Contents

I Stochastic Calculus 4

1 Brownian Motion (Wiener Process) 5

1.1 Definition 5

1.2 Quadratic Variation 6

1.3 BM properties 7

1.4 BM Martingales 7

1.4.1 Martingale 7

1.4.2 BM Martingales 7

1.5 TODO 9

1.6 Problems 9

2 Conditional expectations 10

2.1 Properties 10

3 Ito calculus 12

3.1 Ito calculus 12

3.1.1 Product Rule 12

3.1.2 Inversion Rule 12

3.1.3 Quotient Rule 13

3.2 Fubini's Theorem 13

3.3 Integrals involving Brownian Motion 14

3.4 Time Integrals 14
3.5 Ito Integral 14
3.6 Properties of Ito Integral of Simple Processes 15
3.6.1 Linearity. 16
3.7 Some Important Brownian Motion Integrals 16
3.7.1 $\int_0^t W_s ds$ 16
3.7.2 Ito Isometry 16
3.7.3 CEV Process 16

4 Girsanov Theorem **17**

5 Radon-Nikodym Derivative **18**
5.1 Radon-Nikodym Theorem 18
5.2 Radon-Nikodym Derivative 18

6 BS PDE **19**
6.1 Derivation 19
6.1.1 Steps to derive the equation: 19
6.1.2 Derivation: 20
6.1.3 Solution: 21

7 Fundamental theorems of asset pricing **24**
7.1 Drift of a risky asset under the risk-free probability measure . . 25
7.1.1 Steps 25

8 Black-Scholes formula derivation **27**
8.1 Steps 27
8.2 Derivation 28

9 Things to add **31**

II FX 32

10 Risk neutral measures 33

10.1 Domestic risk neutral measure 33

10.2 Foreign risk-neutral measure 34

11 Things to add - FX 37

III MC 38

12 Things to add - MC 39

IV Volatility modelling 40

13 Thins to add 41

V PDE 42

14 To understand 43

Part I

Stochastic Calculus

Chapter 1

Brownian Motion (Wiener Process)

1.1 Definition

The process $W = (W_t : t > 0)$ is a ***$\mathbb{P}$-adapted Brownian motion (or $\mathbb{P}$-adapted Wiener process)*** *if and only if:*

1. *W_t is* ***continuous*** *and $W_0 = 0$*

2. *the value of W_t is distributed, under $\mathbb{P}$, as a* ***normal random variable*** *$N(0,t)$:*

$$W_t \sim N(0,t) \tag{1.1}$$

3. *the* ***increments*** *$W_{s+t} - W_s$* ***are normally distributed with variance***

$t-s$ *as a* $N(0,t)$ *under* $\mathbb{P}$*:*

$$W_t - W_s \sim N(0, t-s) \tag{1.2}$$

4. ***the increments*** $W_{s+t} - W_s$ *are* ***independent*** *of the behaviour of* W_r *for* $r \leq s$*:*

$$W_{s+t} - W_s \neq f(W_r), \forall r \leq s \tag{1.3}$$

[1]

1.2 Quadratic Variation

Let $f : [0,T] \to \mathbb{R}$ be a function the interval $[0,T]$.

Let us take partition Δ of the interval, that:

$$0 = t_0 < t_1 < ... < t_n = T \tag{1.4}$$

and define its *length* as:

$$l(\Delta) = \max(t_{i+1} - t_i) \tag{1.5}$$

The quadratic variation for the partition Δ is then

$$Q(\Delta) = \sum_{i=0}^{n-1} \left(f\left(t_{i+1}\right) - f\left(t_i\right)\right)^2 \geq 0 \tag{1.6}$$

[1] Through the paper we use W_t and B_t interchangeably

The quadratic variation of the function is then the limit of this as the length of the partition goes to zero.

1.3 BM properties

1.4 BM Martingales

1.4.1 Martingale

A stochastic process $X(t), \forall t \geq 0$ is a martingale if

1. for any t it is integrable (it has a finite expectation):

$$\mathbf{E}(X(t)) < \infty \tag{1.7}$$

2. and for any $s > 0$

$$\mathbb{E}(X(t+s)|\mathscr{F}_t) = X(t) \tag{1.8}$$

1.4.2 BM Martingales

The list is taken from

"Fima Klebaner. Introduction to Stochastic Calculus with Applications. p.64"

- W_t - is a martingale
- $W_t^2 - t$ - is a martingale
- for any u, $e^{uW_t - \frac{u^2}{2}t}$ - is a martingale

Proof: The key idea in establishing the martingale property is that for any function g, the conditional expectation of $g(W_{t+s} - W_t)$ given $\mathscr{F}_s$ equals to the unconditional one.

- ***W_t is a martingale.***

  We have to show that

$$\mathbb{E}(W_s|\mathscr{F}_s) = W_s, \forall s < t \tag{1.9}$$

We can write:

$$W_t = W_s + (W_t - W_s) \tag{1.10}$$

Using the linearity of expectation, we have:

$$\mathbb{E}(W_t|\mathscr{F}_s) = \mathbb{E}(W_s|\mathscr{F}_s) + \mathbb{E}(W_t - W_s|\mathscr{F}_s) \tag{1.11}$$

The first term on the right-hand side is W_s since W_s contains no information not contained in $\mathscr{F}_s$.

Now we show that:

$$\mathbb{E}(W_t - W_s) = 0 \tag{1.12}$$

Recall that

$$(W_t - W_s) \sim N(0, t - s) \tag{1.13}$$

As $W_t - W_s$ is independent of the value of W_s and the path up to time s, so when we condition on information available at time s, we are effectively conditioning on no information. This means that:

$$\mathbb{E}[(W_t - W_s)|\mathscr{F}_s] = \mathbb{E}(W_t - W_s) = 0 \tag{1.14}$$

Thus BM is a martingale.

- Process in the form:

$$dX_t = \sigma(X,t)dW_t \tag{1.15}$$

is a martingale.

1.5 TODO

- BM construction
- Joshi1.page155 - creation of martingales

1.6 Problems

- is $Z = \sqrt{t}N(0,1)$ a BM?
- is $X_t = \rho W_1 + \sqrt{1-\rho^2}\tilde{W}_t$ a BM?

Chapter 2

Conditional expectations

TODO - Joshi1.page155 - creation of martingales.

2.1 Properties

1. Conditional expectation based on **no information** is the ordinary expectation:

$$\mathbb{E}(X|\mathscr{F}_0) = \mathbb{E}(X) \tag{2.1}$$

2. **The Tower Law** - if $s < t$ and we first take the conditional expectation at time t followed by the conditional expectation at time s, then this is the same as taking the conditional expectation at time s:

$$\mathbb{E}(\mathbb{E}(X|\mathscr{F}_t)|\mathscr{F}_s) = \mathbb{E}(X|\mathscr{F}_s) \tag{2.2}$$

3. If we **condition on information that is independent of the value**

of the random variable then we get the same value as conditioning on no information.

Random variable independent of the information. If changing the path up to time s does not affect the value of the random variable, then the random variable is independent of $\mathscr{F}_s$.

So if X is independent, we have:

$$\mathbb{E}(X|\mathscr{F}_s) = \mathbb{E}(X) \tag{2.3}$$

4. If **the random variable is determined by the information in $\mathscr{F}_s$**, then conditioning on that information will have no effect, and therefore:

$$\mathbb{E}(X|\mathscr{F}_s) = X \tag{2.4}$$

Chapter 3

Ito calculus

3.1 Ito calculus

3.1.1 Product Rule

$$d(XY) = XdY + YdX + dXdY \tag{3.1}$$

3.1.2 Inversion Rule

If

$$f = \frac{1}{Y} = Y^{-1} \tag{3.2}$$

then

$$\frac{\partial f}{\partial Y} = -Y^{-2} = -\frac{1}{Y^2} \tag{3.3}$$

$$\frac{\partial^2 f}{\partial Y^2} = 2Y^{-3} = \frac{2}{Y^3} \tag{3.4}$$

and

$$\begin{aligned} df &= \frac{\partial f}{\partial Y} dY + \frac{1}{2}\frac{\partial^2 f}{\partial Y^2} dY^2 \\ &= -\frac{dY}{Y^2} + \frac{dY^2}{Y^3} \end{aligned} \tag{3.5}$$

3.1.3 Quotient Rule

$$d\left(\frac{X}{Y}\right) = \frac{X}{Y}\left[\frac{dX}{X} - \frac{dY}{Y} - \frac{dX}{X}\frac{dY}{Y} + \left(\frac{dY}{Y}\right)^2\right] \tag{3.6}$$

3.2 Fubini's Theorem

If X and Y **σ-finite measure** spaces and if f is a measurable function such that **any one of the three integrals**:

$$\int_X \left(\int_Y |f(x,y)|dy\right) dx \tag{3.7}$$

$$\int_Y \left(\int_X |f(x,y)|dx\right) dy \tag{3.8}$$

$$\int_{X\times Y} |f(x,y)| d(x,y) \tag{3.9}$$

is finite, then

$$\int_{X\times Y} f(x,y)d(x,y) = \int_X \left(\int_Y f(x,y)dy\right)dx \tag{3.10}$$

$$= \int_Y \left(\int_X f(x,y)dx\right)dy \tag{3.11}$$

that is, a **double integral** could be replaced with **iterated integrals**.

3.3 Integrals involving Brownian Motion

There are 2 kinds of integrals involving Brownian motion - **time integrals** and **Ito integrals**.

3.4 Time Integrals

The time integral is the ordinary Riemann integral of a continuous but random function of t with respect to t:

$$Y_t = \int_0^T f(B_t)dt \tag{3.12}$$

3.5 Ito Integral

Suppose that W_t is a Wiener process adapted to the filtration $\mathbb{F}$

Suppose that X_t is another stochastic process that is adapted to the same filtration $\mathbb{F}$.

The Ito integral is defined as:

$$Y_t = \int_0^t X_s dW_s \tag{3.13}$$

The Ito integral is one of the ways to construct a new stochastic process Y_t from old ones X_t and W_t.

It is not possible to define (3.13) unless X_t is adapted.

3.6 Properties of Ito Integral of Simple Processes

A simple function is a function that takes on finitely many values in its range. That is, X_t is such a process for which there exist times $0 = t_0 < t_1 < ... < t_n = T$ and constants $c_0, c_1, ..., c_{n-1}$, such that:

$$X(t) = \begin{cases} c_0 & \text{if } t = 0 \\ c_i & \text{if } t_i < t \leq t_{i+1},\ i = 0, ..., n-1 \end{cases} \tag{3.14}$$

The Ito integral is defined as a sum

$$\int_0^T X(t)dB(t) = \sum_{i=0}^{n-1} c_i \Big(B\big(t_{i+1}\big) - B\big(t_i\big) \Big) \tag{3.15}$$

It is easy to see by using the independence property of Brownian increments that the integral, which is the sum in (3.15) is a Gaussian random variable with mean zero and variance

$$\begin{aligned} Var\Big(XdB\Big) &= \mathbb{E}\left(\sum_{i=0}^{n-1} c_i \Big(B\Big(t_{i+1}\Big) - B\Big(t_i\Big)\Big)\right)^2 \qquad (3.16) \\ &= \mathbb{E}\left(\sum_{i=0}^{n-1} c_i \Big(B\Big(t_{i+1}\Big) - B\Big(t_i\Big)\Big) \cdot \sum_{i=0}^{n-1} c_i \Big(B\Big(t_{i+1}\Big) - B\Big(t_i\Big)\Big)\right) \end{aligned}$$

3.6.1 Linearity.

If X_t and Y_t are simple processes and α and β are some constants, then:

$$\int_0 \tag{3.17}$$

3.7 Some Important Brownian Motion Integrals

3.7.1 $\int_0^t W_s ds$

Noting, that:

$$d\Big(tW_t\Big) = W_t dt + t dW_t \tag{3.18}$$

Therefore:

$$\begin{aligned} \int_0^t W_s ds &= \int_0^t d\Big(sW_s\Big) - \int_0^t s dW_s && (3.19) \\ &= tW_t - \int_0^t s dW_s && (3.20) \\ &= \int_0^t \Big(t - s\Big) dW_t && (3.21) \end{aligned}$$

3.7.2 Ito Isometry

3.7.3 CEV Process

Chapter 4

Girsanov Theorem

If

- W_t *- is a Brownian motion with sample space* Ω *and measure* $\mathbb{P}$*,*
- ν *- is a reasonable function*

then there exists an equivalent measure $\mathbb{Q}$ *on* Ω *such that*

$$\tilde{W}_t = W_t - \nu t \tag{4.1}$$

is a Brownian motion.

Chapter 5

Radon-Nikodym Derivative

5.1 Radon-Nikodym Theorem

If a ***probability measure*** $\mathbb{Q}$ *is* ***absolutely continuous (equivalent)*** *with respect to a probability measure* $\mathbb{P}$*, then it can be written as:*

$$\mathbb{Q} = \int_E f d\mathbb{P} \tag{5.1}$$

5.2 Radon-Nikodym Derivative

By analogy with the first fundamental theorem of calculus, the function f is called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$:

$$f = \frac{d\mathbb{Q}}{d\mathbb{P}} \tag{5.2}$$

A measure change consists of re-weighting the probability of paths. We therefore construct them by multiplying probabilities by a random variable.

Chapter 6

BS PDE

6.1 Derivation

6.1.1 Steps to derive the equation:

1. Take a derivative $C(S,t)$
2. Construct a portfolio consisting of the derivative C ans α stocks S
3. Wright SDE for the portfolio, noting that the α is defined at the beginning of the time interval and therefore is constant during that period
4. Choose α in such a way, that removes the stochastic term
5. Equate the drift of riskless portfolio to the risk-free rate
6. Rearrange the equation

6.1.2 Derivation:

1. Take a derivative $C(S,t)$:

$$C(S,t) \tag{6.1}$$

2. Construct a portfolio consisting of the derivative C ans α stocks S:

$$P(C,S) = C + \alpha S \tag{6.2}$$

3. Wright SDE for the portfolio, noting that the α is defined at the beginning of the time interval and is constant during that period

$$dP = d(C+\alpha S) = \Big[\frac{\partial C(S,t)}{dt} + \mu S\frac{\partial C(S,t)}{dS} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C(S,t)}{\partial S^2} + \alpha\mu S\Big]dt + \sigma S\Big[\frac{\partial C(S,t)}{\partial S} + \alpha\Big]dW_t \tag{6.3}$$

4. Choose α in such a way, that removes the stochastic term:

$$\alpha = -\frac{\partial C(S,t)}{\partial S} \tag{6.4}$$

This will also cancel some drift terms:

$$d(C+\alpha S) = \Big[\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C(S,t)}{\partial S^2}\Big]dt \tag{6.5}$$

5. Equate the drift of riskless portfolio to the risk-free rate:

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C(S,t)}{\partial S^2} = r\Big[C - S\frac{\partial C(S,t)}{\partial S}\Big] \tag{6.6}$$

6. Rearrange the equation:

$$\frac{\partial C(S,t)}{\partial t} + rS\frac{\partial C(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} = rC \tag{6.7}$$

Solution

Steps:

- Change the variables to use $x = \log(S)$ and revert the time to move to the backward time rescaled by the factor $\frac{\sigma^2}{2}$
- Introduce $k = \frac{2r}{\sigma^2}$ and transform the boundary condition
- Change variable $v(x,t) = e^{\alpha x + \beta t} u(x,t)$ and get rid off the terms that are not in the heat equation
- Solve the heat equation
- Transform back

6.1.3 Solution:

The details of the solution could be found at:

"http://www.math.tamu.edu/ stecher/blackScholesHeatEquation.pdf425/Sp12/"

- ***Change variables to use*** $\log(S)$ ***and rescaled reverted time*** τ***:***

$$\begin{aligned} S &= e^x \Rightarrow x = \log(S) & (6.8) \\ \tau &= \frac{\sigma^2}{2}(T-t) & (6.9) \\ C(S,t) &= v(x,\tau) & (6.10) \end{aligned}$$

Then:

$$\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2} \tag{6.11}$$
$$\frac{\partial x}{\partial S} = \frac{1}{S} \tag{6.12}$$

The partial derivatives are:

$$\frac{\partial C}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} \tag{6.13}$$
$$\frac{\partial C}{\partial S} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial v}{\partial x} \tag{6.14}$$
$$\frac{\partial}{\partial S} = \frac{1}{S}\frac{\partial}{\partial x} \tag{6.15}$$
$$\begin{aligned}
\frac{\partial^2 C}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial v}{\partial x}\right) \\
&= -\frac{1}{S^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x}\right) \\
&= -\frac{1}{S^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{S^2}\left(\frac{\partial^2 v}{\partial x^2}\right)
\end{aligned} \tag{6.16}$$

Putting everything together, we get:

$$-\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} + rS\frac{1}{S}\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{S^2}\left(\frac{\partial^2 v}{\partial x^2}\right)\right) = rv \tag{6.17}$$

or

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v \tag{6.18}$$

- ***Introduce $k = \frac{2r}{\sigma^2}$ and transform the boundary condition:***

  Setting:

$$k = \frac{2r}{\sigma^2} \tag{6.19}$$
$$t = \tau \tag{6.20}$$

we get:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, -\infty < x < \infty, 0 \leq t \leq \frac{\sigma^2}{2}T \quad (6.21)$$

$$v(x,0) = C(e^x, T) = f(e^x), -\infty < x < \infty \quad (6.22)$$

- ***Change variable*** $v(x,t) = e^{\alpha x + \beta t}u(x,t)$ ***and get rid off the terms that are not in the heat equation:***

  Setting:

$$v(x,t) = e^{\alpha x + \beta t}u(x,t) = \phi u \quad (6.23)$$

we get:

$$\frac{\partial v}{\partial t} = \beta\phi u + \phi\frac{\partial u}{\partial t} \quad (6.24)$$

$$\frac{\partial v}{\partial x} = \alpha\phi u + \phi\frac{\partial u}{\partial x} \quad (6.25)$$

$$\frac{\partial^2 v}{\partial x^2} = \alpha^2\phi u + 2\alpha\phi\frac{\partial u}{\partial x} + \phi\frac{\partial^2 u}{\partial x^2} \quad (6.26)$$

Placing these expressions into the pde and setting:

$$\alpha = -\frac{1}{2}(k-1) = \frac{\sigma^2 - 2r}{2\sigma^2} \quad (6.27)$$

$$\beta = -\frac{1}{4}(k+1)^2 = -\left(\frac{\sigma^2 + 2r}{2\sigma^2}\right)^2 \quad (6.28)$$

we have:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, 0 \leq t \leq \frac{\sigma^2}{2}T \quad (6.29)$$

$$u(x,0) = e^{-\alpha x}v(x,0) = e^{-\alpha x}f(e^x), -\infty < x < \infty \quad (6.30)$$

Chapter 7

Fundamental theorems of asset pricing

A discrete market is the market with finite state.

1. A discrete market, on a discrete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, is arbitrage-free if, and only if, there exists at least one risk neutral probability measure that is equivalent to the original probability measure $\mathbb{P}$.

2. An arbitrage-free market (S, B) consisting of a collection of stocks S and a risk-free bond B is complete if and only if there exists a unique risk-neutral measure that is equivalent to $\mathbb{P}$ and has numeraire B.

7.1 Drift of a risky asset under the risk-free probability measure

7.1.1 Steps

- Put SDE-s for a stock and a bond
- Write SDE for $f = \frac{S_t}{B_t}$
- Require it to be a martingale

We have:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{7.1}$$

$$dB_t = r B_t dt \tag{7.2}$$

and therefore

$$d\left(\frac{S_t}{B_t}\right) = \frac{dS_t}{B_t} + S_t d\left(\frac{1}{B_t}\right) \tag{7.3}$$

With no additional Ito cross terms since B_t is deterministic.

Since:

$$B_t = B_0 e^{rt} \tag{7.4}$$

we get:

$$d(B_t^{-1}) = -rB_t^{-2} B_t dt = -rB_t^{-1} dt \tag{7.5}$$

Thus

$$d\left(\frac{S_t}{B_t}\right) = (\mu - r)\frac{S_t}{B_t}dt + \sigma\frac{S_t}{B_t}dW_t \tag{7.6}$$

Chapter 8

Black-Scholes formula derivation

8.1 Steps

- Write a value of an option as a discounted payoff expectation at maturity
- Write the distribution of the stock price at maturity
- Put it into the discounted payoff at maturity
- Rewrite the discounted payoff at maturiy using LOTUS (**note x appearing in the integral instead of the normal distribution $N(0,1)$**)
- Change the integration bounds to get rid off the max function under the integral
- Solve the integral bounds for x
- Denote the right-hand side of the inequality as l

- Split the integral into two parts - one with x and with strike K
- Note that the second part is just a normal CDF from l to ∞. Find its value as $N(-l)$ using evenness of CDF and replacing the integral $\int_l^{\infty}$ with $\int_{-\infty}^{-l}$
- Solve the first integral part by making the part in the exponent a full square

8.2 Derivation

- ***Write a value of an option as a discounted payoff expectation at maturity:***

  A price of a call option at time 0 is defined as:

$$C = e^{-rT}\mathbb{E}((S_T - K)_+) \tag{8.1}$$

- ***Write the distribution of the stock price at maturity***:

  In the risk-neutral world we have that:

$$S_t = S_0 exp\bigg(rT - \frac{1}{2}\frac{\sigma^2}{2}T + \sigma\sqrt{T}N(0,1)\bigg) \tag{8.2}$$

- ***Put it into the discounted payoff at maturity***:

  The value of our option is:

$$\frac{B_0}{B_T}\mathbb{E}\Bigg[\bigg(S_0 exp\bigg(rT - \frac{1}{2}\frac{\sigma^2}{2}T + \sigma\sqrt{T}N(0,1)\bigg) - K\bigg)_+\Bigg] \tag{8.3}$$

- ***Rewrite the discounted payoff at maturiy using LOTUS***:

Recalling that the density of $N(0,1)$ is $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, we can write this as:

$$\frac{e^{-rT}}{\sqrt{2\pi}}\int e^{-\frac{x^2}{2}}\left(S_0 exp\left(rT-\frac{1}{2}\frac{\sigma^2}{2}T+\sigma\sqrt{T}x\right)-K\right)_+ dx \tag{8.4}$$

- ***Change the integration bounds to get rid off the max function under the integral***:

  The integral is non-zero if and only if:

$$S_0 exp\left(rT-\frac{1}{2}\frac{\sigma^2}{2}T+\sigma\sqrt{T}x\right)\geq K \tag{8.5}$$

- ***Solve the integral bounds for x:***

  Thus the integral must be taken over:

$$x\geq\frac{\log K/S_0+\frac{1}{2}\sigma^2T-rT}{\sigma\sqrt{T}} \tag{8.6}$$

- ***Denote the right-hand side of the inequality as l***

- ***Split the integral into two parts - one with x and with strike K***:

  Our integral now has two terms. The second simple term is just:

$$\frac{e^{-rT}}{\sqrt{2\pi}}\int_l^\infty e^{-\frac{x^2}{2}}Kdx \tag{8.7}$$

- ***Note that the second part is just a normal CDF from l to ∞. Find its value as $N(-l)$ using evenness of CDF and replacing the integral $\int_l^\infty$ with $\int_{-\infty}^{-l}$***:

  The second term is therefore equal to:

$$e^{-rT}KN\left(\frac{\log S_0/K+rT-\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right) \tag{8.8}$$

- ***Solve the first integral part by making the part in the exponent a full square***:

  Changing the variables:

$$x = \bar{x} + \sigma\sqrt{T} \tag{8.9}$$

  we get:

$$\frac{e^{-rT}}{\sqrt{T}} \int_{l-\sigma\sqrt{T}}^{\infty} e^{-\frac{\bar{x}^2}{2}} S_0 e^{rT} d\bar{x} \tag{8.10}$$

Chapter 9

Things to add

- Quadratic Variation
- American options
- Ito process
- Ito calculus
- Ito integral
- Ito isometry derivation
- Feynman-Kac theorem
- chooser option
- brownian bridge

Part II

FX

Chapter 10

Risk neutral measures

10.1 Domestic risk neutral measure

The domestic investor sees the foreign bond B_t^f as a risky asset, which, denominated in the domestic currency, is valued at $B_t^f S_t$. Where S_t is the FX exchange rate.

Construct the ratio of this against the domestic bond:

$$\begin{aligned} Z_t &= S_t B_t^f / B_t^d \\ &= S_0 exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2 t\right)\right) exp\left(\left(r^f - r^d\right)t\right) \\ &= S_0 exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right) exp((\mu + r^f - r^d)t) \end{aligned} \tag{10.1}$$

To attain the martingale property we require that

$$\mu = \mu^d \equiv r^d - r^f \tag{10.2}$$

(μ^d is used to avoid the confusion below), so under the domestic risk-neutral measure $\mathbb{P}^d$ we write

$$S_t = S_0 exp\left(\sigma W_t^d + \left(r^d - r^f - \frac{1}{2}\sigma^2\right)\right) \quad (10.3)$$

$$dS_t = \left(r^d - r^f\right) S_t dt + \sigma S_t dW_t^d \quad (10.4)$$

The drift change required is:

$$W_t^d = W_t + \frac{\mu - \mu^d}{\sigma} t \quad (10.5)$$

which gives a Radon-Nikodym derivative at time T of

$$\frac{d\mathbb{P}^d}{d\mathbb{P}} = exp\left(- \gamma^d W_T - \frac{1}{2}[\gamma^d]^2 T\right) \quad (10.6)$$

where:

$$\gamma^d = \frac{\mu - \mu^d}{\sigma} \quad (10.7)$$

10.2 Foreign risk-neutral measure

The foreign investor sees the domestic bond B_t^d as the risky asset, which denominated in foreign currency is valued at B_t^d / S_t or, equivalently, $B_t^d \hat{S}_t$, where $\hat{S}_t) = 1/S_t$ denotes the flipped spot rate. By taking the reciprocal we have

$$B_t^d \hat{S}_t = \hat{S}_0 exp\left(- \sigma W_t + \left(\frac{1}{2}\sigma^2 - \mu + r^d\right) t\right) \quad (10.8)$$

Construct the ratio of this quantity divided by the foreign bond B_t^f:

$$\begin{aligned}
\hat{Z} &= \hat{S}_t B_t^d / B_t^f \\
&= \hat{S}_0 exp\bigg(- \sigma W_t + \Big(\frac{1}{2}\sigma^2 - \mu + r^d - r^f\Big) t\bigg) \\
&= \hat{S}_0 exp\bigg(- \sigma W_t - \frac{1}{2}\sigma^2 t\bigg) exp\bigg(\Big(- \mu + r^d - r^f + \sigma^2\Big) t\bigg) \qquad (10.9)
\end{aligned}$$

This indicates that $\hat{Z}$ is a martingale if

$$\mu = \mu^f \equiv r^d - r^f + \sigma^2 \qquad (10.10)$$

So, under the foreign risk-neutral measure $\mathbb{P}^f$ we write:

$$\hat{S}_t = S_0 exp\bigg(- \sigma W_t^f + \Big(\frac{1}{2}\sigma^2 - \mu^f\Big) t\bigg) \qquad (10.11)$$

or, alternatively, we can express the non-flipped sport rate

$$\begin{aligned}
S_t &= exp\bigg(\sigma W_t^f + \Big(\mu^f - \frac{1}{2}\sigma^2\Big) t\bigg) \qquad (10.12) \\
dS_t &= (r^d - r^f + \sigma^2) S_t dt + \sigma S_t dW_t^f \qquad (10.13)
\end{aligned}$$

The drift change required is

$$W_t^f = W_t + \frac{\mu - \mu^f}{\sigma} t \qquad (10.14)$$

which gives a Radon-Nikodym derivative at time T of

$$\frac{d\mathbb{P}^f}{d\mathbb{P}} = exp\bigg(- \gamma^f W_T - \frac{1}{2}[\gamma^f]^2 T\bigg) \qquad (10.15)$$

where

$$\gamma^f = \frac{\mu - \mu^f}{\sigma} \tag{10.16}$$

Chapter 11

Things to add - FX

- Garman-Kolhagen formula derivation
- pricing quotations
- delta quotations
- quanto option
- compo option
- Heston

Part III

MC

Chapter 12

Things to add - MC

- convergence
- brownian bridge
- variance reduction techniques
- sobol and mersenn twister

Part IV

Volatility modelling

Chapter 13

Thins to add

- local vol
- loc-stoch vol

Part V

PDE

Chapter 14

To understand

- Characteristic function
- bbrr

Bibliography

[1] Mark S. Joshi. *The Concepts and Practice of Mathematical Finance*. Cambridge, Second Edition

[2] https://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2007/notes

[3] https://www.math.nyu.edu/faculty/goodman/teaching/DerivSec10/notes