MIT GR Course Problem Set 1

Solutions and Explanations

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Problem1.

(a)Show that the sum of any two orthogonal spacelike vectors is spacelike.

(b)Show that a timelike vector and a null vector cannot be orthogonal.

Solution.

(a)Given two spacelike vector, A^a, B^b satisfiying:

$$\eta_{ab}A^aB^b = 0$$
$$\eta_{ab}A^aA^a > 0$$

Suppose C^c is the sum of A^a, B^b then we can prove.

$$\begin{aligned}\left(A^a + B^b\right) \cdot \left(A^a + B^b\right) &= A^aA_a + 2\eta_{ab}A^aB^b + B^bB_b \\ &= A^aA_a + B^bB_b > 0\end{aligned}$$

So, the sum of any two orthogonal spacelike vectors is spacelike.

(b)Suppose we have a timelike vector v^a and a null vector u^a. Let's assume the two vectors are orthogonal and take the inner product to get

$$\eta_{ab}v^a u^b = -v^0u^0 + v^1u^1 + v^2u^2 + v^3u^3 = 0$$
$$\Rightarrow v^0u^0 = v^1u^1 + v^2u^2 + v^3u^3 \tag{1}$$

Let $\mathbf{v} = (v^1, v^2, v^3)$, $\mathbf{u} = (u^1, u^2, u^3)$.According to Cauchy Inequality, we get

$$|\mathbf{v}||\mathbf{u}| \geq \left|v^1u^1 + v^2u^2 + v^3u^3\right| \tag{2}$$

And since v^a is a timelike vector and u^a is a null vector, we have

$$v^a v_a = -\left(v^0\right)^2 + \mathbf{v}^2 < 0 \tag{3}$$
$$u^a u_a = -\left(u^0\right)^2 + \mathbf{u}^2 = 0 \tag{4}$$

Combining Eq.(1), Eq.(2), Eq.(3) and Eq.(4) to get

$$\left|v^0u^0\right| = \left|v^1u^1 + v^2u^2 + v^3u^3\right| \leq |\mathbf{v}||\mathbf{u}| < \left|v^0\right|\left|u^0\right| \tag{5}$$

Obviously, there is a contradiction in Eq.(5), so our assumption is wrong and the original proposition is valid.

Problem2. In some reference frame, the vector fields $\overrightarrow{U}$ and $\overrightarrow{D}$ have the components

$$U^\alpha \doteq \left(1 + t^2, t^2, \sqrt{2}t, 0\right)$$
$$D^\alpha \doteq \left(x, 5tx, \sqrt{2}t, 0\right)$$

The scalar ρ has the value

$$\rho = x^2 + t^2 - y^2$$

(The relationship "LHS $\doteq$ RHS" means "the object on the left-hand side is represented by the object on the right-hand side in the specified reference frame.")

(a)Show that $\overrightarrow{U}$ is suitable as a 4-velocity. Is $\overrightarrow{D}$?

(b)Find the spatial velocity $\mathbf{v}$ of a particle whose 4-velocity is $\overrightarrow{U}$, for arbitrary t. Describe the motion in the limits $t = 0$ and $t = \infty$.

(c)Find $\partial_\beta U^\alpha$ for all α, β. Show that $U_\alpha \partial_\beta U^\alpha = 0$.(There's a clever way to do this; do it the brute force way instead.)

(d)Find $\partial_\alpha D^\alpha$.

(e)Find $\partial_\beta (U^\alpha D^\beta)$ for all α.

(f)Find $U_\alpha \partial_b eta(U^\alpha D^\beta)$. Why is the answer so similar to (d)?

(g)Calculate $\partial_\alpha \rho$ for all α. Calculate $\partial^\alpha \rho$.

(h)Find $\nabla_{\overrightarrow{U}} \rho$ and $\nabla_{\overrightarrow{D}} \rho$.

Solution.

(a)For $\overrightarrow{U}$ we have

$$U^a U_a = -\left(1+t^2\right)^2 + \left(t^2\right)^2 + \left(\sqrt{2}t\right)^2 + 0^2 = -1$$

so $\overrightarrow{U}$ is a 4-velocity, but for $\overrightarrow{D}$ we have

$$D^a D_a = -x^2 + (5tx)^2 + \left(\sqrt{2}t\right)^2 + 0^2 \neq -1$$

so $\overrightarrow{D}$ is not a 4-velocity.

(b)For a 4-velocity $\overrightarrow{U}$, we can write it in the form

$$U^a = \left(\frac{\partial}{\partial \tau}\right)^a = \left(\frac{\partial}{\partial t}\right)^a \frac{\mathrm{d}t}{\mathrm{d}\tau} + \left(\frac{\partial}{\partial x^i}\right)^a \frac{\mathrm{d}x^i}{\mathrm{d}\tau} \, (i = 1, 2, 3) \tag{6}$$

For a spatial velocity (or a 3-velocity) can be written in the form

$$u^a = \left(\frac{\partial}{\partial x^i}\right)^a \frac{\mathrm{d}x^i}{\mathrm{d}t} = \left(\frac{\partial}{\partial x^i}\right)^a \frac{\mathrm{d}x^i/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau} \tag{7}$$

As for $\overrightarrow{U}$, we know by comparing Eq.(6) that

$$\begin{aligned} U^0 = 1 + t^2 \quad U^1 = t^2 \\ U^2 = \sqrt{2}t \quad U^3 = 0 \end{aligned} \tag{8}$$

Then we can find the spatial velocity $\mathbf{v}$ by Eq.(7) and Eq.(8)

$$\mathbf{v} = \left(\frac{t^2}{1+t^2}, \frac{\sqrt{2}t}{1+t^2}, 0 \right) \tag{9}$$

It's obvious that when $t = 0$ we have $\mathbf{v} = (0, 0, 0)$, which means the particle starts moving at rest. As t increases, we can tell that v^1 gradually increases in the form of a growth rate of $\frac{2t}{(1+t^2)^2}$; However, from the growth rate $\sqrt{2}\frac{1-t^2}{(1+t^2)^2}$ of v^2, we can see that v^2 first increases to a maximum value $\frac{\sqrt{2}}{2}$ and the decreases to 0.

We now consider the limiting behaviour of $\mathbf{v}$ as $t \to \infty$. Analysis of the v^1 expression reveals that its limit approaches 1 as $t \to \infty$, consistent with the 3-velocity always remains below the speed of light. Meanwhile, the v^2 expression demonstrates a limiting value of 0 as $t \to \infty$. The combined analysis of both components indicates that the spatial velocity magnitude of the particle never exceeds the speed of light, monotonically increases in the x-direction, and initially increases followed by a decrease in the y-direction.

(c)Since U^α solely depend on t, only the $\partial_0 U^\alpha$ component of $\partial_\beta U^\alpha$ is non-vanishing

$$\begin{aligned} \partial_0 U^0 = 2t \quad \partial_0 U^1 = 2t \\ \partial_0 U^2 = \sqrt{2} \quad \partial_0 U^3 = 0 \end{aligned} \tag{10}$$

To compute $U_\alpha \partial_\beta U^\alpha$, we first need to determine U^α. By utilizing the Minkowski metric, we obtain

$$\begin{aligned} U_0 = \eta_{00} U^0 = -(1+t^2) \quad U_1 = \eta_{11} U^1 = t^2 \\ U_2 = \eta_{22} U^2 = \sqrt{2}t \quad U_3 = \eta_{33} U^3 = 0 \end{aligned} \tag{11}$$

The value of $U_\alpha \partial_\beta U^\alpha$ can the be determined by combining Eq.(10) and Eq.(11)

$$\begin{aligned} U_\alpha \partial_\beta U^\alpha &= U_\alpha \partial_0 U^\alpha \\ &= -(1+t^2)(2t) + t^2(2t) + \sqrt{2}t \cdot \sqrt{2} + 0 = 0 \end{aligned} \tag{12}$$

(d)Given that D^α depends on exclusively on t and x, the value of $\partial_\alpha D^\alpha$ can be directly evaluated.

$$\partial_\alpha D^\alpha = \partial_0 D^0 + \partial_1 D^1 + \partial_2 D^2 + \partial_3 D^3 = 5t \tag{13}$$

(e)From the results of problems (c) Eq.(10) and (d) Eq.(13), the value of the sought expression $\partial_b eta\big(U^\alpha D^\beta\big)$ can be derived.

$$\begin{aligned} \partial_\beta \big(U^\alpha D^\beta\big) &= U^\alpha \partial_\beta D^\beta + D^\beta \partial_\beta U^\alpha \\ &= U^\alpha \partial_1 D^1 + D^0 \partial_0 U^\alpha \\ &= 5tU^\alpha + x\partial_0 U^\alpha \end{aligned} \tag{14}$$

The explicit components are expressed as follows:

$$\begin{cases} 5t(1+t^2) + 2tx & \alpha = 0, \\ 5t^3 + 2tx & \alpha = 1, \\ 5\sqrt{2}t^2 + \sqrt{2}x & \alpha = 2, \\ 0 & \alpha = 3. \end{cases} \tag{15}$$

(f)Both the value of the sought expression and the reason for the similarity can be derived through the following derivation.

$$\begin{aligned} &U_\alpha \partial_\beta \big(U^\alpha D^\beta\big) = U_\alpha D^\beta \partial_\beta U^\alpha + U_\alpha U^\alpha \partial_\beta D^\beta \\ &\overset{(c)}{=} U_\alpha U^\alpha \partial_\beta D^\beta \overset{(a)}{=} -\partial_\beta U^\beta \overset{(d)}{=} -5t \end{aligned} \tag{16}$$

(g)The sole critical aspect requiring attention is the relation $\partial^\alpha = \eta^{\alpha\beta}\partial_\beta$. Subsequent evaluation through direct substitution yields the crresponding value.

$$\partial_\alpha \rho = \begin{cases} \partial_0 \rho = 2t, \\ \partial_1 \rho = 2x, \\ \partial_2 \rho = -2y, \\ \partial_3 \rho = 0. \end{cases} \tag{17}$$

$$\partial^\alpha \rho = \begin{cases} \partial^0 \rho = -2t, \\ \partial^1 \rho = 2x, \\ \partial^2 \rho = -2y, \\ \partial^3 \rho = 0. \end{cases} \tag{18}$$

(h)The definition of the directional derivative operator $\nabla_{\vec{v}}$ is given as follows

$$\nabla_{\vec{v}} = v^a \partial_a$$

Based on this definition, the value of the sought expression can be explicitly determined.

$$\begin{aligned} &\nabla_{\vec{U}} \rho = U^\alpha \partial_\alpha \rho = \\ &U^0 \partial_0 \rho + U^1 \partial_1 \rho + U^2 \partial_2 \rho + U^3 \partial_3 \rho \\ &= 2t(1+t^2) + 2xt^2 - 2\sqrt{2}yt \end{aligned} \tag{19}$$

$$\begin{aligned} &\nabla_{\vec{D}} \rho = D^\alpha \partial_\alpha \rho = \\ &D^0 \partial_0 \rho + D^1 \partial_1 \rho + D^2 \partial_2 \rho + D^3 \partial_3 \rho \\ &= 2tx + 10tx^2 - 2\sqrt{2}yt \end{aligned} \tag{20}$$

Notes for Problem2.

It can be proven that for any 4-velocity U^α, the relation $U_\alpha \partial_\beta U^\alpha = 0$ holds identically. By employing the abstract index notation, we derive

$$\partial_b (U^a U_a) = U^a \partial_b U_a + U_a \partial_b U^a = 2U_a \partial_b U^a = 0 \tag{21}$$

Within Eq.(21), the expression for 4-acceleration naturally arise

$$A^a = U_b \partial_b U^a$$

Through careful examination of Eq.(21), it is revealed that this formulation explicitly demonstrates the orthogonality between the 4-acceleration and 4-velocity at all points along a particle's worldline in special relativity.

Problem3. Consider a timelike unit 4-vector $\overrightarrow{U}$ and the tensor

$$P_{\alpha\beta} = \eta_{\alpha\beta} + U_\alpha U_\beta$$

Show that this tensor is a projection operator that projects an arbitrary vector $\overrightarrow{V}$ into one orthogonal to $\overrightarrow{U}$. In other words, show that the vector $\overrightarrow{V_\perp}$ whose components are

$$V_\perp^\alpha = P^\alpha_{\ \beta} V^\beta$$

is

(a)orthogonal to $\overrightarrow{U}$

(b)unaffected by further projections:

$$V_{\perp\perp}^\alpha \equiv P^\alpha_{\ \beta} V_\perp^\beta = V_\perp^\alpha$$

(c)Show that $P_{\alpha\beta}$ is the metric for the space of vectors orthogonal to $\overrightarrow{U}$:

$$P_{\alpha\beta} V_\perp^\alpha W_\perp^\beta = \overrightarrow{V_\perp} \cdot \overrightarrow{W_\perp}$$

(d)Show that for an arbitrary nonnull vector $\overrightarrow{q}$, the projection tensor is given by

$$P_{\alpha\beta}(q^\alpha) = \eta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^\gamma q_\gamma}$$

Do we need a projection tensor for null vectors?

Solution.

(a)To demonstrate the orthogonality between $V_{\perp}^{\alpha}$ and the 4-velocity $\overrightarrow{U}$, it suffices to compute their inner product:

$$\begin{aligned} &V_{\perp}^{\alpha}U_{\alpha} = P_{\beta}^{\alpha}V^{\beta}U_{\alpha} = \eta^{\alpha\gamma}P_{\gamma\beta}V^{\beta}U_{\alpha} \\ &= \eta^{\alpha\gamma}(\eta_{\gamma\beta} + U_{\gamma}U_{\beta})V^{\beta}U_{\alpha} = \eta_{\beta}^{\alpha}V^{\beta}U_{\alpha} + U^{\alpha}U_{\beta}V^{\beta}U_{\alpha} \\ &= U_{\beta}V^{\beta} - U_{\beta}V^{\beta} = 0 \end{aligned} \tag{22}$$

So, $V_{\perp}^{\alpha}$ is orthogonal to $\overrightarrow{U}$.

(b)To establish the original proposition, it suffices to demonstrate that the problem is reducible to

$$P_{\beta}^{\alpha}P_{\gamma}^{\beta} = P_{\gamma}^{\alpha} \tag{23}$$

Substituting the definition of tensor yields:

$$\begin{aligned} &\left(\eta_{\beta}^{\alpha} + U^{\alpha}U_{\beta}\right)\left(\eta_{\gamma}^{\beta} + U^{\beta}U_{\gamma}\right) \\ &= \eta_{\gamma}^{\alpha} + U^{\alpha}U_{\gamma} + U^{\alpha}U_{\gamma} - U^{\alpha}U_{\gamma} \\ &= \eta_{\gamma}^{\alpha} + U^{\alpha}U_{\gamma} = P_{\gamma}^{\alpha} \end{aligned} \tag{24}$$

thus conclusively proving the original proposition.

(c)For any two vectors $\overrightarrow{V_{\perp}}$ and $\overrightarrow{W_{\perp}}$ residing in the subspace orthogonal to the 4-velocity $\overrightarrow{U}$, the action of the tensor $P_{\alpha\beta}$ yields:

$$\begin{aligned} &P_{\alpha\beta}V_{\perp}^{\alpha}W_{\perp}^{\beta} = (\eta_{\alpha\beta} + U_{\alpha}U_{\beta})V_{\perp}^{\alpha}W_{\perp}^{\beta} \\ &= V_{\perp\beta}W_{\perp}^{\beta} + U_{\alpha}U_{\beta}V_{\perp}^{\alpha}W_{\perp}^{\beta} = V_{\perp\beta}W_{\perp}^{\beta} \end{aligned} \tag{25}$$

Evidently, the final term in the equation corresponds precisely to the inner product of the two vectors. This conclusively demonstrates $P_{\alpha\beta}$ serves as the metric tensor for the 4-velocity-orthogonal vector subspace.

(d)Close examination of the target relation reveals that the term $\frac{q_{\alpha}q_{\beta}}{q^{\gamma}q_{\gamma}}$ represents the squared norm of a normalized nonnull vector. Let this normalized vector be denoted as n^α, which reduces the expression to:

$$P_{\alpha\beta} = \eta_{\alpha\beta} \mp n_\alpha n_\beta \tag{26}$$

(select "-" when $n^\alpha n_\alpha = +1$, adopt "+" when $n^\alpha n_\alpha = -1$)

The proof thus reduces to verifying that the tensor $P_{\alpha\beta}$ constracted in this way satisfies all defining properties of a projection tensor. For an arbitrary vector v^α we have

$$P^\alpha_\beta v^\beta = \eta^\alpha_\beta v^\beta \mp n^\alpha n_\beta v^\beta \tag{27}$$

$$\Rightarrow v^\alpha = P^\alpha_\beta v^\beta \pm n^\alpha \big(n_\beta v^\beta\big) \tag{28}$$

The term $n^\alpha\big(n_\beta v^\beta\big)$ in the final expression of Eq.(28) evidently demonstrates that this vector is collinear with n^α. To prove that $P_{\alpha\beta}$ constitutes a projection tensor, it is equivalent to verify the orthogonality condition $\big(P^\alpha_\beta v^\beta\big)n_\alpha = 0$. The demonstration proceeds as follows:

$$\big(P^\alpha_\beta v^\beta\big)n_\alpha = \big(v^\alpha \mp n^\alpha n_\beta v^\beta\big)n_\alpha = n_\alpha v^\alpha \mp n_\alpha n^\alpha n_\beta v^\beta \tag{29}$$

since Eq.(29) choose the $\mp$ sign as the Eq.(26) does, this implies that when $n_\alpha n^\alpha = +1$, we have $n_\alpha v^\alpha - n_\beta v^\beta = 0$; when $n_\alpha n^\alpha = -1$, we have $n_\alpha v^\alpha - n_\beta v^\beta = 0$. This completes the rigorous demonstration of the original proposition. For the reason why we do not need a projection tensor for null vectors, please read the Notes.

Notes for Problem3.

Actually, tensor P_{ab} is the induced metric on the hypersurface which is orthogonal to the vector n^a. I'm more comfortable to write it as h_{ab}. As for vector n^a, it's in fact a normalized normal vector of the hypersurface. When n^a is a **spacelike** vector, we say that the hypersurface is **timelike**; when n^a is a **spacelike** vector, we say that the hypersurface is **timelike**. In this sense, it's more convenient to prove why we don't need this tensor for null vectors. Strictly speaking, it is not that we "dispense with" defining a projection tensor for null vectors, but rather that null vectors inherently preclude the definition of a projection tensor. This fundamental limitation arises because the induced metric on a null hypersurface is degenerate, thereby rendering the existence of a projection tensor mathematically inadmissible. The formal proof unfolds as follows:

Suppose there is a non-degenerate induced metric h_{ab} on the null hypersurface. For an arbitrary point q in the hypersurface, we can define the induced tanget space W_q on that point. Since there exists a null vector $n^a \in W_q$ on the null hypersurface, for any non-zero vector w^a in W_q, the following relation holds:

$$h_{ab}n^a w^b = 0 \tag{30}$$

(n^a is both a null vector and the normal vector on the hypersurface)

In Eq.(30) we see the contradiction for the metric h_{ab} because there exists a non-vanishing vector n^a satisfiying Eq.(30). So our original guess is wrong, which means the induced metric on the null hypersurface is degenerate.

Problem4. Let $\Lambda_B(\mathbf{v})$ be a Lorentz boost associated with 3-velocity $\mathbf{v}$. Consider

$$\Lambda \equiv \Lambda_B(\mathbf{v_1}) \cdot \Lambda_B(\mathbf{v_2}) \cdot \Lambda_B(-\mathbf{v_1}) \cdot \Lambda_B(-\mathbf{v_2})$$

where $\mathbf{v_1} \cdot \mathbf{v_2} = 0$. Assume $v_1 \ll 1$, $v_2 \ll 1$.

Show that Λ is a rotation. What is the axis of rotation? What is the angle of rotation?

Solution. Within the Lorentz group framework, the group structure is characterized by boost generators K_i and rotation generators J_i, whose algebraic relations are governed by the commutator structure:

$$[K_i, K_j] = -\epsilon_{ijk} K_k \tag{31}$$

$$[J_i, J_j] = \epsilon_{ijk} J_k \tag{32}$$

$$[J_i, K_j] = \epsilon_{ijk} K_k \tag{33}$$

Under the low-velocity approximation ($v \ll 1$), the Lorentz boost operation expands as:

$$\Lambda_B(v) = \mathrm{e}^{vK} = 1 + vK + \frac{v_1^2}{2} K + \mathscr{O}(v^3) \tag{34}$$

Given the orthogonality of 3-velocities $\mathbf{v_1}$ and $\mathbf{v_2}$, we may without loss of generality orient $\mathbf{v_1}$ along the x-axis and $\mathbf{v_2}$ along the y-axis. This configuration induces:

$$\begin{aligned}
\Lambda &= \Lambda_B(\mathbf{v_1}) \cdot \Lambda_B(\mathbf{v_2}) \cdot \Lambda_B(-\mathbf{v_1}) \cdot \Lambda_B(-\mathbf{v_2}) \\
&= \left(1 + v_1 K_x + \frac{v_1^2}{2} K_x^2 + \mathscr{O}(v_1^3)\right)\left(1 + v_2 K_y + \frac{v_2^2}{2} K_y^2 + \mathscr{O}(v_2^3)\right) \\
&\cdot \left(1 - v_1 K_x + \frac{v_1^2}{2} K_x^2 + \mathscr{O}(v_1^3)\right)\left(1 - v_2 K_y + \frac{v_2^2}{2} K_y^2 + \mathscr{O}(v_2^3)\right) \\
&\approx I + v_1 v_2 [K_x, K_y] = I - v_1 v_2 J_z
\end{aligned} \tag{35}$$

This operational scheme corresponds to a z-axis rotation matrix $R_z = 1 - \theta J_z$, where the rotation angle $\theta = v_1 v_2$ (dimensionless in natural units) manifests clockwise spatial rotation.

Problem5. "Superluminal" motion

The quasar 3C 273 emits relativistic blobs of plasma from near the massive black hole at its center. The blobs travel at speed v along a jet making an angle θ with respect to the line of sight of the observer. Projected onto the sky, the blobs appear to travel perpendicular to the line of sight with angular speed v_{app}/r where r is the distance to 3C 273 as and v_{app} is the apparent speed.

(a)Show that

$$v_{\text{app}} = \frac{v \sin\theta}{1 - v\cos\theta}$$

.

(b)For a given value of v, what value of θ maximizes v_{app}? What is the corresponding maximal value of v_{app}? Can this be greater than the speed of light? If so, is special relativity violated?

(c)For 3C 273, $v_{\text{app}} \simeq 10c$. What is the largest possible value of θ (in degrees)?

Solution.

(a)As depicted in Fig. 1, suppose a plasma blob is ejected from point A with velocity v at time $t = 0$, traverses a duration Δt to reach point B. Observer O measures the angle between the AB trajectory and the line of sight as θ Since photons emitted from point A at t_1 arrive at O, while those emitted from point B at t_2 reach O, the geometric-temporal relation yields:

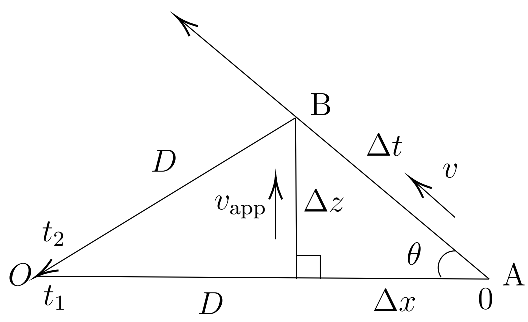


Fig. 1: relativistic jet

$$\Delta z = v \Delta t \sin\theta \tag{36}$$

$$\Delta x = v \Delta t \cos\theta \tag{37}$$

$$t_1 = D + \Delta x \tag{38}$$

$$t_2 = D \tag{39}$$

$$\Delta t' = t_1 - t_2 \tag{40}$$

Given the apparent velocity definition $v_{\text{app}} = \frac{\Delta z}{\Delta t'}$ combined with Eq.(36) to Eq.(40), we can get:

$$v_{\text{app}} = \frac{\Delta z}{\Delta t'} = \frac{v \sin\theta}{1 - v\cos\theta} \tag{41}$$

thus establishing the apparent superluminal motion relation.

(b)For a given velocity v, differentiation of the apparent velocity expression reveals its maximum value occurs at $\theta = \arccos v$:

$$v_{\text{app}}^{\max} = \frac{v\sqrt{1-v^2}}{1-v^2} = \frac{v}{\sqrt{1-v^2}} \tag{42}$$

As $v \to 1$, $v_{\text{app}} \to \infty$. This apparent superluminal motion arises purely from relativistic projection effects, where v_{app} represents a geometrical artifact rather than physical particle velocity. Crucially, the true plasma blob velocity v remains subluminal, and no actual information or energy transfer exceeds light speed, thereby fully preserving relativistic causality.

(c)Given the apparent velocity $v_{\text{app}} = 10$, we substitute this value into the definitional equation:

$$v = \frac{10}{\sin\theta + 10\cos\theta} \tag{43}$$

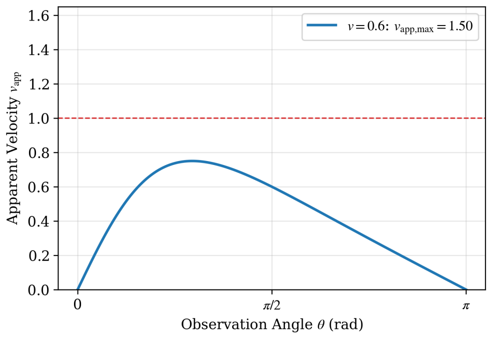
With the constraint $v \leq 1$, to determine the maximum attainable angle θ, we utilize the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$. Solving the coupled equations yields:

$$101\cos^2\theta - 200\cos\theta + 99 = 0 \tag{44}$$

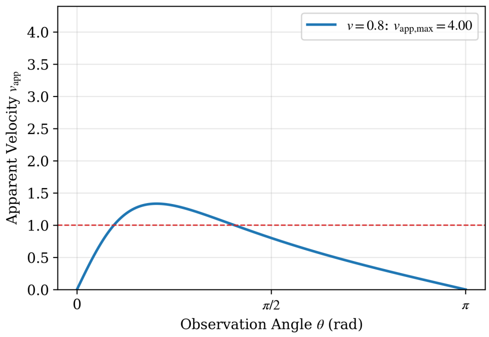
Solving this quadratic equation, we get $\theta = \arccos\left(\frac{99}{101}\right) \approx 11.4°$

Notes for Problem5.

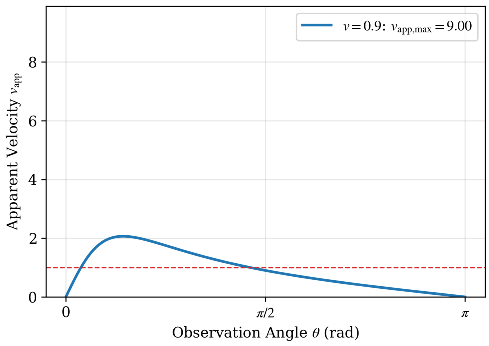
This geometrically-induced superluminal phenomenon presents intriguing implications. To quantitatively visualize these effects, we employ Python-generated numerical simulations, producing the series of diagrams from Fig. 2a to Fig. 3



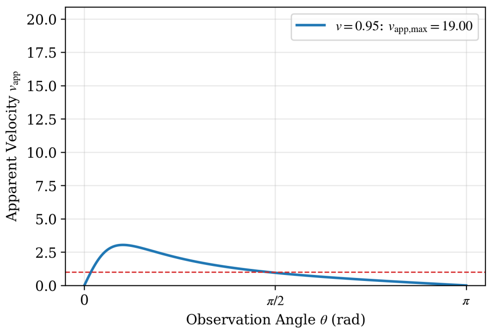
(a) $v = 0.6$



(b) $v = 0.8$



(c) $v = 0.9$



(d) $v = 0.95$

Fig. 2: v_{app} v.s. θ for different v in (a)-(d)

Critical observations from the diagrams include:

1. **Subluminal Constraint Regime**: At specific fixed velocities v, the apparent velocity v_{app} remains strictly subluminal regardless of observation angle θ (exemplified in Fig. 2a).

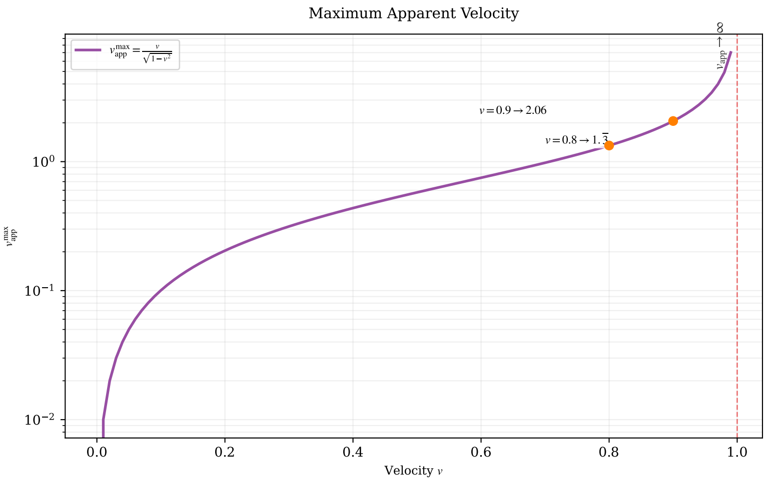


Fig. 3: $v_{\text{app, max}}$ v.s. v

2. **Threshold Condition**: Through rigorous analysis, we establish that superluminal apparent motion occurs if and only if $v \geq \frac{1}{\sqrt{2}} \approx 0.714$

Problem6. GZK cutoff in the cosmic ray spectrum

(a)Calculate the threshold energy of a nucleon N for it to undergo the reaction $\gamma + N \to N + \pi^0$, where γ represents a microwave background photon of energy kT with $T = 2.73K$. Assume the collision is head-on and take the nucleon and pion masses to be 938 MeV and 135 MeV, respectively.

(b)Explain why one might expect to observe very few cosmic rays of energy above $\sim 10^{11}$ GeV.

(c)This expectation is called the Griesen-Zatsepin-Kuzmin (GZK) cutoff. Modern observations show no sharp cutoff; there may even be evidence for an *upturn* in cosmic ray flux at these energies. Can you suggest a mechanism by which the GZK cutoff can be avoided?

Solution. (a)To determine the nuclear reaction energy threshold, we analyze the critical kinematic condition where the total system energy precisely equals the combined rest mass of the reaction products during a head-on collision. Applying energy-momentum conservation laws:

$$E = m_{\text{final}}$$

Substituting the relativistic energy-momentum relation $E_N^2 = p_N^2 + m_N^2$:

$$\begin{aligned} &(E_N + E_\gamma)^2 - (p_N - p_\gamma)^2 = (m_N + m_\pi)^2 \\ &\Rightarrow 2E_\gamma(E_N + p_N) = 2m_N m_\pi + m_\pi \end{aligned} \tag{45}$$

Under the high-energy approximation $p_N \gg m_N$:

$$\begin{aligned} E_N &= \frac{m_N m_\pi}{2E_\gamma} \approx \frac{938 \times 135 \times 1.60 \times 10^{-19}}{2 \times 2.73 \times 1.38 \times 10^{-23} \times 10^{-6}} \text{MeV} \\ &\approx 2.7 \times 10^{14}\,\text{MeV} \end{aligned} \tag{46}$$

(b)The pronounced attenuation of ultra-high-energy cosmic ray flux arises due to the catastrophic energy loss through photopion production processes $\gamma + N \to N + \pi^0$ where cosmic rays interact with cosmic microwave background (CMB) photons. This mechanism, known as the Greisen-Zatsepin-Kuzmin (GZK) cutoff, effectively suppresses particle fluxes beyond the critical energy threshold $E_{\text{GZK}} \approx 5 \times 10^{19}\,\text{eV}$, resulting in the extreme scarcity of observed particles exceeding this energy limit.

(c)The GZK cutoff may be circumvented under two scenarios:

1. **Proximity to cosmic ray sources**, where insufficient energy is lost to microwave background photons.

2. **Heavy nuclear composition** (e.g., iron nuclei), which possess higher photodisintegration threshold energies.