# PH30110 – 2022

*Coursework 1 – 24270*

## Q1a.

The second order ODE given in the question can be split into it's $\vec{x}$ and $\vec{y}$ components and further split by defining new quantities $\vec{u}$ and $\vec{v}$ equal to the time derivative of $\vec{x}$ and $\vec{y}$ – i.e., velocities in the x and y direction (equations *1* through *4* for reference).

$$\ddot{x} = -\left(\frac{GM}{x^3}\right)\vec{x} \qquad \textit{Eq. 1}$$

$$\ddot{y} = -\left(\frac{GM}{y^3}\right)\vec{y} \qquad \textit{Eq. 2}$$

$$\dot{x} = \vec{u} \quad , \quad \dot{u} = -\left(\frac{GM}{x^3}\right)\vec{x} \qquad \textit{Eq. 3}$$

$$\dot{y} = \vec{v} \quad , \quad \dot{v} = -\left(\frac{GM}{y^3}\right)\vec{y} \qquad \textit{Eq. 4}$$

The original ODE is now four simultaneous 1st order ODEs and can be solved using a basic fourth order Runge-Kutte function (RK4). As can be seen in Q1a.py, equations for $\frac{du}{dt}$ and $\frac{dv}{dt}$ are defined first, then called in the RK4 function to solve the four ODEs in question.

Taking the known orbital period of the comet[1], a one-dimensional array of t values was created set 1 day apart over 76 years. The ODEs are run through the RK4 function at each interval using a for loop, starting at aphelion (which in the chosen reference frame falls at $y = 0$, so that $v_x = 0$ and $v_y = 880\ m/s$), and the position/velocity vectors are calculated and inserted into empty arrays. The position values can be plotted against each other to produce a phase space diagram of the orbit – see figure 1.

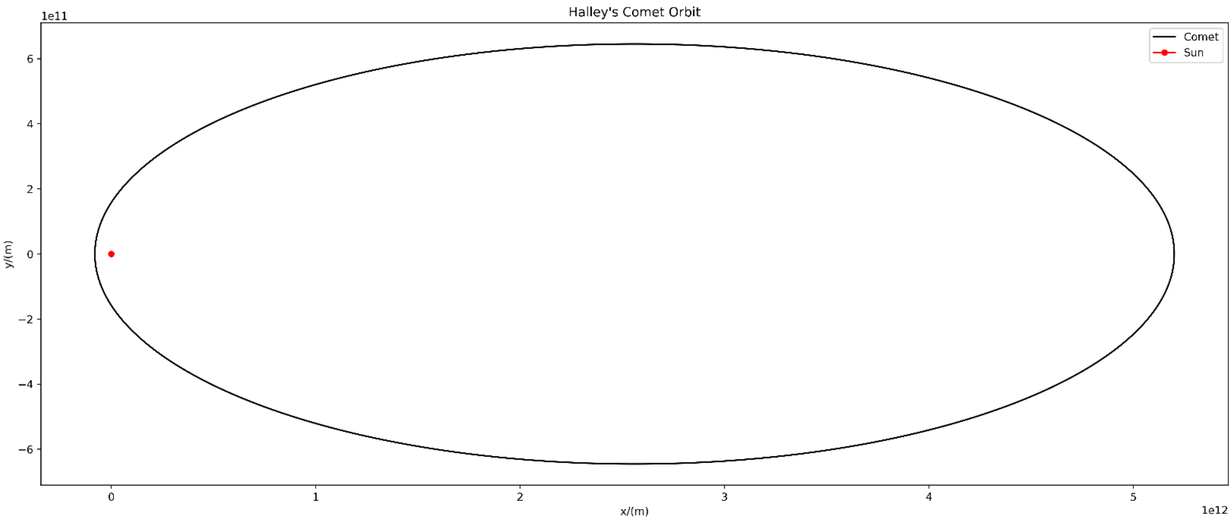


***Figure 1:*** *A phase space diagram of the orbit of Halley's comet, output of the Q1a.py code.*

Through analysis of this phase plot, one can determine the perihelion to be ≈ $8.0076\times10^{10}$ m. Comparing this value to the known average perihelion distance[2], a percentage error of 9.26% is observed.

## Q1b.

Upon further analysis of the values returned by the RK4 function, the initial timestep of 1 day fails to keep up with the speed of the comet about the perihelion, visualised by the markers in figure 2.

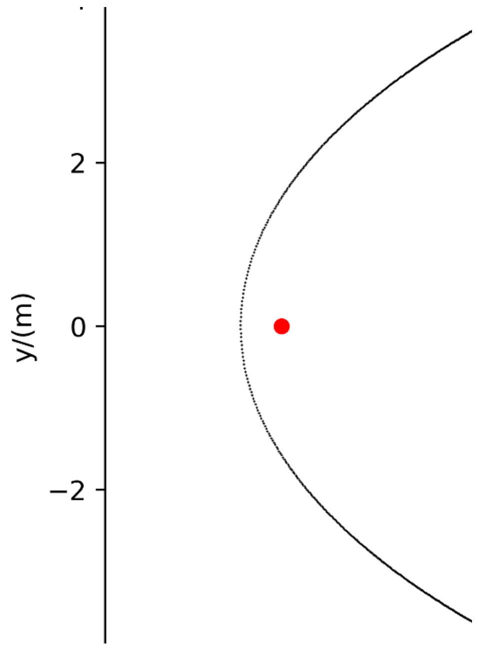


***Figure 2:*** *Output using same code as Q1a, with small markers in place of solid line to highlight the inaccuracy at comet perihelion.*

In order to amend this, whilst also streamlining the programme in its entirety, an adaptive timestep size was introduced.

$$\varepsilon = \frac{1}{30}|k_1 - k_2| \qquad \textit{Eq. 5}$$

To quantify the error in RK4 estimates about the perihelion, equation 5 was implemented into each RK step and ε values plotted against time (figure 3, left).

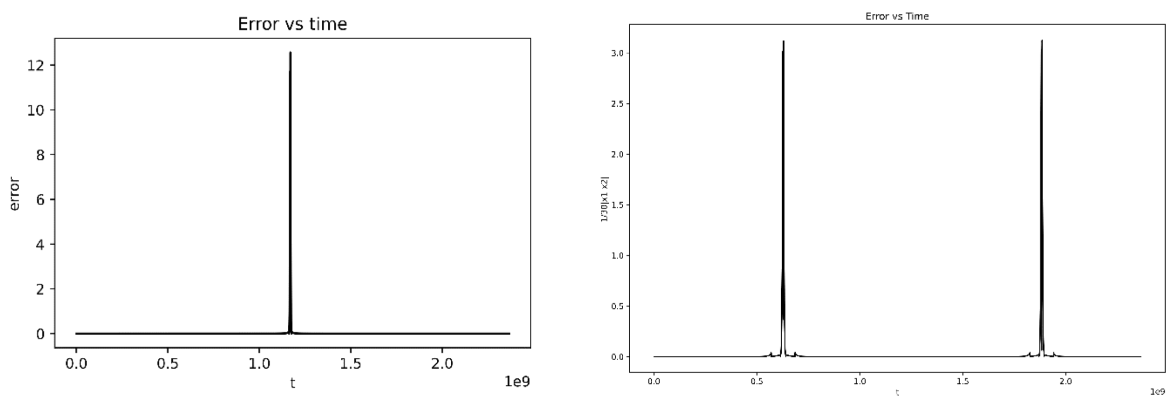

***Figure 3:*** *Error values at and around perihelion before(left) and after(right) adaptive step size was implemented.*

An if statement was inserted into the original for loop, stating that the new position/velocity values must be recalculated with the timestep halved if $\varepsilon > 0.01$. Similarly, to reduce unnecessary calculations in sections of the orbit where the comet moves relatively slowly, a second if statement was written in, doubling the timestep if $\varepsilon < 0.001$.

The results of these improvements are displayed in figure 3 (left). The adaptive timestep cut the error value at the perihelion by a factor of 3 and the programme was able to compute almost 2 full orbits using the same initial t array size. The effect is immediately recognisable looking at distance between markers about the perihelion when comparing figures 2 and 4.

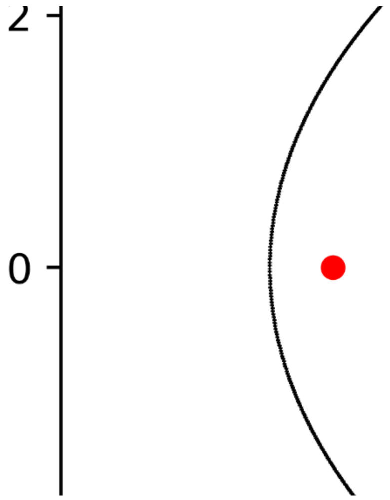


***Figure 4:*** *Output with implemented adaptive step size using same marker/marker size parameters as figure 2.*

## Q2a.

In this section, M is defined as 1 solar mass M⊙.

Setup of a 3-body gravitational system with one body fixed and of mass M requires calculation of initial orbital conditions. Given values for the semi-major axes and the mass ratios of the two outer bodies:

$$a_1 = 2.52AU \quad , \quad a_2 = 5.24AU$$
$$m_1 = 0.001M \quad , \quad m_2 = 0.04M$$

The period $P_i$ of the outer bodies can be found by utilising Kepler III:

$$P_i = 2\pi\sqrt{\frac{a_i^3}{GM}} \qquad \textit{Eq. 6}$$

And assuming circular orbits, one can make use of the equation for uniform circular motion to calculate initial velocities:

$$v_i = \frac{2\pi a_i}{P_i} \qquad \textit{Eq. 7}$$

The main function in Q2a.py packs these initial values into a one-dimensional array, which is then unpacked in the RK4 function. The RK4 function itself was designed slightly differently to accommodate the gravitation equation functions for a 3-body problem where ($m_1$, $m_2$ << M) (eq. 8 & 9).

$$\ddot{r}_1 = -\frac{GM}{r_1^3}\overrightarrow{r_1} + \frac{Gm_2}{r_{21}^3}\overrightarrow{r_{21}} \qquad \textit{Eq. 8}$$

$$\ddot{r}_2 = -\frac{GM}{r_2^3}\overrightarrow{r_2} + \frac{Gm_1}{r_{21}^3}\overrightarrow{r_{21}} \qquad \textit{Eq. 9}$$

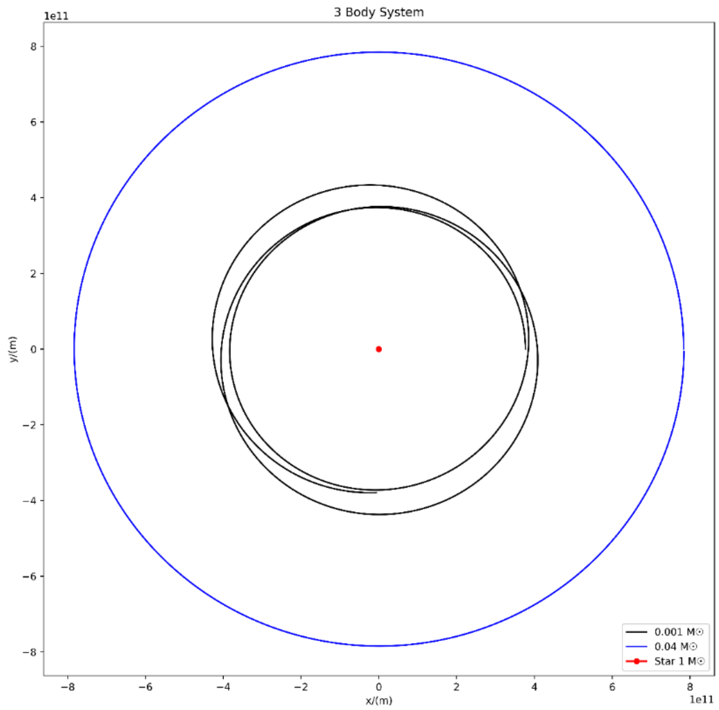


**Figure 5:** Output of code Q2a. The trajectories are pictured over 1 orbital period of the more massive body.

Plotting the phase space trajectories of the 2 smaller masses produces figure 5. The force of $m_2$ on $m_1$ causes $m_1$ to deviate periodically from its circular orbit by small but observable amounts, visible in figure 6.

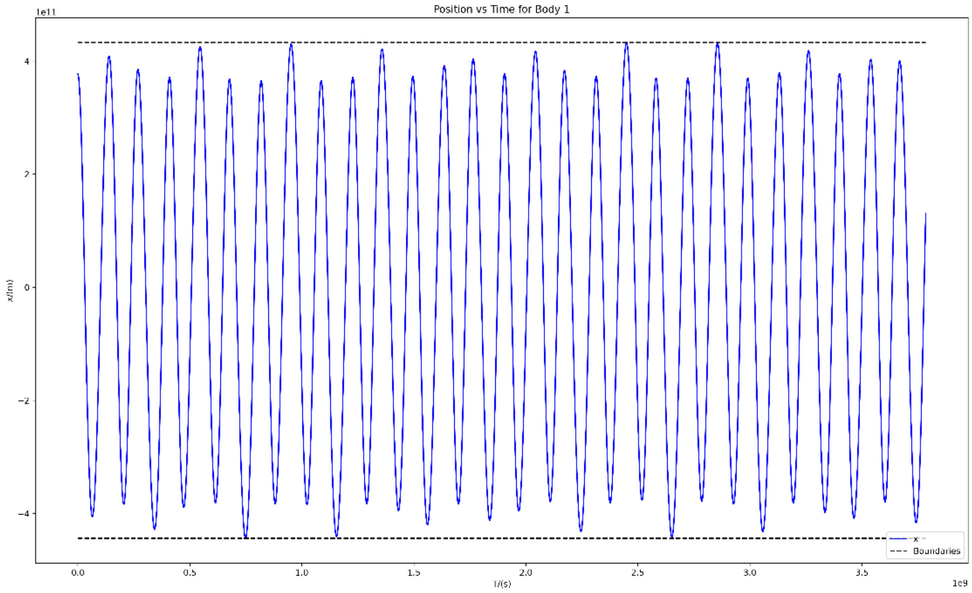


***Figure 6:*** *x value vs time for Body 1 over 10 orbital periods of Body 2*

The gravitational effect of body $m_2$ on body $m_1$ is too small to see in figure 5, but it is there – see figure 7.

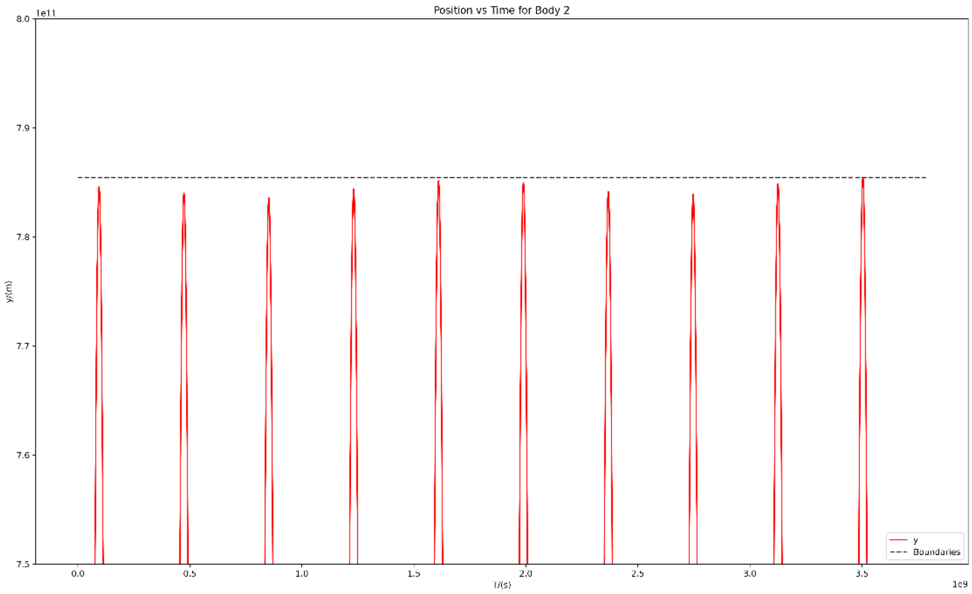


***Figure 7:*** *y position against time for Body 2, between the y limits of 7.5e11 and 8e11.*

## Q2b.

Given that:

$$\vec{r}_{cm}(t) = \sum_i \frac{m_i}{M_{total}} \vec{r}_i(t) \qquad \textit{Eq. 10}$$

the code used in Q2a.py can be adapted to a true Jupiter, Saturn, Sun 3-body problem with all three masses orbiting around a common centre of mass (CoM) which is itself following a trajectory.

Firstly, the initial parameters of the outer masses were changed to match those of Jupiter and Saturn, accounting for the eccentricity of their orbits in the initial speed $v_{aphelion}$ calculation.

To plot the trajectory of the CoM a separate function Rcm that uses the sum in eq. 10 to calculate the position vectors $\overrightarrow{x_{cm}}(t)$ and $\overrightarrow{y_{cm}}(t)$ for every RK4 step. These values are added to another empty array and plotted as a phase space diagram. The output of Q2b.py can be seen in figures 8 and 9.

In figure 9 it is seen the barycentre's initial position lies $\sim 1.15 \times 10^9 m$ outside the centre of the Sun, which is also its maximum distance from the Sun's centre. This makes sense physically as the initial positions of Saturn and Jupiter are aligned along the $x$ axis, so the maximum force pulling the barycentre away from the Sun will be at $t = 0$. This also means the barycentre's trajectory is periodic, completing one period every time Jupiter and Saturn align along the $x$ axis.

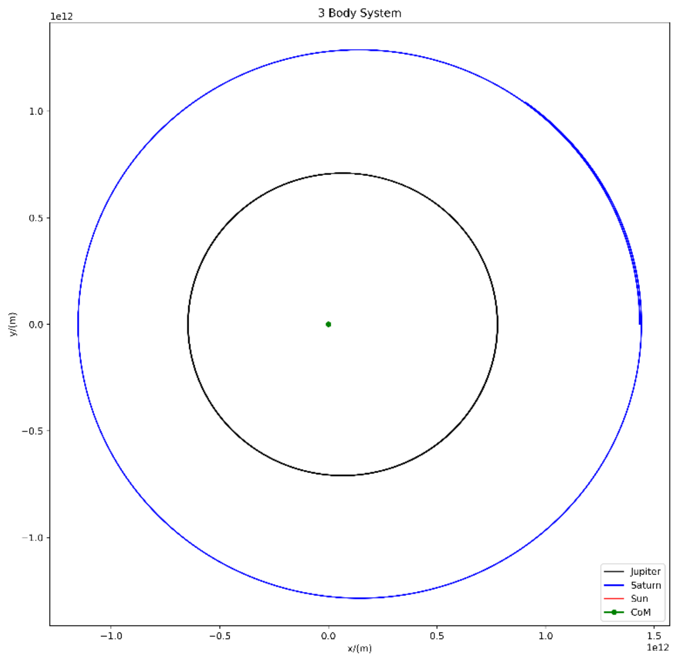


***Figure 8:*** *Phase space diagram of the orbital trajectories of Saturn, Jupiter, and the Sun around a common centre of mass over one period of Saturn.*

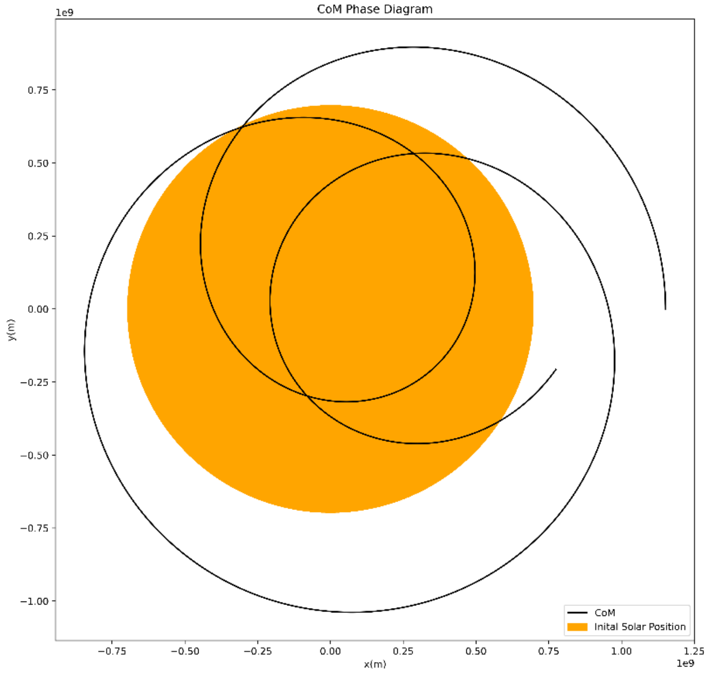


***Figure 9:*** *Zoomed in view of initial sun position and centre of mass fluctuation over one period of Saturn.*

**References.**


[1] - NASA Solar System Exploration. 2022. *In Depth | 1P/Halley – NASA Solar System Exploration*. [online] Available at: https://solarsystem.nasa.gov/asteroids-comets-and-meteors/comets/1p-halley/in-depth/#:~:text=Halley's%20orbit%20period%20is%2C%20on,gravitational%20effects%20of%20the%20planets.

[2] - Tugarinov, D., 2009. *Comet Halley*. [online] Ucl.ac.uk. Available at: https://www.ucl.ac.uk/~zcape78/CometHalley.html