Math183 - Homework 7

due Wednesday, December 4th at 11:59PM

Problem 1.

Important: For $Z \sim N(0,1)$, remember that z_α is defined to be the quantity such that

$$P(Z > z_\alpha) = \alpha$$

i.e., the area under the $N(0,1)$ distribution *to the right* of z_α is exactly equal to α.

1. (12 points) Let $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu_X, \sigma_X^2)$ and let $Y_1, Y_2, \ldots, Y_m \overset{\text{iid}}{\sim} N(\mu_Y, \sigma_Y^2)$.

For the following pairs of assertions:

1. Indicate whether they constitute a valid hypothesis test, and
2. Why they do (or don't) constitute a valid hypothesis test.

Assertions:

- $H_0 : \mu_Y = 100 \quad \text{vs.} \quad H_a : \overline{Y} \neq 100$ (2 points)
- $H_0 : \mu_X = 100 \quad \text{vs.} \quad H_a : \mu_X < 100$ (2 points)
- $H_0 : \mu_X = 100 \quad \text{vs.} \quad H_a : \mu_X > 100$ (2 points)
- $H_0 : \max(X_1, X_2, \ldots, X_n) = 100 \quad \text{vs.} \quad H_a : \max(X_1, X_2, \ldots, X_n) < 100$ (2 points)
- $H_0 : p = 0.25 \quad \text{vs.} \quad H_a : p = 0.25$ (5 points)
- $H_0 : \mu_X - \mu_Y = 25 \quad \text{vs.} \quad H_a : \mu_X - \mu_Y \neq 100$ (2 points)

Solution:

1.

- **No**, this is not a valid hypothesis test.
- The alternative hypothesis H_a is about the sample mean $\overline{Y}$, a statistic, rather than the population mean μ_Y, a parameter. Hypotheses should be about population parameters.

2.

- **Yes**, this is a valid hypothesis test.
- Both H_0 and H_a are about the population parameter μ_X, and they are mutually exclusive and collectively exhaustive.

3.

- **Yes**, this is a valid hypothesis test.
- Similar to the previous case, both hypotheses concern the population parameter μ_X and cover all possibilities.

4.

- **No**, this is not a valid hypothesis test.

- The hypotheses are about the sample maximum $\max(X_1, \ldots, X_n)$, a statistic, instead of a population parameter. Valid hypotheses should involve population parameters.

5.

- **No**, this is not a valid hypothesis test.
- The null and alternative hypotheses are identical ($p = 0.25$), providing no basis for testing. Valid hypothesis tests require mutually exclusive hypotheses.

6.

- **No**, this is not a valid hypothesis test.
- The hypotheses are not mutually exclusive and collectively exhaustive. Values like $\mu_X - \mu_Y = 50$ satisfy neither H_0 nor H_a, making the test invalid.

Problem 2.

For each of the following scenarios: write down your assumptions about the distribution of the data, then write down the null hypotheses H_0 and the alternate hypotheses H_a which enable testing the main question of interest, e.g., $X_1, X_2, \ldots, X_n \stackrel{\text{iid}}{\sim} D(\theta)$ and $H_0 : \theta = \theta_0$ vs. $H_a : \theta \neq \theta_0$.

(a) **(5 points)** A school counselor believes that less than 60% of students participate in extracurricular activities. To test this hypothesis, she conducts a survey of a random sample of 200 students in the school, asking whether they participate in any extracurricular activities.

(b) **(5 points)** A researcher wants to know if the proportion of students who own a smartphone is different between high school and middle school students. He collects data by randomly sampling 150 high school students and 150 middle school students and asks them whether they own a smartphone.

(c) **(5 points)** A comprehensive national survey found that people read, on average, 5 books every 3 months. A skeptical teacher thinks that this is an over-estimate when it comes to the college student subpopulation. So, she collects data by asking every student in her class to report the number of books they have read during the fall quarter.

(d) **(5 points)** A nutritionist believes that the average daily calorie intake of teenagers is more than 500 calories higher than that of children. To test this hypothesis, she conducts a study where she asks a random sample of 100 teenagers and 120 children to record their daily calorie intake for a week.

Solution:

(a) **Assumptions:**

$X_1, X_2, \ldots, X_{200} \stackrel{\text{iid}}{\sim}$ Bernoulli(p), where $X_i = 1$ if student i participates in extracurricular activities, and $X_i = 0$ otherwise.

Hypotheses:

$$H_0 : p = 0.60 \quad \text{vs.} \quad H_a : p < 0.60$$

(b) **Assumptions:**

High school students:

$X_1, X_2, \ldots, X_{150} \stackrel{\text{iid}}{\sim}$ Bernoulli(p_1), where $X_i = 1$ if student i owns a smartphone.

Middle school students:

$Y_1, Y_2, \ldots, Y_{150} \stackrel{\text{iid}}{\sim}$ Bernoulli(p_2), where $Y_i = 1$ if student i owns a smartphone.

Hypotheses:

$$H_0 : p_1 = p_2 \quad \text{vs.} \quad H_a : p_1 \neq p_2$$

(c) **Assumptions:**

$X_1, X_2, \ldots, X_n \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$, where X_i is the number of books read by student i during the fall quarter.

Hypotheses:

$$H_0 : \mu = 5 \quad \text{vs.} \quad H_a : \mu < 5$$

(d) **Assumptions:**

Teenagers:

$X_1, X_2, \ldots, X_{100} \overset{\text{iid}}{\sim} N(\mu_{\text{teen}}, \sigma^2_{\text{teen}})$, where X_i is the average daily calorie intake of teenager i.

Children:

$Y_1, Y_2, \ldots, Y_{120} \overset{\text{iid}}{\sim} N(\mu_{\text{child}}, \sigma^2_{\text{child}})$, where Y_i is the average daily calorie intake of child i.

Hypotheses:

$$H_0 : \mu_{\text{teen}} - \mu_{\text{child}} = 500 \quad \text{vs.} \quad H_a : \mu_{\text{teen}} - \mu_{\text{child}} > 500$$

Problem 3.

For the hypothesis testing of the population mean, suppose the test statistic $\widehat{T}$ has a standard normal $N(0,1)$ distribution when H_0 is true. Calculate the Type-I error probability α for each of the following situations:

(a) **(5 points)** $H_a : \mu > \mu_0$, and the rejection region is $R(\alpha, T) = (1.88, \infty)$.

(b) **(5 points)** $H_a : \mu < \mu_0$, and the rejection region is $R(\alpha, T) = (-\infty, -2.75]$.

(c) **(5 points)** $H_a : \mu \neq \mu_0$, and the rejection region is $R(\alpha, T) = (-\infty, -2.88] \cup (2.88, \infty)$.

Solution:

(a) **Hypotheses:** $H_0 : \mu = \mu_0$ vs. $H_a : \mu > \mu_0$

Rejection Region: $R(\alpha, T) = (1.88, \infty)$

Type I Error Probability:

$$\alpha = P(T > 1.88 \mid H_0 \text{ true}) = P(Z > 1.88)$$

Since $Z \sim N(0,1)$,

$$\alpha = 1 - P(Z \leq 1.88) = 1 - 0.9699 = 0.0301$$

(b) **Hypotheses:** $H_0 : \mu = \mu_0$ vs. $H_a : \mu < \mu_0$

Rejection Region: $R(\alpha, T) = (-\infty, -2.75]$

Type I Error Probability:

$$\alpha = P(T \leq -2.75 \mid H_0 \text{ true}) = P(Z \leq -2.75)$$

Using symmetry,

$$P(Z \leq -2.75) = 1 - P(Z \leq 2.75) = 1 - 0.9970 = 0.0030$$

(c) **Hypotheses:** $H_0 : \mu = \mu_0$ vs. $H_a : \mu \neq \mu_0$

Rejection Region: $R(\alpha, T) = (-\infty, -2.88] \cup (2.88, \infty)$

Type I Error Probability:

$$\alpha = P(T \leq -2.88 \text{ or } T > 2.88 \mid H_0 \text{ true}) = 2 \times P(Z > 2.88)$$

$$P(Z > 2.88) = 1 - P(Z \leq 2.88) = 1 - 0.9980 = 0.0020$$

Therefore,

$$\alpha = 2 \times 0.0020 = 0.0040$$

Problem 4.

The melting point of each of $n = 16$ samples $X_1, X_2, \ldots, X_n$ of a brand of hydrogenated vegetable oil was determined, resulting in $\overline{X} = 94.32$. Assume that the distribution of $X_1, X_2, \ldots, X_n$ is $N(\mu, 1.20^2)$ normal with known $\sigma = 1.20$.

(a) **(5 points)** For the hypotheses $H_0 : \mu = 95$ vs. $H_a : \mu \neq 95$, calculate the rejection region $R(\alpha, \theta)$ at level $\alpha = 0.01$.

(b) **(5 points)** For the same test with $\alpha = 0.01$, what is the probability of a Type-II error when the true $\mu = 94$ under H_a?

(c) **(5 points)** Compute the p-value of the two-tailed hypothesis test, and conclude whether this p-value agrees with your conclusion from part (a).

Solution:

(a) **Test Statistic:**

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{94.32 - 95}{1.20/\sqrt{16}} = \frac{-0.68}{0.30} = -2.267$$

Critical Value at $\alpha = 0.01$:

$$z_{\alpha/2} = z_{0.005} = 2.5758$$

Rejection Region:

$$R(\alpha, \theta) = \left\{ \overline{X} \leq \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \text{ or } \overline{X} \geq \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \right\}$$

Calculating the boundaries:

$$\mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 95 - 2.5758 \times 0.30 = 94.227$$

$$\mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 95 + 2.5758 \times 0.30 = 95.773$$

Thus,

$$R(\alpha, \theta) = \{\overline{X} \leq 94.227 \text{ or } \overline{X} \geq 95.773\}$$

(b) **Type II Error Probability when** $\mu = 94$:

$$\beta = P(94.227 < \overline{X} < 95.773 \mid \mu = 94)$$

Since $\overline{X} \sim N\left(94, \left(\frac{1.20}{\sqrt{16}}\right)^2\right) = N(94, 0.30^2)$, standardize:

$$Z_1 = \frac{94.227 - 94}{0.30} = 0.757$$

$$Z_2 = \frac{95.773 - 94}{0.30} = 5.909$$

Calculate β:

$$\beta = P(0.757 < Z < 5.909) \approx 1 - P(Z \leq 0.757)$$

From standard normal tables, $P(Z \leq 0.757) \approx 0.7757$, so:

$$\beta \approx 1 - 0.7757 = 0.2243$$

(c) **Test Statistic:**

$$Z = \frac{94.32 - 95}{0.30} = -2.267$$

P-value:

$$p = 2 \times P(Z \leq -2.267) = 2 \times 0.0117 = 0.0234$$

Since $p = 0.0234 > \alpha = 0.01$, we fail to reject H_0.

Conclusion: The p-value supports the conclusion from part (a).

Problem 5.

In a study to assess cardiovascular health, researchers measured heart rate recovery after moderate exercise. For $n = 10$ athletes and $m = 11$ non-athletes, the summary statistics for the average recovery rate (measured as *decrease in beats/minute in a five minute window*) is given in Table 1.

Sample	Number of Samples	Sample Mean ($\hat{\mu}$)	Sample Variance ($\hat{\sigma}^2$)
Athletes	$n = 10$	$\overline{X} = 0.64$	$\hat{\sigma}_X^2 = 0.2$
Non-athletes	$m = 11$	$\overline{Y} = 2.05$	$\hat{\sigma}_Y^2 = 0.4$

Table 1: Cardiovascular data summary

(a) **(5 points)** Consider testing $H_0 : \mu_X - \mu_Y = -1.0$ vs. $H_a : \mu_X - \mu_Y < -1.0$. Describe, in words, what H_a says.

(b) **(5 points)** At level $\alpha = 0.01$, find the level-α rejection region $R(\alpha, \theta)$.

(c) **(5 points)** For the same $\alpha = 0.01$, find the lower $(1 - \alpha)$ confidence interval $CI_{\text{lower}}(\alpha, \theta)$ for $\theta = \mu_X - \mu_Y$.

(d) **(5 points)** What is the relationship between the rejection region $R(\alpha, \theta)$ and the lower confidence interval $CI_{\text{lower}}(\alpha, \theta)$?

(e) **(5 points)** What is the probability of a Type-II error when the actual difference between μ_X and μ_Y is $\mu_X - \mu_Y = -1.2$?

(f) **(5 points)** Find the p-value for a hypothesis test of H_0 vs. H_a and conclude what your decision will be at level $\alpha = 0.1$.

Solution:

(a) H_a states that the mean recovery rate for athletes is more than 1.0 units less than that for non-athletes; that is, athletes recover significantly more slowly than non-athletes.

(b) **Test Statistic:**

$$t = \frac{(\overline{X} - \overline{Y}) - (\mu_0)}{\sqrt{\frac{\hat{\sigma}_X^2}{n} + \frac{\hat{\sigma}_Y^2}{m}}}$$

Calculate the difference in sample means:

$$\overline{X} - \overline{Y} = 0.64 - 2.05 = -1.41$$

Compute the standard error (SE):

$$SE = \sqrt{\frac{0.2}{10} + \frac{0.4}{11}} = \sqrt{0.02 + 0.03636} = \sqrt{0.05636} = 0.2373$$

Compute the test statistic:

$$t = \frac{(-1.41) - (-1.0)}{0.2373} = \frac{-0.41}{0.2373} = -1.728$$

Degrees of Freedom:

$$\nu = \frac{\left(\frac{\hat{\sigma}_X^2}{n} + \frac{\hat{\sigma}_Y^2}{m}\right)^2}{\frac{\left(\frac{\hat{\sigma}_X^2}{n}\right)^2}{n-1} + \frac{\left(\frac{\hat{\sigma}_Y^2}{m}\right)^2}{m-1}} = \frac{(0.05636)^2}{\frac{(0.02)^2}{9} + \frac{(0.03636)^2}{10}} = \frac{0.003174}{0.00004444 + 0.0001322} = \frac{0.003174}{0.00017664} \approx 17.97 \approx 18$$

Critical Value at $\alpha = 0.01$:

$$t_{\alpha,\nu} = t_{0.01,18} = -2.552$$

Rejection Region:

$$R(\alpha, \theta) = \{t \leq -2.552\}$$

So, we reject H_0 if $t \leq -2.552$.

(c) **Lower Confidence Limit:**

$$\text{Lower Limit} = (\overline{X} - \overline{Y}) - t_{\alpha,\nu} \times SE = -1.41 - (-2.552) \times 0.2373 = -1.41 + 0.606 = -0.804$$

Lower Confidence Interval:

$$CI_{\text{lower}}(\alpha, \theta) = (-\infty, -0.804)$$

This is the lower 99% confidence interval for $\theta = \mu_X - \mu_Y$.

(d) The null value $\mu_X - \mu_Y = -1.0$ lies within the lower confidence interval $(-\infty, -0.804)$ because $-1.0 > -0.804$. Therefore, we **fail to reject** H_0 at the $\alpha = 0.01$ significance level. This aligns with the rejection region $R(\alpha, \theta)$, where our test statistic $t = -1.728$ does not fall into the rejection region $t \leq -2.552$. Thus, both the confidence interval and the rejection region lead to the same conclusion.

(e) To find the probability of a Type-II error (β) when $\mu_X - \mu_Y = -1.2$, we proceed as follows:

Non-centrality Parameter:

$$\delta = \frac{(\mu_X - \mu_Y) - \mu_0}{SE} = \frac{(-1.2) - (-1.0)}{0.2373} = \frac{-0.2}{0.2373} = -0.8437$$

Type-II Error Probability:

$$\beta = P(T > t_{\alpha,\nu} \mid \delta)$$

Since $t_{\alpha,\nu} = -2.552$ and $\delta = -0.8437$, we calculate:

$$\beta = P(T > -2.552 \mid \delta = -0.8437)$$

Using statistical software or non-central t-distribution tables with $\nu = 18$, we find:

$$\beta \approx 0.956$$

Therefore, the probability of a Type-II error when $\mu_X - \mu_Y = -1.2$ is approximately 95.6%.

(f) **P-value Calculation:**

$$p = P(T \leq t \mid H_0) = P(T \leq -1.728 \mid \nu = 18)$$

Using a t-distribution table or software:

$$p \approx 0.0499$$

Since $p < \alpha = 0.1$, we **reject** H_0 at the $\alpha = 0.1$ significance level. This indicates that there is sufficient evidence at the 10% level to conclude that the mean recovery rate difference is less than -1.0.

Problem 6.

A sample of $n = 300$ urban adult residents of CA revealed that 120 favorably approved of the incumbent president's job performance, whereas a sample of $m = 180$ rural residents yielded 75 who favorably approved of the incumbent president. We are interested in testing whether or not there is a difference in perception of the incumbent president's performance across the two groups.

(a) **(5 points)** Let $X_1, X_2, \ldots, X_n$ be the responses of the urban residents and $Y_1, Y_2, \ldots, Y_m$ be the responses of the rural residents. In the setting of this problem, describe the distributions these random variables are sampled from.

(b) **(5 points)** Identify the main parameter of interest, θ.

(c) **(5 points)** Write down the expression for the statistic $\hat{\theta}$, which is our best guess for the population parameter θ.

(d) **(5 points)** Write down the null and alternative hypothesis for the question.

(e) **(5 points)** What would the ideal rejection region look like for rejecting H_0 in favor of H_a?

(f) **(5 points)** Assuming the null hypothesis H_0 is true, what is the sampling distribution of $\hat{\theta}$? *[Hint: Use CLT approximation for the sampling distribution of the statistic $\hat{\theta}$ which we have encountered in class earlier. You just need to write down what this sampling distribution is under H_0.]*

(g) **(15 points)** By setting α to be the Type-I error probability, write down the final expression for the rejection region $R(\alpha, \theta)$ in terms of $z_{\alpha/2}$.

(h) **(5 points)** Fixing $\alpha = 0.01$, find the level-α rejection region $R(\alpha, \theta)$.

(i) **(5 points)** What is your final decision based on the level $\alpha = 0.01$ hypothesis test?

(j) **(5 points)** Compute the p-value for the hypothesis test, and specify what your decision will be if you were to, instead, perform a level $\alpha = 0.05$ hypothesis test.

Solution:

(a) Each response X_i (urban) and Y_i (rural) is a Bernoulli random variable:

$$X_i \sim \text{Bernoulli}(p_U), \quad Y_i \sim \text{Bernoulli}(p_R)$$

where p_U is the probability an urban resident approves, and p_R is the probability a rural resident approves.

(b) The main parameter of interest is the difference in approval proportions:

$$\theta = p_U - p_R$$

(c) The statistic $\hat{\theta}$ is the difference in sample proportions:

$$\hat{\theta} = \hat{p}_U - \hat{p}_R = \frac{\sum_{i=1}^n X_i}{n} - \frac{\sum_{i=1}^m Y_i}{m}$$

$$\hat{p}_U = \frac{120}{300} = 0.4, \quad \hat{p}_R = \frac{75}{180} \approx 0.4167$$

(d) Null and alternative hypotheses:

$$H_0 : p_U = p_R \quad \text{vs.} \quad H_a : p_U \neq p_R$$

(e) The ideal rejection region is:

$$R(\alpha, \theta) = \left\{ \hat{\theta} : |\hat{\theta}| > z_{\alpha/2} \times \text{SE} \right\}$$

where SE is the standard error under H_0.

(f) Under H_0, $\hat{\theta}$ is approximately normally distributed:

$$\hat{\theta} \sim N\left(0, \sigma_{\hat{\theta}}^2\right)$$

where

$$\sigma_{\hat{\theta}}^2 = p(1-p)\left(\frac{1}{n} + \frac{1}{m}\right)$$

and p is the common proportion under H_0.

(g) The rejection region is:

$$R(\alpha, \theta) = \left\{ |\hat{\theta}| > z_{\alpha/2} \times \sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n} + \frac{1}{m}\right)} \right\}$$

where $\hat{p} = \frac{120+75}{300+180} = 0.40625$.

(h) Compute the standard error:

$$\text{SE} = \sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n} + \frac{1}{m}\right)} = \sqrt{0.40625 \times 0.59375\left(\frac{1}{300} + \frac{1}{180}\right)}$$

Calculate the combined sample size term:

$$\frac{1}{300} + \frac{1}{180} = 0.003333 + 0.005556 = 0.008889$$

Calculate $\hat{p}(1-\hat{p})$:

$$\hat{p}(1-\hat{p}) = 0.40625 \times 0.59375 = 0.2419434$$

Compute the standard error:

$$\text{SE} = \sqrt{0.2419434 \times 0.008889} = \sqrt{0.0021527} = 0.0464$$

Determine the critical value for $\alpha = 0.01$:

$$z_{\alpha/2} = z_{0.005} = 2.576$$

Compute the rejection region boundary:

$$\text{Rejection region:} \quad |\hat{\theta}| > z_{\alpha/2} \times \text{SE} = 2.576 \times 0.0464 = 0.1195$$

Therefore, the level-α rejection region $R(\alpha, \theta)$ is:

$$R(\alpha, \theta) = \left\{ |\hat{\theta}| > 0.1195 \right\}$$

(i) Calculated difference:

$$\hat{\theta} = 0.4 - 0.4167 = -0.0167$$

Since $|\hat{\theta}| = 0.0167 < 0.1029$, we fail to reject H_0 at $\alpha = 0.01$.

(j) Compute the test statistic:

$$Z = \frac{\hat{\theta}}{\text{SE}} = \frac{-0.0167}{0.0464} = -0.3596$$

Calculate the p-value:

$$p = 2 \times P(Z \leq -0.3596) = 2 \times (1 - P(Z \leq 0.3596)) = 2 \times (1 - 0.6398) = 2 \times 0.3602 = 0.7204$$

(Note: The value $P(Z \leq 0.3596) = 0.6398$ is obtained from standard normal distribution tables or a calculator.)

Decision at $\alpha = 0.05$:

Since $p = 0.7204 > 0.05$, we **fail to reject** H_0 at the $\alpha = 0.05$ significance level.

Problem 7.

In a study to estimate the average height of adult male basketball players, a researcher wants to test if the average height is *greater than* 200cm. Prior studies indicate that the variance in height is 16cm^2.

(a) **(10 points)** Write down any assumptions about the data and identify the setting of the problem.

(b) **(10 points)** From part (a), identify the relevant population parameter, θ, and the sample statistic, $\hat{\theta}$, the researcher will use to make any statistical inference.

(c) **(10 points)** The researcher wants to compute a *two-sided* 99% confidence interval for the sample statistic θ. If they want the margin of error to be 0.01cm, what is the minimum number of samples needed?

(d) **(10 points)** In part (c), the researcher uses a two-sided confidence interval. In words, describe why/why not this type of a confidence interval is appropriate for the research question they wish to investigate.

(e) **(10 points)** Write down the appropriate null and alternate hypotheses for the question.

(f) **(10 points)** The researcher aims to have a power of 80% to detect an actual average height of 202cm. What sample size is required for this test at a $\alpha = 0.01$ significance level?

Solution:

(a) **Assumptions:**

- The heights of adult male basketball players are independent and identically distributed (i.i.d.).
- The heights follow a normal distribution: $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$. - The variance is known and equal to $\sigma^2 = 16\,\text{cm}^2$.

Setting:

Testing whether the population mean height μ is greater than 200 cm.

(b) **Population Parameter:**

$\theta = \mu$ (the true mean height).

Sample Statistic:

$\hat{\theta} = \overline{X}$ (the sample mean height).

(c) **Margin of Error:**

For a two-sided 99% confidence interval:

$$E = z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}}$$

Given:

$$E = 0.01\,\text{cm}, \quad \sigma = 4\,\text{cm}, \quad \alpha = 0.01$$

Critical value:

$$z_{\alpha/2} = z_{0.005} = 2.576$$

Solving for n:

$$n = \left(\frac{z_{\alpha/2} \times \sigma}{E}\right)^2 = \left(\frac{2.576 \times 4}{0.01}\right)^2 = (1030.4)^2 = 1{,}061{,}723.84$$

Therefore, the minimum sample size needed is:

$$n = 1{,}061{,}724$$

(d) Since the researcher is interested in whether the mean height is *greater than* 200 cm, a one-sided confidence interval would be more appropriate. Using a two-sided interval is unnecessary and increases the required sample size.

(e) **Hypotheses:**

$$H_0 : \mu = 200\,\text{cm} \quad \text{vs.} \quad H_a : \mu > 200\,\text{cm}$$

(f) To achieve 80% power ($\beta = 0.20$) to detect $\mu = 202\,\text{cm}$ at $\alpha = 0.01$:

- Significance level critical value:

$$z_\alpha = z_{0.99} = 2.326$$

- Power critical value:

$$z_{1-\beta} = z_{0.84} = 0.8416$$

- Required sample size:

$$n = \left(\frac{z_\alpha + z_{1-\beta}}{\frac{\mu_1 - \mu_0}{\sigma}} \right)^2$$

Where $\mu_0 = 200\,\text{cm}$, $\mu_1 = 202\,\text{cm}$, $\sigma = 4\,\text{cm}$.

Calculating:

$$n = \left(\frac{2.326 + 0.8416}{\frac{2}{4}} \right)^2 = \left(\frac{3.1676}{0.5} \right)^2 = (6.3352)^2 = 40.08$$

Therefore, the required sample size is:

$$n = 41$$

Problem 8.

A clinical trial is needed to compare the efficacy of a new diabetes drug X in comparison to the baseline Y. Prior pilot studies found the standard deviations for both drugs to be $\sigma_X = 10.0$ units and $\sigma_Y = 12.0$ units. The FDA requires there to be a reduction of $5\mu g/ml$ in blood sugar to be considered "innovation" in order to release the drug into the market. Furthermore, all results need to be reported at a statistical significance level of $\alpha = 0.01$.

(a) **(10 points)** State the main assumptions in this problem and identify the problem setting.

(b) **(10 points)** Identify the population parameter θ and sample statistic $\hat{\theta}$ the researchers are interested in.

(c) **(10 points)** Identify the null and alternate hypotheses for this problem which will enable the researchers to make the necessary statistical inference.

(d) **(10 points)** The units for the standard deviation are intentionally left as *units*. What units should these be for this problem to make sense?

(e) **(10 points)** Suppose the researchers choose to recruit n volunteers for the research study and randomly split half of them to the two groups, i.e., $n/2$ to take drug X and the remaining $n/2$ volunteers to take the drug Y. What is the minimum sample size, n, needed to detect if the new drug improves on the baseline with power 90%?

Solution:

(a) **Assumptions:**

- The blood sugar reductions for both drugs are independent and normally distributed.
- Blood sugar reductions for drug X have variance $\sigma_X^2 = 100\,(\mu g/ml)^2$.
- Blood sugar reductions for drug Y have variance $\sigma_Y^2 = 144\,(\mu g/ml)^2$.
- Participants are randomly assigned to each group, ensuring independent samples.

Problem Setting: Comparing the mean blood sugar reductions between the new drug X and the baseline drug Y, conducting a one-sided two-sample test for the difference of means with known variances.

(b) **Population Parameter:** $\theta = \mu_X - \mu_Y$ (the difference in mean blood sugar reductions).

Sample Statistic: $\hat{\theta} = \overline{X} - \overline{Y}$ (the difference in sample means).

(c) **Null Hypothesis:** $H_0 : \mu_X - \mu_Y \leq 5\,\mu g/ml$

Alternative Hypothesis: $H_a : \mu_X - \mu_Y > 5\,\mu g/ml$

(d) **Units for Standard Deviation:** The standard deviations should be in $\mu g/ml$ to match the units of blood sugar reduction measurements.

(e) To find the minimum sample size n needed for 90% power at $\alpha = 0.01$:

Given:

- $\sigma_X = 10\,\mu g/ml$
- $\sigma_Y = 12\,\mu g/ml$
- Significance level $\alpha = 0.01$ (one-sided test)
- Desired power $1 - \beta = 0.90$ ($\beta = 0.10$)
- Difference to detect $\delta = 5\,\mu g/ml$ (assuming $\mu_X - \mu_Y = 10\,\mu g/ml$ under H_a)

Critical Values:

$$z_\alpha = z_{0.99} = 2.326, \quad z_{1-\beta} = z_{0.90} = 1.2816$$

Combined Standard Deviation:

$$\sigma_{\text{combined}} = \sqrt{\sigma_X^2 + \sigma_Y^2} = \sqrt{10^2 + 12^2} = \sqrt{100 + 144} = \sqrt{244} \approx 15.62\,\mu g/ml$$

Sample Size Calculation:

$$n = 2\left(\frac{z_\alpha + z_{1-\beta}}{\delta/\sigma_{\text{combined}}}\right)^2$$

$$n = 2\left(\frac{2.326 + 1.2816}{5/15.62}\right)^2$$

$$n = 2\left(\frac{3.6076}{0.320}\right)^2$$

$$n = 2\,(11.27375)^2$$

$$n = 2 \times 127.04 = 254.08$$

Conclusion: The minimum sample size required is $n = 256$ participants (rounded up to the next even integer for equal group sizes).

Participants per Group: 128 in drug X group and 128 in drug Y group.