1 Stochastic Calculus

1.1 Brownian Motion

The process $W = (W_t : t > 0)$ is a $\mathbb{P}$-Brownian motion if and only if:

- *W_t is continuous and $W_0 = 0$*
- *the value of W_t is distributed, under $\mathbb{P}$, as a normal random variable $N(0,t)$:*

$$W_t \sim N(0,t) \tag{1}$$

- *the increments $W_{s+t} - W_s$ are normally distributed with variance $t-s$ as a $N(0,t)$ under $\mathbb{P}$:*

$$W_t - W_s \sim N(0,t-s) \tag{2}$$

- *the increments $W_{s+t} - W_s$ are independent of the behaviour of W_r for $r \leq s$:*

$$W_{s+t} - W_s \neq f(W_r), \forall r \leq s \tag{3}$$

1.1.1 BM Martingales

Martingale A stochastic process $X(t), t \geq 0$ is a martingale if for any t it is integrable, $\mathbf{E}(X(t)) < \infty$, and for any $s > 0$

$$\mathbb{E}(X(t+s)|\mathscr{F}_t) \quad = \quad X(t) \tag{4}$$

List The list is taken from
"Fima Klebaner. Introduction to Stochastic Calculus with Applications. p.64"

- W_t - is a martingale
- $W_t^2 - t$ - is a martingale
- for any u, $e^{uW_t - \frac{u^2}{2}t}$ - is a martingale

Proof: The key idea in establishing the martingale property is that for any function g, the conditional expectation of $g(W_{t+s} - W_t)$ given $\mathscr{F}_s$ equals to the unconditional one.

- ***W_t is a martingale.***
  We have to show that

$$\mathbb{E}(W_s|\mathscr{F}_s) \quad = \quad W_s, \forall s < t \tag{5}$$

  We can write:

$$W_t = W_s + (W_t - W_s) \tag{6}$$

Using the linearity of expectation, we have:

$$\mathbb{E}(W_t|\mathscr{F}_s) = \mathbb{E}(W_s|\mathscr{F}_s) + \mathbb{E}(W_t - W_s|\mathscr{F}_s) \quad (7)$$

The first term on the right-hand side is W_s since W_s contains no information not contained in $\mathscr{F}_s$.
Now we show that:

$$\mathbb{E}(W_t - W_s) = 0 \quad (8)$$

Recall that

$$(W_t - W_s) \sim N(0, t - s) \quad (9)$$

As $W_t - W_s$ is independent of the value of W_s and the path up to time s, so when we condition on information available at time s, we are effectively conditioning on no information. This means that:

$$\mathbb{E}[(W_t - W_s)|\mathscr{F}_s] = \mathbb{E}(W_t - W_s) = 0 \quad (10)$$

Thus BM is a martingale.

- Process in the form:

$$dX_t = \sigma(X, t)dW_t \quad (11)$$

is a martingale.

TODO - check how to construct BM.
Problems to solve:

- is $Z = \sqrt{t}N(0, 1)$ a BM?
- is $X_t = \rho W_1 + \sqrt{1 - \rho^2}\tilde{W}_t$ a BM?

1.2 Conditional expectations

TODO - Joshi1.page155 - creation of martingales.

1.2.1 Properties

- Conditional expectation based on no information is the ordinary expectation:

$$\mathbb{E}(X|\mathscr{F}_0) = \mathbb{E}(X) \quad (12)$$

- The Tower Law - if $s < t$ and we first take the conditional expectation at time t followed by the conditional expectation at time s, then this is the same as taking the conditional expectation at time s:

$$\mathbb{E}(\mathbb{E}(X|\mathscr{F}_t)|\mathscr{F}_s) = \mathbb{E}(X|\mathscr{F}_s) \quad (13)$$

- If we condition on information that is independent of the value of the random variable then we get the same value as conditioning on no information. Random variable is independent of the information, if changing the path up to time s does not affect the value of the random variable, then the random variable is independent of $\mathscr{F}_s$.So if X is independent, we have:

$$\mathbb{E}(X|\mathscr{F}_s) = \mathbb{E}(X) \tag{14}$$

- If the random variable is determined by the information in $\mathscr{F}_s$, then conditioning on that information will have no effect, and therefore:

$$\mathbb{E}(X|\mathscr{F}_s) = X \tag{15}$$

1.3 Ito calculus

1.4 Ito calculus

1.4.1 Product Rule

$$d(XY) = XdY + YdX + dXdY \tag{16}$$

1.4.2 Ratio rule

$$d\left(\frac{X}{Y}\right) = \tag{17}$$

1.4.3 Ito isometry

1.4.4 CEV Process

1.5 Girsanov Theorem

If

- W_t *- is a Brownian motion with sample space* Ω *and measure* $\mathbb{P}$*,*
- ν *- is a reasonable function*

then there exists an equivalent measure $\mathbb{Q}$ *on* Ω *such that*

$$\tilde{W}_t = W_t - \nu t \tag{18}$$

is a Brownian motion.

1.6 Radon-Nikodym Derivative

1.6.1 Radon-Nikodym Theorem

If a probability measure $\mathbb{Q}$ *is absolutely continuous (equivalent) with respect to a probability measure* $\mathbb{P}$*, then it can be written as:*

$$\mathbb{Q} = \int_E f d\mathbb{P} \tag{19}$$

1.6.2 Radon-Nikodym Derivative

By analogy with the first fundamental theorem of calculus, the function f is called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$:

$$f = \frac{d\mathbb{Q}}{d\mathbb{P}} \tag{20}$$

A measure change consists of re-weighting the probability of paths. We therefore construct them by multiplying probabilities by a random variable.

1.7 BS PDE

1.7.1 Derivation

Steps to derive the equation:

1. Take a derivative $C(S,t)$
2. Construct a portfolio consisting of the derivative C ans α stocks S
3. Wright SDE for the portfolio, noting that the α is defined at the beginning of the time interval and therefore is constant during that period
4. Choose α in such a way, that removes the stochastic term
5. Equate the drift of riskless portfolio to the risk-free rate
6. Rearrange the equation

Derivation:

1. Take a derivative $C(S,t)$:

$$C(S,t) \tag{21}$$

2. Construct a portfolio consisting of the derivative C ans α stocks S:

$$P(C,S) = C + \alpha S \tag{22}$$

3. Wright SDE for the portfolio, noting that the α is defined at the beginning of the time interval and is constant during that period

$$\begin{aligned} dP = d(C + \alpha S) = \Big[\frac{\partial C(S,t)}{dt} + \mu S \frac{\partial C(S,t)}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + \alpha\mu S\Big]dt \\ + \sigma S\Big[\frac{\partial C(S,t)}{\partial S} + \alpha\Big]dW_t \end{aligned} \tag{23}$$

4. Choose α in such a way, that removes the stochastic term:

$$\alpha = -\frac{\partial C(S,t)}{\partial S} \tag{24}$$

This will also cancel some drift terms:

$$d(C + \alpha S) = \Big[\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2}\Big] dt \tag{25}$$

5. Equate the drift of riskless portfolio to the risk-free rate:

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} = r\Big[C - S\frac{\partial C(S,t)}{\partial S}\Big] \tag{26}$$

6. Rearrange the equation:

$$\frac{\partial C(S,t)}{\partial t} + rS\frac{\partial C(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} = rC \tag{27}$$

1.7.2 Solution

Steps:

- Change the variables to use $x = \log(S)$ and revert the time to move to the backward time rescaled by the factor $\frac{\sigma^2}{2}$
- Introduce $k = \frac{2r}{\sigma^2}$ and transform the boundary condition
- Change variable $v(x,t) = e^{\alpha x + \beta t} u(x,t)$ and get rid off the terms that are not in the heat equation
- Solve the heat equation
- Transform back

Solution: The details of the solution could be found at:
"http://www.math.tamu.edu/ stecher/blackScholesHeatEquation.pdf425/Sp12/"

- ***Change variables to use*** $\log(S)$ ***and rescaled reverted time*** τ***:***

$$S = e^x \Rightarrow x = \log(S) \tag{28}$$

$$\tau = \frac{\sigma^2}{2}(T - t) \tag{29}$$

$$C(S,t) = v(x,\tau) \tag{30}$$

Then:

$$\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2} \tag{31}$$

$$\frac{\partial x}{\partial S} = \frac{1}{S} \tag{32}$$

The partial derivatives are:

$$\begin{aligned}
\frac{\partial C}{\partial t} &= -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} && (33)\\
\frac{\partial C}{\partial S} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial v}{\partial x} && (34)\\
\frac{\partial}{\partial S} &= \frac{1}{S}\frac{\partial}{\partial x} && (35)\\
\frac{\partial^2 C}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial v}{\partial x}\right)\\
&= -\frac{1}{S^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x}\right)\\
&= -\frac{1}{S^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{S^2}\left(\frac{\partial^2 v}{\partial x^2}\right) && (36)
\end{aligned}$$

Putting everything together, we get:

$$-\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} + rS\frac{1}{S}\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{S^2}\left(\frac{\partial^2 v}{\partial x^2}\right)\right) = rv \qquad (37)$$

or

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v \qquad (38)$$

- ***Introduce $k = \frac{2r}{\sigma^2}$ and transform the boundary condition:***
  Setting:

$$\begin{aligned}
k &= \frac{2r}{\sigma^2} && (39)\\
t &= \tau && (40)
\end{aligned}$$

we get:

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, -\infty < x < \infty, 0 \le t \le \frac{\sigma^2}{2}T && (41)\\
v(x,0) &= C(e^x, T) = f(e^x), -\infty < x < \infty && (42)
\end{aligned}$$

- ***Change variable $v(x,t) = e^{\alpha x + \beta t}u(x,t)$ and get rid off the terms that are not in the heat equation:***
  Setting:

$$v(x,t) = e^{\alpha x + \beta t}u(x,t) = \phi u \qquad (43)$$

we get:

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \beta\phi u + \phi\frac{\partial u}{\partial t} && (44)\\
\frac{\partial v}{\partial x} &= \alpha\phi u + \phi\frac{\partial u}{\partial x} && (45)\\
\frac{\partial^2 v}{\partial x^2} &= \alpha^2\phi u + 2\alpha\phi\frac{\partial u}{\partial x} + \phi\frac{\partial^2 u}{\partial x^2} && (46)
\end{aligned}$$

Placing these expressions into the pde and setting:

$$\alpha = -\frac{1}{2}(k-1) = \frac{\sigma^2 - 2r}{2\sigma^2} \quad (47)$$

$$\beta = -\frac{1}{4}(k+1)^2 = -\left(\frac{\sigma^2 + 2r}{2\sigma^2}\right)^2 \quad (48)$$

we have:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, 0 \leq t \leq \frac{\sigma^2}{2}T \quad (49)$$

$$u(x,0) = e^{-\alpha x} v(x,0) = e^{-\alpha x} f(e^x), -\infty < x < \infty \quad (50)$$

1.8 Fundamental theorems of asset pricing

A discrete market is the market with finite state.

1. A discrete market, on a discrete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, is arbitrage-free if, and only if, there exists at least one risk neutral probability measure that is equivalent to the original probability measure $\mathbb{P}$.
2. An arbitrage-free market (S, B) consisting of a collection of stocks S and a risk-free bond B is complete if and only if there exists a unique risk-neutral measure that is equivalent to $\mathbb{P}$ and has numeraire B.

1.9 Drift of a risky asset under the risk-free probability measure

Steps

- Put SDE-s for a stock and a bond
- Write SDE for $f = \frac{S_t}{B_t}$
- Require it to be a martingale

We have:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (51)$$

$$dB_t = r B_t dt \quad (52)$$

and therefore

$$d\left(\frac{S_t}{B_t}\right) = \frac{dS_t}{B_t} + S_t d\left(\frac{1}{B_t}\right) \quad (53)$$

With no additional Ito cross terms since B_t is deterministic.
Since:

$$B_t = B_0 e^{rt} \quad (54)$$

we get:

$$d(B_t^{-1}) = -rB_t^{-2}B_t dt = -rB_t^{-1}dt \tag{55}$$

Thus

$$d\left(\frac{S_t}{B_t}\right) = (\mu - r)\frac{S_t}{B_t}dt + \sigma\frac{S_t}{B_t}dW_t \tag{56}$$

1.10 Black-Scholes formula derivation

1.10.1 Steps

- Write a value of an option as a discounted payoff expectation at maturity
- Write the distribution of the stock price at maturity
- Put it into the discounted payoff at maturity
- Rewrite the discounted payoff at maturiy using LOTUS (**note x appearing in the integral instead of the normal distribution $N(0,1)$**)
- Change the integration bounds to get rid off the max function under the integral
- Solve the integral bounds for x
- Denote the right-hand side of the inequality as l
- Split the integral into two parts - one with x and with strike K
- Note that the second part is just a normal CDF from l to ∞. Find its value as $N(-l)$ using evenness of CDF and replacing the integral $\int_l^\infty$ with $\int_{-\infty}^{-l}$
- Solve the first integral part by making the part in the exponent a full square

1.11 Derivation

- ***Write a value of an option as a discounted payoff expectation at maturity:***
  A price of a call option at time 0 is defined as:

$$C = e^{-rT}\mathbb{E}((S_T - K)_+) \tag{57}$$

- ***Write the distribution of the stock price at maturity***:
  In the risk-neutral world we have that:

$$S_t = S_0 exp\left(rT - \frac{1}{2}\frac{\sigma^2}{2}T + \sigma\sqrt{T}N(0,1)\right) \tag{58}$$

- ***Put it into the discounted payoff at maturity***:
The value of our option is:

$$\frac{B_0}{B_T}\mathbb{E}\Bigg[\Bigg(S_0 exp\Bigg(rT - \frac{1}{2}\frac{\sigma^2}{2}T + \sigma\sqrt{T}N(0,1)\Bigg) - K\Bigg)_+\Bigg] \tag{59}$$

- ***Rewrite the discounted payoff at maturiy using LOTUS***:
Recalling that the density of $N(0,1)$ is $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, we can write this as:

$$\frac{e^{-rT}}{\sqrt{2\pi}}\int e^{-\frac{x^2}{2}}\Bigg(S_0 exp\Bigg(rT - \frac{1}{2}\frac{\sigma^2}{2}T + \sigma\sqrt{T}x\Bigg) - K\Bigg)_+ dx \tag{60}$$

- ***Change the integration bounds to get rid off the max function under the integral***:
The integral is non-zero if and only if:

$$S_0 exp\Bigg(rT - \frac{1}{2}\frac{\sigma^2}{2}T + \sigma\sqrt{T}x\Bigg) \geq K \tag{61}$$

- ***Solve the integral bounds for x:***
Thus the integral must be taken over:

$$x \geq \frac{\log K/S_0 + \frac{1}{2}\sigma^2 T - rT}{\sigma\sqrt{T}} \tag{62}$$

- ***Denote the right-hand side of the inequality as l***

- ***Split the integral into two parts - one with x and with strike K***:
Our integral now has two terms. The second simple term is just:

$$\frac{e^{-rT}}{\sqrt{2\pi}}\int_l^\infty e^{-\frac{x^2}{2}}K dx \tag{63}$$

- ***Note that the second part is just a normal CDF from l to ∞. Find its value as $N(-l)$ using evenness of CDF and replacing the integral $\int_l^\infty$ with $\int_{-\infty}^{-l}$***:
The second term is therefore equal to:

$$e^{-rT}KN\Bigg(\frac{\log S_0/K + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\Bigg) \tag{64}$$

- ***Solve the first integral part by making the part in the exponent a full square***:
Changing the variables:

$$x \;\; = \;\; \bar{x} + \sigma\sqrt{T} \tag{65}$$

we get:

$$\frac{e^{-rT}}{\sqrt{T}}\int_{l-\sigma\sqrt{T}}^\infty e^{-\frac{\bar{x}^2}{2}}S_0 e^{rT} d\bar{x} \tag{66}$$

1.12 Things to add

- American options
- ito process
- ito calculus
- Feynman-Kac theorem
- chooser option

2 FX

2.1 Risk neutral measures

2.1.1 Domestic risk neutral measure

The domestic investor sees the foreign bond B_t^f as a risky asset, which, denominated in the domestic currency, is valued at $B_t^f S_t$. Where S_t is the FX exchange rate.
Construct the ratio of this against the domestic bond:

$$\begin{aligned} Z_t &= S_t B_t^f / B_t^d \\ &= S_0 exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2 t \right) \right) exp\left(\left(r^f - r^d \right) t \right) \\ &= S_0 exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t \right) exp((\mu + r^f - r^d)t) \end{aligned} \tag{67}$$

To attain the martingale property we require that

$$\mu = \mu^d \equiv r^d - r^f \tag{68}$$

(μ^d is used to avoid the confusion below), so under the domestic risk-neutral measure $\mathbb{P}^d$ we write

$$S_t = S_0 exp\left(\sigma W_t^d + \left(r^d - r^f - \frac{1}{2}\sigma^2 \right) \right) \tag{69}$$

$$dS_t = \left(r^d - r^f \right) S_t dt + \sigma S_t dW_t^d \tag{70}$$

The drift change required is:

$$W_t^d = W_t + \frac{\mu - \mu^d}{\sigma} t \tag{71}$$

which gives a Radon-Nikodym derivative at time T of

$$\frac{d\mathbb{P}^d}{d\mathbb{P}} = exp\left(-\gamma^d W_T - \frac{1}{2}[\gamma^d]^2 T \right) \tag{72}$$

where:

$$\gamma^d = \frac{\mu - \mu^d}{\sigma} \tag{73}$$

2.1.2 Foreign risk-neutral measure

The foreign investor sees the domestic bond B_t^d as the risky asset, which denominated in foreign currency is valued at B_t^d/S_t or, equivalently, $B_t^d\hat{S}_t$, where $\hat{S}_t) = 1/S_t$ denotes the flipped spot rate. By taking the reciprocal we have

$$B_t^d\hat{S}_t \quad = \quad \hat{S}_0 exp\Bigg(-\sigma W_t + \Bigg(\frac{1}{2}\sigma^2 - \mu + r^d\Bigg)t\Bigg) \tag{74}$$

Construct the ratio of this quantity divided by the foreign bond B_t^f:

$$\begin{aligned}
\hat{Z} &= \hat{S}_t B_t^d / B_t^f \\
&= \hat{S}_0 exp\Bigg(-\sigma W_t + \Bigg(\frac{1}{2}\sigma^2 - \mu + r^d - r^f\Bigg)t\Bigg) \\
&= \hat{S}_0 exp\Bigg(-\sigma W_t - \frac{1}{2}\sigma^2 t\Bigg) exp\Bigg(\Bigg(-\mu + r^d - r^f + \sigma^2\Bigg)t\Bigg)
\end{aligned} \tag{75}$$

This indicates that $\hat{Z}$ is a martingale if

$$\mu = \mu^f \equiv r^d - r^f + \sigma^2 \tag{76}$$

So, under the foreign risk-neutral measure $\mathbb{P}^f$ we write:

$$\hat{S}_t \quad = \quad S_0 exp\Bigg(-\sigma W_t^f + \Bigg(\frac{1}{2}\sigma^2 - \mu^f\Bigg)t\Bigg) \tag{77}$$

or, alternatively, we can express the non-flipped sport rate

$$S_t \quad = \quad exp\Bigg(\sigma W_t^f + \Bigg(\mu^f - \frac{1}{2}\sigma^2\Bigg)t\Bigg) \tag{78}$$

$$dS_t \quad = \quad (r^d - r^f + \sigma^2) S_t dt + \sigma S_t dW_t^f \tag{79}$$

The drift change required is

$$W_t^f \quad = \quad W_t + \frac{\mu - \mu^f}{\sigma} t \tag{80}$$

which gives a Radon-Nikodym derivative at time T of

$$\frac{d\mathbb{P}^f}{d\mathbb{P}} \quad = \quad exp\Bigg(-\gamma^f W_T - \frac{1}{2}[\gamma^f]^2 T\Bigg) \tag{81}$$

where

$$\gamma^f \quad = \quad \frac{\mu - \mu^f}{\sigma} \tag{82}$$

2.2 Things to add - FX

- Garman-Kolhagen formula derivation
- pricing quotations
- delta quotations
- quanto option
- compo option
- Heston

3 MC

3.1 Things to add - MC

- convergence
- brownian bridge
- variance reduction techniques
- sobol and mersenn twister

4 Volatility modelling

4.1 Thins to add

- local vol
- loc-stoch vol

5 PDE

6 To understand

- Characteristic function
- bbrr