\mathbf{v}$  and  $\mathbf{w}$  implicitly dependent on  $\mathbf{q}$ ,

$$(\mathbf{rot}_{\mathbf{q}} \mathbf{v}) \mathbf{w} \text{ for } (\nabla \times \mathbf{v}(t, \cdot))(q(t)) \times \mathbf{w}(t, q(t)).$$

**Reference frames** Script capital letters denote reference frames. For all frames  $\mathcal{F}$ ,  $Q^\mathcal{F}$  is the space of displacements from the origin in frame  $\mathcal{F}$  (representing positions in space), a three-dimensional real inner product space associated with  $\mathcal{F}$ . Variables  $\mathbf{q}^\mathcal{F}$  or  $\mathbf{q}_i^\mathcal{F}$  have values in  $Q^\mathcal{F}$ .

Reference frames are defined in relation to each other by invertible transformations; thus if  $\mathcal{G}$  is defined by  $\mathbf{q}^\mathcal{G} := \mathbf{g}(\mathbf{q}^\mathcal{F})$ , a function  $f$  that depends on  $\mathbf{q}^\mathcal{F}$  can be taken as a function that depends on  $\mathbf{q}^\mathcal{G}$ , and differentiated accordingly.

For example, for  $f$  implicitly dependent on  $\mathbf{q}^\mathcal{F}$  as well as  $t$ , we may write

$$\frac{d f}{d \mathbf{q}^\mathcal{G}} = \frac{d f}{d \mathbf{q}^\mathcal{F}} \frac{d \mathbf{q}^\mathcal{F}}{d \mathbf{q}^\mathcal{G}},$$

which is implicitly

$$d(f(t, \cdot) \circ \mathbf{g}(t, \cdot)^{-1})(\mathbf{q}^\mathcal{G}(t)) = d(f(t, \cdot))(\mathbf{g}(t, \mathbf{q}^\mathcal{G}(t))) d \mathbf{g}(t, \cdot)^{-1}.$$

## 1 Geometric potential

Let  $\mathbf{q}^{\mathcal{F}}$  be a field of free-falling trajectories such that  $\dot{\mathbf{q}}^{\mathcal{F}} = 0$  at time  $t$ ; the field  $\ddot{\mathbf{q}}^{\mathcal{F}}$  is the field of *geometric accelerations at rest*.

In all reference frames considered, the geometric accelerations at rest have a constant curl throughout space, thus, at time  $t$ ,

$$\mathbf{rot}_{\mathbf{q}^{\mathcal{F}}} \ddot{\mathbf{q}}^{\mathcal{F}} = \mathbf{A}^{\mathcal{F}}$$

for some  $\mathbf{A}^{\mathcal{F}}$  which does not depend on  $\mathbf{q}^{\mathcal{F}}$ .

The *geometric potential*  $V^{\mathcal{F}}$  of a frame  $\mathcal{F}$  is defined on  $Q^{\mathcal{F}}$  at time  $t$  by the equation

$$\ddot{\mathbf{q}}^{\mathcal{F}} = -\frac{dV^{\mathcal{F}}}{d\mathbf{q}^{\mathcal{F}}} + \frac{1}{2}\mathbf{A}^{\mathcal{F}}\mathbf{q}^{\mathcal{F}}. \quad (1.1)$$

The geometric potential  $V^{\mathcal{F}}$  implicitly depends on  $\mathbf{q}^{\mathcal{F}}$  as well as  $t$ .

The acceleration  $\ddot{\mathbf{q}}^{\mathcal{F}}$  of a free-falling trajectory  $\mathbf{q}^{\mathcal{F}}$  is the *geometric acceleration*, which, for a given frame, depends on time, position, and velocity. The gradient of the geometric potential is the *rotation-free geometric acceleration at rest*; it depends on time and position. If the geometric acceleration at rest is irrotational, it is equal to the rotation-free geometric acceleration at rest.

Note that unless the geometric acceleration at rest is irrotational, the geometric potential depends on the choice of the origin of  $\mathcal{F}$ .

## 2 Inertial frame

Let  $\mathcal{I}$  be an inertial frame. Then for all free-falling trajectories  $\mathbf{q}^{\mathcal{I}}$ ,

$$\ddot{\mathbf{q}}^{\mathcal{I}} = -\frac{dV^{\mathcal{I}}}{d\mathbf{q}^{\mathcal{I}}}. \quad (2.1)$$

## 3 Rotating frame

Consider the rotating reference frame  $\mathcal{R}$  defined by

$$\mathbf{q}^{\mathcal{R}} := \mathbf{R}(\mathbf{q}^{\mathcal{I}} - \mathbf{q}_0^{\mathcal{I}}). \quad (3.1)$$

Velocities in  $\mathcal{R}$  are related to velocities in  $\mathcal{I}$  as follows:

$$\dot{\mathbf{q}}^{\mathcal{R}} = -\mathbf{R}\dot{\mathbf{q}}_0^{\mathcal{I}} + \boldsymbol{\omega}\mathbf{q}^{\mathcal{R}} + \mathbf{R}\dot{\mathbf{q}}^{\mathcal{I}}, \quad (3.2)$$

where

$$\dot{\mathbf{R}} = \boldsymbol{\omega}\mathbf{R}.$$

Accelerations in  $\mathcal{R}$  are related to accelerations in  $\mathcal{I}$  as follows:

$$\begin{aligned} \ddot{\mathbf{q}}^{\mathcal{R}} &= -\boldsymbol{\omega}\mathbf{R}\dot{\mathbf{q}}_0^{\mathcal{I}} - \mathbf{R}\ddot{\mathbf{q}}_0^{\mathcal{I}} + \dot{\boldsymbol{\omega}}\mathbf{q}^{\mathcal{R}} + \boldsymbol{\omega}\dot{\mathbf{q}}^{\mathcal{R}} + \boldsymbol{\omega}\mathbf{R}\dot{\mathbf{q}}^{\mathcal{I}} + \mathbf{R}\ddot{\mathbf{q}}^{\mathcal{I}} \\ &= -\boldsymbol{\omega}\mathbf{R}\dot{\mathbf{q}}_0^{\mathcal{I}} - \mathbf{R}\ddot{\mathbf{q}}_0^{\mathcal{I}} + \dot{\boldsymbol{\omega}}\mathbf{q}^{\mathcal{R}} + \boldsymbol{\omega}\dot{\mathbf{q}}^{\mathcal{R}} + \boldsymbol{\omega}(\dot{\mathbf{q}}^{\mathcal{R}} - \boldsymbol{\omega}\mathbf{q}^{\mathcal{R}} + \mathbf{R}\dot{\mathbf{q}}_0^{\mathcal{I}}) + \mathbf{R}\ddot{\mathbf{q}}^{\mathcal{I}} \\ &= -\mathbf{R}\ddot{\mathbf{q}}_0^{\mathcal{I}} + \underbrace{\dot{\boldsymbol{\omega}}\mathbf{q}^{\mathcal{R}}}_{\text{Euler}} + \underbrace{2\boldsymbol{\omega}\dot{\mathbf{q}}^{\mathcal{R}}}_{\text{Coriolis}} - \underbrace{\boldsymbol{\omega}^2\mathbf{q}^{\mathcal{R}}}_{\text{centrifugal}} + \mathbf{R}\ddot{\mathbf{q}}^{\mathcal{I}}. \end{aligned} \quad (3.3)$$

Observe that for a free-falling trajectory,

$$\mathbf{R}\ddot{\mathbf{q}}^{\mathcal{I}} = -\mathbf{R}\frac{dV^{\mathcal{I}}}{d\mathbf{q}^{\mathcal{I}}} = -\left(\frac{dV^{\mathcal{I}}}{d\mathbf{q}^{\mathcal{I}}}\mathbf{R}^{-1}\right)^{\top} = -\left(\frac{dV}{d\mathbf{q}^{\mathcal{R}}}\frac{d\mathbf{q}^{\mathcal{I}}}{d\mathbf{q}^{\mathcal{R}}}\right)^{\top} = -\frac{dV^{\mathcal{I}}}{d\mathbf{q}^{\mathcal{R}}}.$$

At rest in  $\mathcal{R}$ , i.e., for  $\dot{\mathbf{q}}^{\mathcal{R}} = 0$ , we have

$$\begin{aligned} \ddot{\mathbf{q}}^{\mathcal{R}} &= -\mathbf{R}\ddot{\mathbf{q}}_0^{\mathcal{I}} + \dot{\boldsymbol{\omega}}\mathbf{q}^{\mathcal{R}} - \boldsymbol{\omega}^2\mathbf{q}^{\mathcal{R}} + \mathbf{R}\ddot{\mathbf{q}}^{\mathcal{I}} \\ &= \dot{\boldsymbol{\omega}}\mathbf{q}^{\mathcal{R}} - \left(\frac{d}{d\mathbf{q}^{\mathcal{R}}}\left(\mathbf{q}^{\mathcal{R}} \cdot \mathbf{R}\ddot{\mathbf{q}}_0^{\mathcal{I}} - \frac{(\mathbf{q}^{\mathcal{R}})^{\top}\boldsymbol{\omega}^{\top}\boldsymbol{\omega}\mathbf{q}^{\mathcal{R}}}{2} + V^{\mathcal{I}}\right)\right)^{\top}, \end{aligned} \quad (3.4)$$

so that the geometric potential is

$$V^{\mathcal{R}} = \mathbf{q}^{\mathcal{R}} \cdot \mathbf{R}\ddot{\mathbf{q}}_0^{\mathcal{I}} - \frac{\boldsymbol{\omega}\mathbf{q}^{\mathcal{R}} \cdot \boldsymbol{\omega}\mathbf{q}^{\mathcal{R}}}{2} + V^{\mathcal{I}}. \quad (3.5)$$

## 4 Rotating-pulsating frame

The rotating-pulsating reference frame  $\mathcal{P}$  is defined by

$$\mathbf{q}^{\mathcal{P}} := \frac{\mathbf{q}^{\mathcal{R}}}{r}. \quad (4.1)$$

For velocities in  $\mathcal{P}$ , we have

$$\dot{\mathbf{q}}^{\mathcal{P}} = -\frac{1}{r^2} \dot{r} \mathbf{q}^{\mathcal{R}} + \frac{1}{r} \dot{\mathbf{q}}^{\mathcal{R}} = -\frac{1}{r} \dot{r} \mathbf{q}^{\mathcal{P}} + \frac{1}{r} \dot{\mathbf{q}}^{\mathcal{R}}. \quad (4.2)$$

For accelerations in  $\mathcal{P}$ ,

$$\begin{aligned} \ddot{\mathbf{q}}^{\mathcal{P}} &= \frac{1}{r^2} \ddot{r}^2 \mathbf{q}^{\mathcal{P}} - \frac{1}{r} \ddot{r} \mathbf{q}^{\mathcal{P}} - \frac{1}{r} \dot{r} \dot{\mathbf{q}}^{\mathcal{P}} - \frac{1}{r^2} \dot{r} \dot{\mathbf{q}}^{\mathcal{R}} + \frac{1}{r} \ddot{\mathbf{q}}^{\mathcal{R}} \\ &= \frac{1}{r^2} \ddot{r}^2 \mathbf{q}^{\mathcal{P}} - \frac{1}{r} \ddot{r} \mathbf{q}^{\mathcal{P}} - \frac{1}{r} \dot{r} \dot{\mathbf{q}}^{\mathcal{P}} - \frac{1}{r} \dot{r} \left( \dot{\mathbf{q}}^{\mathcal{P}} + \frac{1}{r} \dot{r} \mathbf{q}^{\mathcal{P}} \right) + \frac{1}{r} \ddot{\mathbf{q}}^{\mathcal{R}} \\ &= -\frac{1}{r} \ddot{r} \mathbf{q}^{\mathcal{P}} - \frac{2\dot{r}}{r} \dot{\mathbf{q}}^{\mathcal{P}} + \frac{1}{r} \ddot{\mathbf{q}}^{\mathcal{R}} \\ &= -\frac{1}{r} \ddot{r} \mathbf{q}^{\mathcal{P}} - \frac{2\dot{r}}{r} \dot{\mathbf{q}}^{\mathcal{P}} + \frac{1}{r} \left( \dot{\boldsymbol{\omega}} \mathbf{q}^{\mathcal{R}} + 2\boldsymbol{\omega} \dot{\mathbf{q}}^{\mathcal{R}} - \frac{dV^{\mathcal{R}}}{d\mathbf{q}^{\mathcal{R}}} \right) \\ &= -\frac{1}{r} \ddot{r} \mathbf{q}^{\mathcal{P}} - \frac{2\dot{r}}{r} \dot{\mathbf{q}}^{\mathcal{P}} + \dot{\boldsymbol{\omega}} \mathbf{q}^{\mathcal{P}} + 2\boldsymbol{\omega} \left( \dot{\mathbf{q}}^{\mathcal{P}} + \frac{\dot{r}}{r} \mathbf{q}^{\mathcal{P}} \right) - \frac{1}{r^2} \frac{dV^{\mathcal{R}}}{d\mathbf{q}^{\mathcal{P}}} \\ &= \left( 2\boldsymbol{\omega} - \frac{2\dot{r}}{r} \mathbb{1} \right) \dot{\mathbf{q}}^{\mathcal{P}} + \left( 2\frac{\dot{r}}{r} \boldsymbol{\omega} + \dot{\boldsymbol{\omega}} \right) \mathbf{q}^{\mathcal{P}} - \frac{1}{r} \ddot{r} \mathbf{q}^{\mathcal{P}} - \frac{1}{r^2} \frac{dV^{\mathcal{R}}}{d\mathbf{q}^{\mathcal{P}}} \\ &= \left( 2\boldsymbol{\omega} - \frac{2\dot{r}}{r} \mathbb{1} \right) \dot{\mathbf{q}}^{\mathcal{P}} + \underbrace{\left( 2\frac{\dot{r}}{r} \boldsymbol{\omega} + \dot{\boldsymbol{\omega}} \right)}_{\mathbf{A}^{\mathcal{P}}} \mathbf{q}^{\mathcal{P}} - \underbrace{\left( \frac{d}{d\mathbf{q}^{\mathcal{P}}} \left( \frac{\dot{r} \mathbf{q}^{\mathcal{P}} \cdot \mathbf{q}^{\mathcal{P}}}{2r} + \frac{V^{\mathcal{R}}}{r^2} \right) \right)}_{\mathbf{V}^{\mathcal{P}}} \end{aligned} \quad (4.4)$$

## 5 Rotating-pulsating frame of the Kepler problem

Consider a system consisting of two point masses with time-independent gravitational parameters  $\mu_1$  and  $\mu_2$ , subject to Newtonian gravity. A test mass is then subject to the potential

$$\begin{aligned} V^{\mathcal{J}} &= -\frac{\mu_1}{|\mathbf{q}_1^{\mathcal{J}} - \mathbf{q}^{\mathcal{J}}|} - \frac{\mu_2}{|\mathbf{q}_2^{\mathcal{J}} - \mathbf{q}^{\mathcal{J}}|} \\ &= -\frac{\mu_1}{|\mathbf{q}_1^{\mathcal{R}} - \mathbf{q}^{\mathcal{R}}|} - \frac{\mu_2}{|\mathbf{q}_2^{\mathcal{R}} - \mathbf{q}^{\mathcal{R}}|} \\ &= -\frac{1}{r} \left( \frac{\mu_1}{|\mathbf{q}_1^{\mathcal{P}} - \mathbf{q}^{\mathcal{P}}|} + \frac{\mu_2}{|\mathbf{q}_2^{\mathcal{P}} - \mathbf{q}^{\mathcal{P}}|} \right). \end{aligned}$$

$\underbrace{\hspace{10em}}_{=V^{\mathcal{P}}}$

Let  $\mathbf{q}_0^{\mathcal{J}}$  be the barycentre,

$$\mathbf{q}_0^{\mathcal{J}} := \frac{\mu_1 \mathbf{q}_1^{\mathcal{J}} + \mu_2 \mathbf{q}_2^{\mathcal{J}}}{\mu_1 + \mu_2}.$$

We have  $\ddot{\mathbf{q}}_0^{\mathcal{J}} = \mathbf{0}$ . Let  $\mathbf{R}$  be such that  $\mathbf{q}_1^{\mathcal{R}}$  and  $\mathbf{q}_2^{\mathcal{R}}$  both lie on the  $x$ -axis, with  $\mathbf{q}_1^{\mathcal{R}}$  on the negative side, and such that  $\boldsymbol{\omega}$  is in the positive  $x \wedge y$  direction.

Let  $\mathbf{r} := \mathbf{q}_1^{\mathcal{J}} - \mathbf{q}_2^{\mathcal{J}}$ , and  $r := |\mathbf{r}|$ , so that

$$\mathbf{q}_1^{\mathcal{R}} = \begin{pmatrix} -\frac{\mu_2}{\mu_1 + \mu_2} r \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{q}_2^{\mathcal{R}} = \begin{pmatrix} \frac{\mu_1}{\mu_1 + \mu_2} r \\ 0 \\ 0 \end{pmatrix},$$

and, in the pulsating frame,

$$\mathbf{q}_1^{\mathcal{P}} = \begin{pmatrix} -\frac{\mu_2}{\mu_1 + \mu_2} \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{q}_2^{\mathcal{P}} = \begin{pmatrix} \frac{\mu_1}{\mu_1 + \mu_2} \\ 0 \\ 0 \end{pmatrix}.$$

If the eccentricity vanishes,  $\dot{\boldsymbol{\omega}} = \mathbf{0}$ , so that the Euler force vanishes, and the geometric acceleration at rest in  $\mathcal{R}$  is irrotational. Further,  $r$  is constant, thus so are  $\mathbf{q}_1^{\mathcal{R}}$  and  $\mathbf{q}_2^{\mathcal{R}}$ , and therefore the geometric potential  $V^{\mathcal{R}}$  is constant. The critical points of  $V^{\mathcal{R}}$  are thus fixed; they are the *Lagrange points*. However, when the eccentricity does not vanish, the Euler force appears, and the critical points of the geometric potential are not fixed.

Observe that  $\omega r^2$  is the areal velocity of the Kepler problem, so that

$$\frac{d}{dt} \omega r^2 = \dot{\omega} r^2 + 2\omega \dot{r} r = 0, \text{ and therefore } \dot{\omega} + 2\frac{\dot{r}}{r}\omega = 0.$$

Since the rotational axis is invariant in the Kepler problem,

$$\mathbf{A}^{\mathcal{P}} = \dot{\boldsymbol{\omega}} + 2\frac{\dot{r}}{r}\boldsymbol{\omega} = \mathbf{0},$$

i.e., the geometric acceleration at rest is irrotational: the pulsation of the reference frame eliminates the Euler force.

Further, observe that, since  $\ddot{\mathbf{r}} = -\frac{\mu_1 + \mu_2}{r^2}\hat{\mathbf{r}}$ , we have

$$\ddot{r} - r\omega^2 = \ddot{\mathbf{r}} \cdot \hat{\mathbf{r}} = -\frac{\mu_1 + \mu_2}{r^2}.$$

Consider now the geometric potential

$$\begin{aligned} V^{\mathcal{P}} &= \frac{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}{2r} + \frac{V^{\mathcal{R}}}{r^2} \\ &= \frac{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}{2r} - \frac{\boldsymbol{\omega} \mathbf{q}^{\mathcal{R}} \cdot \boldsymbol{\omega} \mathbf{q}^{\mathcal{R}}}{2r^2} + \frac{V^{\mathcal{R}}}{r^2} \\ &= \frac{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}{2r^2} - \frac{\boldsymbol{\omega} \mathbf{q}^{\mathcal{P}} \cdot \boldsymbol{\omega} \mathbf{q}^{\mathcal{P}}}{2} + \frac{V^{\mathcal{R}}}{r^3}. \end{aligned}$$

For  $\mathbf{q}^{\mathcal{P}}$  in the  $xy$  plane,

$$\begin{aligned} V^{\mathcal{P}} &= \left( \frac{\dot{r}}{r} - \omega^2 \right) \frac{(q^{\mathcal{P}})^2}{2} + \frac{V^{\mathcal{R}}}{r^3} \\ &= \frac{1}{r^3} \left( -\frac{(\mu_1 + \mu_2)(q^{\mathcal{P}})^2}{2} + V^{\mathcal{R}} \right). \end{aligned}$$

Thus, in the  $xy$  plane, the geometric potential, while not constant, varies only by multiplication by a position-independent scalar; in particular, its critical points are fixed.