LEVEL J

J 1-10 : Expansion of Polynomial Products

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

$(a+b)(a-b) = a^2 - b^2$

$(x+a)(x+b) = x^2 + (a+b)x + ab$

$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

J 11-60 : Factorisation

$acx^2 + (ad+bc)x + bd = (ax+b)(cx+d)$

$$2x^2+5x-12=(2x-3)(x+4)$$

$$\begin{array}{ccc} 2 & \diagdown\!\!\!\!\diagup & -3 \\ 1 & & 4 \end{array}$$

$a^2 \pm 2ab + b^2 = (a \pm b)^2$

$a^2 - b^2 = (a+b)(a-b)$

"Taking out" the common factor, e.g.
$a(x+y) + x + y = (x+y)(a+1)$

$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

<u>Extra notes:</u>

$(x-y)^2 = (y-x)^2$ $\quad$ $(x-y)^4 = (y-x)^4$

$(x-y) = -(y-x)$ $\quad$ $(x-y)^3 = -(y-x)^3$

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For complicated quadratic expressions with multiple variables (e.g. *x*, *y*, *z*),

1) Arrange terms in descending powers of *x*
2) Factorise terms which are independent of *x*
3) Factorise in quadratic terms, e.g. $(x+\blacksquare)(x+\triangle)$
4) Simplify content in the brackets

$$\begin{aligned}
&2a^2+2b^2+c^2+4ab+3ac+3bc\\
&=2a^2+(4b+3c)a+(2b^2+3bc+c^2)\\
&=2a^2+(4b+3c)a+(2b+c)(b+c)\\
&=[2a+(2b+c)][a+(b+c)]\\
&=(2a+2b+c)(a+b+c)
\end{aligned}$$

$$\left[\begin{array}{cccc} 2 & \diagdown\!\!\!\!\diagup & 2b+c & \rightarrow 2b+c \\ 1 & & b+c & \rightarrow \underline{2b+2c} \\ & & & 4b+3c \end{array}\right]$$

Useful trick 1:

$$\begin{aligned}
&x^4-6x^2+1\\
&=\boxed{x^4-2x^2+1}\boxed{-4x^2}\\
&=\boxed{(x^2-1)^2}\boxed{-(2x)^2}\\
&=(x^2-1+2x)(x^2-1-2x)\\
&=(x^2+2x-1)(x^2-2x-1)
\end{aligned}$$

Useful trick 2:

$$\begin{aligned}
&(x^2+x-5)(x^2+2x-5)-12x^2\\
&=(A+x)(A+2x)-12x^2 \quad [\text{where } A=x^2-5]\\
&=A^2+3xA-10x^2\\
&=(A+5x)(A-2x)\\
&=(x^2+5x-5)(x^2-2x-5)
\end{aligned}$$

Useful trick 3:

Arrange terms in descending powers of the variable which has the highest power of 2 or 1.

$$\begin{aligned}
&x^3+(2a+1)x^2+(a^2+2a-1)x+(a^2-1)\\
&=(x+1)a^2+2x(x+1)a+x^3+x^2-x-1
\end{aligned}$$

$$\begin{aligned}
&ax^2-a^3-a^2b+ab^2+b^3-bx^2\\
&=(a-b)x^2-(a^3-b^3)-ab(a-b)
\end{aligned}$$

J 61-70 : Fractional Expressions

To reduce fractional expressions, first **factorise** the numerator and denominator

$$\frac{-x^2+5x-6}{x^2-7x+12}=\frac{-(x-3)(x-2)}{(x-4)(x-3)}=-\frac{x-2}{x-4}$$

To combine two or more fractional expressions (adding/subtracting), first seek the LCM

$$\frac{x-5}{x+5}+\frac{x+5}{x-5}=\frac{(x-5)^2+(x+5)^2}{(x+5)(x-5)}=\frac{2(x^2+25)}{(x+5)(x-5)}$$

Simplifying complex fractions by eliminating their denominators

$$\frac{1+\dfrac{1}{a}}{1-\dfrac{1}{a}}=\frac{\left(1+\dfrac{1}{a}\right)a}{\left(1-\dfrac{1}{a}\right)a}=\frac{a+1}{a-1}$$
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J 71-90 : Irrational Numbers

Rationalise the denominators

$$\frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}} = \frac{\sqrt{15}}{5}$$

$$\frac{1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{\sqrt{3}-1}{2}$$

Using the fact that $(\sqrt{a}\pm\sqrt{b})^2 = a + b \pm 2\sqrt{ab}$

$\Rightarrow \sqrt{a+b\pm 2\sqrt{ab}} = \sqrt{a}\pm\sqrt{b}$ where $a > b$

$$\sqrt{\underset{\underset{5+2}{\uparrow}}{7}+2\sqrt{\underset{\underset{5\times 2}{\uparrow}}{10}}} = \sqrt{5}+\sqrt{2}$$

$$\sqrt{\underset{\underset{5+3}{\uparrow}}{8}-2\sqrt{\underset{\underset{5\times 3}{\uparrow}}{15}}} = \sqrt{5}-\sqrt{3}$$

Note: The above formula is only possible when there is a "**2**" next to the square root.

e.g.

$$\sqrt{2+\sqrt{3}} = \sqrt{\frac{4+2\sqrt{3}}{2}} = \frac{\sqrt{4+2\sqrt{3}}}{\sqrt{2}}$$
$$= \frac{\sqrt{3}+1}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{2}$$

The concept of principal square root is analogous to that of absolute value

When $a \geqslant 0$, (a is 0 or a positive number), $\sqrt{a^2} = a$

When $a < 0$, (a is a negative number), $\sqrt{a^2} = -a$

When $a \geqslant b$, $\sqrt{(a-b)^2} = a-b$

When $a < b$, $\sqrt{(a-b)^2} = -(a-b)$

J 91-110 : Quadratic Equations

Technique 1: Factorisation

$6x^2 - x = 12$

[Sol] $6x^2 - x - 12 = 0$

$(3x+4)(2x-3) = 0$

$x = -\frac{4}{3}, \frac{3}{2}$

Technique 2: Solve as "linear equation"

$x^2 - 18 = 0$

$x^2 = 18$

$x = \pm 3\sqrt{2}$

Technique 3: Completing the square

$x^2 - 5x - 7 = 0$

[Sol] $x^2 - 5x = 7$

$$x^2 - 5x + \left(\frac{5}{2}\right)^2 = 7 + \left(\frac{5}{2}\right)^2$$
$$\left(x - \frac{5}{2}\right)^2 = \frac{53}{4}$$
$$x - \frac{5}{2} = \pm\frac{\sqrt{53}}{2}$$
$$x = \frac{5\pm\sqrt{53}}{2}$$

Technique 4: Quadratic Formula

Quadratic Formula I

When $ax^2+bx+c=0$,
$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Quadratic Formula II

When $ax^2+2b'x+c=0$,
$$x = \frac{-b'\pm\sqrt{b'^2-ac}}{a}$$

J 111-120 : Complex Numbers

$\sqrt{-1} = i$ e.g. $\sqrt{-5} = \sqrt{5}i$

$i^2 = -1$

Imaginary numbers/complex numbers work the same way as algebra/real numbers.

$$\sqrt{-4}\times\sqrt{-25} = 2i\times 5i = 10i^2 = -10$$

$$(2+i)(3+2i) = 6+7i+2i^2 = 4+7i$$

$$i^3 = i^2\cdot i = (-1)\cdot i = -i$$
$$i^4 = (i^2)^2 = (-1)^2 = 1$$

Avoid having complex numbers in the denominator

$$\frac{3-i}{3+2i} = \frac{(3-i)(3-2i)}{(3+2i)(3-2i)}$$
$$= \frac{9-9i+2i^2}{9+4}$$
$$= \frac{7-9i}{13}$$

J 121-130 : Discriminant

The discriminant tells us what type of solutions/ roots a quadratic equation has

For $ax^2+bx+c=0$,
the discriminant is

$$D=b^2-4ac$$

For $ax^2+2b'x+c=0$,
use

$$\frac{D}{4}=b'^2-ac$$

$D>0 \Leftrightarrow$ There are 2 different real solutions.

$D=0 \Leftrightarrow$ There is 1 repeated real solution.

$D<0 \Leftrightarrow$ There are 2 different complex (conjugate) solutions.

J 131-140 : Root-Coefficient Relationships

Given $ax^2+bx+c=0$ $(a\neq 0)$,
if the roots are α and β,

$$\alpha+\beta=-\frac{b}{a} \qquad \alpha\beta=\frac{c}{a}$$

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Extra Notes: Given $\boldsymbol{\alpha+\beta}$ and $\boldsymbol{\alpha\beta}$, one might appreciate the following identities.

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$

$$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$$

$$\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2(\alpha\beta)^2$$

Alternatively, one can use

$$\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$$

J 141-150 : Simultaneous Equations

Useful trick 1: Substitution

$$\begin{cases} y=2x-1 & \cdots\cdots① \\ x^2+y^2=29 & \cdots\cdots② \end{cases}$$

[Sol] Substituting ① into ②,

$$x^2+(2x-1)^2=29$$

Useful trick 2: Eliminate "troublesome" terms

$$\begin{cases} x^2+y^2-2x+2y=7 & \cdots\cdots① \\ x^2+y^2+4x-4y=1 & \cdots\cdots② \end{cases}$$

[Sol] From ①−②,

$$-6x+6y=6$$

Therefore, $x=y-1$ ……③

Substituting ③ into ①

Useful trick 3: Obtain relationships between x and y from equation (1)

$$\begin{cases} (x-y)(x+2y)=0 & \cdots\cdots① \\ x^2-xy+2y^2=16 & \cdots\cdots② \end{cases}$$

[Sol] From ①, $x=y$ or $x=-2y$

$$\begin{cases} x=y & \cdots③ \\ x^2-xy+2y^2=16 & \cdots② \end{cases} \quad \Bigg| \quad \begin{cases} x=-2y & \cdots③' \\ x^2-xy+2y^2=16 & \cdots② \end{cases}$$

Useful trick 4: Apply root-coefficient relationships

$$\begin{cases} x+y=9 & \cdots\cdots① \\ xy=7 & \cdots\cdots② \end{cases}$$

[Sol] x and y are the roots of

$$t^2-9t+7=0$$

Therefore, $t=\dfrac{9\pm\sqrt{53}}{2}$

Useful trick 5: Use identities for $\boldsymbol{x+y}$ and $\boldsymbol{xy}$

$$\begin{cases} xy=3 & \cdots\cdots① \\ x^2+y^2=10 & \cdots\cdots② \end{cases}$$

[Sol] From ②,

$$(x+y)^2-2xy=10 \quad \cdots\cdots③$$

J 151-160 : Polynomial Division

Algorithm similar to long division for numbers.
Leave a space for terms with coefficient equal zero.

Dividend = Divisor × Quotient + Remainder

$$\underbrace{(x^3-x^2-1)}_{\text{Dividend}}\div\underbrace{(x-2)}_{\text{Divisor}}$$

$$=\underbrace{x^2+x+2}_{\text{Quotient}} \quad \text{Remainder } 3$$

This relationship may be written as

$$x^3-x^2-1=(x-2)(x^2+x+2)+3.$$

Dividend = Divisor × Quotient + Remainder
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J 161-170 : Remainder Theorem

$P(x)$ is a polynomial,

The Remainder Theorem

When $P(x)$ is divided by $x-a$, the remainder is $P(a)$.

The Remainder Theorem can only be used when dividing by a linear expression, i.e. $ax + b$

Important Note:

Dividing by a **first**-degree expression (a linear), the *remainder* must be of degree **zero** (constant).

Dividing by a **second**-degree expression (a quadratic), the *remainder* must be of degree **one or less** ($ax + b$).

Dividing by a **third**-degree expression (a cubic), the *remainder* must be of degree **two or less** ($ax^2 + bx + c$).

…

Dividing by an nth-degree expression, the remainder must be of degree (n – 1) or less.

J 171-180 : Factor Theorem

$P(x)$ is a polynomial,

The Factor Theorem

$P(x)$ has a factor $(x-a) \Leftrightarrow P(a)=0$

($\Leftrightarrow$ means "if and only if")

Useful tricks when guessing the value of *a*:

List out all factors of the term with the highest power (e.g. x^3) and the constant term.

Try out different combinations until you obtain $P(a) = 0$. In the example below, one might need to try $\pm1, \pm2, \pm5, \pm10, \pm\frac{1}{2}, \pm\frac{5}{2}$.

$P(x) = 2x^3 - 3x^2 - x - 10$

1 2   1 2 5 10

This may be factorised as $(2x-\bigcirc)(x^2+\triangle x+\square)$ or $(x-\bigcirc)(2x^2+\triangle x+\square)$

$P\left(\frac{5}{2}\right) = 2\times\left(\frac{5}{2}\right)^3 - 3\times\left(\frac{5}{2}\right)^2 - \frac{5}{2} - 10$

$= 0$

Try $\pm\frac{1}{2}, \pm\frac{5}{2}$.

Therefore, $(2x-5)$ is a factor of $P(x)$

J 181-200 : Proof of Identities and Inequalities

If an identity is held true for all values of x, we can:

Compare coefficients

$(ax+b)(x+1) = 3x^2+5x+2$

[Sol] $ax^2+(a+b)x+b = 3x^2+5x+2$

$$\begin{cases} a=3 & \cdots① \\ a+b=5 & \cdots② \\ b=2 & \cdots③ \end{cases}$$

Substitute appropriate values

$x^2 = a(x-1)(x-2)+b(x-1)+c$

[Sol] When $x=1$, $\quad 1=c \quad \cdots①$

When $x=2$, $\quad 4=b+c \quad \cdots②$

When $x=3$, $\quad 9=2a+2b+c \cdots③$

There are 2 methods to prove an **identity**. Example: Given $x + y = 1$, show that $x^2 + y = y^2 + x$.

Method 1: $\quad y = 1 - x$

$LHS = x^2 + (1 - x) = x^2 - x + 1$

$RHS = (1 - x)^2 + x = x^2 - x + 1$

$\therefore LHS = RHS$

Method 2:

$$\begin{aligned} LHS - RHS &= x^2 + y - y^2 - x \\ &= (x-y)(x+y) - (x-y) \\ &= (x-y)(x+y-1) = 0 \end{aligned}$$

$\therefore LHS = RHS$

To prove an **inequality**, we can:

Complete the square ($\because$ a square is non-negative)

Prove $x^2 > x-1$.

[Sol] $(\text{LHS})-(\text{RHS}) = x^2-x+1$

$$= \left(x-\frac{1}{2}\right)^2 + \frac{3}{4}$$

Since $\left(x-\frac{1}{2}\right)^2 \geqslant 0$, $\quad \left(x-\frac{1}{2}\right)^2 + \frac{3}{4} \geqslant \frac{3}{4} > 0$

Therefore, $x^2 > x-1$

Use the arithmetic mean-geometric mean inequality

When $a>0$ and $b>0$, then $\frac{a+b}{2} \geqslant \sqrt{ab}$

LHS = RHS when $a=b$.