Networks and Random Processes

Problem sheet 2

Sheet counts $60/100$ homework marks, [x] indicates weight of the question.
Please put solutions in my pigeon hole or give them to me by **Friday, 28.10.2016, 2pm**.

2.1 Kingman's coalescent [11]

Consider a system of L well mixed, coalescing particles. Each of the $\binom{L}{2}$ pairs of particles coalesces independently with rate 1. This can be interpreted as generating an ancestral tree of L individuals in a population model, tracing back to a single common ancestor.

(a) Let N_t be the number of particles at time t with $N_0 = L$. Give the transition rates of the process $(N_t : t \geq 0)$ on the state space $\{1, \ldots, L\}$, write down the generator $(\mathcal{L}f)(n)$ for $n \in \{1, \ldots, L\}$ and the master equation.
Is the process ergodic? Does it have absorbing states? Give all stationary distributions.

(b) Show that the mean time to absorption is given by $\mathbb{E}(T) = 2\big(1 - \frac{1}{L}\big)$.

(c) Write the generator of the rescaled process N_t/L and Taylor expand up to second order. Show that the slowed down, rescaled process $X_t^L := \frac{1}{L} N_{t/L} \to X_t$ converges to the process $(X_t : t \geq 0)$ with generator

$$\bar{\mathcal{L}}f(x) = -\frac{x^2}{2}\, f'(x) \quad \text{and state space} \quad (0, 1] \quad \text{with} \quad X_0 = 1\ .$$

Convince yourself that this process is 'deterministic', i.e. $X_t = \mathbb{E}(X_t)$ for all $t \geq 0$, and compute X_t explicitly. How is your result compatible with the result from (b)?

(d) Generate sample paths of the process X_t^L for $L = 10, 100, 1000$ and compare to the solution X_t from (c) in a single plot.

2.2 Geometric Brownian motion [11]

Let $(X_t : t \geq 0)$ be a Brownian motion with constant drift on $\mathbb{R}$ with generator

$$(\mathcal{L}f)(x) = \mu\, f'(x) + \frac{1}{2}\sigma^2\, f''(x)\ , \quad \mu \in \mathbb{R}, \sigma^2 > 0\ ,$$

with initial condition $X_0 = 0$. **Geometric Brownian motion** is defined as

$$(Y_t : t \geq 0) \quad \text{with} \quad Y_t = e^{X_t}\ .$$

(a) Show that $(Y_t : t \geq 0)$ is a diffusion process on $[0, \infty)$ and compute its generator.

(b) Use the evolution equation of expectation values of test functions $f : \mathbb{R} \to \mathbb{R}$

$$\frac{d}{dt}\mathbb{E}\big[f(Y_t)\big] = \mathbb{E}\big[\mathcal{L}f(Y_t)\big]\ ,$$

to derive ODEs for the mean $m(t) := \mathbb{E}[Y_t]$ and the variance $v(t) := \mathbb{E}[Y_t^2] - m(t)^2$.
Is $(Y_t : t \geq 0)$ a Gaussian process?

(c) Under which conditions on μ and σ^2 is $(Y_t : t \geq 0)$ a martingale? Justify your answer. Under which conditions on μ and σ^2 does the process have a stationary distribution, and what is it? Does it converge to the stationary distribution with $Y_0 = 1$?

(d) For $\sigma^2 = 1$ choose $\mu = 1/2$ and two other values $\mu < 1/2$ and $\mu > 1/2$. Simulate and plot a sample path of the process with $Y_0 = 1$ up to time $t = 10$, by numerically integrating the corresponding SDE with time steps $\Delta t = 0.1$ and 0.01.

2.3 Moran model and Wright-Fisher diffusion [14]

Consider a fixed population of L individuals. At time $t = 0$ each individuum i has a different type $X_0(i)$, for simplicity we simply put $X_0(i) = i$. In continuous time, each individuum independently, with rate 1, imposes its type on another randomly chosen individuum (or equivalently, kills it and puts its own offspring in its place).

(a) Give the state space of the Markov chain $(X_t : t \geq 0)$. Is it irreducible?
What are the stationary distributions?

(b) Let $N_t = \sum_{i=1}^{L} \delta_{X_t(i),k}$ be the number of individuals of a given type $k \in \{1, \ldots, L\}$ at time t, with $N_0 = 1$.
Is $(N_t : t \geq 0)$ a Markov process? Give the state space and the generator.
Is the process irreducible? What are the stationary distributions?
What is the limiting distribution as $t \to \infty$ for the initial condition $N_0 = 1$?

(c) From now on consider general initial conditions $N_0 = n \in \{0, \ldots L\}$.
Show that N_t is a martingale.
Compute $m_2(t) := \mathbb{E}[N_t^2]$. What happens in the limit $t \to \infty$?
Use this to compute the absorption probabilities as a function of the initial condition n, and to deduce how the absorption time scales with the system size L.

(d) Consider the rescaled process $M_t^L := \frac{1}{L} N_{tL^\alpha}$ on the state space $[0, 1]$.
For which value of $\alpha > 0$ does M_t^L have a (non-trivial) scaling limit $(M_t : t \geq 0)$? Compute the generator of this process, write down the Fokker-Planck equation and show that it is a martingale. (The scaling limit is called **Wright-Fisher diffusion**).

(e) For the limit process $(M_t : t \geq 0)$ in (d) compute $m(t) := \mathbb{E}[M_t]$ and $v(t) := \mathbb{E}[M_t^2] - m(t)^2$. Is it a Gaussian process?

(f) Simulate the **original process** $(X_t : t \geq 0)$ and plot N_t for each type as a function of time in a single plot. Reasonable parameter values are to be specified in class.