Contents

Part I

Stochastic Calculus

Chapter 1

Brownian Motion

1.1 Definition

The process $W = (W_t : t > 0)$ is a $\mathbb{P}$***-Brownian motion*** *if and only if:*

1. *W_t is* ***continuous*** *and $W_0 = 0$*
2. *the value of W_t is distributed, under $\mathbb{P}$, as a* ***normal random variable*** *$N(0, t)$:*

$$W_t \sim N(0, t) \tag{1.1}$$

3. *the* ***increments*** *$W_{s+t} - W_s$* ***are normally distributed with variance*** *$t - s$ as a $N(0, t)$ under $\mathbb{P}$:*

$$W_t - W_s \sim N(0, t - s) \tag{1.2}$$

4. ***the increments*** *$W_{s+t} - W_s$ are* ***independent*** *of the behaviour of W_r for*

$r \leq s$:

$$W_{s+t} - W_s \neq f(W_r), \forall r \leq s \tag{1.3}$$

1.2 Quadratic Variation

1.3 BM properties

1.4 BM Martingales

1.4.1 Martingale

A stochastic process $X(t), \forall t \geq 0$ is a martingale if

1. for any t it is integrable (it has a finite expectation):

$$\mathbf{E}(X(t)) < \infty \tag{1.4}$$

2. and for any $s > 0$

$$\mathbb{E}(X(t+s)|\mathscr{F}_t) \quad = \quad X(t) \tag{1.5}$$

1.4.2 BM Martingales

The list is taken from

"Fima Klebaner. Introduction to Stochastic Calculus with Applications. p.64"

- W_t - is a martingale

- $W_t^2 - t$ - is a martingale

- for any u, $e^{uW_t - \frac{u^2}{2}t}$ - is a martingale

Proof: The key idea in establishing the martingale property is that for any function g, the conditional expectation of $g(W_{t+s} - W_t)$ given $\mathscr{F}_s$ equals to the unconditional one.

- ***W_t is a martingale.***

  We have to show that

$$\mathbb{E}(W_s|\mathscr{F}_s) = W_s, \forall s < t \tag{1.6}$$

We can write:

$$W_t = W_s + (W_t - W_s) \tag{1.7}$$

Using the linearity of expectation, we have:

$$\mathbb{E}(W_t|\mathscr{F}_s) = \mathbb{E}(W_s|\mathscr{F}_s) + \mathbb{E}(W_t - W_s|\mathscr{F}_s) \tag{1.8}$$

The first term on the right-hand side is W_s since W_s contains no information not contained in $\mathscr{F}_s$.

Now we show that:

$$\mathbb{E}(W_t - W_s) = 0 \tag{1.9}$$

Recall that

$$(W_t - W_s) \sim N(0, t - s) \tag{1.10}$$

As $W_t - W_s$ is independent of the value of W_s and the path up to time s, so when we condition on information available at time s, we are effectively conditioning on no information. This means that:

$$\mathbb{E}[(W_t - W_s)|\mathscr{F}_s] = \mathbb{E}(W_t - W_s) = 0 \tag{1.11}$$

Thus BM is a martingale.

- Process in the form:

$$dX_t = \sigma(X,t)dW_t \tag{1.12}$$

is a martingale.

1.5 TODO

- BM construction
- Joshi1.page155 - creation of martingales

1.6 Problems

- is $Z = \sqrt{t}N(0,1)$ a BM?
- is $X_t = \rho W_1 + \sqrt{1-\rho^2}\tilde{W}_t$ a BM?

Chapter 2

Conditional expectations

TODO - Joshi1.page155 - creation of martingales.

2.1 Properties

1. Conditional expectation based on **no information** is the ordinary expectation:

$$\mathbb{E}(X|\mathscr{F}_0) = \mathbb{E}(X) \tag{2.1}$$

2. **The Tower Law** - if $s < t$ and we first take the conditional expectation at time t followed by the conditional expectation at time s, then this is the same as taking the conditional expectation at time s:

$$\mathbb{E}(\mathbb{E}(X|\mathscr{F}_t)|\mathscr{F}_s) = \mathbb{E}(X|\mathscr{F}_s) \tag{2.2}$$

3. If we **condition on information that is independent of the value**

of the random variable then we get the same value as conditioning on no information.

Random variable independent of the information. If changing the path up to time s does not affect the value of the random variable, then the random variable is independent of $\mathscr{F}_s$.

So if X is independent, we have:

$$\mathbb{E}(X|\mathscr{F}_s) = \mathbb{E}(X) \tag{2.3}$$

4. If **the random variable is determined by the information in $\mathscr{F}_s$**, then conditioning on that information will have no effect, and therefore:

$$\mathbb{E}(X|\mathscr{F}_s) = X \tag{2.4}$$

Chapter 3

Ito calculus

3.1 Ito calculus

3.1.1 Product Rule

$$d(XY) = XdY + YdX + dXdY \tag{3.1}$$

3.1.2 Inversion Rule

If

$$f = \frac{1}{Y} = Y^{-1} \tag{3.2}$$

then

$$\frac{\partial f}{\partial Y} = -Y^{-2} = -\frac{1}{Y^2} \tag{3.3}$$

$$\frac{\partial^2 f}{\partial Y^2} = 2Y^{-3} = \frac{2}{Y^3} \tag{3.4}$$

and

$$\begin{aligned} df &= \frac{\partial f}{\partial Y} dY + \frac{1}{2}\frac{\partial^2 f}{\partial Y^2} dY^2 \\ &= -\frac{dY}{Y^2} + \frac{dY^2}{Y^3} \end{aligned} \tag{3.5}$$

3.1.3 Quotient Rule

$$d\left(\frac{X}{Y}\right) = \frac{X}{Y}\left[\frac{dX}{X} - \frac{dY}{Y} - \frac{dX}{X}\frac{dY}{Y} + \left(\frac{dY}{Y}\right)^2\right] \tag{3.6}$$

3.1.4 Ito Isometry

3.1.5 CEV Process

Chapter 4

Girsanov Theorem

If

- W_t *- is a Brownian motion with sample space* Ω *and measure* $\mathbb{P}$*,*
- ν *- is a reasonable function*

then there exists an equivalent measure $\mathbb{Q}$ *on* Ω *such that*

$$\tilde{W}_t = W_t - \nu t \tag{4.1}$$

is a Brownian motion.

Chapter 5

Radon-Nikodym Derivative

5.1 Radon-Nikodym Theorem

If a ***probability measure*** $\mathbb{Q}$ *is* ***absolutely continuous (equivalent)*** *with respect to a probability measure* $\mathbb{P}$*, then it can be written as:*

$$\mathbb{Q} = \int_E f d\mathbb{P} \tag{5.1}$$

5.2 Radon-Nikodym Derivative

By analogy with the first fundamental theorem of calculus, the function f is called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$:

$$f = \frac{d\mathbb{Q}}{d\mathbb{P}} \tag{5.2}$$

A measure change consists of re-weighting the probability of paths. We therefore construct them by multiplying probabilities by a random variable.

Chapter 6

BS PDE

6.1 Derivation

6.1.1 Steps to derive the equation:

1. Take a derivative $C(S, t)$
2. Construct a portfolio consisting of the derivative C ans α stocks S
3. Wright SDE for the portfolio, noting that the α is defined at the beginning of the time interval and therefore is constant during that period
4. Choose α in such a way, that removes the stochastic term
5. Equate the drift of riskless portfolio to the risk-free rate
6. Rearrange the equation

6.1.2 Derivation:

1. Take a derivative $C(S,t)$:

$$C(S,t) \tag{6.1}$$

2. Construct a portfolio consisting of the derivative C ans α stocks S:

$$P(C,S) = C + \alpha S \tag{6.2}$$

3. Wright SDE for the portfolio, noting that the α is defined at the beginning of the time interval and is constant during that period

$$dP = d(C+\alpha S) = \Big[\frac{\partial C(S,t)}{dt} + \mu S\frac{\partial C(S,t)}{dS} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C(S,t)}{\partial S^2} + \alpha\mu S\Big]dt + \sigma S\Big[\frac{\partial C(S,t)}{\partial S} + \alpha\Big]dW_t \tag{6.3}$$

4. Choose α in such a way, that removes the stochastic term:

$$\alpha = -\frac{\partial C(S,t)}{\partial S} \tag{6.4}$$

This will also cancel some drift terms:

$$d(C+\alpha S) = \Big[\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C(S,t)}{\partial S^2}\Big]dt \tag{6.5}$$

5. Equate the drift of riskless portfolio to the risk-free rate:

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C(S,t)}{\partial S^2} = r\Big[C - S\frac{\partial C(S,t)}{\partial S}\Big] \tag{6.6}$$

6. Rearrange the equation:

$$\frac{\partial C(S,t)}{\partial t} + rS\frac{\partial C(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} = rC \tag{6.7}$$

Solution

Steps:

- Change the variables to use $x = \log(S)$ and revert the time to move to the backward time rescaled by the factor $\frac{\sigma^2}{2}$
- Introduce $k = \frac{2r}{\sigma^2}$ and transform the boundary condition
- Change variable $v(x,t) = e^{\alpha x + \beta t} u(x,t)$ and get rid off the terms that are not in the heat equation
- Solve the heat equation
- Transform back

6.1.3 Solution:

The details of the solution could be found at:

"http://www.math.tamu.edu/ stecher/blackScholesHeatEquation.pdf425/Sp12/"

- ***Change variables to use*** $\log(S)$ ***and rescaled reverted time*** τ***:***

$$\begin{aligned} S &= e^x \Rightarrow x = \log(S) && (6.8) \\ \tau &= \frac{\sigma^2}{2}(T-t) && (6.9) \\ C(S,t) &= v(x,\tau) && (6.10) \end{aligned}$$

Then:

$$\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2} \tag{6.11}$$

$$\frac{\partial x}{\partial S} = \frac{1}{S} \tag{6.12}$$

The partial derivatives are:

$$\frac{\partial C}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} \tag{6.13}$$

$$\frac{\partial C}{\partial S} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial v}{\partial x} \tag{6.14}$$

$$\frac{\partial}{\partial S} = \frac{1}{S}\frac{\partial}{\partial x} \tag{6.15}$$

$$\begin{aligned}
\frac{\partial^2 C}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial v}{\partial x}\right) \\
&= -\frac{1}{S^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x}\right) \\
&= -\frac{1}{S^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{S^2}\left(\frac{\partial^2 v}{\partial x^2}\right)
\end{aligned} \tag{6.16}$$

Putting everything together, we get:

$$-\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} + rS\frac{1}{S}\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{S^2}\left(\frac{\partial^2 v}{\partial x^2}\right)\right) = rv \tag{6.17}$$

or

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v \tag{6.18}$$

- ***Introduce $k = \frac{2r}{\sigma^2}$ and transform the boundary condition:***

  Setting:

$$k = \frac{2r}{\sigma^2} \tag{6.19}$$

$$t = \tau \tag{6.20}$$

we get:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, -\infty < x < \infty, 0 \le t \le \frac{\sigma^2}{2}T \quad (6.21)$$

$$v(x,0) = C(e^x, T) = f(e^x), -\infty < x < \infty \quad (6.22)$$

- ***Change variable*** $v(x,t) = e^{\alpha x + \beta t}u(x,t)$ ***and get rid off the terms that are not in the heat equation:***

  Setting:

$$v(x,t) = e^{\alpha x + \beta t}u(x,t) = \phi u \quad (6.23)$$

we get:

$$\frac{\partial v}{\partial t} = \beta \phi u + \phi \frac{\partial u}{\partial t} \quad (6.24)$$

$$\frac{\partial v}{\partial x} = \alpha \phi u + \phi \frac{\partial u}{\partial x} \quad (6.25)$$

$$\frac{\partial^2 v}{\partial x^2} = \alpha^2 \phi u + 2\alpha\phi \frac{\partial u}{\partial x} + \phi \frac{\partial^2 u}{\partial x^2} \quad (6.26)$$

Placing these expressions into the pde and setting:

$$\alpha = -\frac{1}{2}(k-1) = \frac{\sigma^2 - 2r}{2\sigma^2} \quad (6.27)$$

$$\beta = -\frac{1}{4}(k+1)^2 = -\left(\frac{\sigma^2 + 2r}{2\sigma^2}\right)^2 \quad (6.28)$$

we have:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, 0 \le t \le \frac{\sigma^2}{2}T \quad (6.29)$$

$$u(x,0) = e^{-\alpha x}v(x,0) = e^{-\alpha x}f(e^x), -\infty < x < \infty \quad (6.30)$$

Chapter 7

Fundamental theorems of asset pricing

A discrete market is the market with finite state.

1. A discrete market, on a discrete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, is arbitrage-free if, and only if, there exists at least one risk neutral probability measure that is equivalent to the original probability measure $\mathbb{P}$.

2. An arbitrage-free market (S, B) consisting of a collection of stocks S and a risk-free bond B is complete if and only if there exists a unique risk-neutral measure that is equivalent to $\mathbb{P}$ and has numeraire B.

7.1 Drift of a risky asset under the risk-free probability measure

7.1.1 Steps

- Put SDE-s for a stock and a bond
- Write SDE for $f = \frac{S_t}{B_t}$
- Require it to be a martingale

We have:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (7.1)$$

$$dB_t = r B_t dt \quad (7.2)$$

and therefore

$$d\left(\frac{S_t}{B_t}\right) = \frac{dS_t}{B_t} + S_t d\left(\frac{1}{B_t}\right) \quad (7.3)$$

With no additional Ito cross terms since B_t is deterministic.

Since:

$$B_t = B_0 e^{rt} \quad (7.4)$$

we get:

$$d(B_t^{-1}) = -rB_t^{-2} B_t dt = -rB_t^{-1} dt \quad (7.5)$$

Thus

$$d\left(\frac{S_t}{B_t}\right) = (\mu - r)\frac{S_t}{B_t}dt + \sigma\frac{S_t}{B_t}dW_t \tag{7.6}$$

Chapter 8

Black-Scholes formula derivation

8.1 Steps

- Write a value of an option as a discounted payoff expectation at maturity
- Write the distribution of the stock price at maturity
- Put it into the discounted payoff at maturity
- Rewrite the discounted payoff at maturiy using LOTUS (**note x appearing in the integral instead of the normal distribution $N(0, 1)$**)
- Change the integration bounds to get rid off the max function under the integral
- Solve the integral bounds for x
- Denote the right-hand side of the inequality as l

- Split the integral into two parts - one with x and with strike K
- Note that the second part is just a normal CDF from l to ∞. Find its value as $N(-l)$ using evenness of CDF and replacing the integral $\int_l^\infty$ with $\int_{-\infty}^{-l}$
- Solve the first integral part by making the part in the exponent a full square

8.2 Derivation

- ***Write a value of an option as a discounted payoff expectation at maturity:***

  A price of a call option at time 0 is defined as:

$$C = e^{-rT}\mathbb{E}((S_T - K)_+) \tag{8.1}$$

- ***Write the distribution of the stock price at maturity***:

  In the risk-neutral world we have that:

$$S_t = S_0 exp\left(rT - \frac{1}{2}\frac{\sigma^2}{2}T + \sigma\sqrt{T}N(0,1)\right) \tag{8.2}$$

- ***Put it into the discounted payoff at maturity***:

  The value of our option is:

$$\frac{B_0}{B_T}\mathbb{E}\left[\left(S_0 exp\left(rT - \frac{1}{2}\frac{\sigma^2}{2}T + \sigma\sqrt{T}N(0,1)\right) - K\right)_+\right] \tag{8.3}$$

- ***Rewrite the discounted payoff at maturiy using LOTUS***:

Recalling that the density of $N(0,1)$ is $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, we can write this as:

$$\frac{e^{-rT}}{\sqrt{2\pi}}\int e^{-\frac{x^2}{2}}\left(S_0exp\left(rT-\frac{1}{2}\frac{\sigma^2}{2}T+\sigma\sqrt{T}x\right)-K\right)_+dx \tag{8.4}$$

- ***Change the integration bounds to get rid off the max function under the integral***:

  The integral is non-zero if and only if:

$$S_0exp\left(rT-\frac{1}{2}\frac{\sigma^2}{2}T+\sigma\sqrt{T}x\right)\geq K \tag{8.5}$$

- ***Solve the integral bounds for x:***

  Thus the integral must be taken over:

$$x\geq\frac{\log K/S_0+\frac{1}{2}\sigma^2T-rT}{\sigma\sqrt{T}} \tag{8.6}$$

- ***Denote the right-hand side of the inequality as l***

- ***Split the integral into two parts - one with x and with strike K***:

  Our integral now has two terms. The second simple term is just:

$$\frac{e^{-rT}}{\sqrt{2\pi}}\int_l^\infty e^{-\frac{x^2}{2}}Kdx \tag{8.7}$$

- ***Note that the second part is just a normal CDF from l to ∞. Find its value as $N(-l)$ using evenness of CDF and replacing the integral $\int_l^\infty$ with $\int_{-\infty}^{-l}$***:

  The second term is therefore equal to:

$$e^{-rT}KN\left(\frac{\log S_0/K+rT-\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right) \tag{8.8}$$

- ***Solve the first integral part by making the part in the exponent a full square***:

  Changing the variables:

$$x = \bar{x} + \sigma\sqrt{T} \tag{8.9}$$

  we get:

$$\frac{e^{-rT}}{\sqrt{T}} \int_{l-\sigma\sqrt{T}}^{\infty} e^{-\frac{\bar{x}^2}{2}} S_0 e^{rT} d\bar{x} \tag{8.10}$$

Chapter 9

Things to add

- American options
- Ito process
- Ito calculus
- Feynman-Kac theorem
- chooser option
- brownian bridge

Part II

FX

Chapter 10

Risk neutral measures

10.1 Domestic risk neutral measure

The domestic investor sees the foreign bond B_t^f as a risky asset, which, denominated in the domestic currency, is valued at $B_t^f S_t$. Where S_t is the FX exchange rate.

Construct the ratio of this against the domestic bond:

$$\begin{aligned} Z_t &= S_t B_t^f / B_t^d \\ &= S_0 exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2 t\right)\right) exp\left(\left(r^f - r^d\right)t\right) \\ &= S_0 exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right) exp((\mu + r^f - r^d)t) \end{aligned} \tag{10.1}$$

To attain the martingale property we require that

$$\mu = \mu^d \equiv r^d - r^f \tag{10.2}$$

(μ^d is used to avoid the confusion below), so under the domestic risk-neutral measure $\mathbb{P}^d$ we write

$$S_t = S_0 exp\bigg(\sigma W_t^d + \Big(r^d - r^f - \frac{1}{2}\sigma^2\Big)\bigg) \tag{10.3}$$

$$dS_t = \Big(r^d - r^f\Big)S_t dt + \sigma S_t dW_t^d \tag{10.4}$$

The drift change required is:

$$W_t^d = W_t + \frac{\mu - \mu^d}{\sigma} t \tag{10.5}$$

which gives a Radon-Nikodym derivative at time T of

$$\frac{d\mathbb{P}^d}{d\mathbb{P}} = exp\bigg(- \gamma^d W_T - \frac{1}{2}[\gamma^d]^2 T\bigg) \tag{10.6}$$

where:

$$\gamma^d = \frac{\mu - \mu^d}{\sigma} \tag{10.7}$$

10.2 Foreign risk-neutral measure

The foreign investor sees the domestic bond B_t^d as the risky asset, which denominated in foreign currency is valued at B_t^d/S_t or, equivalently, $B_t^d \hat{S}_t$, where $\hat{S}_t) = 1/S_t$ denotes the flipped spot rate. By taking the reciprocal we have

$$B_t^d \hat{S}_t = \hat{S}_0 exp\bigg(- \sigma W_t + \Big(\frac{1}{2}\sigma^2 - \mu + r^d\Big)t\bigg) \tag{10.8}$$

Construct the ratio of this quantity divided by the foreign bond B_t^f:

$$\begin{aligned}
\hat{Z} &= \hat{S}_t B_t^d / B_t^f \\
&= \hat{S}_0 exp\bigg(-\sigma W_t + \Big(\frac{1}{2}\sigma^2 - \mu + r^d - r^f\Big)t\bigg) \\
&= \hat{S}_0 exp\bigg(-\sigma W_t - \frac{1}{2}\sigma^2 t\bigg) exp\bigg(\Big(-\mu + r^d - r^f + \sigma^2\Big)t\bigg) \qquad (10.9)
\end{aligned}$$

This indicates that $\hat{Z}$ is a martingale if

$$\mu = \mu^f \equiv r^d - r^f + \sigma^2 \qquad (10.10)$$

So, under the foreign risk-neutral measure $\mathbb{P}^f$ we write:

$$\hat{S}_t = S_0 exp\bigg(-\sigma W_t^f + \Big(\frac{1}{2}\sigma^2 - \mu^f\Big)t\bigg) \qquad (10.11)$$

or, alternatively, we can express the non-flipped sport rate

$$\begin{aligned}
S_t &= exp\bigg(\sigma W_t^f + \Big(\mu^f - \frac{1}{2}\sigma^2\Big)t\bigg) \qquad (10.12) \\
dS_t &= (r^d - r^f + \sigma^2) S_t dt + \sigma S_t dW_t^f \qquad (10.13)
\end{aligned}$$

The drift change required is

$$W_t^f = W_t + \frac{\mu - \mu^f}{\sigma} t \qquad (10.14)$$

which gives a Radon-Nikodym derivative at time T of

$$\frac{d\mathbb{P}^f}{d\mathbb{P}} = exp\bigg(-\gamma^f W_T - \frac{1}{2}[\gamma^f]^2 T\bigg) \qquad (10.15)$$

where

$$\gamma^f = \frac{\mu - \mu^f}{\sigma} \tag{10.16}$$

Chapter 11

Things to add - FX

- Garman-Kolhagen formula derivation
- pricing quotations
- delta quotations
- quanto option
- compo option
- Heston

Part III

MC

Chapter 12

Things to add - MC

- convergence
- brownian bridge
- variance reduction techniques
- sobol and mersenn twister

Part IV

Volatility modelling

Chapter 13

Thins to add

- local vol
- loc-stoch vol

Part V

PDE

Chapter 14

To understand

- Characteristic function
- bbrr