1 Analytical solution of the score-based Bayesian rating approach

1.1 General

Our starting point is a game with the outcome Δs. Involved are n teams and m players.

The general idea is to find the probability for the skills:

$$p(\mathbf{l}) = \prod_i p(l_i)\,, \tag{1}$$

which are assumed to be independent.

The probability distribution $p((l_i; \mu_i, \sigma_i)$ is estimated from the game outcomes. A game outcome is measured in the score difference Δs = score(team1) - score(team2). After every game, the conditional probability

$$p(\mathbf{l}|\Delta s) \tag{2}$$

is used as the new estimate for the skills.

Using the Bayes theorem, we can write

$$p(\mathbf{l}|\Delta s) \propto p(\Delta s|\mathbf{l})p(\mathbf{l}) \tag{3}$$

This equation can be alternatively stated as a Hidden Markov model using following considerations

- the player performance p_i is uniquely determined by the player's skill l_i
- the individual players performances determine the overall team performance t_i
- the differences in the team performances are described as the performance differences d
- the performance difference determines the game outcome, i.e. the score difference Δs

Thus, we can look at processes

$$p(\Delta s|d)p(d|\mathbf{t})p(\mathbf{t}|\mathbf{p})p(\mathbf{p}|\mathbf{l})p(\mathbf{l}) \tag{4}$$

Using such intermediate processes, one can describe $p(\mathbf{l}|\Delta s)$ by integrating out the intermediate variables d, $\mathbf{t}$ and $\mathbf{p}$. Thus, we get

$$p(\mathbf{l}|\Delta s) = \int_{-\infty}^{\infty} dd \int d^n t \int d^m p\ p(\Delta s|d)p(d|\mathbf{t})p(\mathbf{t}|\mathbf{p})p(\mathbf{p}|\mathbf{l})p(\mathbf{l}). \tag{5}$$

1.2 Probabilities

The skills l_i are assumed to be independent and Gaussian distributed, i.e.

$$p(\mathbf{l}) = \prod_i \mathcal{N}(l_i; \mu_i, \sigma_i) \tag{6}$$

The game outcome Δs is assumed to arise from the differences in the team performances

$$p(\Delta s|d) = \mathcal{N}(s; d, \gamma) \tag{7}$$

where γ is the variance of the game outcome (should easily be obtained from existing game data).

In a smilier way, the probability $p(\Delta s|d)$ can be described as

$$p(\Delta s|d) = \mathcal{N}(s; d, \beta). \tag{8}$$

1.3 Teams and team performances

Skills can be additive or non-additive. Additive skills are seen in games like Halo or soccer. In such games, more players in a team lead to a higher overall skill.

Quite contrary is the case in games like several card games. Here, more players does not mean more skill in a team. A good example is the German game Skat. Here, three players compete and, interestingly, the player who plays alone has a higher winning probability.

As will be seen below, the additive and non-additive skills will be described slightly different.

The probabilities $p(d|\mathbf{t})$ and $p(\mathbf{t}|\mathbf{p})$ depend on the time size and I will give examples how to describe them.

Lets focus on the case of two teams t_1 and t_2. Then the difference in team performances is simply

$$p(d|\mathbf{t}) = \delta[d - (t_1 - t_2)] \tag{9}$$

where $\delta(..)$ is the Dirac-delta function.

How to obtain the team performances will be illustrated on an example:

Example: 1 vs 3

Team 1 is comprised of player i. Team 2 is comprised of players j, k and l. Hence, we describe the probability in the case of additive skills:

$$p(\mathbf{t}|\mathbf{p}) = \delta(t_1 - p_i)\delta[t_2 - (t_j + t_k + t_l)]. \tag{10}$$

In the case of non-additive skills, this writes

$$p(\mathbf{t}|\mathbf{p}) = \delta(t_1 - p_i)\delta(t_2 - (t_j + t_k + t_l)/3). \tag{11}$$

Since all functions are either assumed as Gaussian or Dirace delta functions, one can now explicitly solve the integrals and obtain $p(\mathbf{l}|\Delta s)$. To obtain the individual update $p(l_i|\Delta s)$, it remains to integrate out all other skill variables l_j, $j \neq i$.

1.4 Analytical solution

A general solution can be found in the case of team 1 vs team 2. To define such a solution in a general way, we need the following parameters:

- n_{rmpl} is number of players in own team
- n_{pop} is number of players in opponents team
- $n = n_{rmpl} + n_{\text{pop}}$ is the overall number of players participating in the game
- $n_{\max} = \max(n_{rmpl}, n_{rmpop})$ is the number of players in the largest team (analogously $n_{\min}$.
- $\Delta n = n_{\max} - n_{\min}$ is the difference between the team sizes.
- σ and μ are the standard deviation and mean, respectively, of the player's skill distribution.
- σ_{pl}^2 and μ_{pl} are the sum of the teams variances and mean values, respectively
- σ_{opp}^2 and μ_{opp} are the sum of the opponent teams variances and mean values, respectively
- again, Δs = score(player) - score(opponent)

The update are here presented as precision $\pi = 1/\sigma^2$ and precision adjusted mean $\tau = \mu/\sigma^2$.

In the case of additive skills, the update is given as

$$\pi_i^{\text{new}} = \frac{1}{\sigma^2} + \frac{1}{n\beta^2 + \gamma^2 + \sigma_{\text{opp}}^2 + \sigma_{\text{pl}}^2 - \sigma^2} \tag{12}$$

$$\tau_i^{\text{new}} = \frac{\mu_i}{\sigma^2} + \frac{\Delta s - \mu_{\text{pl}} + \mu + \mu_{\text{opp}}}{n\beta^2 + \gamma^2 + \sigma_{\text{opp}}^2 + \sigma_{\text{pl}}^2 - \sigma^2} \tag{13}$$

In the case of non-additive skills we obtain

$$\pi_i^{\text{new}} = \frac{1}{\sigma^2} + \frac{1}{n_{\text{pl}}} \frac{1}{\frac{2+\Delta n}{n_{\max}}\beta^2 + \gamma^2 + \frac{\sigma_{\text{opp}}^2}{n_{\text{opp}}} + \frac{\sigma_{\text{pl}}^2 - \sigma^2}{n_{\text{pl}}}} \tag{14}$$

$$\tau_i^{\text{new}} = \frac{1}{\sigma^2} + \frac{1}{n_{\text{pl}}^2} \frac{\Delta s - (\mu_{\text{pl}} - \mu)/n_{\text{pl}} + \mu_{\text{opp}}/n_{\text{opp}}}{\frac{2+\Delta n}{n_{\max}}\beta^2 + \gamma^2 + \frac{\sigma_{\text{opp}}^2}{n_{\text{opp}}} + \frac{\sigma_{\text{pl}}^2 - \sigma^2}{n_{\text{pl}}}} \tag{15}$$