Numerically Modelling the Trajectory of a Golf Ball

Investigating the effect of the elevation at which a golf ball is hit

Peter Baldry - COSC2500 Project – Semester 2, 2017

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Introduction

Numerical methods play an important role in many different disciplines. In terms of sport, the most beneficial numerical methods involve solving differential equations. The motivation for this project comes from Golf and a specific topic that is relevant at the current time. Many golf professionals on the PGA Tour (Professional Golfers Association Tour) are unable to comprehend, or put a number to, the effect of hitting a ball from greater elevations on their ball flight. The PGA Tour has held golf tournaments in Mexico City, which is renowned for its elevation above sea level. The professionals measure how far they hit the ball at home and often find that they hit the ball further in Mexico City. In this project, a mathematical model for the trajectory of a golf ball in $\mathbb{R}^3$ will be formulated and this model will be solved using various numerical methods. The software MATLAB will be used to implement the methods and investigate the effect of elevation on golf ball trajectory. The numerical solutions will be verified against data from one of the leading commercial indoor and outdoor golf simulators for ball flight tracking and analysis, TrackMan.

Theory

There are two sections of theory which will govern the mathematical models used to describe the trajectory of a golf ball. The first section involves calculating the initial velocity of the golf ball. The second section will deal with the forces acting on the ball during flight.

Section 1: Initial velocity of the golf ball

To calculate the initial velocity of the ball, a combination of the equations of the conservation of kinetic energy and the conservation of momentum will be used. The conservation of kinetic energy in the process of hitting a golf ball is given by (1) in the equations below. The conservation of momentum in the process of hitting a golf ball is given by (2) in the equations below. (Arnold, n.d.):

$$\frac{1}{2}MV_0^2 = \frac{1}{2}MV^2 + \frac{1}{2}mv^2 \qquad (1)$$

$$MV_0 = MV + mv \qquad (2)$$

where,

$M = mass\ of\ the\ golf\ club$ (kg)
$m = mass\ of\ the\ golf\ ball$ (kg)
$V_0 = velocity\ of\ golf\ club\ prior\ to\ hitting\ ball$ (m/s)
$V = velocity\ of\ golf\ club\ after\ to\ hitting\ ball$ (m/s)
$v = velocity\ of\ golf\ ball$ (m/s)

Equations (1) and (2) can be rearranged and solved as (3):

$$v = \frac{2V}{1 + \frac{m}{M}} \qquad (3)$$

However, not all the kinetic energy is conserved due to factors such as heat loss (Arnold, n.d.). Furthermore, that a coefficient of restitution can be included in the equation to deal with this in equation (4), the complete equation which approximately describes the initial velocity of the ball after impact (Arnold, n.d.):

$$v = \frac{(1 + C_R)V}{1 + \frac{m}{M}} \qquad (4)$$

where,

$M = mass\ of\ the\ golf\ club$ (kg)
$m = mass\ of\ the\ golf\ ball$ (kg)
$V_0 = velocity\ of\ golf\ club\ prior\ to\ hitting\ ball$ (m/s)
$V = velocity\ of\ golf\ club\ after\ to\ hitting\ ball$ (m/s)
$v = velocity\ of\ golf\ ball$ (m/s)
$C_R = coefficient\ of\ restitution$

The coefficient of restitution is approximately 0.83 for a golf shot with a driver (Golfclub-technology.com, 2017). It is important to note that equation (4) is an approximation for the initial velocity of the ball. It disregards several factors such as the quality of the strike (how close to the middle of the club the ball is hit), the path of the strike and factors such as the brand of golf club, which can differ depending on the weight distribution in the head of the club.

Section 2: The forces acting on the golf ball in flight

Once the ball has been hit, the ball is subject to various forces in the x, y and z directions. These forces will be covered as comprehensively as possible to accurately model the trajectory of the ball. The forces acting on the ball in $\mathbb{R}^2$ can be visually denoted in the Figure 1:

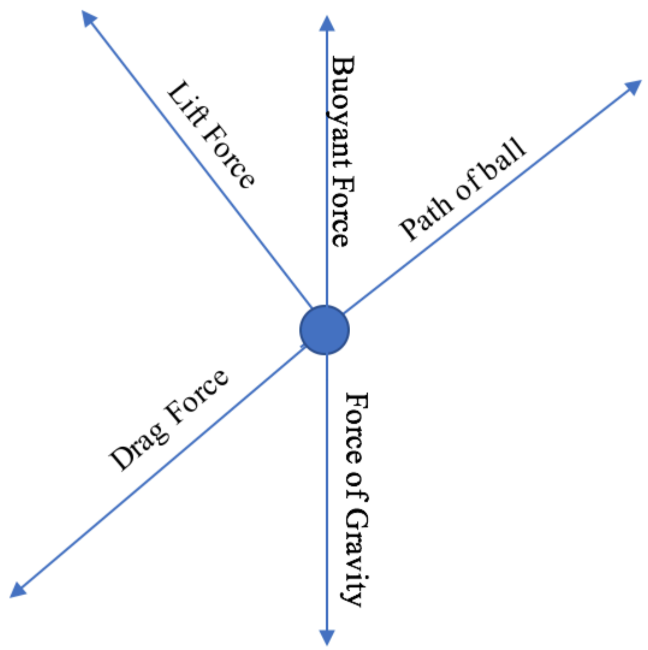

Figure 1: The forces acting on a golf ball in flight ($\mathbb{R}^2$)

Force due to gravity:

The force due to gravity acts in a vertical direction and follows the negative y axis. Its magnitude given by the following equation, based on Newton's second law:

$$F_g = mg$$

where,

$$m = mass\ of\ golf\ ball\ (kg)$$
$$g = acceleration\ due\ to\ gravity\ (9.81ms^{-2})$$

Drag force:

The drag force acting upon the golf ball is directly opposing the motion (the path) of the ball (Figure 1). The drag force is dependent on several factors, such as the cross-sectional area of the ball, the instantaneous velocity of the ball, a drag coefficient and air density. The force of drag on a golf ball is given by the following equation (Cross, n.d.):

$$F_D = \frac{1}{2}C_D p v^2 A$$

where,

$$C_D = drag\ coefficient\ specific\ for\ a\ golf\ ball$$
$$p = air\ density\ (kgm^{-3})$$
$$v = instantaneous\ velocity\ (ms^{-1})$$
$$A = cross - sectional\ area\ of\ the\ ball\ (m^2)$$

Several experiments conducted in a joint investigation by the USGA (United States Golf Association) and the R&A (Royal and Ancient Golf Club of St Andrews) found that the following equation for the coefficient of drag fit the experimental data accurately ($R^2 = 0.94$). Source: Second Report on the Study of Spin Generation, 2007:

$$C_D = 0.1403 - 0.3406 * R * \log(R) + 0.3747 * R^{1.5}$$

where,

$$R = spin\ ratio$$

Lift force:

The lift force acting on the golf ball acts in a direction orthogonal to the flight path of the ball and acts in a positive y direction (Figure 1). The lift force, like the drag force, is dependent on the air density, instantaneous velocity and the cross-sectional area of the ball, as well as a coefficient of lift. The force of acting on the golf ball is given as such (Cross, n.d.):

$$F_L = \frac{1}{2}C_L p v^2 A$$

where,

$$C_L = lift\ coefficient\ specific\ for\ a\ golf\ ball$$
$$p = air\ density\ (kgm^{-3})$$
$$v = instantaneous\ velocity\ (ms^{-1})$$
$$A = cross - sectional\ area\ of\ the\ ball\ (m^2)$$

Several experiments conducted in a joint investigation by the USGA (United States Golf Association) and the R&A (Royal and Ancient Golf Club of St Andrews) found that the following equation for the coefficient of lift fit the experimental data accurately ($R^2 = 0.92$). Source: Second Report on the Study of Spin Generation, 2007:

$$C_L = 0.3996 + 0.1583 * \log(R) + 0.03790 * R^{-0.5}$$

where,

$$R = spin\ ratio$$

Buoyant force:

The buoyant force acts in a direction opposite to the force of gravity (along the positive y axis) and is highly dependent on the density of the air. The equation for the buoyant force is as follows (Theory.uwinnipeg.ca, 2017):

$$F_B = pgV$$

where,

$p = air\ density\ (kgm^{-3})$
$V = volume\ of\ the\ ball\ (m^3)$
$g = acceleration\ due\ to\ gravity\ (9.81ms^{-2})$

The buoyant force is often disregarded during flight because the air density is dependent on the elevation, and the change of elevation throughout the trajectory of the golf ball is so little that the air density won't change by any significant magnitude. The force can have a significant effect when comparing the initial elevation of the shot (ie. Sea Level vs Mexico City), as elevation can differ by kilometres in magnitude.

Development of Mathematical Models

The mathematical models which describe the trajectory of the golf ball are based on Newton's Second Law:

$$F = ma$$

Applying this in the y direction, where acceleration is given by the second derivative of y:

$$F_y = m\frac{d^2y}{dt^2} \qquad (5)$$

The left-hand side of (5) can be expressed as the sum of forces acting in the y direction:

$$F_y = \sum Forces\ acting\ in\ y\ direction$$

$$F_y = F_{B(y)} + F_{L(y)} + F_{D(y)} + F_{G(y)}$$

$$F_y = pgV + \frac{1}{2}C_L pA(v)^2 cos(\theta) - \frac{1}{2}C_D pA(v)^2 sin(\theta) - mg$$

$$F_y = pgV + \frac{1}{2}C_L pA(v)^2 \frac{\frac{dx}{dt}}{v} - \frac{1}{2}C_D pA(v)^2 \frac{\frac{dy}{dt}}{v} - mg$$

$$F_y = pgV + \frac{1}{2}C_L pAv\frac{dx}{dt} - \frac{1}{2}C_D pAv\frac{dy}{dt} - mg \qquad (6)$$

Substituting (5) into (6):

$$m\frac{d^2y}{dt^2} = pgV + \frac{1}{2}\,C_L pAv\frac{dx}{dt} - \frac{1}{2}C_D pAv\frac{dy}{dt} - mg$$

$$\frac{d^2y}{dt^2} = \frac{pgV + \frac{1}{2}\,C_L pAv\frac{dx}{dt} - \frac{1}{2}C_D pAv\frac{dy}{dt} - mg}{m}$$

$$\frac{d^2y}{dt^2} = -\frac{pAv}{m}(-gV - \frac{1}{2}\,C_L\frac{dx}{dt} + \frac{1}{2}C_D\frac{dy}{dt}) - g \qquad (7)$$

Applying Newton's Second Law to the x direction yields:

$$F_x = m\frac{d^2x}{dt^2} \qquad (8)$$

The left-hand side of (8) can be expressed as the sum of forces acting in the x direction:

$$F_x = \sum Forces\ acting\ in\ x\ direction$$

$$F_x = F_{L(x)} + F_{D(x)}$$

$$F_x = -\frac{1}{2}\,C_L pA(v)^2 sin(\theta) - \frac{1}{2}C_D pA(v)^2 cos(\theta)$$

$$F_x = -\frac{1}{2}\,C_L pA(v)^2\frac{\frac{dy}{dt}}{v} - \frac{1}{2}C_D pA(v)^2\frac{\frac{dx}{dt}}{v}$$

$$F_x = -\frac{1}{2}\,C_L pAv\frac{dy}{dt} - \frac{1}{2}C_D pAv\frac{dx}{dt} \qquad (9)$$

Substituting (9) into (8):

$$m\frac{d^2x}{dt^2} = -\frac{1}{2}\,C_L pAv\frac{dy}{dt} - \frac{1}{2}C_D pAv\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{-\frac{1}{2}\,C_L pAv\frac{dy}{dt}}{m} - \frac{\frac{1}{2}C_D pAv\frac{dx}{dt}}{m}$$

$$\frac{d^2x}{dt^2} = -\frac{pAv}{2m}\left(C_L\frac{dy}{dt} + C_D\frac{dx}{dt}\right) \qquad (10)$$

Applying Newton's Second Law to the z direction yields:

$$F_z = m\frac{d^2z}{dt^2}$$

$$F_z = \sum Forces\ acting\ in\ z\ direction$$

Note: For the main model, cross wind and side spin will not be considered (so the acceleration is 0 in the z direction). The z component is only implemented for completeness of the 3-dimensional trajectory.

$$\frac{d^2z}{dt^2} = 0 \qquad (10)$$

Therefore, the following set of motion equations form the mathematical model for the trajectory of a golf ball.

$$\frac{d^2y}{dt^2} = -\frac{pAv}{m}\left(-gV - \frac{1}{2}\,C_L\frac{dx}{dt} + \frac{1}{2}C_D\frac{dy}{dt}\right) - g$$

$$\frac{d^2x}{dt^2} = -\frac{pAv}{2m}\left(C_L\frac{dy}{dt} + C_D\frac{dx}{dt}\right)$$

$$\frac{d^2z}{dt^2} = 0$$

Assumptions

The number of assumptions used was kept as limited as possible when considering the development of the mathematical model. However, several assumptions had to be upheld to keep the mathematical models simple enough.

i. The air density is considered constant throughout flight. Although the ball changes in altitude during flight, resulting in a change in air density, the magnitude of the greatest change would be very small.
ii. For the main model, wind from any direction is disregarded. Later in the report, crosswinds may be investigated.
iii. The effect of the dimples on the golf ball are accounted for by the coefficients of drag and lift.
iv. The acceleration due to gravity is constant, even though in reality it may be changing by very small amounts during the flight of the ball.
v. More in depth initial data such as swing path and attack angle were disregarded in the calculation of the initial velocity of the golf ball.
vi. The coefficient of restitution for the impact of the club on the ball was that of a driver, 0.83. This was assumed to be approximately equal for other clubs.

Numerical Solution

Finite difference method

The finite difference method is a numerical method commonly used to solve differential equations. It generates a set of discrete points which approximately describe the solution to the differential equation. To solve the set of motion equations above, the second order finite central difference equation must be derived. This can be done starting with the Taylor Series approximation:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2 f''(x)}{2} + \frac{h^3 f'''(x)}{6} + \frac{h^4 f''''(x)}{24} + \cdots \quad (11)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2 f''(x)}{2} - \frac{h^3 f'''(x)}{6} + \frac{h^4 f''''(x)}{24} - \cdots \quad (12)$$

Computing (11) + (12) gives:

$$f(x+h) + f(x-h) = 2f(x) + \frac{2h^2 f''(x)}{2} + \frac{2h^4 f''''(x)}{24} + \cdots$$

Which can be rearranged to:

$$f(x+h) = 2f(x) + h^2 f''(x) - f(x-h) + \frac{h^4 f''''(x)}{12} + \cdots$$

For sufficient values of h:

$$f(x+h) \approx 2f(x) + h^2 f''(x) - f(x-h)$$

This can be translated into an explicit formula:

$$f(x)_{i+1} \approx 2f(x)_i + h^2 f''(x) - f(x)_{i-1} \quad (13)$$

Equation (13) can then be used to calculate each successive point, i+1, over a step size h. The step size h must be chosen such that the error is minimized. There are two main sources of error in equation (13). If the step size h is too large, truncation error can become a problem. It would seem intuitive to make the step size h the smallest possible value to reduce truncation error. However, once the step size decreases to a certain size, decreasing it further will result in rounding error. The reason for this is that for very small h, the approximation will be subtracting almost identical numbers, resulting in a loss of significant digits (Sauer, 2017). The following equation gives an approximate h value which minimizes the total error (Sauer, 2017):

$$h \approx \varepsilon^{\frac{1}{n+1}} \quad (14)$$

where $\varepsilon = machine\ epsilon$
$n = order\ of\ approximation$

Evaluating equation (14) with n=2 gives approximately $h = 6.0555 \times 10^{-6}$.

MATLAB Code

The code was created as a MATLAB function:

```
% Golf Ball Trajectory
% Computes trajectory of ball
% Author: Peter Baldry
% Date: 2017
% Version: 1
% Input: p = air density (kg/m^3)
%        Vc = velocity of club at impact (km/hr),
%        rpm = initial spin on ball (rev/minute),
%        la = launch angle (degrees)
% Output: Plot of trajectory
% Example Usage:
% balltrajectory(1.225, 180, 2500, 10)

function sol = balltrajectory(p,Vc,rpm,la)
Vc = 0.277777777777778*Vc; %Velocity of club
launchangle = la *pi/180; %Launchangle
Mb = 0.040; %Mass Ball
Mc = 0.200; %Mass Club
Cr = 0.83; %Coefficient of restitution
A = 0.0013; %Cross Sectional Area of Ball
g = 9.81; %Acceleration due to gravity
V = 4.07*10^-5; %Volume of ball
d=0.0427; %Diameter of ball
omega = 2*pi*rpm/60; %Spin
h=6.0555e-06; %Time increments

%Initial velocity of ball
Vb = Mc*Vc*(1+Cr)/(Mb+Mc);

%Initial Velocity in x direction
v0x = Vb*cos(launchangle);


%Initial Velocity in y direction
v0y = Vb*sin(launchangle);

x1=[0, v0x*h]; %Initial x value
y1=[0, v0y*h]; %Initial y value
z1=[50, 50]; %Initial z value


%Implementation of finite difference method
while y1(end) > 0

        %Velocity in x direction
        vx = (x1(end) - x1(end-1))/h;
```

```
        %Velocity in y direction
        vy = (y1(end) - y1(end-1))/h;
        %Velocity
        v = sqrt(vx^2 +vy^2);

        %Spin Ratio
        R = (omega*d/2)/v;

        %Coefficient of Drag
        Cd = 0.1403 - 0.3406*R*log(R) + 0.3747*R^1.5;

        %Coefficient of Lift
        Cl = 0.3996 + 0.1583 * log(R) + 0.03790 * R^(-0.5);

        %Calculate x value for this step
        x = -p*A*v/(2*Mb) * (Cl*vy + Cd*vx)*h^2 + 2* x1(end) -
x1(end-1);
        x1(end+1) = x;

        %Calculate y value for this step
        y = (-p*A*v/(Mb) * (-g*V - 0.5*Cl*vx+0.5*Cd*vy) -
g)*h^2 + 2* y1(end) - y1(end-1);
        y1(end+1) =y;

        %Calculate z value for this step
        z = 50;
        z1(end+1) =z;

end
figure (1);
set(gcf, 'Units', 'Normalized', 'OuterPosition', [0 0 1 1]);
trajplot= plot3(x1,y1,z1,'LineWidth',2);
grid on;
xlabel('Distance (m)','FontSize',20)
ylabel('Height (m)','FontSize',20)
zlabel('Horizontal Deviation (m)','FontSize',20)
title('Trajectory of a golf ball','FontSize',20)
camproj('Perspective')
xt = get(gca, 'XTick');
set(gca, 'FontSize', 16)
yt = get(gca, 'YTick');
set(gca, 'FontSize', 16)
zt = get(gca, 'ZTick');
set(gca, 'FontSize', 16)
set(gca,'zdir','reverse')
set(gca,'xdir','reverse')
ylim([0 70])
xlim([0 (x1(end)+50)])
zlim([0 100])
view(180.1, -85.9)
end
```

Other numerical methods, justification of choice & discussion of code

The set of second order differential equations which describe the trajectory of the ball in flight are complex and therefore difficult to solve analytically. Combining numerical methods and computational techniques, the solution to complex sets of initial value problems can be found. There are several general methods of solving differential equations. Such methods include:

- Euler's method (similar to the finite difference method that was used)
- Further finite difference methods
- Runge-Kutta methods
- MATLAB's ode45 (Dormand-Prince method)

Methods such as the 4th order Runge-Kutta method and MATLAB's ode45 are the most accurate methods and therefore desired methods to solve initial value problems (Sauer, 2017). However, considering the complexity of the set of second order equations that were developed in this project, the simplest method that was deemed to be accurate enough was the central finite difference method. The main reason for this is that there are several variables in the acceleration equation which are changing. For example, the coefficients of lift and drag are changing dependent on the spin ratio which is changing depending on the velocity and the spin in revolutions per minute. Also, the velocity components are changing dependent on each other and the velocity in the direction of path of the ball and possibly the angle of elevation (which is changing), depending on how the model is set up.

By using the central finite difference method, only one calculation per step, per acceleration equation is required (so, the model consists of x,y,z components of acceleration = 3 calculations per step). Of course, the variables such as coefficients of lift, drag etc, are also calculated each step. Compare this to a more accurate method such as the 4th order Runge-Kutta method, where 5 calculations are made for each acceleration equation (so, the model consists of x,y,z components of acceleration = 3*5 = 15 calculations per step), plus all the variables that need to be updated. This totals to a more complex program consisting of many calculations per step. Using the 4th order Runge-Kutta method also requires each second order equation (ie x,y,z accelerations) to be broken up into first order equations. However, the central finite difference method does not require this break up, working with the second order equations by simply substituting the acceleration into the general derived formula.

The result of the decision to use the finite difference method is a fairly accurate solution (see the results and comparison to real data on the following pages) that uses code which is easily manipulate. For example, if a crosswind was to be added, the magnitude of the acceleration acting on the ball would be added in the z direction and the code would run without error. If the crosswind was to be added into another method, such as the Runge-Kutta method, many calculations in the code would have to be altered accordingly.

Overall, despite the range of higher order methods which can solve in a more accurate manner, the flexibility of the code when implementing the finite difference method and the sufficient accuracy of the method makes it a much more appropriate method to use.

MATLAB example output solution trajectory

An example solution trajectory was made using the example usage comment at the beginning of the code. This used some relatively standard initial data.

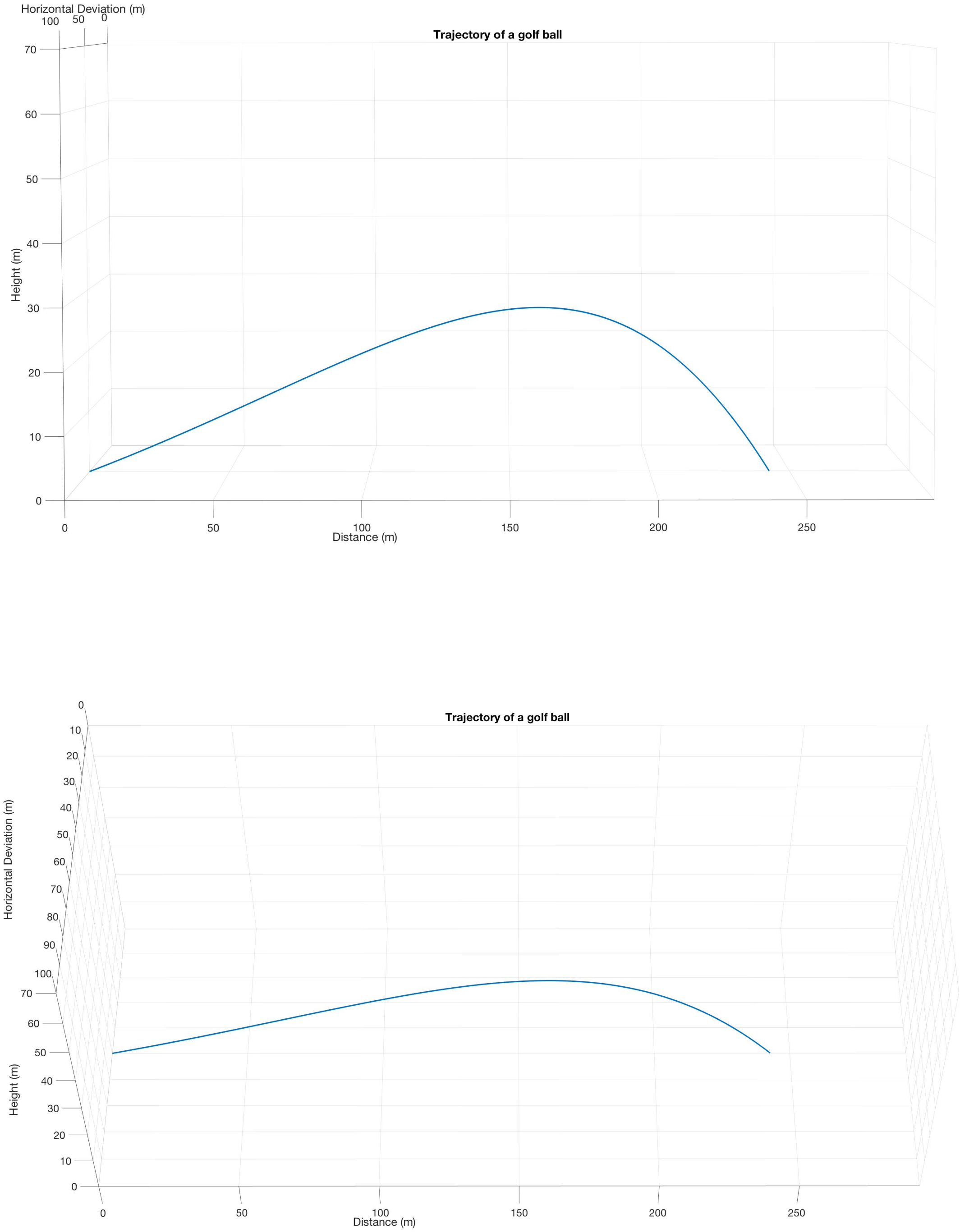

Figure 2: Example solution from different perspectives

Impact of spin on trajectory

Implementing some standard input values, but varying the spin:

>>balltrajectory(1.225, 180, 2500, 10)

>>balltrajectory(1.225, 180, 3000, 10)

>>balltrajectory(1.225, 180, 2000, 10)

When combining the three, it produced the following plot:

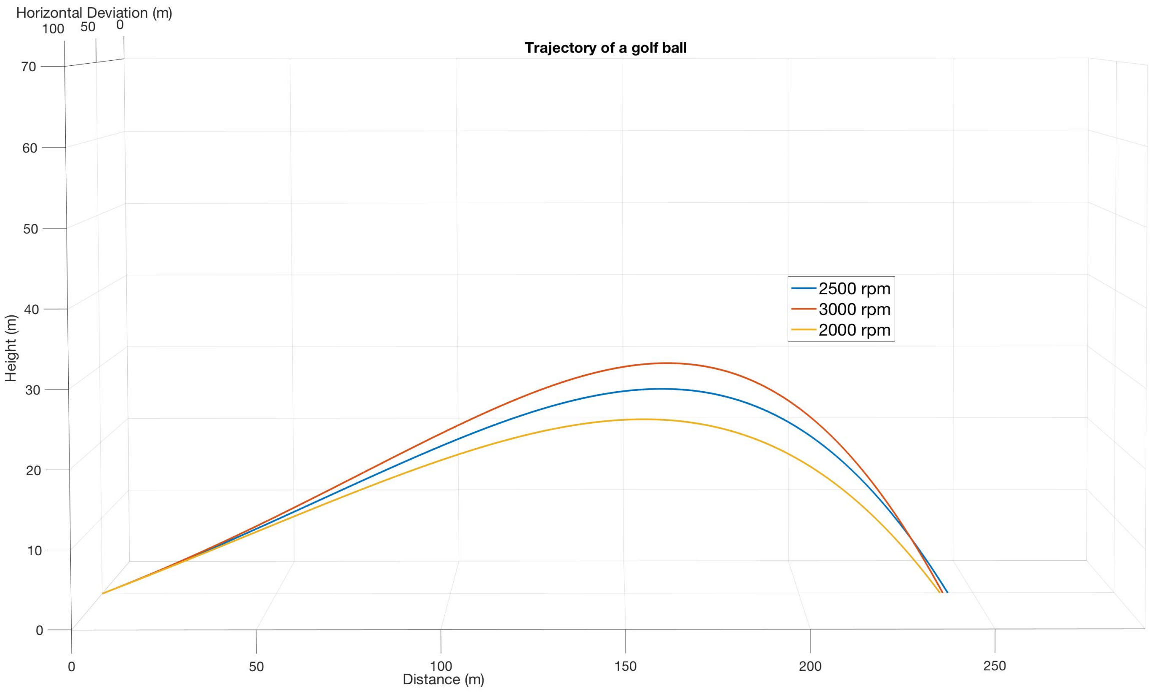

Figure 3: Effect of initial spin on trajectory

From Figure 3, increasing the spin rate at impact results in a higher trajectory in the model as it would realistically. Note that the total carry is approximately the same for all three spin rates. This emulates the prospect of having a driver which creates higher or lower ball spin rates. This is a problem many professional golfers face. Some believe that the lower the spin on the ball when hitting a driver, the further the ball will go. This may be partially true because the backspin when the ball lands will be reduced, causing the ball to potentially roll further. Another factor that golf professional's encounter is the dynamic structure of the PGA Tours schedule. Each tournament, a new course is used. Some courses may require high ball flights (ie. if the greens are hard, it would be better to have a higher ball flight and spin to stop the ball quicker). Some courses may require a lower ball flight (ie. particularly in Europe, where high winds are common, so it is better to have a lower, more piercing ball flight). The spin can be dependent on the type of driver used, so professionals could alter the driver depending on the course.

Impact of the velocity of the club on trajectory

Implementing some standard input values, but varying the velocity of the club:

```
>>balltrajectory(1.225, 180, 2500, 10)

>>balltrajectory(1.225, 160, 2500, 10)

>>balltrajectory(1.225, 200, 2500, 10)
```

When combining the three, it produced the following plot:

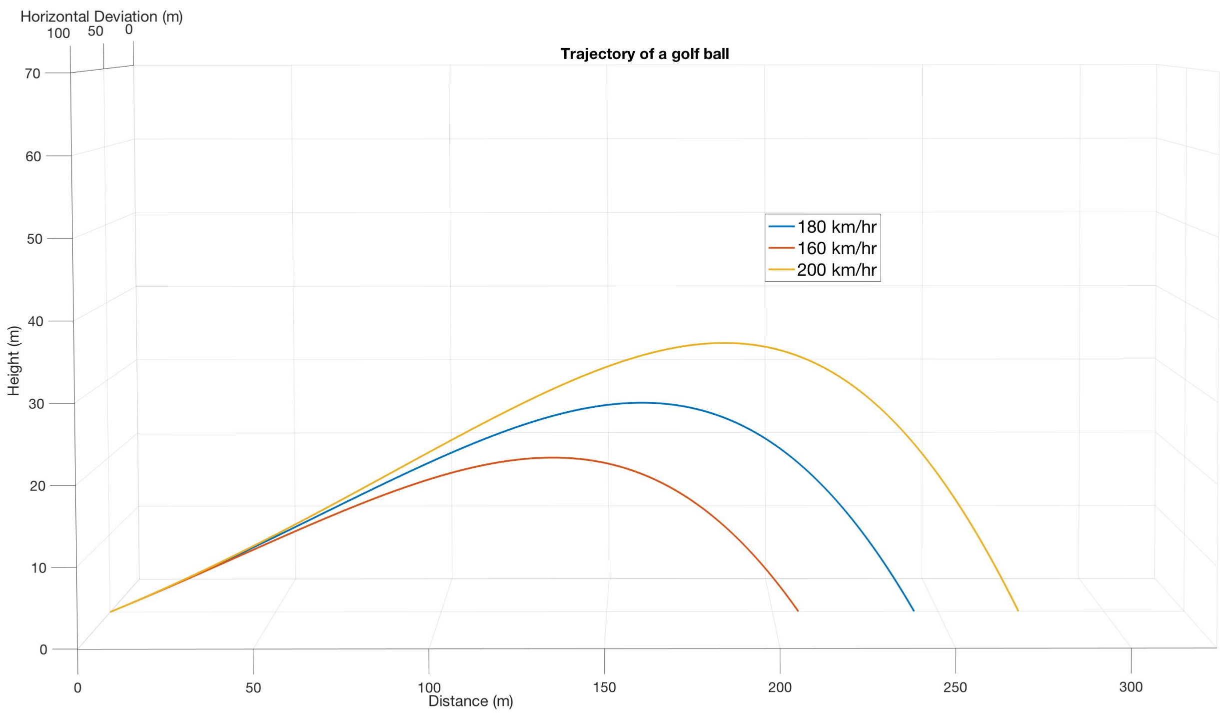

Figure 4: Effect of the velocity of the club on trajectory

From Figure 4, the importance of the velocity of the golf club on the apex and total carry of the ball trajectory is shown. The faster the initial velocity of the club, the greater the velocity of the ball and thus the higher and further the ball travels. Golf professionals are always aware that the initial velocity of the ball is crucial to maximising distance and height of the ball flight. There are several ways that this can be achieved. A faster club head speed (which is generated by physical strength, swing mechanics and other factors), will generate a faster ball speed. However, there are also factors such as the quality of the strike (ie middle of the club) and a phenomenon known as smash factor, which can vary using different brands and types of drivers.

Impact of the initial launch angle on trajectory

Implementing some standard input values, but varying the initial launch angle:

```
>>balltrajectory(1.225, 180, 2500, 8)

>>balltrajectory(1.225, 180, 2500, 10)

>>balltrajectory(1.225, 180, 2500, 12)
```

When combining the three, it produced the following plot:

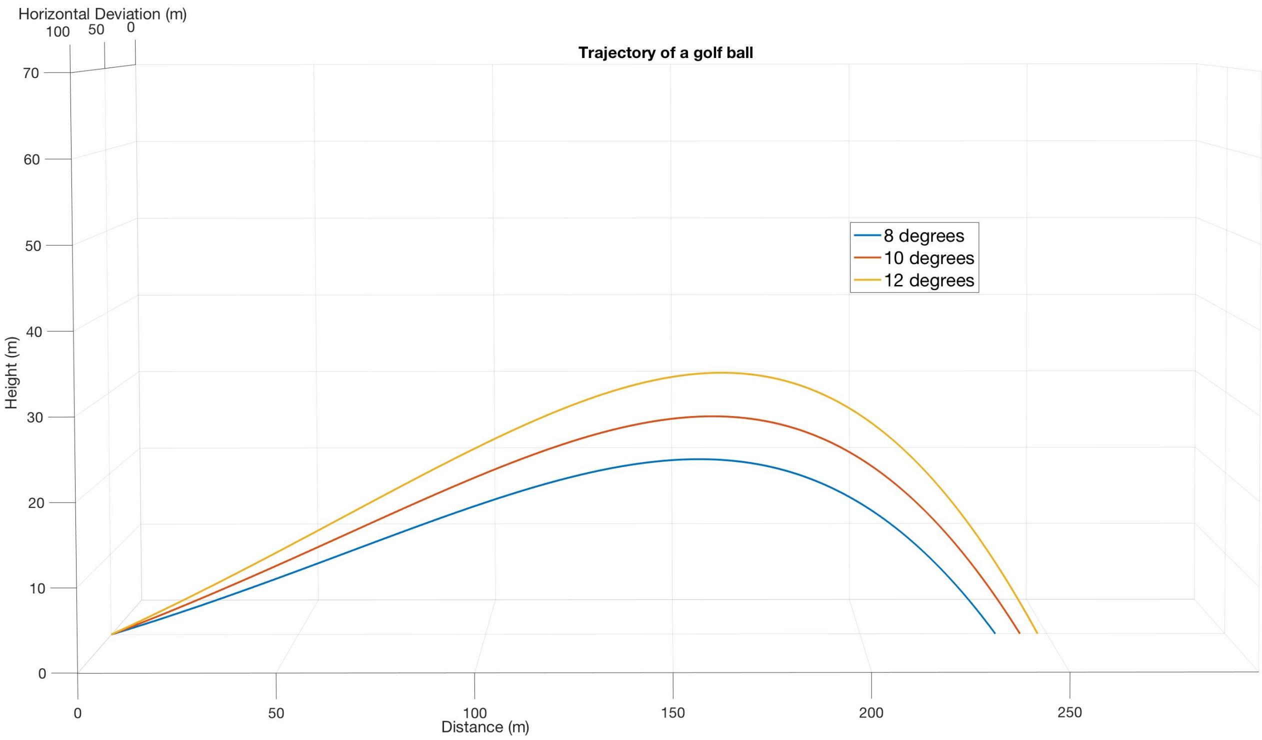

Figure 5: Effect of launch angle on trajectory

The launch angle is a pivotal component to the trajectory of the ball. As the launch angle is increased, the maximum height and carry distances increase as well (Figure 5). The launch angle is primarily dependent on swing mechanics which produce the angle of attack to the ball and thus the launch angle. However, the launch angle is also heavily impacted by the loft of the club used.

Impact of the air density (& elevation) on trajectory

Implementing some standard input values, but varying the air density:

```
>>balltrajectory(1.0, 180, 2500, 10)
```

```
>>balltrajectory(1.5, 180, 2500, 10)
```

```
>>balltrajectory(1.25, 180, 2500, 10)
```

When combining the three, it produced the following plot:

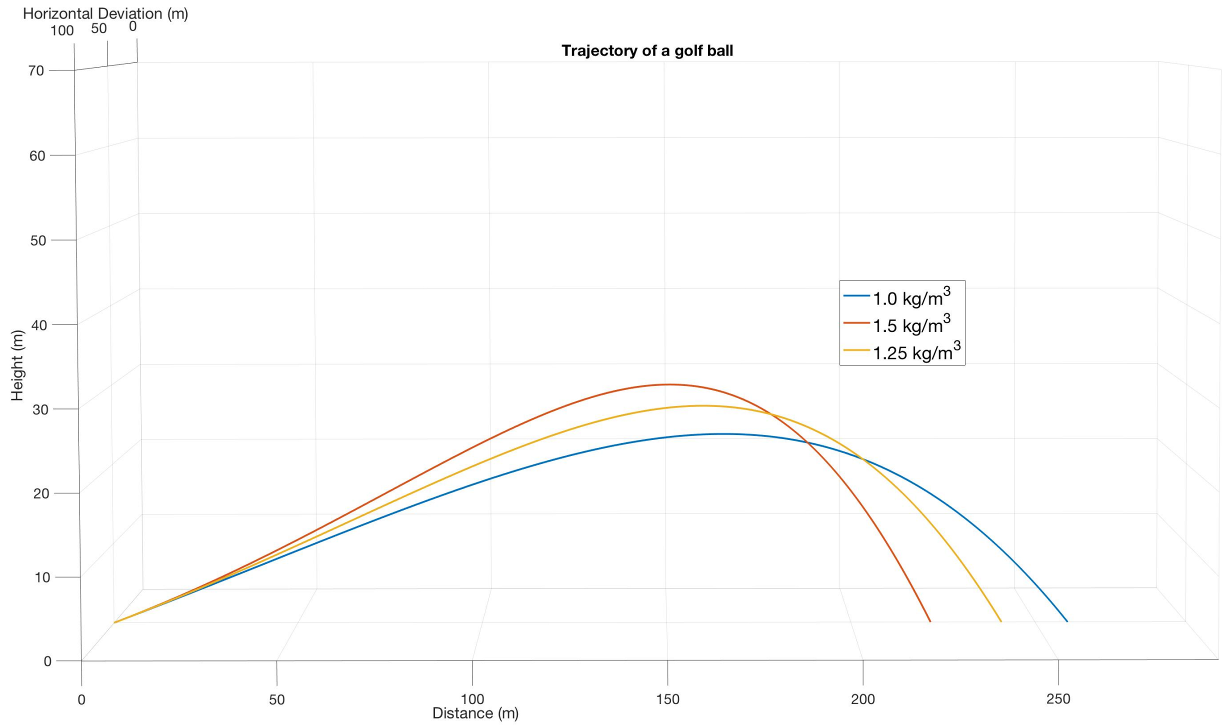

Figure 6: Effect of air density on trajectory

Figure 6 gives visual evidence that the air density at which the ball is travelling in has a significant influence on the trajectory of the ball. As the density of the air is increased, the ball reaches a greater height but decreases in total carry. The shot simulated in not so dense air (blue) results in a decrease in maximum height (ie. a lower trajectory) and an increase in total carry. This result provides general evidence that golf professionals playing in locations where the air is denser than usual can expect a loss in carry distance. On the contrary, an increase in carry distance can be expected when playing in conditions where the air is less dense than usual.

The mean elevation in Mexico City is 2,250 m (Geo-mexico.com, 2017), which correlates to an air density of approximately 1.001 kgm^{-3}(Engineeringtoolbox.com, 2017). Investigating the difference in trajectory when a professional golfer uses a driver at sea level vs New Mexico:

```
>>balltrajectory(1.225, 180, 2500, 10)
>>balltrajectory(1.001, 180, 2500, 10)
```

When combining the two, it produced the following plot:

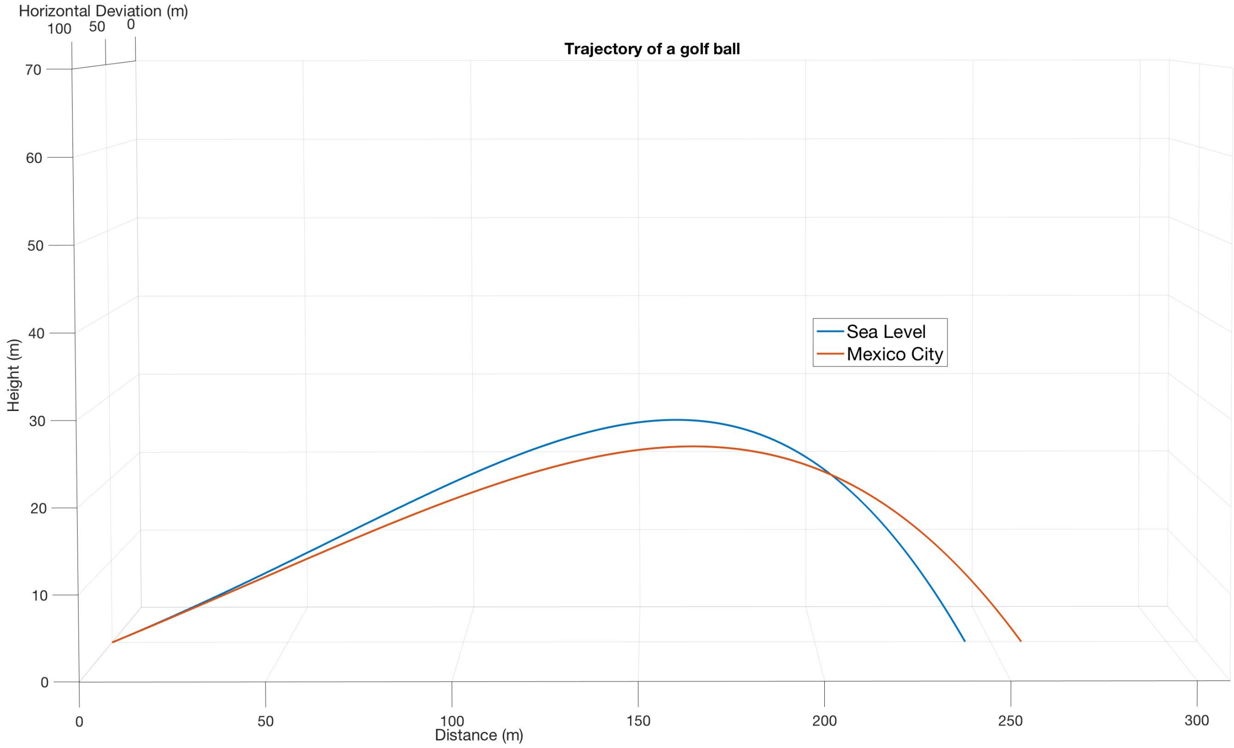

Figure 7: Ball trajectory at Sea Level vs Mexico City

The code was then altered to output the carry distances:

Carry (Sea Level) = 242.7883m

Carry (Mexico City) = 258.7644m

The model predicts that using standard tour data, a golf ball hit by a driver will carry further by just under 16 metres. This is a significant increase in distance. Also, it can be observed in Figure 7 that the maximum height that the ball reaches is lower in the lower air density of Mexico City compared to sea level air density. Although the bounce and run (ie how far the ball travels after landing) is not calculated in this model, it could be hypothesised that the lower ball flight implies the ball travelling further after landing than the higher ball flight. This is subject to many other variables such as spin and how soft the ground is, so it is difficult to comment on. The results from the model have shown that the ball does carry further in Mexico City. However, the accuracy of the solution has not been verified yet, so no conclusion can be made until this is completed.

Comparison to real data & validity of results

In this section, the developed model will be tested against real data sourced from TrackMan software. TrackMan is one of the leading commercial software used to simulate golf ball trajectory.

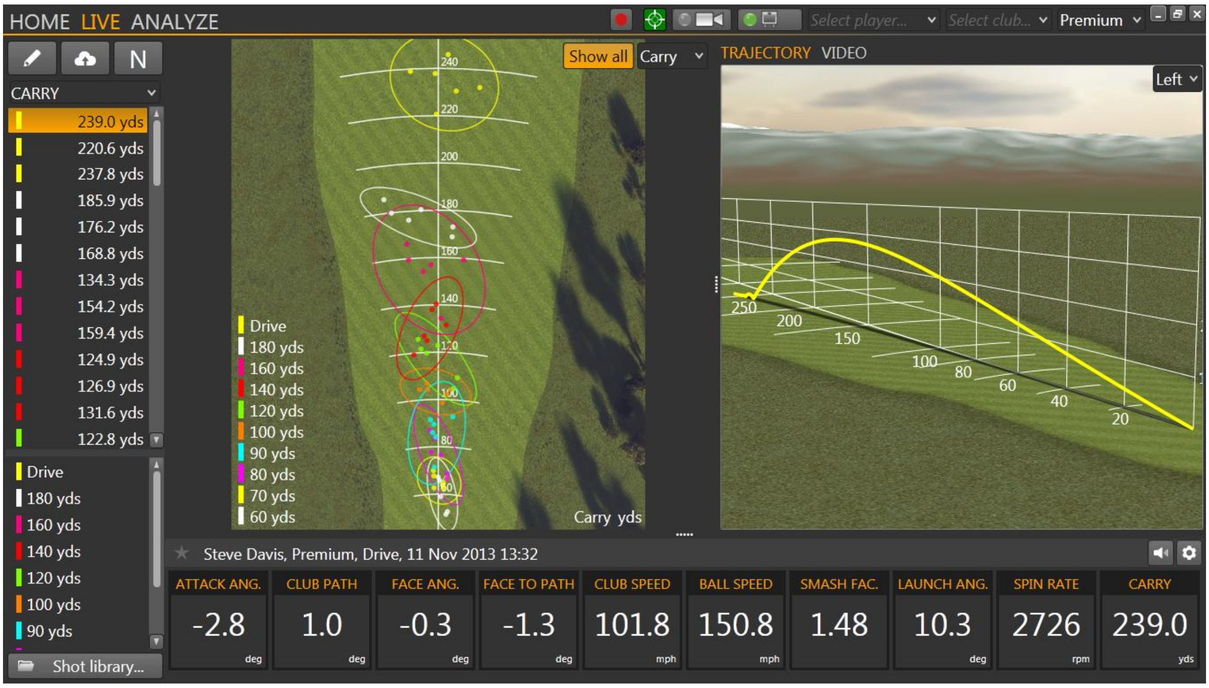

Figure 8: Screenshot of TrackMan example simulation (Westridge Golf Centre, 2017)

Seen in Figure 8, TrackMan uses sophisticated sensors to detect numerous initial conditions, such as attack angle, club path, face angle, face to path angle, club speed, ball speed, smash factor (a ratio between ball speed and club speed), launch angle and spin rate. The software also includes a premium addition with even more initial conditions. The software then, almost instantaneously, computes the trajectory of the ball and a simulation of the trajectory is made. The carry, apex, landing angle and other features are calculated from these initial conditions and the corresponding simulation.

The mathematical model made in this project takes into consideration many of these initial conditions, but certainly not all. It could be hypothesised that the accuracy of the model will be slightly compromised due to this lack of initial conditions.

It would be an achievement for the model to give solutions anywhere close to the TrackMan data. The typical cost of Trackman 4 is as such (Craig, 2017):

Indoor Model: $19,000

Outdoor Model: $25,000

The cost is generally a good indicator of the accuracy and time spent developing the TrackMan model and software. However, it would be reasonable to suggest that a large source of the cost would be the sophisticated sensors which can detect the initial conditions.

Trackman collected PGA Tour average data (Figure 9) (below).

TRACKMAN PGA TOUR AVERAGES WWW.TRACKMANGOLF.COM

	Club Speed (mph)	Attack Angle (deg)	Ball Speed (mph)	Smash Factor	Launch Ang. (deg)	Spin Rate (rpm)	Max Height (yds)	Land Angle (deg)	Carry (yds)
Driver	113	-1.3°	167	1.48	10.9°	2686	32	38°	275
3-wood	107	-2.9°	158	1.48	9.2°	3655	30	43°	243
5-wood	103	-3.3°	152	1.47	9.4°	4350	31	47°	230
Hybrid 15-18°	100	-3.5°	146	1.46	10.2°	4437	29	47°	225
3 Iron	98	-3.1°	142	1.45	10.4°	4630	27	46°	212
4 Iron	96	-3.4°	137	1.43	11.0°	4836	28	48°	203
5 Iron	94	-3.7°	132	1.41	12.1°	5361	31	49°	194
6 Iron	92	-4.1°	127	1.38	14.1°	6231	30	50°	183
7 Iron	90	-4.3°	120	1.33	16.3°	7097	32	50°	172
8 Iron	87	-4.5°	115	1.32	18.1°	7998	31	50°	160
9 Iron	85	-4.7°	109	1.28	20.4°	8647	30	51°	148
PW	83	-5.0°	102	1.23	24.2°	9304	29	52°	136

Please be aware that the location (altitude) and weather conditions have not been taken into consideration for the above data. Besides these reservations, the data is based on a large sample size and gives a good indication of key numbers for tour professionals.

Figure 9: PGA Tour (Professional Golfers Association Tour) average data (TrackMan Golf, 2017)

The MATLAB code / mathematical model created in this project will now be compared to this data using the following commands, where the initial conditions inputted to the MATLAB function are identical to the data above.

- Note that some of the values in Figure 9 have differing units to the inputs for the program – these were converted as seen below).
- Note that the altitude was assumed to be sea level.

```
balltrajectory(1.225, 181.856, 2686, 10.9)
balltrajectory(1.225, 172.2, 3655, 9.2)
balltrajectory(1.225, 165.762, 4250, 9.4)
balltrajectory(1.225, 160.934, 4437, 10.2)
balltrajectory(1.225, 157.716, 4630, 10.4)
balltrajectory(1.225, 154.497, 4836, 11)
balltrajectory(1.225, 151.278, 5361, 12.1)
balltrajectory(1.225, 148.06, 6231, 14.1)
balltrajectory(1.225, 144.84, 7097, 16.3)
balltrajectory(1.225, 140.013, 7998, 18.1)
balltrajectory(1.225, 136.794, 8647, 20.4)
balltrajectory(1.225, 133.576, 9304, 24.2)
```

The program was altered such that carry distances were outputted. These outputs, subject to identical initial conditions to the TrackMan data, were compared to the TrackMan outputs.

Club	TrackMan Carry (m)	Project Model Carry (m)	Error
Driver	251.46	247.4836	-3.9764
3-Wood	222.19	224.3864	2.1964
5-Wood	210.31	211.4545	1.1445
Hybrid	205.74	204.1304	-1.6096
3 Iron	193.83	198.4758	4.6458
4 Iron	185.62	193.1131	7.4931
5 Iron	177.39	186.0519	8.6619
6 Iron	167.34	176.7868	9.4468
7 Iron	157.27	167.2055	9.9255
8 Iron	146.30	156.0694	9.7694
9 Iron	135.33	147.8344	12.5044
PW	124.36	137.8991	13.5391

Table 1: TrackMan Carry vs Project Solution Carry

From Table 1, the error in the model is evident and follows a trend. The error for the Driver, 3-wood, 5-wood and hybrid (the first 4 entries) are all within an absolute error of 4 metres. For the 5-wood and hybrid, the absolute error is below 2 metres. This is very accurate. However, as the club choice decreases towards the pitching wedge, the absolute error increases. The maximum error occurs with the pitching wedge at 13.5391 metres. This is relatively significant. It would be reasonable to suggest that the trend should be the opposite. The shorter the shot goes, the less room for inaccuracy. The further the shot goes, the more inaccurate the model should be. However, this is not the case. Since the accuracy is sufficient with the longer shots, it indicates that the model lacks the flexibility of using higher lofted clubs. The are several reasons why this may be the case:

1. Higher lofted clubs produce higher lofted ball flights, which could be impacted by forces with greater magnitudes, so small differences between the actual forces on the ball and the emulated forces on the ball in this model will be magnified.

2. The coefficient of restitution used was relevant for a driver. In principle, it would be reasonable to suggest that the further the properties of a particular club differ comparatively to a driver (ie, the further down Table 1), the greater the error will be. This is generally true in Table 1.

3. The overall error could be due to the numerical solution used (Finite Difference method). The finite difference method, as stated previously, is prone to both round off error and truncation error. This error was minimized by using a step size ‘h’ which sufficed to minimize the error.

Computation time & efficiency

To find the efficiency of the code, two checkpoints were made in the code to determine the computation time for each section:

- The computation time of the solution using the finite difference method
- The computation time of the plot
- The overall time taken for the code to output the solution

The MATLAB timer functions tic() and toc() were used. Since computation time is not always the same every time, an average of 10 runs were used:

Component of code	Average computation time (s)
Implementation of finite difference method	0.6094
Plot of solution	0.1314
Overall computation time	0.7120

Table 2: Computation time of the code

From Table 2, the overall average computation time is 0.7120 seconds. This is a reasonable computation time. One reason for this is the plot of the trajectory (0.1314 seconds), which cannot be improved. A reason which can be improved is the implementation of the finite difference method in the code (0.6094 seconds). The individual calculations themselves do not take long, however it is the while loop which slows down the method. This is because each step, these calculations are made. This means that the step size is crucial to the computation time. In this case, a relatively small step size was used ($h = 6.0555 \times 10^{-6}$) because this step size minimised the error. If the computation time was a problem, this step size could be increased (to reduce the amount of calculations) such that the computation time would decreased. However, in doing this, the accuracy of the solution will be compromised. For this reason, it is reasonable to leave the step size which minimizes error and has an acceptable run time.

Conclusion

The mathematical model generated based on the theory was successfully solved using a numerical solution and computational methods to produce the intended trajectory of a golf ball. The model was complex enough to produce, with the aid of computational and numerical methods, relatively accurate results compared to TrackMan. The numerical method used (finite difference method), provided the added benefit of simple and flexible code, while providing an average computation time of less than 1 second. In respect to the original question, the model predicted that with the same initial properties (ie. club head speed, launch angle, etc…), a shot made with a driver by a simulated professional golfer will carry approximately 16 metres further in Mexico City compared to at sea level. This figure can be considered as a relatively true estimate of the extra carry distance that can be achieved in Mexico City, considering the accuracy of the model.

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