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**COVID-19: THE SUSCEPTIBLE INFECTED
REMOVED (SIR) MODEL WITH VACCINATIONS**

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Abstract: This paper investigates the effect of province-wide vaccination in Ontario, Canada on the spread of the novel pandemic, COVID-19. An SIR model is designed, such that a vaccination term is considered. The analysis includes finding the points of equilibrium and discussing their stability, as well as finding the portion of the population that needs to be vaccinated for the reproductive number to be less than one. The equilibrium points occur when there are no infections, resulting in a stable manifold since the system stops evolving when no infected people remain. The other equilibrium point occurs when the susceptible population vanishes, which corresponds to everyone being infected. With a basic reproductive number of 3 and a 95% vaccine efficacy, approximately 70% of the population needs to be vaccinated for the spread of COVID-19 to effectively decline. The effectiveness in increasing daily vaccine doses is also investigated, showing how the peak in infections decreases significantly with increasing vaccine rollout. The threshold number of infectives is determined to be 200 daily infections for the pandemic to be under control.

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1) INTRODUCTION

Coronaviruses are a family of diseases that cause respiratory tract infections. The recent outbreak of the virus near Wuhan City, Hubei Province, China in December 2019 has been globally recognized as COVID-19 and has been labelled as a pandemic [1]. COVID-19 was declared a global pandemic by the World Health Organization (WHO) on March 11, 2020, with many countries going into national lockdowns, including Canada.

The way COVID-19 spreads from an infected person to others is through respiratory droplets and aerosols. These aerosols occur when a droplet from an infected person is so small that it lingers in the air under certain circumstances [4]. According to the WHO, the basic reproductive number, which is the average expected number of additional cases that one case would generate, was estimated to be at $R_0 = 1.4$ to 4, depending on geographical location. It is known that if $R_0 < 1$, then the disease will gradually fade away [5]. Outbreaks occurring in provinces across Canada usually exhibit non-linear behavior which requires the use of dynamical systems to capture the full picture of the spread. Susceptible, Infected, and Removed (SIR) models are therefore used to model the spread of COVID-19 in Canada where control parameters may be used [2]. COVID-19 was first detected in Canada on January 25, 2020, with community spreading occurring as early as April 2020 across almost all provinces [3]. The total number of COVID-19 related cases reported in Ontario on April 1, 2020 was 2,392 with the death toll reported at 37 deceased. Since then, the number of tests taking place has increased significantly and the number of COVID-19 related infections have increased dramatically and overwhelmed intensive care units (ICUs) at hospitals around the province.

This report will focus on the spread of COVID-19 within the province of Ontario. Data obtained will be based on government testing reported on their website [6]. For the sake of simplicity, the control parameter used in this report will be a single-dose vaccine with an arbitrary efficacy of 95% versus the virus. It is also assumed that only one strain of the virus exists, and all data will be dealt with as such. The use of dynamical systems to analyze the results of numerical simulations will be the main emphasis throughout this paper.

2) METHODOLOGY & ANALYSIS

A common model for simple disease research and investigations is called the Susceptible, Infected, and Removed (SIR) model. This is a compartmental model that is composed of three compartments all governed by a set of non-linear differential equations given as follows:

- $S(t)$ is the susceptible population which includes the portion of the population that can be infected by COVID-19.
- $I(t)$ is the infected population which is the portion of the population that have contracted the virus and can infect the susceptible population $S(t)$, therefore moving them to the infected compartment $I(t)$.
- $R(t)$ is the portion of the population that cannot infect others and are also no longer susceptible, therefore they are removed from the system. $R(t)$ includes people who have died, been vaccinated, or have survived the virus who cannot get reinfected.

The system to be analyzed in this project is a set of three equations \dot{S} , \dot{I} and \dot{R} which are time derivatives of $S(t)$, $I(t)$ and $R(t)$, respectively.

$$\dot{S} = -\beta S(t)I(t) - \epsilon\eta \quad (1)$$

$$\dot{I} = \beta S(t)I(t) - (\delta + \gamma)I(t) \quad (2)$$

$$\dot{R} = (\delta + \gamma)I(t) + \epsilon\eta \quad (3)$$

where β is the transmission rate, ϵ is the efficacy of the vaccine, η is the vaccination rate, δ is the death rate, and γ is the recovery rate.

Each equation above describes how people are moving through the system of compartments as a function of time. The $\beta S(t)I(t)$ term shows how people from the susceptibility group become infected at a rate of β and are then moved to the infected group. This is shown by the negative sign in equation (1) and a positive sign in equation (2). Hence, the rate of the susceptible population over time is decreasing and that of the infected population is increasing. The $\epsilon\eta$ term in is the rate at which the susceptible population gets vaccinated and hence will no longer become susceptible, moving them to the removed population. This is shown in equation (1) with a negative sign associated to the $\epsilon\eta$ term and a positive sign in equation (3). The $(\delta + \gamma)I(t)$ term combines both the recovered and dead population of the infectives, which then moves to the removed population since they can no longer be reinfected or infect others. This is shown with the

negative sign associated with the $(\delta + \gamma)I(t)$ term in equation (2) and the positive sign associated with it in equation (3).

For simplicity, the total population is assumed to be constant which leads to $(S + I + R)' = 0$. The initial conditions are

1. $S(0) = S_0 = N - I$, where N is the total population.
2. $I(0) = I_0 = I$, E.g. one person is infected and starts the spread
3. $R(0) = R_0 = 0$, E.g. no one has recovered or died from the virus yet

Therefore, the initial conditions combined with the condition that the total population is constant leads to

$$S + I + R = S_0 + I_0 + R_0 = N \quad (4)$$

where $S_0 + I_0 + R_0$ represents the total population.

The susceptible population is decreasing over time, so $S(t) \leq S_0$. Substituting this inequality into equation (2) yields

$$\frac{dI}{dt} < I(t)[\beta S_0 - (\delta + \gamma)] \quad (5)$$

A pandemic occurs when $I(t) > I_0$, i.e. $\beta S_0 - (\delta + \gamma) > 0 \rightarrow S_0 > \frac{\delta + \gamma}{\beta} = I/q$. Here, q is known as the contact ratio. Further multiplying both sides by β and dividing by $(\delta + \gamma)$ yields the basic reproduction ratio

$$R_0 = \beta S_0 / (\delta + \gamma) \quad (6)$$

where $R_0 > 1$ signals an epidemic is in progress. Let $r = \beta$ and $a = \delta + \gamma$, it is widely assumed for the case of COVID-19 and initial condition $S_0 = I$, equation (6) becomes $R_0 = \frac{r S_0}{a} = \frac{r}{a} \approx 3$.

Hence, the required condition for COVID-19 to start slowing down its spread is

$$\frac{r S^*}{a} < 1 \rightarrow S^* < \frac{a}{r} = \frac{I}{3} \quad (7)$$

where S^* is the susceptible population and $\frac{I}{3}$ refers to the portion of the population that is still susceptible. Following this, the required vaccinated population is then

$$S^* + V = I \rightarrow V = I - S^* \rightarrow V < \frac{2}{3} \quad (8)$$

where V is the vaccinated population and $\frac{2}{3}$ refers to the portion of the population that needs to be vaccinated, assuming the vaccine is 100% effective, in order for the spread to slow down.

The portion of the population that needs to be vaccinated for a $\eta = 95\%$ effective vaccine is $V > \frac{1}{\eta} (1 - \frac{1}{R_0}) \approx 70\%$. Hence, the effectively vaccinated population is $V_{eff} = \eta V \approx 67\%$.

Equilibrium solutions

Solving this system conventionally is mathematically rigorous. Integrating dI/dt to find the maximum number of infections is tedious, hence, it is more efficient to use equations (1)-(4) where it is possible to find the maximum infectives as well as to find equilibrium solutions and analyze their stability. For the analysis, the $S(t)$ & $I(t)$ functions will be used so that the $R(t)$ function is determined from them. First, the system will be analyzed without the vaccination term η . For $\dot{S} = 0$ and $\dot{I} = 0$, the following conditions are true

$$\begin{aligned} S' = 0 &= -\beta SI \\ I' = 0 &= \beta SI - (\delta + \gamma)I \end{aligned}$$

which has a trivial solution $I=0$. With respect to the model, this means that whenever $I=0$ the pandemic is considered over. This is because when there are no infectious people, the virus can no longer be transmitted to others. Therefore, the axis along $I=0$ is a stable manifold since the system stops evolving after that point and tends towards the origin. Another stable line to consider is the $S=0$ line, this corresponds to everyone having been infected. After reaching the $S=0$ or $I=0$ lines, all the solutions are then tending towards the origin in the S - I plane, hence, the origin is a stable node that attracts solutions. This makes sense since pandemics that cannot reinfect will not be able to continue forever and must at some point end. This justifies the solution that the origin is a stable attracting node.

Eigenvalues of the SIR model

The model can be described by the following linear system

$$\begin{bmatrix} 0 & -\beta S \\ 0 & \beta S - (\delta + \gamma) \end{bmatrix}$$

which would be the form of the matrix, A , with eigenvalues $\lambda_1 = 0, \lambda_2 = \beta S - (\delta + \gamma)$. $\lambda_2 < 0$ if $\beta S < (\delta + \gamma)$, and $\lambda_2 > 0$ if $\beta S > (\delta + \gamma)$. Since S is a function of time, the eigenvalue can change value over time. This means that the line at $S = (\delta + \gamma)/\beta$ acts as a turning

point for the S-I phase plot. The infection derivative goes from a positive to a negative slope as it passes line S. This means that the number of infected people will begin to decline after this period, marking it as the point of maximum infections. This can be seen in the phase plot below.

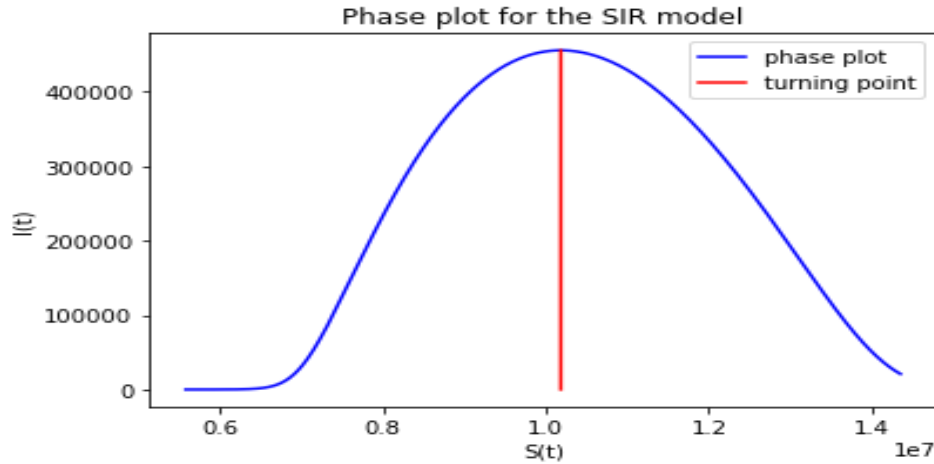


Figure 1 - The red line labelled turning point is given by $S = (\delta + \gamma)/\beta$. Plugging the values into the equation yields $S = (1.088 \cdot 10^{-3} + 0.0858)/8.534 \cdot 10^{-9} = 10,181,392$. Therefore, when 10,181,392 susceptible people are reached, the number of infected people is at a maximum, after which they will begin to decrease. This ends the “wave” of the coronavirus.

This result agrees with the current COVID-19 data and shows the accuracy of the SIR model and its agreement with the analytic calculations.

Data and modelling

Currently, Ontario is undergoing another rise in infection numbers. The current governmental data estimates an effective reproduction number of $R_0 = 1.18$. The current daily death rate and the current recovery rate are assumed to be 23 and 1814 people per day, respectively. Rearranging equation (6) yields $\beta = R_0(\delta + \gamma)/S_0$, which can be used to calculate the transmission rate of COVID-19.

The latest data puts the population of Ontario at 14,734,014. As of the writing of this report, it is estimated that the total number of recoveries from COVID-19 in Ontario are 352,420 people and the total number of deaths is estimated at 7,389 people. This puts the susceptible starting

population at 14,353,330, the removed starting population at 359,809, and the infected starting population, or active cases, at 20,875. The parameters are estimated to be

1. $\beta = 8.534 \cdot 10^{-9}$, is the rate of spread of COVID-19
2. $\epsilon = 0.0002$, is the rate of vaccine distribution
3. $\eta = 0.95$, is the efficacy of the vaccine
4. $\delta = 1.088 \cdot 10^{-3}$, is the death rate from COVID-19
5. $\gamma = 0.0858$, is the recovery rate from COVID-19

The β, δ, γ values can be determined by solving equations based on initial conditions for the $S'(0)$, $I'(0)$, $R'(0)$. Meaning, taking the daily death rates, recovery rates, and infection rates with the corresponding $S(0)$, $I(0)$, $R(0)$ it is possible to solve for the parameters. Plugging these values into the SIR model and plotting, yields the following graphs.

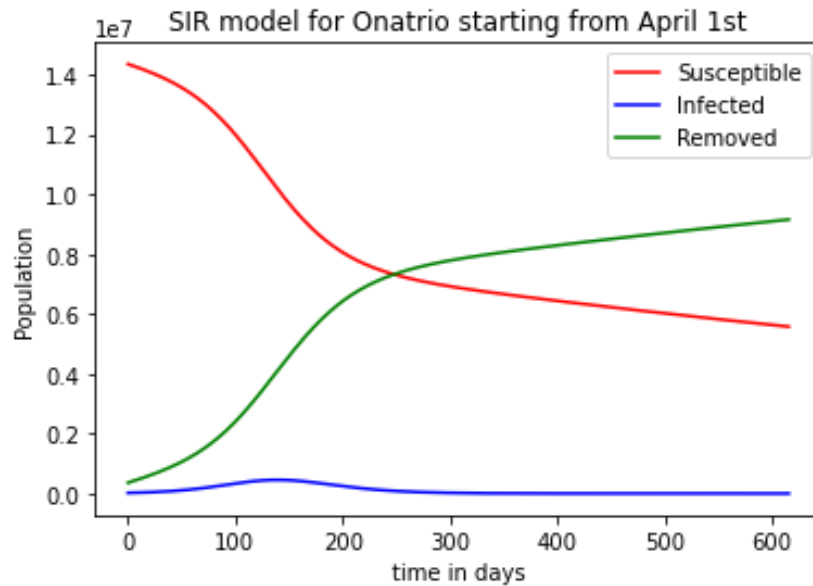


Figure 2 - The SIR model with the vaccination term based on Ontario's numbers starting from April 1, 2020 predicts a small maximum in the number of active infected cases.

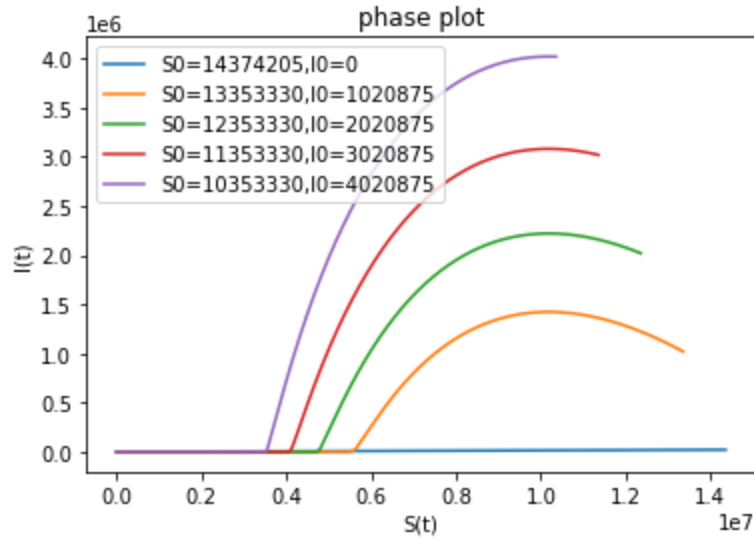


Figure 3 - The phase plot in the S-I plane is for the SIR model for varying initial conditions. Looking at the graph from right to left, the number of infectives increases until the reproductive ratio has reached a value of one, after which they decrease. The lower the susceptible population, the faster the solution will go to equilibrium.

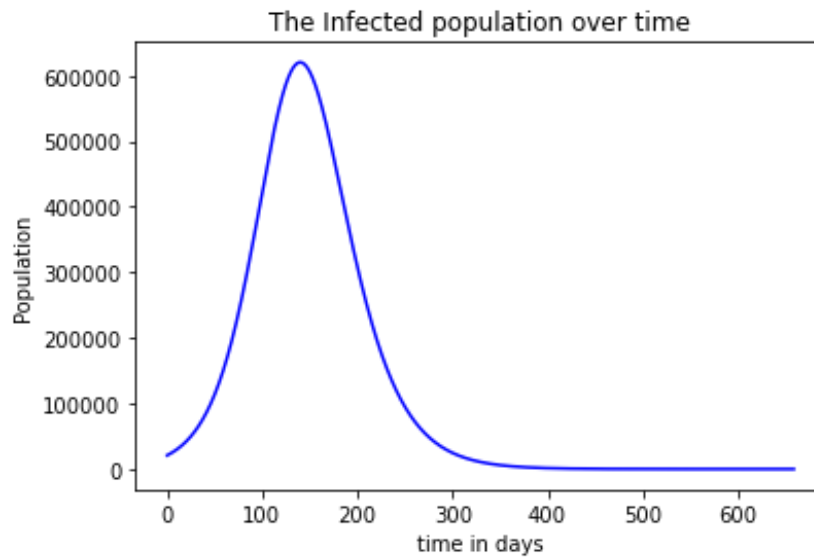


Figure 4 - The diagram shows how the infectives over time increases until the maximum is reached. This plot is for the current vaccine roll-out rate of $\epsilon = 0.0002$.

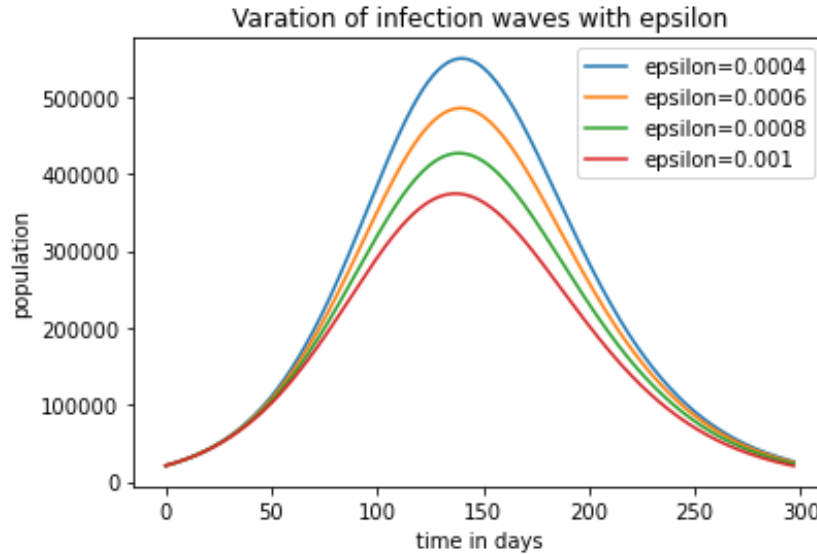


Figure 5 - This graph shows how altering the vaccination roll-out rate to higher values than the current rate of 0.0002, significantly decreases the number of infectives overall. For $\epsilon = 0.001$, the peak number of infectives is 375,000 at approximately day 135. For $\epsilon = 0.0002$ (the current rate) the peak number of infectives is 620,000 and occurs around day 145. This is a 65% difference between the two peaks.

From figure 5, the threshold of 200 daily infection cases is required for the virus to be contained and prevent another wave from happening. For the current vaccine roll-out rate, this number will be reached in 465 days, while it takes 440 days for the $\epsilon = 0.001$ roll-out rate to reach the 200 daily infections level. Hence, vaccinating 0.1% of the Ontario population daily, shortens the pandemic by almost one month.

3) CONCLUSION

During the week of March 29, 2021, Ontario was experiencing what seemed to be its third wave of COVID-19. The sharp increases in new cases, of approximately 2000 to 2500 cases over the course of 5 days, was a point of concern for public health officials. Figure 5 highlights the slow roll-out of the COVID-19 vaccine by the provincial government. This, in addition to loosely placed lockdown rules have spiraled the province into a third wave of the spread of the pandemic. A final comment on the current situation, if no changes are made to the vaccination program, Ontario could very soon see a peak of over 600,000 overall cases from the approximately 408,000 overall cases presently. From figure 5, the difference in peaks between a roll-out rate of $\epsilon = 0.001$ and a roll-out rate of $\epsilon = 0.0002$ is 65% within a span of 10 days, which would overflow the ICUs and devastate the healthcare system.

When will the pandemic end? If the threshold of 200 daily infections is reached, then this will signify the beginning of the end. Although lockdowns can temporarily contain the virus, an effective vaccine distribution program would help end the pandemic end faster, cause less deaths, flatten the curve and reduce the stress on healthcare services.

4) APPENDIX

Code for the SIR model simulation

```
import numpy as np
import matplotlib.pyplot as plt
N=14734014 # total population for normalizing the graphs
b=8.534e-9 # this beta seems to work reasonably well
a=0.0869 # same reason as b
n=150000 # vaccinated individuals per day
dt=0.001 # time scale between euler steps
t=np.arange(0,730,dt) # time vector for plotting
I=np.zeros((len(t))) # empty vector for storing values
S=np.zeros((len(t))) # same as above
R=np.zeros((len(t))) # same as above
cumul_I=np.zeros((len(t)))
cumul_R=np.zeros((len(t)))
S[0]=14353330 # initial conditions from April 1st 2020
I[0]=20875 #
R[0]=359809 #
print(np.size(I))
print(np.size(S))
print(np.size(R))
for x in range(1,len(t)): # for loop doing euler method of solving the System
    S[x]=dt*(-b*S[x-1]*I[x-1]-0.95*0.0007*S[x-1])+S[x-1] # euler step solutions
    I[x]=dt*(b*S[x-1]*I[x-1]-a*I[x-1])+I[x-1]
    R[x]=dt*(a*I[x-1]+0.95*0.0007*S[x-1])+R[x-1]
    if S[x]<0: # checks to see if susceptible goes negative
        S=[0]
    if I[x]<0.5: # checks to see if no infected are left
        I[x]=0 # If there are then the simulation ends
        S=S[:x]
        I=I[:x]
        R=R[:x]
        t=t[:x]
        print('Pandemic has ended')
        print(x)
        break
plt.plot(t,S,'r',label='Susceptible') # all plots overlaid
plt.plot(t,I,'b',label='Infected')
plt.plot(t,R,'g',label='Removed')
plt.xlabel('time in days')
plt.ylabel('Population')
plt.title('SIR model for Onatrio starting from April 1st')
plt.legend()
plt.plot(S,I) #NOTE separate the blocks of code to produce graphs
plt.xlabel('S(t)')
plt.ylabel('I(t)')
plt.title('Phase plot for the SIR model')
```

5) REFERENCES

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