Rotational Motion of a Rigid Reference Frame

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This document describes the computations that are performed by the class `RigidReferenceFrame` and its subclasses to determine the rotational motion (rotation, angular velocity, and angular accelation) of a rigid frame.

Definitions

We are considering in this section a rigid reference frame that is defined by two bodies B_1 and B_2 at positions q_1 and q_2, respectively. A basis of the reference frame is defined by three vectors:

- the *fore* vector $\boldsymbol{F}$ which is along the axis $q_2 - q_1$;
- the *normal* vector $\boldsymbol{N}$ which is orthogonal to $\boldsymbol{F}$ and is such that the velocity of the frame, $\dot{q_2} - \dot{q_1}$ is in the plane $(\boldsymbol{F}, \boldsymbol{N})$;
- the *binormal* vector $\boldsymbol{B}$ which is orthogonal to $\boldsymbol{F}$ and $\boldsymbol{N}$ such that $(\boldsymbol{F}, \boldsymbol{N}, \boldsymbol{B})$ forms a direct trihedron.

There are obviously many possible choices for $(\boldsymbol{F}, \boldsymbol{N}, \boldsymbol{B})$. In practice, it is convenient to choose $\boldsymbol{B}$ before $\boldsymbol{N}$ so that the basis is defined exclusively using vector products:

$$\begin{cases} \boldsymbol{F} &= \boldsymbol{r} \\ \boldsymbol{B} &= \boldsymbol{r} \wedge \dot{\boldsymbol{r}} \\ \boldsymbol{N} &= \boldsymbol{B} \wedge \boldsymbol{F} \end{cases} \tag{1}$$

where we have defined $\boldsymbol{r} \coloneqq q_2 - q_1$. Is trivial to check that these definitions satisfy the properties above, and in particular that they determine a direct orthogonal basis. The corresponding orthonormal basis is:

$$\begin{cases} \boldsymbol{f} &= \dfrac{\boldsymbol{F}}{|\boldsymbol{F}|} \\ \boldsymbol{b} &= \dfrac{\boldsymbol{B}}{|\boldsymbol{B}|} \\ \boldsymbol{n} &= \dfrac{\boldsymbol{N}}{|\boldsymbol{N}|} \end{cases} \tag{2}$$

These vectors are sufficient to define the rotation of the reference frame at any point in time.

Derivatives of normalized vectors

In what follows, we will need to derive the elements of the trihedron $(\boldsymbol{f}, \boldsymbol{n}, \boldsymbol{b})$. To help with this we prove two formulæ that define the first and second derivative of $\boldsymbol{V}/|\boldsymbol{V}|$ based on that of $\boldsymbol{V}$.

The first derivative is:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\boldsymbol{V}}{|\boldsymbol{V}|} &= \frac{|\boldsymbol{V}|\dot{\boldsymbol{V}} - \frac{\mathrm{d}\,|\boldsymbol{V}|}{\mathrm{d}\,t}\boldsymbol{V}}{|\boldsymbol{V}|^2} \\
&= \frac{|\boldsymbol{V}|\dot{\boldsymbol{V}} - \frac{(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})}{|\boldsymbol{V}|}\boldsymbol{V}}{|\boldsymbol{V}|^2} \\
&= \frac{|\boldsymbol{V}|^2\dot{\boldsymbol{V}} - (\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\boldsymbol{V}}{|\boldsymbol{V}|^3} \qquad (3)
\end{aligned}$$

The second derivative is somewhat more complicated:

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}\frac{\boldsymbol{V}}{|\boldsymbol{V}|} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{|\boldsymbol{V}|^2\dot{\boldsymbol{V}} - (\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\boldsymbol{V}}{|\boldsymbol{V}|^3}\right) \\
&= \frac{|\boldsymbol{V}|^3\frac{\mathrm{d}}{\mathrm{d}\,t}\left(|\boldsymbol{V}|^2\dot{\boldsymbol{V}} - (\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\boldsymbol{V}\right) - 3|\boldsymbol{V}|(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\left(|\boldsymbol{V}|^2\dot{\boldsymbol{V}} - (\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\boldsymbol{V}\right)}{|\boldsymbol{V}|^6} \\
&= \frac{2(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\dot{\boldsymbol{V}} + |\boldsymbol{V}|^2\ddot{\boldsymbol{V}} - \left(|\dot{\boldsymbol{V}}|^2 + (\boldsymbol{V}\cdot\ddot{\boldsymbol{V}})\right)\boldsymbol{V} - (\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\dot{\boldsymbol{V}}}{|\boldsymbol{V}|^3} - 3\frac{|\boldsymbol{V}|^3(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})\dot{\boldsymbol{V}} - |\boldsymbol{V}|(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})^2\boldsymbol{V}}{|\boldsymbol{V}|^6} \\
&= \frac{\ddot{\boldsymbol{V}}}{|\boldsymbol{V}|} - 2\dot{\boldsymbol{V}}\frac{(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})}{|\boldsymbol{V}|^3} - \boldsymbol{V}\frac{|\dot{\boldsymbol{V}}|^2 + (\boldsymbol{V}\cdot\ddot{\boldsymbol{V}})}{|\boldsymbol{V}|^3} + 3\boldsymbol{V}\frac{(\boldsymbol{V}\cdot\dot{\boldsymbol{V}})^2}{|\boldsymbol{V}|^5} \qquad (4)
\end{aligned}$$

Angular velocity

To compute the angular velocity, we derive the equations (1) and obtain:

$$\begin{cases}
\dot{\boldsymbol{F}} &= \dot{\boldsymbol{r}} \\
\dot{\boldsymbol{B}} &= \boldsymbol{r}\wedge\ddot{\boldsymbol{r}} \\
\dot{\boldsymbol{N}} &= \dot{\boldsymbol{B}}\wedge\boldsymbol{F} + \boldsymbol{B}\wedge\dot{\boldsymbol{F}}
\end{cases} \qquad (5)$$

Injecting these expressions in the derivative formula (3) makes it possible to compute the trihedron of the derivatives $(\dot{\boldsymbol{f}}, \dot{\boldsymbol{n}}, \dot{\boldsymbol{b}})$ of (2). The angular velocity is then written:

$$\boldsymbol{\omega} = (\dot{\boldsymbol{n}}\cdot\boldsymbol{b})\boldsymbol{f} + (\dot{\boldsymbol{b}}\cdot\boldsymbol{f})\boldsymbol{n} + (\dot{\boldsymbol{f}}\cdot\boldsymbol{n})\boldsymbol{b}$$

Angular acceleration

To compute the angular acceleration, we derive the equations (5) and obtain:

$$\begin{cases}
\ddot{\boldsymbol{F}} &= \ddot{\boldsymbol{r}} \\
\ddot{\boldsymbol{B}} &= \dot{\boldsymbol{r}}\wedge\ddot{\boldsymbol{r}} + \boldsymbol{r}\wedge\boldsymbol{r}^{(3)} \\
\ddot{\boldsymbol{N}} &= \ddot{\boldsymbol{B}}\wedge\boldsymbol{F} + 2\dot{\boldsymbol{B}}\wedge\dot{\boldsymbol{F}} + \boldsymbol{B}\wedge\ddot{\boldsymbol{F}}
\end{cases} \qquad (6)$$

Injecting these expressions in the second derivative formula (4) makes it possible to compute the trihedron of the second derivatives $(\ddot{\boldsymbol{f}}, \ddot{\boldsymbol{n}}, \ddot{\boldsymbol{b}})$ of (2). The angular acceleration is then written:

$$\begin{aligned}
\dot{\boldsymbol{\omega}} = (\ddot{\boldsymbol{n}}\cdot\boldsymbol{b})\boldsymbol{f} + (\dot{\boldsymbol{n}}\cdot\dot{\boldsymbol{b}})\boldsymbol{f} + (\dot{\boldsymbol{n}}\cdot\boldsymbol{b})\dot{\boldsymbol{f}} \\
+ (\ddot{\boldsymbol{b}}\cdot\boldsymbol{f})\boldsymbol{n} + (\dot{\boldsymbol{b}}\cdot\dot{\boldsymbol{f}})\boldsymbol{n} + (\dot{\boldsymbol{b}}\cdot\boldsymbol{f})\dot{\boldsymbol{n}} \\
+ (\ddot{\boldsymbol{f}}\cdot\boldsymbol{n})\boldsymbol{b} + (\dot{\boldsymbol{f}}\cdot\dot{\boldsymbol{n}})\boldsymbol{b} + (\dot{\boldsymbol{f}}\cdot\boldsymbol{n})\dot{\boldsymbol{b}}
\end{aligned}$$