MIT GR Course Problem Set 2

Solutions and Explanations

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Problem1. Show that the number density of dust measured by an observer whoes 4-velocity is $\vec{U}$ is given by $n = -\vec{N} \cdot \vec{U}$, where $\vec{N}$ is the matter current 4-velocity.

Solution. By definition we have the matter current 4-velocity $N^a = nU^a$, we have

$$\boxed{-N^a U_a = -nU^a U_a = n} \tag{1}$$

Problem2. Take the limit of continuity equation for $|\mathbf{v}| \ll 1$ to get $\partial n/\partial t + \partial(nv^i)/\partial x^i = 0$

Solution. For $|\mathbf{v}| \ll 1$, the Lorentz factor satisfies $\gamma \approx 1$. Therefore the 4-velocity can be expressed as $\vec{U} = (1, \mathbf{v})$.

According to the continuity equation $\partial_\alpha N^\alpha = 0$ and $N^\alpha = nU^\alpha$, we have

$$\begin{aligned} &\partial_\alpha (nU^\alpha) = 0 \\ \Rightarrow &\partial_t \big(nU^t\big) + \partial_i \big(nU^i\big) = 0 \\ \Rightarrow &\boxed{\frac{\partial n}{\partial t} + \frac{\partial (nv^i)}{\partial x^i} = 0} \end{aligned} \tag{2}$$

Problem3. In an inertial frame $\mathscr{O}$, calculate the components of the stress-energy tensors of the following systems:

(a) A group of prticles all moving with the same 3-velocity $\mathbf{v} = \beta\vec{e}_x$ as seen in $\mathscr{O}$. Let the rest-mass desnity of these particles be ρ_0, as measured in their own rest frame. Assume a sufficiently high density of particles to enable treating them as a continuum.

(b) A ring of N similar particles of rest mass m rotating counter-clockwise in the x-y plane about the origin of $\mathscr{O}$, at a radius a from this point, with an angular velocity ω. The ring is a torus of circular cross-section $\delta a \ll a$, within which the particles are uniformly distributed with a high enough density for the continuum approximation to apply. Do not include the stress-energy of whatever forces keep them in orbit. Part of this calculation will relate ρ_0 of part (a) to N, a, δa, and ω.

(c) Two such rings of particles, one rotating clockwise and the other counter-clockwise, at the same radius a. The particles do not collide or otherwise interact in any way.

Solution.

(a) First, in the rest frame of the group of these particles, the stress-energy tensor $T^{\mu\nu}$ assumes the simplified form $T^{\mu\nu} = \rho_0 U^\mu U^\nu$, or in matrix:

$$\begin{pmatrix} \rho_0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

Second, given these particles move in the same 3-velocity $\mathbf{v} = \beta\vec{e}_x$ as seen in $\mathscr{O}$, their 4-velocity in this frame becomes $\vec{U'} = (\gamma, \gamma\beta, 0, 0)$. Consequently, the stress-energy tensor in frame $\mathscr{O}$ assumes the form:

$$\boxed{T'^{\mu\nu} = \rho_0 U'^{\mu} U'^{\nu} = \gamma^2 \rho_0 \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & \beta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \tag{3}$$

(b) First, given the torus volume $V = 2\pi a \pi(\delta a)^2$, the rest-mass density ρ_0 is determined as:

$$\rho_0 = \frac{Nm}{2\pi^2 a(\delta a)^2} \tag{4}$$

Next, for particles undergoing counter-clockwise rotation with angular velocity ω, the corresponding 4-velocity components at the angle ϕ are derived via Cartesian coordinates:

$$\vec{U} = \gamma(1, -a\omega\sin(\phi), a\omega\cos(\phi), 0), \quad \gamma = \left(1 - a^2\omega^2\right)^{-\frac{1}{2}} \tag{5}$$

Finally, combining the system's symetry and the strss-energy tensor definition $T^{\mu\nu} = \rho_0 U^{\mu} U^{\nu}$, angular averaging over ϕ yields:

$$\begin{aligned} \left\langle T^{00} \right\rangle &= \gamma^2 \rho_0 \\ \left\langle T^{11} \right\rangle &= \gamma^2 \rho_0 \cdot \frac{1}{2} a^2 \omega^2 \\ \left\langle T^{22} \right\rangle &= \gamma^2 \rho_0 \cdot \frac{1}{2} a^2 \omega^2 \\ \left\langle T^{12} \right\rangle &= \left\langle T^{21} \right\rangle = \langle \sin\phi\cos\phi \rangle = 0 \end{aligned} \tag{6}$$

Combining **??**, **??** and **??** to get the final stress-energy tensor assuming the matrix representation:

$$\boxed{T^{\mu\nu} = \gamma^2 \rho_0 \begin{pmatrix} 1 & & & \\ & \frac{1}{2} a^2 \omega^2 & & \\ & & \frac{1}{2} a^2 \omega^2 & \\ & & & 0 \end{pmatrix}} \tag{7}$$

(c) Since the two rings are counter-rotating and the constituent particles exhibit no interactions, the stress-energy tensor for each ring independently assumes the form specified in **??**. The composite system's stress-energy tensor is therefore given by linear superposition:

$$\boxed{T^{\mu\nu} = 2\gamma^2\rho_0 \begin{pmatrix} 1 & & & \\ & \frac{1}{2}a^2\omega^2 & & \\ & & \frac{1}{2}a^2\omega^2 & \\ & & & 0 \end{pmatrix}} \tag{8}$$

Notes for Problem3.

In problem (a), the stress-energy tensor $T'^{\mu\nu}$ can be derived via an alternative approach distinct from **??**:

$$T'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\sigma}}\frac{\partial x'^{\nu}}{\partial x^{\rho}}T^{\sigma\rho} = \frac{\partial x'^{\mu}}{\partial x^{0}}\frac{\partial x'^{\nu}}{\partial x^{0}}T^{00} \tag{9}$$

Evidently, the application of tensor transformation formula in **??** yields identical result.

Problem4. Use the identity $\partial_\nu T^{\mu\nu} = 0$ to prove the following results for a bounded system (i.e., a system for which $T^{\mu\nu} = 0$ beyond some bounded region of space):

(a) $\partial_t \int T^{0\alpha}\,\mathrm{d}^3x = 0$. This expresses conservation of energy and momentum.

(b) $\partial_t^2 \int T^{00}x^i x^j\,\mathrm{d}^3x = 2\int T^{ij}\,\mathrm{d}^3x$. This result is a version of the virial theorem; it will come in quite handy when we derive the quadrupole formula for gravitational radiation.

(c) $\partial_t^2 \int T^{00}(x^i x_i)^2\,\mathrm{d}^3x = 4\int T^i_i x^j x_j\,\mathrm{d}^3x + 8\int T^{ij}x_i x_j\,\mathrm{d}^3x$. No pithy wisdom for this one.

Solution. Utilizing the identity $\partial_\nu T^{\mu\nu} = 0$, we derive the following equations:

$$\begin{cases} \partial_t T^{00} + \partial_k T^{0k} = 0, & \mu = 0 \\ \partial_t T^{0i} + \partial_k T^{ik} = 0, & \mu = i \end{cases} \qquad (i, j, k = 1, 2, 3) \tag{10}$$

Later we will use **??** for multiple times, so it's best to put these relations before the steps of the solutions.

(a) For the $\mu = 0$ component in **??**, integration to get

$$\partial_t \int T^{00} \, \mathrm{d}^3 x = - \int \partial_k T^{0k} \, \mathrm{d}^3 x \tag{11}$$

Applying Gauss's law we get $\int \partial_k T^{0k} \, \mathrm{d}^3 x = \oint T^{0k} \, \mathrm{d}S_k$. Given that it is a bounded system, T^{0k} vanishes at spatial infinity, leading to:

$$\boxed{\partial_t \int T^{00} \, \mathrm{d}^3 x = 0 \quad \text{(Conservation of Energy)}} \tag{12}$$

Similarly, for $\mu = i$ we have:

$$\boxed{\partial_t \int T^{0i} \, \mathrm{d}^3 x = 0 \quad \text{(Conservation of Momentum)}} \tag{13}$$

This completes the proof of $\partial_t \int T^{0\alpha} \, \mathrm{d}^3 x = 0 \quad (\alpha = 0, 1, 2, 3)$.

(b) According to **??** we have:

$$\partial_t \int T^{00} x^i x^j \, \mathrm{d}^3 x = -\partial_k \int T^{0k} x^i x^j \, \mathrm{d}^3 x$$

Applying integration by parts yields:

$$\begin{aligned} &- \int \partial_k T^{0k} x^i x^j \, \mathrm{d}^3 x = \int T^{0k} \partial_k \left(x^i x^j \right) \mathrm{d}^3 x - 0 \\ &= \int T^{0k} \left(x^j \delta^i_k + x^i \delta^j_k \right) \mathrm{d}^3 x = \int \left(T^{0i} x^j + T^{0j} x^i \right) \mathrm{d}^3 x \end{aligned}$$

Subsequently taking the time derivative once and combining **??** when $\mu = i$ provides:

$$\begin{aligned} &\partial_t \int \left(T^{0i} x^j + T^{0j} x^i \right) \mathrm{d}^3 x = \int \left(\partial_k T^{ik} x^j + \partial_k T^{jk} x^i \right) \mathrm{d}^3 x \\ &= \int \left(T^{ik} \delta^j_k + T^{jk} \delta^i_k \right) \mathrm{d}^3 x = 2 \int T^{ij} \, \mathrm{d}^3 x \end{aligned}$$

Thus, we prove the identity $\boxed{\partial_t^2 \int T^{00} x^i x^j \, \mathrm{d}^3x = 2 \int T^{ij} \, \mathrm{d}^3x}$.

(c) According to **??** we have:

$$\partial_t \int T^{00} \left(x^i x_i\right)^2 \mathrm{d}^3x = -\partial_k \int T^{0k} \left(x^i x_i\right)^2 \mathrm{d}^3x$$

Applying integration by parts yields:

$$\begin{aligned}
&- \partial_k \int T^{0k} \left(x^i x_i\right)^2 \mathrm{d}^3x = \int T^{0k} \partial_k \left(x^i x_i\right)^2 \mathrm{d}^3x \\
&= \int T^{0k} \cdot 2(x^i x_i)(x^i \delta_{ik} + x_i \delta^i_K) \, \mathrm{d}^3x = \int T^{0k} \cdot 2(x^i x_i) \cdot 2x_k \, \mathrm{d}^3x \\
&= 4 \int T^{0k} (x^i x_i) x_k \, \mathrm{d}^3x
\end{aligned}$$

Subsequently taking the time derivative once and combining **??** when $\mu = i$ provides:

$$\begin{aligned}
&4 \int \partial_t T^{0k} (x^i x_i) x_k \, \mathrm{d}^3x = -4 \int \partial_j T^{jk} x_k (x^i x_i) \, \mathrm{d}^3x \\
&= 4 \int T^{jk} \partial_j \left[x_k (x^i x_i)\right] \mathrm{d}^3x \\
&= 4 \int T^{jk} x^i x_i \delta_{jk} \, \mathrm{d}^3x + 8 \int T^{jk} x_k x_j \, \mathrm{d}^3x \\
&= 4 \int T^j_j x^i x_i \, \mathrm{d}^3x + 8 \int T^{jk} x_j x_k \, \mathrm{d}^3x
\end{aligned}$$

Regroup the index to get the identity $\partial_t^2 \int T^{00} (x^i x_i)^2 \, \mathrm{d}^3x = 4 \int T^i_i x^j x_j \, \mathrm{d}^3x + 8 \int T^{ij} x_i x_j \, \mathrm{d}^3x$.

Problem5.

The vector potential $\overrightarrow{A} \doteq (A^0, \mathbf{A})$ generates the electromagnetic field tensor via

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(a) Show that the electric and magnetic fields in a special Lorentz frame are given by

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$
$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla} A^0.$$

(b) Show that the Maxwell's equations hold if and only if

$$\partial_\mu \partial^\mu A^\alpha - \partial^\alpha \partial_\mu A^\mu = -4\pi J^\alpha$$

(c) Show that a gauge transformation of the form

$$A_\mu^{\text{new}} = A_\mu^{\text{old}} + \partial_\mu \phi$$

leaves the field tensor unchanged.

(d) Show that one can adjust the gauge so that

$$\partial_\mu A^\mu = 0$$

Show that Maxwell's equations take on a particularly simple form with this gauge choice. Use the operator $\Box \equiv \partial_\mu \partial^\mu$ to simplify your result.

Solution. The components of electromagnetic field tensor in a particular reference frame assume the matrix form:

$$(F_{\mu\nu}) = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix} \tag{14}$$

(a) By utilizing the Levi-Civita symbol ϵ_{ijk}, the curl of the vector $\mathbf{A}$ can be reformulated in the following tensorial notation:

$$\boldsymbol{\nabla} \times \mathbf{A} = \epsilon^{ijk} \partial_i A_j$$

By definition, the electric and magnetic field vectors can be expressed by $F_{\mu\nu}$:

$$\begin{aligned} E_\mu =& F_{\mu 0} && (15) \\ B_\mu =& \frac{1}{2}\epsilon_{0\mu\nu\sigma}F^{\nu\sigma} && (16) \end{aligned}$$

where $\mu, \nu, \sigma = 1, 2, 3$. Since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F^{ij} = F_{ij}$, combining **??** and **??** yields:

$$\begin{aligned} E_\mu =& \partial_\mu A_0 - \partial_0 A_\mu && (17) \\ B_\mu =& \frac{1}{2}\epsilon_{0\mu\nu\sigma}(\partial_\nu A_\sigma - \partial_\sigma A_\nu) = \epsilon_{0\mu\nu\sigma}\partial_{[\nu}A_{\sigma]} = \epsilon_{0\mu\nu\sigma}\partial_\nu A_\sigma && (18) \end{aligned}$$

Because of $A_0 = -A^0$, the electric and magnetic field vectors assume the more familiar form:

$$\boxed{\begin{aligned} E_\mu =& -\partial_\mu A^0 - \partial_0 A_\mu \doteq -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla} A^0 \\ B_\mu =& \epsilon_{0\mu\nu\sigma}\partial_\nu A_\sigma \doteq \boldsymbol{\nabla} \times \mathbf{A} \end{aligned}} \qquad (19)$$

(b) Starting from the 4-dimensional Maxwell's equations

$$\partial^\mu F_{\mu\nu} = -4\pi J_\nu$$

we raise indices via the Minkowski metric tensor $\eta^{\mu\nu}$ to obtain the contravariant form:

$$\partial_\mu F^{\mu\nu} = -4\pi J^\nu$$

Substituting the field tensor definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ yields:

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = -4\pi J^\nu$$

Performing the index substitution $\nu \to \alpha$, we arrive at the final form:

$$\boxed{\partial_\mu\partial^\mu A^\alpha - \partial^\alpha\partial_\mu A^\mu = 4\pi J^\alpha} \qquad (20)$$

(c) We can easily prove this gauge transformation formula by substitution:

$$\begin{aligned} &\boxed{F_{\mu\nu}^{\text{new}} = \partial_\mu A_\nu^{\text{new}} - \partial_\nu A_\mu^{\text{new}}} = \partial_\mu\big(A_\nu^{\text{old}} + \partial_\nu\phi\big) - \partial_\nu\big(A_\mu^{\text{old}} + \partial_\mu\phi\big) \\ &\partial_\mu A_\nu^{\text{old}} + \partial_\mu\partial_\nu\phi - \partial_\nu A_\mu^{\text{old}} + \partial_\nu\partial_\mu\phi = \boxed{\partial_\mu A_\nu^{\text{old}} - \partial_\nu A_\mu^{\text{old}} = F_{\mu\nu}^{\text{old}}} \end{aligned} \tag{21}$$

(d) Under the gauge transformation $A^\mu = A'^\mu + \partial^\mu\phi$, the divergence of the 4-potential transforms as:

$$\partial_\mu A^\mu = \partial_\mu A'^\mu + \partial_\mu\partial^\mu\phi$$

Employing the Lorentz gauge condition $\partial_\mu A^\mu = 0$, we obtain a wave equation for the gauge function ϕ:

$$\partial_\mu\partial^\mu\phi = -\partial_\mu A'^\mu$$

Provided A' satisfies the integrability condition $\partial_\nu(\partial_\mu A'^\mu) = \partial_\mu(\partial_\nu A'^\mu)$, there always exists a non-trivial solution ϕ for given initial and boundary conditions. This guarantees the existence of a gauge function ϕ that enforces $\partial_\mu A^\mu = 0$. Consequently, **??** simplifies to the form:

$$\boxed{\Box A^\alpha = -4\pi J^\alpha \quad \text{with} \quad \Box \equiv \partial_\mu\partial^\mu} \tag{22}$$

Problem6. An astronaut has acceleration g in the x direction (in other words, the magnitude of his 4-acceleration, $\sqrt{\vec{a}\cdot\vec{a}}$ is g). This astronaut assigns coordinates $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ to spacetime as follows:

First, the astronaut defines spatial coordinates to be $(\bar{x}, \bar{y}, \bar{z})$, and sets the time coordinates $\bar{t}$ to be his own proper time.

Second, at $\bar{t} = 0$, the astronaut assigns $(\bar{x}, \bar{y}, \bar{z})$ to coincide with the Euclidean coordinates (x, y, z) of inertial reference frame that momentarily coincides with his motion. (In other words, though the astronaut is not inertial — he is accelerating — there is an inertial frame that, at $\bar{t} = 0$, is momentarily at rest with respect to him. This is the frame used to assign $(\bar{x}, \bar{y}, \bar{z})$ at $\bar{t} = 0$) Observers who remain at fixed values of the spatial coordinates $(\bar{x}, \bar{y}, \bar{z})$ are called coordinate-stationary observers (CSOs). Note that proper time for these observers is not neccessarily $\bar{t}$! —- we cannot assume that the CSOs' clocks remain synchronized with the clocks of the astronaut. Assume that some function A converts between coordinate time $\bar{t}$ and proper time at the location of CSO:

$$A = \frac{\mathrm{d}\bar{t}}{\mathrm{d}\tau}$$

The function A is evaluated at a CSO's location and thus can in principle depend on all four coordinates $\bar{t},\ \bar{x},\ \bar{y},\ \bar{z}$.

Finally, the astronaut requires that the worldlines of CSOs must be orthogonal to the hypersurface $\bar{t} =$ constant, and that for each $\bar{t}$ there exists an inertial frame, momentarily at rest with respect to the astronaut, in which all events with $\bar{t} =$ constant are simultaneous.

It is easy to see that $\bar{y} = y$ and $\bar{z} = z$; henceforth we drop this coordinates from the problem.

(a) What is the 4-velocity of the astronaut, as a function of $\bar{t}$, in the initial inertial frame [the frame that uses coordinates $(t,\ x,\ y,\ z)$]?(Hint: by considering the conditions on $\vec{u} \cdot \vec{u}$, $\vec{u} \cdot \vec{a}$, and $\vec{a} \cdot \vec{a}$, you should be able to find simple forms for u^t and u^x.)

(b) Imagine that each coordinate-stationary observer carries a clock. What is the 4-velocity of each clock in the intial inertial frame?

(c) Explain why $A(\bar{x}, \bar{t})$ cannot depend on time. In other words, why can we put $A(\bar{x}, \bar{t}) = A(\bar{x})$?(Hint: consider the coordinate system that a different CSO may set up.)

(d) Find an explicit solution for the coordinate transformation $x(\bar{t}, \bar{x})$

and $t(\bar{t}, \bar{x})$.

(e) Show that the line element $ds^2 = -dt^2 + dx^2 = -(1 + g\bar{x})^2 d\bar{t}^2 + d\bar{x}^2$.

Solution. (a) Since the astronaut's 4-velocity possesses only a spatial component along x direction, we may model their motion in a 1+1-dimensional spacetime. Let the 4-velocity be $\vec{U} = (u_t, u_x)$. The normalization condition of 4-velocity gives:

$$U^\alpha U_\alpha = -u_t^2 + u_x^2 = -1 \tag{23}$$

Give the astronaut's 4-acceleration magnitude g, the 4-acceleration satisfies:

$$A^\alpha A_\alpha = -a_t^2 + a_x^2 = g^2 \tag{24}$$

Orthogonality between 4-velocity and 4-acceleration requires:

$$A^\alpha U_\alpha = -a_t u_t + a_x u_x = 0 \tag{25}$$

Boundary Condition:

- At $\bar{t} = 0$: $u_t = 0$, $u_x = 0$;
- For $\bar{t} > 0$: $u_x > 0$

Combining the Boundary Condition and **??** - **??** we get the solution:

$$\boxed{\begin{aligned} \vec{U} =& (\cosh(g\bar{t}), \sinh(g\bar{t})) \\ \vec{A} =& (g\sinh(g\bar{t}), g\cosh(g\bar{t})) \end{aligned}} \tag{26}$$

Problems (b)-(e) fundamentally constitute different facets of a unified physical scenario. The subsequent analysis will synthesize these components through an integrated treatment, deliberately omitting distinct problem identifiers to emphasize their intrinsic coherence.

First, let's explain the reason why $A(\bar{x}, \bar{t})$ cannot depend on time, as to say $A(\bar{x}, \bar{t}) = A(\bar{x})$.

1. **Coordinate Consistency:** If A depends on $\bar{t}$, coordinate-stationary observers (CSOs) at different times would establish divergent time synchronization standards, thereby precluding the cunstruction of a globally consistent coordinate system.

2. **Symmetry Consideration:** While the astronaut's fixed acceleration direction preserves spatial translational symmetry, explicit time dependence in the metric components breaks this symmetry.

3. **Orthogonality Constraints:** The requierment that the CSOs' worldlines remain orthogonal to constant-$\bar{t}$ hypersurfaces enforces A to depend exclusively on spatial coordinates.

Second, integrate **??** to get astronaut's equations of motion:

$$t = \frac{1}{g}\sinh(g\bar{t}) \tag{27}$$

$$x = \frac{1}{g}\cosh(g\bar{t}) - \frac{1}{g} \tag{28}$$

Assuming that CSOs' equations of motion are similar to **??** and **??**, but are parameterized by $\bar{x}$ in the following way:

$$t = T(\bar{x})\sinh(g\bar{t}) \tag{29}$$

$$x = X(\bar{x})\cosh(g\bar{t}) + Y(\bar{x}) \tag{30}$$

Obviously, CSOs' 4-velocity components assume the form:

$$\begin{aligned} u_t &= gAT\cosh(g\bar{t}) \\ u_x &= gAX\sinh(g\bar{t}) \end{aligned} \quad \text{with} \quad A = \frac{\mathrm{d}\bar{t}}{\mathrm{d}\tau} \tag{31}$$

According to **??**, we obtain:

$$\begin{aligned} &g^2A^2(T^2\cosh^2(g\bar{t}) - X^2\sinh^2(g\bar{t})) = 1 \\ &\Rightarrow \boxed{\begin{cases} T = & X \\ X = & \frac{1}{gA} \end{cases}} \end{aligned} \tag{32}$$

The reason why **??** can be solved like this is that A, T, X are all independent to $\bar{t}$. Now, let's what will happen when we try to obtain the expression of the line element.

$$\begin{aligned}
-dt^2 + dx^2 = &-g^2T^2\cosh^2(g\bar{t})d\bar{t}^2 - \left(\frac{\mathrm{d}T}{\mathrm{d}\bar{x}}\right)^2\sinh(g\bar{t})d\bar{x}^2 \\
&- 2gT\frac{\mathrm{d}T}{\mathrm{d}\bar{x}}\cosh(g\bar{t})\sinh(g\bar{t})d\bar{t}d\bar{x} + g^2X^2\sinh^2(g\bar{t})d\bar{t}^2 \\
&+ \left(\frac{\mathrm{d}X}{\mathrm{d}\bar{x}}\right)^2\cosh(g\bar{t})d\bar{x}^2 + \left(\frac{\mathrm{d}Y}{\mathrm{d}\bar{x}}\right)^2 d\bar{x}^2 + 2\frac{\mathrm{d}X}{\mathrm{d}\bar{x}}\frac{\mathrm{d}Y}{\mathrm{d}\bar{x}}\cosh(g\bar{t})d\bar{x}^2 \\
&= 2gX\frac{\mathrm{d}X}{\mathrm{d}\bar{x}}\cosh(g\bar{t})\sinh(g\bar{t})d\bar{t}d\bar{x} + 2gX\frac{\mathrm{d}Y}{\mathrm{d}y}\sinh(g\bar{t})d\bar{t}d\bar{x} \\
&= -\big(g^2T^2\cosh^2(g\bar{t}) - g^2X^2\sinh(g\bar{t})\big)d\bar{t}^2 - \left(\frac{\mathrm{d}T}{\mathrm{d}\bar{x}}\right)^2\sinh^2(g\bar{t})d\bar{x}^2 \\
&+ \left(\frac{\mathrm{d}X}{\mathrm{d}\bar{x}}\right)\cosh^2(g\bar{t})d\bar{x}^2 + \left(\frac{\mathrm{d}Y}{\mathrm{d}\bar{x}}\right)^2 d\bar{x}^2 + 2\frac{\mathrm{d}X}{\mathrm{d}\bar{x}}\frac{\mathrm{d}Y}{\mathrm{d}\bar{x}}\cosh(g\bar{t})d\bar{x}^2 \\
&+ 2gX\frac{\mathrm{d}X}{\mathrm{d}\bar{x}}\cosh(g\bar{t})\sinh(g\bar{t})d\bar{t}d\bar{x} + 2gX\frac{\mathrm{d}Y}{\mathrm{d}\bar{x}}\sinh(g\bar{t})d\bar{t}d\bar{x} \\
&- 2gT\frac{\mathrm{d}T}{\mathrm{d}\bar{x}}\cosh(g\bar{t})\sinh(g\bar{t})d\bar{t}d\bar{x}
\end{aligned} \tag{33}$$

Combining **??** and according to the **Symmetry Consideration** part, the metric tensor exhibits no explicit dependence on the coordinate $\bar{t}$ and contains no temporal-spatial cross terms, which means:

$$\begin{cases} \frac{X}{\cosh(g\bar{t})}\frac{\mathrm{d}Y}{\mathrm{d}\bar{x}} = 0 \\ -dt^2 + dx^2 = -g^2X^2d\bar{t}^2 + \left(\frac{\mathrm{d}X}{\mathrm{d}\bar{x}}\right)^2 d\bar{x}^2 \end{cases} \tag{34}$$

It therefore follows that Y must be a constant. To determine its explicit form, considering the following boundary conditions:

- **Initial Inertial Frame Alignment:** At $\bar{t} = 0$, the CSOs' coordinates coincide with the initial inertial coordinate $\bar{x}$ yielding:

$$X + Y = \bar{x} \tag{35}$$

- **Observer Self-Consistency Condition:** When $\bar{x} = 0$ (the astronaut serves as his own CSO), the coordinate must reduce to the astronaut's known equations of motion:

$$-Y\cosh(g\bar{t}) + Y = \frac{1}{g}\cosh(g\bar{t}) - \frac{1}{g} \tag{36}$$

Combining **??** and **??** to obtain:

$$\boxed{\begin{aligned} X =& \bar{x} + \frac{1}{g} \\ Y =& -\frac{1}{g} \end{aligned}} \tag{37}$$

By **??** we obtain:

$$\boxed{A = \frac{1}{1 + g\bar{x}}} \tag{38}$$

Now we can write the completed version of CSOs equations of motion:

$$\boxed{\begin{aligned} t =& \left(\bar{x} + \frac{1}{g}\right)\sinh(g\bar{t}) \\ x =& \left(\bar{x} + \frac{1}{g}\right)\cosh(g\bar{t}) - \frac{1}{g} \\ u_t =& g\left(\bar{x} + \frac{1}{g}\right)\frac{1}{1 + g\bar{x}}\cosh(g\bar{t}) = \cosh(g\bar{t}) \\ u_x =& g\left(\bar{x} + \frac{1}{g}\right)\frac{1}{q + g\bar{x}}\sinh(g\bar{t}) = \sinh(g\bar{t}) \end{aligned}} \tag{39}$$

Henceforth, the line element assumes the simplified form:

$$\boxed{ds^2 = -dt^2 + dx^2 = -(1 + g^2\bar{x})^2 d\bar{t}^2 + d\bar{x}^2} \tag{40}$$

This spacetime is called the Rindler spacetime and the metric is called Rindler metric.