LEVEL I

I 1-10 : Review up to H

The review includes the operations of fractions, simplifying algebraic expressions, simplifying monomials & polynomials, solving literal equations.

I 11-30 : Multiplication of Polynomials

We expand binomial product as follows

$$(a+b)(x+y) = \underset{①}{ax} + \underset{②}{ay} + \underset{③}{bx} + \underset{④}{by}$$

The right side is the result of steps ①, ②, ③ and ④ where each term inside the first brackets is multiplied by each term inside the second brackets. ('Expand' means to remove brackets by multiplying out.)

Note: After expanding and simplifying, we must arrange the terms in order of descending degree in the specific variable.

$$\begin{aligned} &(x-7)(5-x+2x^2) \\ =\ &(x-7)(2x^2-x+5) \\ =\ &2x^3-x^2+5x-14x^2+7x-35 \\ =\ &2x^3-15x^2+12x-35 \end{aligned}$$

Formulas:

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$(a+b)(a-b) = a^2 - b^2$

$(x+a)(x+b) = x^2 + (a+b)x + ab$

$(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$

$(2x-3)^2 = (2x)^2 - 2 \cdot 2x \cdot 3 + 3^2 = 4x^2 - 12x + 9$

$(x^3+2)(x^3-2) = (x^3)^2 - 2^2 = x^6 - 4$

$(x+3y)(x-5y) = x^2 + (3-5)xy - 15y^2 = x^2 - 2xy - 15y^2$

$(3x+5)(x-6) = 3x^2 + (-18+5)x - 30 = 3x^2 - 13x - 30$

I 31-80 : Factorisation

$ax + ay = a(x+y)$ ⬅ "Taking out" the common factor

$a^2 + 2ab + b^2 = (a+b)^2$

$a^2 - 2ab + b^2 = (a-b)^2$

} Perfect Square Formulas

$a^2 - b^2 = (a+b)(a-b)$ ⬅ Difference of Squares Formula

$x^2 + (a+b)x + ab = (x+a)(x+b)$

$3x^2yz^3 - 6x^2y^3z^2 = 3x^2yz^2(z-2y^2)$

$4a(x+y) - 12b(x+y) = 4(x+y)(a-3b)$

$ax^2 - 6axy + 9ay^2 = a(x^2-6xy+9y^2) = a(x-3y)^2$

$9x^4y^2 - 121 = (3x^2y+11)(3x^2y-11)$

$2x^2 - 28x + 80 = 2(x^2-14x+40) = 2(x-10)(x-4)$

To factorise quadratic polynomial, we will use the "cross-multiplication method".

$10x^2 - 7x - 12 = \mathbf{(5x+4)(2x-3)}$

$$\begin{matrix} 5 & & 4 \\ & \times & \\ 2 & & -3 \end{matrix}$$

If you can do this mentally, it is not necessary to write this step.

$5x^2 - 2xy - 3y^2 = \mathbf{(5x+3y)(x-y)}$

$$\begin{matrix} 5 & & 3y \\ & \times & \\ 1 & & -y \end{matrix}$$

$6x^2 - xy - 12y^2 = \mathbf{(3x+4y)(2x-3y)}$

$$\begin{matrix} 3 & & 4y \\ & \times & \\ 2 & & -3y \end{matrix}$$

In factorisation, treat common factors as single units

$$\begin{aligned} a(x-2y)^2 + 2a^2(x-2y) &= a(x-2y)[(x-2y)+2a] \\ &= \mathbf{a(x-2y)(x-2y+2a)} \end{aligned}$$

$$\begin{aligned} (a+b)^2 - (c-d)^2 &= [(a+b)+(c-d)][(a+b)-(c-d)] \\ &= \mathbf{(a+b+c-d)(a+b-c+d)} \end{aligned}$$

$$\begin{aligned} a^2 - 6a(2b-3c) + 9(2b-3c)^2 &= [a-3(2b-3c)]^2 \\ &= \mathbf{(a-6b+9c)^2} \end{aligned}$$

$$\begin{aligned} (x-5)^2 + (x-5) - 12 &= [(x-5)+4][(x-5)-3] \\ &= (x-5+4)(x-5-3) \\ &= \mathbf{(x-1)(x-8)} \end{aligned}$$

Sometimes, we may need to rearrange the terms

When the power is even, the sign of the term remains the same:

$(b-a)^2 = (a-b)^2, \quad (b-a)^4 = (a-b)^4$

When the power is odd, the sign of the term changes:

$(b-a) = -(a-b), \quad (b-a)^3 = -(a-b)^3$

$$\begin{aligned} x(a-b)^2 + 2y(b-a)^3 &= x(a-b)^2 - 2y(a-b)^3 \\ &= (a-b)^2[x-2y(a-b)] \\ &= \mathbf{(a-b)^2(x-2ay+2by)} \end{aligned}$$

$$\begin{aligned} 5x^2y(x-y)^3 - 10xy^3(y-x)^2 &= 5x^2y(x-y)^3 - 10xy^3(x-y)^2 \\ &= 5xy(x-y)^2[x(x-y)-2y^2] \\ &= 5xy(x-y)^2(x^2-xy-2y^2) \\ &= \mathbf{5xy(x-y)^2(x-2y)(x+y)} \end{aligned}$$

Factorisation by grouping

$$\begin{aligned} 2ax - 3by + 3bx - 2ay &= 2ax - 2ay + 3bx - 3by \\ &= 2a(x-y) + 3b(x-y) \\ &= \mathbf{(x-y)(2a+3b)} \end{aligned}$$

$$\begin{aligned} x^3 + x^2 - 4x - 4 &= x^2(x+1) - 4(x+1) \\ &= (x+1)(x^2-4) \\ &= \mathbf{(x+1)(x+2)(x-2)} \end{aligned}$$

I 81-110 : Square Roots

$\sqrt{a}$ is the positive square root of a number, a.

$$\sqrt{144} = 12 \qquad \sqrt{\frac{9}{16}} = \frac{3}{4} \qquad \sqrt{0.0025} = 0.05$$

Note: *Consult a knowledgeable instructor if you have difficulty understanding the algorithm on I89a.*

When simplifying square roots, first identify the largest square factor (e.g. 4, 9, 16, 25, 36, 49, …)

$$\sqrt{48} = \sqrt{16}\sqrt{3} = 4\sqrt{3} \qquad \sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$$
$$\sqrt{72} = \sqrt{36}\sqrt{2} = 6\sqrt{2} \qquad \sqrt{76} = \sqrt{4}\sqrt{19} = 2\sqrt{19}$$

Prime factorisation may be helpful for simplifying large radicands

$$\sqrt{432} = \sqrt{2^4 \times 3^3} = \sqrt{2^4}\sqrt{3^3} = 4 \times 3\sqrt{3} = 12\sqrt{3}$$
$$\sqrt{1050} = \sqrt{2 \times 3 \times 5^2 \times 7} = \sqrt{2} \times \sqrt{3} \times 5 \times \sqrt{7} = 5\sqrt{42}$$

Other techniques include factorising in a way so that we have $\sqrt{a} \times \sqrt{a} = a$ (where $a > 0$).

$$\sqrt{18}\sqrt{24} = \sqrt{3}\sqrt{6} \cdot \sqrt{6}\sqrt{4} = \sqrt{3} \cdot 6 \cdot 2 = 12\sqrt{3}$$
$$\sqrt{15}\sqrt{20}\sqrt{24} = \sqrt{3}\sqrt{5} \cdot 2\sqrt{5} \cdot 2\sqrt{3}\sqrt{2} = 4 \cdot 3 \cdot 5 \cdot \sqrt{2} = 60\sqrt{2}$$

Simplify each term before adding/ subtracting

$$\sqrt{75} + 2\sqrt{108} - 3\sqrt{3} = 5\sqrt{3} + 12\sqrt{3} - 3\sqrt{3}$$
$$= 14\sqrt{3} \quad \Leftarrow \quad 5x + 12x - 3x = 14x$$

Multiplication using formulas from I 21-30

$$(7 + 2\sqrt{3})(7 - 2\sqrt{3}) = 7^2 - (2\sqrt{3})^2 = 49 - 12 = 37$$
$$(3\sqrt{5} - 2)^2 = 45 - 2 \cdot 3\sqrt{5} \cdot 2 + 4 = 49 - 12\sqrt{5}$$
$$(3\sqrt{2} + 2)(5\sqrt{2} + 3) = 30 + (9 + 10)\sqrt{2} + 6 = 36 + 19\sqrt{2}$$

When simplifying fractions involving square roots, always rationalise the denominators.

$$\frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}$$
$$\frac{\sqrt{2}}{\sqrt{6}} = \frac{\sqrt{2} \times \sqrt{6}}{\sqrt{6} \times \sqrt{6}} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$$
$$\sqrt{\frac{25}{18}} = \frac{\sqrt{25}}{\sqrt{18}} = \frac{5}{3\sqrt{2}} = \frac{5 \times \sqrt{2}}{3\sqrt{2} \times \sqrt{2}} = \frac{5\sqrt{2}}{6}$$

Examples of adding and subtracting:

$$\sqrt{\frac{5}{4}} - \sqrt{5} + \sqrt{\frac{4}{5}} = \frac{\sqrt{5}}{2} - \sqrt{5} + \frac{2}{\sqrt{5}}$$
$$= \frac{\sqrt{5}}{2} - \sqrt{5} + \frac{2\sqrt{5}}{5}$$
$$= -\frac{\sqrt{5}}{10} \quad \Leftarrow \quad \frac{1}{2}x - x + \frac{2}{5}x = -\frac{1}{10}x$$

I 111-140 : Quadratic Equations

Technique 1: Factorisation

$$x^2 - 3x = 0$$
$$x(x - 3) = 0$$
$$x = 0, 3$$

$$4x^2 - x - 18 = 0$$
$$(4x - 9)(x + 2) = 0$$
$$x = \frac{9}{4}, -2$$

$$-2x^2 + 48 = 4x$$
$$-2x^2 - 4x + 48 = 0$$
$$x^2 + 2x - 24 = 0$$
$$(x + 6)(x - 4) = 0$$
$$x = -6, 4$$

Technique 2: Completing the square

$$x^2 - 6x - 1 = 0$$
$$x^2 - 6x = 1$$
$$(x - 3)^2 - 9 = 1$$
$$(x - 3)^2 = 10$$
$$x - 3 = \pm\sqrt{10}$$
$$x = 3 \pm \sqrt{10}$$

$$2x^2 - 3x - 1 = 0$$
$$x^2 - \frac{3}{2}x = \frac{1}{2}$$
$$\left(x - \frac{3}{4}\right)^2 - \frac{9}{16} = \frac{1}{2}$$
$$\left(x - \frac{3}{4}\right)^2 = \frac{17}{16}$$
$$x - \frac{3}{4} = \pm\frac{\sqrt{17}}{4}$$
$$x = \frac{3 \pm \sqrt{17}}{4}$$

Technique 3: Quadratic Formula

$$\boxed{\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.}$$

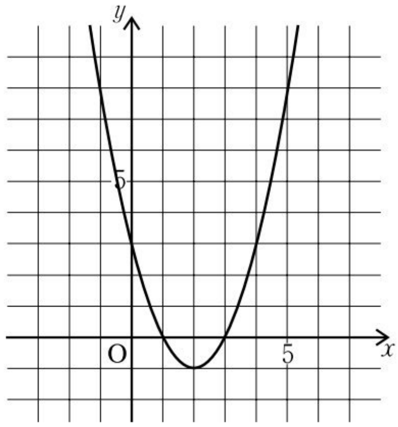
$$3x^2 + 10x + 1 = 0$$
$$x = \frac{-10 \pm \sqrt{100 - 4 \times 3 \times 1}}{6}$$
$$= \frac{-10 \pm 2\sqrt{22}}{6}$$
$$= \frac{-5 \pm \sqrt{22}}{3}$$

I 141-170 : Graphs of Quadratic Functions

Construct a table of points (x, y) and draw the parabola.

$y = x^2 - 4x + 3$

x	y
−1	8
0	3
1	0
2	−1
3	0
4	3
5	8



Given a quadratic function of the form $y = a(x-p)^2 + q$, the vertex is (p, q). We determine the vertex of a quadratic function by completing the square.

$$\begin{aligned} y &= 3x^2 + 7x + 1 \\ &= 3\left(x^2 + \frac{7}{3}x\right) + 1 \\ &= 3\left[\left(x + \frac{7}{6}\right)^2 - \frac{49}{36}\right] + 1 \\ &= 3\left(x + \frac{7}{6}\right)^2 - 3 \times \frac{49}{36} + 1 \\ &= 3\left(x + \frac{7}{6}\right)^2 - \frac{37}{12} \qquad \text{Vertex } \left(-\frac{7}{6}, -\frac{37}{12}\right) \end{aligned}$$

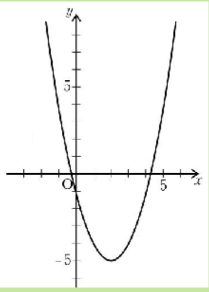
To graph a quadratic function, we determine the (i) vertex, (ii) y-intercept, (iii) x-intercept *by* (i) completing squares, (ii) set x = 0, solve for y, (iii) set y = 0, solve for x.

$\boxed{y = x^2 - 4x - 1}$

Vertex: $(2, -5)$

y-intercept: $(0, -1)$

x-intercept: $(2+\sqrt{5}, 0), (2-\sqrt{5}, 0)$



We can find the points of intersection of a parabola and a line by solving simultaneous equations.

Find the coordinates of the points of intersection of the parabola $y = x^2$ and the line $y = x + 2$.

At the points of intersection,

$$\begin{aligned} x^2 &= x + 2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= 2, -1 \end{aligned}$$

When $x = 2$, $y = 4$

When $x = -1$, $y = 1$

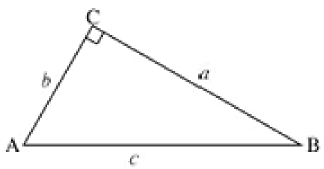
Points of intersection: $(2, 4)$, $(-1, 1)$

I 171-200 : The Pythagorean Theorem

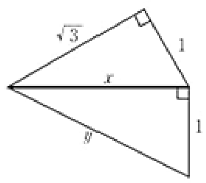
The Pythagorean Theorem

In a right-angled triangle, the square of the length of the hypotenuse (c) equals the sum of the squares of the lengths of the other two sides (a and b).

$$c^2 = a^2 + b^2$$



This is a very important theorem!



Using the Pythagorean Theorem,

$x^2 = 1^2 + (\sqrt{3})^2 = 4$

Since $x > 0$, $x = 2$

(Since the length cannot be a negative number.)

Also, $y^2 = 1^2 + 2^2 = 5$

Since $y > 0$, $y = \sqrt{5}$

$$\begin{cases} x = 2 \\ y = \sqrt{5} \end{cases}$$

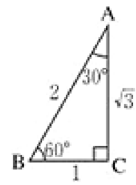
Formula for Area of a Triangle

$$Area = \frac{1}{2} \times b \times h$$

where b = base length and h = altitude

Ratios of Special Triangles

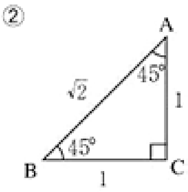
①



In a right-angled triangle with 30° and 60° angles, the ratio of the length of each side to each other is

$1 : \sqrt{3} : 2$

②



In a right-angled triangle with 45° angles, which is called an *isosceles right-angled triangle*, the ratio of the length of each side to each other is

$1 : 1 : \sqrt{2}$

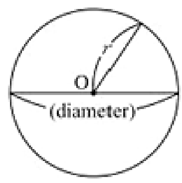
Theorem: If $a : b = c : d$, then $ad = bc$.

Formula for Circumference & Area of a Circle, Area of a Sector

For the circle with radius r,

① its circumference, C, is given by

$C = 2\pi r$

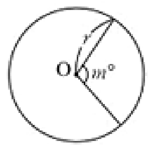


② its area, A, is given by

$A = \pi r^2$

For a circle with radius r, the area, A, of a sector whose central angle measures $m°$ is given by

$$A = \frac{m}{360}\pi r^2$$



We can find the distance between two points on the Cartesian plane using the Pythagorean Theorem.

A(1, 1), B(5, 4)

When forming a right-angled triangle with hypotenuse AB, the length of AC is 4 and the length of BC is 3.

Using the Pythagorean Theorem,

$(AB)^2 = 4^2 + 3^2 = 25$

Since AB > 0, AB = 5