Contents

Part I

Stochastic Calculus

Chapter 1

Brownian Motion

1.1 Definition

The process $W = (W_t : t > 0)$ is a $\mathbb{P}$-Brownian motion if and only if:

- *W_t is continuous and $W_0 = 0$*

- *the value of W_t is distributed, under $\mathbb{P}$, as a normal random variable $N(0, t)$:*

$$W_t \sim N(0, t) \tag{1.1}$$

- *the increments $W_{s+t} - W_s$ are normally distributed with variance $t - s$ as a $N(0, t)$ under $\mathbb{P}$:*

$$W_t - W_s \sim N(0, t - s) \tag{1.2}$$

- *the increments $W_{s+t} - W_s$ are independent of the behaviour of W_r for*

$r \leq s$:

$$W_{s+t} - W_s \neq f(W_r), \forall r \leq s \tag{1.3}$$

1.2 BM Martingales

1.2.1 Martingale

A stochastic process $X(t), t \geq 0$ is a martingale if for any t it is integrable, $\mathbf{E}(X(t)) < \infty$, and for any $s > 0$

$$\mathbb{E}(X(t+s)|\mathscr{F}_t) = X(t) \tag{1.4}$$

1.2.2 BM Martingales

The list is taken from

"Fima Klebaner. Introduction to Stochastic Calculus with Applications. p.64"

- W_t - is a martingale

- $W_t^2 - t$ - is a martingale

- for any u, $e^{uW_t - \frac{u^2}{2}t}$ - is a martingale

Proof: The key idea in establishing the martingale property is that for any function g, the conditional expectation of $g(W_{t+s} - W_t)$ given $\mathscr{F}_s$ equals to the unconditional one.

- W_t *is a martingale.*

We have to show that

$$\mathbb{E}(W_s|\mathscr{F}_s) = W_s, \forall s < t \tag{1.5}$$

We can write:

$$W_t = W_s + (W_t - W_s) \tag{1.6}$$

Using the linearity of expectation, we have:

$$\mathbb{E}(W_t|\mathscr{F}_s) = \mathbb{E}(W_s|\mathscr{F}_s) + \mathbb{E}(W_t - W_s|\mathscr{F}_s) \tag{1.7}$$

The first term on the right-hand side is W_s since W_s contains no information not contained in $\mathscr{F}_s$.

Now we show that:

$$\mathbb{E}(W_t - W_s) = 0 \tag{1.8}$$

Recall that

$$(W_t - W_s) \sim N(0, t - s) \tag{1.9}$$

As $W_t - W_s$ is independent of the value of W_s and the path up to time s, so when we condition on information available at time s, we are effectively conditioning on no information. This means that:

$$\mathbb{E}[(W_t - W_s)|\mathscr{F}_s] = \mathbb{E}(W_t - W_s) = 0 \tag{1.10}$$

Thus BM is a martingale.

- Process in the form:

$$dX_t \quad = \quad \sigma(X, t)dW_t \qquad\qquad (1.11)$$

is a martingale.

TODO - check how to construct BM.

Problems to solve:

- is $Z = \sqrt{t}N(0, 1)$ a BM?

- is $X_t = \rho W_1 + \sqrt{1 - \rho^2}\tilde{W}_t$ a BM?

Chapter 2

Conditional expectations

TODO - Joshi1.page155 - creation of martingales.

2.1 Properties

- Conditional expectation based on no information is the ordinary expectation:

$$\mathbb{E}(X|\mathscr{F}_0) \quad = \quad \mathbb{E}(X) \tag{2.1}$$

- The Tower Law - if $s < t$ and we first take the conditional expectation at time t followed by the conditional expectation at time s, then this is the same as taking the conditional expectation at time s:

$$\mathbb{E}(\mathbb{E}(X|\mathscr{F}_t)|\mathscr{F}_s) \quad = \quad \mathbb{E}(X|\mathscr{F}_s) \tag{2.2}$$

- If we condition on information that is independent of the value of the

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random variable then we get the same value as conditioning on no infor-
mation. Random variable is independent of the information, if changing
the path up to time s does not affect the value of the random variable,
then the random variable is independent of $\mathscr{F}_s$. So if X is independent, we
have:

$$\mathbb{E}(X|\mathscr{F}_s) \;\; = \;\; \mathbb{E}(X) \qquad\qquad (2.3)$$

- If the random variable is determined by the information in $\mathscr{F}_s$, then con-
ditioning on that information will have no effect, and therefore:

$$\mathbb{E}(X|\mathscr{F}_s) \;\; = \;\; X \qquad\qquad (2.4)$$

Chapter 3

Ito calculus

3.1 Ito calculus

3.1.1 Product Rule

$$d(XY) = X\,dY + Y\,dX + dX\,dY \tag{3.1}$$

3.1.2 Ratio rule

$$d\left(\frac{X}{Y}\right) \quad = \tag{3.2}$$

3.1.3 Ito isometry

3.1.4 CEV Process

Chapter 4

Girsanov Theorem

If

- *W_t - is a Brownian motion with sample space Ω and measure $\mathbb{P}$,*

- *ν - is a reasonable function*

then there exists an equivalent measure $\mathbb{Q}$ on Ω such that

$$\tilde{W}_t = W_t - \nu t \qquad (4.1)$$

is a Brownian motion.

Chapter 5

Radon-Nikodym Derivative

5.1 Radon-Nikodym Theorem

*If a **probability measure** $\mathbb{Q}$ is **absolutely continuous (equivalent)** with respect to a probability measure $\mathbb{P}$, then it can be written as:*

$$\mathbb{Q} = \int_E f d\mathbb{P} \tag{5.1}$$

5.2 Radon-Nikodym Derivative

By analogy with the first fundamental theorem of calculus, the function f is called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$:

$$f = \frac{d\mathbb{Q}}{d\mathbb{P}} \tag{5.2}$$

A measure change consists of re-weighting the probability of paths. We therefore construct them by multiplying probabilities by a random variable.

Chapter 6

BS PDE

6.1 Derivation

6.1.1 Steps to derive the equation:

1. Take a derivative $C(S, t)$

2. Construct a portfolio consisting of the derivative C ans α stocks S

3. Wright SDE for the portfolio, noting that the α is defined at the beginning of the time interval and therefore is constant during that period

4. Choose α in such a way, that removes the stochastic term

5. Equate the drift of riskless portfolio to the risk-free rate

6. Rearrange the equation

6.1.2   Derivation:

1. Take a derivative $C(S, t)$:

$$C(S, t) \tag{6.1}$$

2. Construct a portfolio consisting of the derivative C ans α stocks S:

$$P(C, S) = C + \alpha S \tag{6.2}$$

3. Wright SDE for the portfolio, noting that the α is defined at the beginning of the time interval and is constant during that period

$$dP = d(C + \alpha S) = \left[\frac{\partial C(S, t)}{dt} + \mu S \frac{\partial C(S, t)}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + \alpha \mu S \right] dt$$
$$+ \sigma S \left[\frac{\partial C(S, t)}{\partial S} + \alpha \right] dW_t \tag{6.3}$$

4. Choose α in such a way, that removes the stochastic term:

$$\alpha = -\frac{\partial C(S, t)}{\partial S} \tag{6.4}$$

This will also cancel some drift terms:

$$d(C + \alpha S) = \left[\frac{\partial C(S, t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} \right] dt \tag{6.5}$$

5. Equate the drift of riskless portfolio to the risk-free rate:

$$\frac{\partial C(S, t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} = r \left[C - S \frac{\partial C(S, t)}{\partial S} \right] \tag{6.6}$$

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6. Rearrange the equation:

$$\frac{\partial C(S,t)}{\partial t} + rS\frac{\partial C(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} = rC \qquad (6.7)$$

Solution

Steps:

- Change the variables to use $x = \log(S)$ and revert the time to move to the backward time rescaled by the factor $\frac{\sigma^2}{2}$

- Introduce $k = \frac{2r}{\sigma^2}$ and transform the boundary condition

- Change variable $v(x,t) = e^{\alpha x + \beta t} u(x,t)$ and get rid off the terms that are not in the heat equation

- Solve the heat equation

- Transform back

6.1.3   Solution:

The details of the solution could be found at:

"http://www.math.tamu.edu/ stecher/blackScholesHeatEquation.pdf425/Sp12/"

- ***Change variables to use*** $\log(S)$ ***and rescaled reverted time*** τ***:***

$$S \;=\; e^x \Rightarrow x = \log(S) \qquad (6.8)$$
$$\tau \;=\; \frac{\sigma^2}{2}(T - t) \qquad (6.9)$$
$$C(S,t) \;=\; v(x,\tau) \qquad (6.10)$$

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Then:

$$\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2} \tag{6.11}$$

$$\frac{\partial x}{\partial S} = \frac{1}{S} \tag{6.12}$$

The partial derivatives are:

$$\frac{\partial C}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} \tag{6.13}$$

$$\frac{\partial C}{\partial S} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial v}{\partial x} \tag{6.14}$$

$$\frac{\partial}{\partial S} = \frac{1}{S}\frac{\partial}{\partial x} \tag{6.15}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial v}{\partial x}\right)$$

$$= -\frac{1}{S^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x}\right)$$

$$= -\frac{1}{S^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{S^2}\left(\frac{\partial^2 v}{\partial x^2}\right) \tag{6.16}$$

Putting everything together, we get:

$$-\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} + rS\frac{1}{S}\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{S^2}\left(\frac{\partial^2 v}{\partial x^2}\right)\right) = rv \tag{6.17}$$

or

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v \tag{6.18}$$

- **Introduce $k = \frac{2r}{\sigma^2}$ and transform the boundary condition:**

  Setting:

$$k = \frac{2r}{\sigma^2} \tag{6.19}$$

$$t = \tau \tag{6.20}$$

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we get:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, -\infty < x < \infty, 0 \leq t \leq \frac{\sigma^2}{2}T \quad (6.21)$$

$$v(x,0) = C(e^x, T) = f(e^x), -\infty < x < \infty \quad (6.22)$$

- **Change variable $v(x,t) = e^{\alpha x + \beta t}u(x,t)$ and get rid off the terms that are not in the heat equation:**

Setting:

$$v(x,t) = e^{\alpha x + \beta t}u(x,t) = \phi u \quad (6.23)$$

we get:

$$\frac{\partial v}{\partial t} = \beta \phi u + \phi \frac{\partial u}{\partial t} \quad (6.24)$$

$$\frac{\partial v}{\partial x} = \alpha \phi u + \phi \frac{\partial u}{\partial x} \quad (6.25)$$

$$\frac{\partial^2 v}{\partial x^2} = \alpha^2 \phi u + 2\alpha \phi \frac{\partial u}{\partial x} + \phi \frac{\partial^2 u}{\partial x^2} \quad (6.26)$$

Placing these expressions into the pde and setting:

$$\alpha = -\frac{1}{2}(k-1) = \frac{\sigma^2 - 2r}{2\sigma^2} \quad (6.27)$$

$$\beta = -\frac{1}{4}(k+1)^2 = -\left(\frac{\sigma^2 + 2r}{2\sigma^2}\right)^2 \quad (6.28)$$

we have:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, 0 \leq t \leq \frac{\sigma^2}{2}T \quad (6.29)$$

$$u(x,0) = e^{-\alpha x}v(x,0) = e^{-\alpha x}f(e^x), -\infty < x < \infty \quad (6.30)$$

Chapter 7

Fundamental theorems of asset pricing

A discrete market is the market with finite state.

1. A discrete market, on a discrete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, is arbitrage-free if, and only if, there exists at least one risk neutral probability measure that is equivalent to the original probability measure $\mathbb{P}$.

2. An arbitrage-free market (S, B) consisting of a collection of stocks S and a risk-free bond B is complete if and only if there exists a unique risk-neutral measure that is equivalent to $\mathbb{P}$ and has numeraire B.

7.1 Drift of a risky asset under the risk-free probability measure

7.1.1 Steps

- Put SDE-s for a stock and a bond

- Write SDE for $f = \frac{S_t}{B_t}$

- Require it to be a martingale

We have:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{7.1}$$

$$dB_t = r B_t dt \tag{7.2}$$

and therefore

$$d\left(\frac{S_t}{B_t}\right) = \frac{dS_t}{B_t} + S_t d\left(\frac{1}{B_t}\right) \tag{7.3}$$

With no additional Ito cross terms since B_t is deterministic.

Since:

$$B_t = B_0 e^{rt} \tag{7.4}$$

we get:

$$d(B_t^{-1}) = -r B_t^{-2} B_t dt = -r B_t^{-1} dt \tag{7.5}$$

Thus

$$d\left(\frac{S_t}{B_t}\right) \;=\; (\mu - r)\frac{S_t}{B_t}dt + \sigma\frac{S_t}{B_t}dW_t \tag{7.6}$$

Chapter 8

Black-Scholes formula derivation

8.1 Steps

- Write a value of an option as a discounted payoff expectation at maturity

- Write the distribution of the stock price at maturity

- Put it into the discounted payoff at maturity

- Rewrite the discounted payoff at maturiy using LOTUS (**note x appearing in the integral instead of the normal distribution $N(0,1)$**)

- Change the integration bounds to get rid off the max function under the integral

- Solve the integral bounds for x

- Denote the right-hand side of the inequality as l

- Split the integral into two parts - one with x and with strike K

- Note that the second part is just a normal CDF from l to ∞. Find its value as $N(-l)$ using evenness of CDF and replacing the integral $\int_l^\infty$ with $\int_{-\infty}^{-l}$

- Solve the first integral part by making the part in the exponent a full square

8.2 Derivation

- **Write a value of an option as a discounted payoff expectation at maturity:**

  A price of a call option at time 0 is defined as:

  $$C \quad = \quad e^{-rT}\mathbb{E}((S_T - K)_+) \tag{8.1}$$

- **Write the distribution of the stock price at maturity**:

  In the risk-neutral world we have that:

  $$S_t \quad = \quad S_0 exp\left(rT - \frac{1}{2}\frac{\sigma^2}{2}T + \sigma\sqrt{T}N(0,1)\right) \tag{8.2}$$

- **Put it into the discounted payoff at maturity**:

  The value of our option is:

  $$\frac{B_0}{B_T}\mathbb{E}\left[\left(S_0 exp\left(rT - \frac{1}{2}\frac{\sigma^2}{2}T + \sigma\sqrt{T}N(0,1)\right) - K\right)_+\right] \tag{8.3}$$

- **Rewrite the discounted payoff at maturiy using LOTUS**:

Recalling that the density of $N(0,1)$ is $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, we can write this as:

$$\frac{e^{-rT}}{\sqrt{2\pi}}\int e^{-\frac{x^2}{2}}\left(S_0 exp\left(rT - \frac{1}{2}\frac{\sigma^2}{2}T + \sigma\sqrt{T}x\right) - K\right)_+ dx \qquad (8.4)$$

- **Change the integration bounds to get rid off the max function under the integral**:

  The integral is non-zero if and only if:

$$S_0 exp\left(rT - \frac{1}{2}\frac{\sigma^2}{2}T + \sigma\sqrt{T}x\right) \geq K \qquad (8.5)$$

- **Solve the integral bounds for x:**

  Thus the integral must be taken over:

$$x \geq \frac{\log K/S_0 + \frac{1}{2}\sigma^2 T - rT}{\sigma\sqrt{T}} \qquad (8.6)$$

- **Denote the right-hand side of the inequality as l**

- **Split the integral into two parts - one with x and with strike K:**

  Our integral now has two terms. The second simple term is just:

$$\frac{e^{-rT}}{\sqrt{2\pi}}\int_l^\infty e^{-\frac{x^2}{2}}K dx \qquad (8.7)$$

- **Note that the second part is just a normal CDF from l to ∞. Find its value as $N(-l)$ using evenness of CDF and replacing the integral $\int_l^\infty$ with $\int_{-\infty}^{-l}$:**

  The second term is therefore equal to:

$$e^{-rT}KN\left(\frac{\log S_0/K + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \qquad (8.8)$$

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- *Solve the first integral part by making the part in the exponent a full square*:

Changing the variables:

$$x = \bar{x} + \sigma\sqrt{T} \qquad (8.9)$$

we get:

$$\frac{e^{-rT}}{\sqrt{T}} \int_{l-\sigma\sqrt{T}}^{\infty} e^{-\frac{\bar{x}^2}{2}} S_0 e^{rT} d\bar{x} \qquad (8.10)$$

Chapter 9

Things to add

- American options

- ito process

- ito calculus

- Feynman-Kac theorem

- chooser option

Part II

FX

Chapter 10

Risk neutral measures

10.1 Domestic risk neutral measure

The domestic investor sees the foreign bond B_t^f as a risky asset, which, denominated in the domestic currency, is valued at $B_t^f S_t$. Where S_t is the FX exchange rate.

Construct the ratio of this against the domestic bond:

$$
\begin{aligned}
Z_t &= S_t B_t^f / B_t^d \\
&= S_0 exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2 t\right)\right) exp\left(\left(r^f - r^d\right)t\right) \\
&= S_0 exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right) exp((\mu + r^f - r^d)t) \qquad (10.1)
\end{aligned}
$$

To attain the martingale property we require that

$$
\mu = \mu^d \equiv r^d - r^f \qquad (10.2)
$$

(μ^d is used to avoid the confusion below), so under the domestic risk-neutral measure $\mathbb{P}^d$ we write

$$S_t = S_0 exp\left(\sigma W_t^d + \left(r^d - r^f - \frac{1}{2}\sigma^2\right)\right) \qquad (10.3)$$

$$dS_t = \left(r^d - r^f\right)S_t dt + \sigma S_t dW_t^d \qquad (10.4)$$

The drift change required is:

$$W_t^d = W_t + \frac{\mu - \mu^d}{\sigma}t \qquad (10.5)$$

which gives a Radon-Nikodym derivative at time T of

$$\frac{d\mathbb{P}^d}{d\mathbb{P}} = exp\left(-\gamma^d W_T - \frac{1}{2}[\gamma^d]^2 T\right) \qquad (10.6)$$

where:

$$\gamma^d = \frac{\mu - \mu^d}{\sigma} \qquad (10.7)$$

10.2 Foreign risk-neutral measure

The foreign investor sees the domestic bond B_t^d as the risky asset, which denominated in foreign currency is valued at B_t^d/S_t or, equivalently, $B_t^d \hat{S}_t$, where $\hat{S}_t) = 1/S_t$ denotes the flipped spot rate. By taking the reciprocal we have

$$B_t^d \hat{S}_t = \hat{S}_0 exp\left(-\sigma W_t + \left(\frac{1}{2}\sigma^2 - \mu + r^d\right)t\right) \qquad (10.8)$$

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Construct the ratio of this quantity divided by the foreign bond B_t^f:

$$\begin{aligned}
\hat{Z} &= \hat{S}_t B_t^d / B_t^f \\
&= \hat{S}_0 exp\left(-\sigma W_t + \left(\frac{1}{2}\sigma^2 - \mu + r^d - r^f\right)t\right) \\
&= \hat{S}_0 exp\left(-\sigma W_t - \frac{1}{2}\sigma^2 t\right) exp\left(\left(-\mu + r^d - r^f + \sigma^2\right)t\right) \quad (10.9)
\end{aligned}$$

This indicates that $\hat{Z}$ is a martingale if

$$\mu = \mu^f \equiv r^d - r^f + \sigma^2 \qquad (10.10)$$

So, under the foreign risk-neutral measure $\mathbb{P}^f$ we write:

$$\hat{S}_t = S_0 exp\left(-\sigma W_t^f + \left(\frac{1}{2}\sigma^2 - \mu^f\right)t\right) \qquad (10.11)$$

or, alternatively, we can express the non-flipped sport rate

$$S_t = exp\left(\sigma W_t^f + \left(\mu^f - \frac{1}{2}\sigma^2\right)t\right) \qquad (10.12)$$

$$dS_t = (r^d - r^f + \sigma^2)S_t dt + \sigma S_t dW_t^f \qquad (10.13)$$

The drift change required is

$$W_t^f = W_t + \frac{\mu - \mu^f}{\sigma}t \qquad (10.14)$$

which gives a Radon-Nikodym derivative at time T of

$$\frac{d\mathbb{P}^f}{d\mathbb{P}} = exp\left(-\gamma^f W_T - \frac{1}{2}[\gamma^f]^2 T\right) \qquad (10.15)$$

where

$$\gamma^f \quad = \quad \frac{\mu - \mu^f}{\sigma} \qquad\qquad (10.16)$$

Chapter 11

Things to add - FX

- Garman-Kolhagen formula derivation

- pricing quotations

- delta quotations

- quanto option

- compo option

- Heston

Part III

MC

Chapter 12

Things to add - MC

- convergence

- brownian bridge

- variance reduction techniques

- sobol and mersenn twister

Part IV

Volatility modelling

Chapter 13

Thins to add

- local vol

- loc-stoch vol

Part V

PDE

Chapter 14

To understand

- Characteristic function

- bbrr