LEVEL N

N 1-10 : Arithmetic Sequences

The **general term** of an arithmetic sequence $\{a_n\}$ with 1st term a and common difference d is

$$a_n = a + (n-1)d$$

Find the 1st term a and the general term of the arithmetic sequence $\{a_n\}$ whose common difference is -2 and 6th term is 8.

[Sol] $a_6 = a + 5\cdot(-2) = 8$

$\therefore\ a = 18$

Also, $a_n = 18 + (n-1)\cdot(-2) = -2n + 20$

Find the 55th term of the arithmetic sequence $\{a_n\}$ whose 15th term is 77 and 42nd term is 239.

Let a be the 1st term and d be the common difference.

$$\begin{cases} a_{15} = a + 14d = 77 & \cdots ① \\ a_{42} = a + 41d = 239 & \cdots ② \end{cases}$$

From ① and ②, $a = -7,\ d = 6$

$\therefore\ a_{55} = -7 + 54\cdot 6 = \mathbf{317}$

Let S_n be the **sum** of an arithmetic sequence with 1st term a, common difference d, last term l and number of terms n.

$$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n[2a+(n-1)d]$$

Find the general term of the arithmetic sequence $\{a_n\}$ whose sum of the first 5 terms is 100 and whose sum of the first 10 terms is 150.

Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_5 = \frac{1}{2}\cdot 5[2a+(5-1)d] = 100 & \cdots ① \\ S_{10} = \frac{1}{2}\cdot 10[2a+(10-1)d] = 150 & \cdots ② \end{cases}$$

From ① and ②, $a = 24,\ d = -2$

$\therefore\ a_n = 24 + (n-1)\cdot(-2) = \mathbf{-2n+26}$

N 11-20 : Geometric Sequences

The **general term** of a geometric sequence $\{a_n\}$ with 1st term a and common ratio r is

$$a_n = ar^{n-1}$$

Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose 3rd term is 12 and whose sum of the first 3 terms is 21.

$$\begin{cases} ar^2 = 12 & \cdots ① \\ a + ar + ar^2 = 21 & \cdots ② \end{cases}$$

From ②, $a(1+r+r^2) = 21$

$12(1+r+r^2) = 21r^2 \qquad \therefore\ r = -\frac{2}{3},\ 2$

(i) When $r = -\frac{2}{3}$, from ①, $a = 27$

(ii) When $r = 2$, from ①, $a = 3$

From (i) and (ii), $\mathbf{a = 27,\ r = -\frac{2}{3}}$ or $\mathbf{a = 3,\ r = 2}$

Let S_n be the **sum** of a geometric sequence with 1st term a, common ratio r and number of terms n.

When $r \neq 1$, $S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$

When $r = 1$, $S_n = na$

Given the geometric sequence whose 1st term is 6 and common ratio is -2, find the value of n for which the sum of the first n terms is -30.

Let S_n be the sum of the first n terms.

$S_n = \frac{6[1-(-2)^n]}{1-(-2)} = -30$

$(-2)^n = 16$

$\therefore\ \mathbf{n = 4}$

Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose sum of the first 3 terms is 6 and whose sum of the first 6 terms is -42. (The common ratio is a real number.)

Let S_n be the sum of the first n terms.

(i) When $r = 1$, $S_3 = 3a = 6$, $S_6 = 6a = -42$

No a can satisfy these conditions simultaneously.

(ii) When $r \neq 1$,

$$\begin{cases} S_3 = \frac{a(r^3-1)}{r-1} = 6 & \cdots ① \\ S_6 = \frac{a(r^6-1)}{r-1} = -42 & \cdots ② \end{cases}$$

From ① and ②, $6(r^3+1) = -42$

$\therefore\ r = -2$

From ①, $a = 2$

From (i) and (ii), $\mathbf{a = 2,\ r = -2}$

N 21-40 : Various Sequences

Summation Properties:

$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$ (where c is a constant)

Summation Formulas:

$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$, $\sum_{k=1}^{n} c = nc$

$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$

$\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1} \quad (r \neq 1)$

Examples:

$$\begin{aligned}
&\sum_{k=1}^{n}(6k^2 + 2k + 4^{k+1}) \\
&= 6\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 16\cdot 4^{k-1} \\
&= 6\cdot\frac{1}{6}n(n+1)(2n+1) + 2\cdot\frac{1}{2}n(n+1) + \frac{16(4^n-1)}{4-1} \\
&= n(n+1)[(2n+1)+1] + \frac{4^{n+2}-16}{3} \\
&= \mathbf{2n(n+1)^2 + \frac{4^{n+2}-16}{3}}
\end{aligned}$$

$$\sum_{k=1}^{2n}(2k^3+3k^2+6^{k-1})$$
$$=2\sum_{k=1}^{2n}k^3+3\sum_{k=1}^{2n}k^2+\sum_{k=1}^{2n}6^{k-1}$$
$$=2\left[\frac{1}{2}\cdot 2n(2n+1)\right]^2+3\cdot\frac{1}{6}\cdot 2n(2n+1)(2\cdot 2n+1)+\frac{6^{2n}-1}{6-1}$$
$$=n(2n+1)[2n(2n+1)+(4n+1)]+\frac{6^{2n}-1}{5}$$
$$=\mathbf{n(2n+1)(4n^2+6n+1)+\frac{6^{2n}-1}{5}}$$

Sequence of Differences and General term

Let $\{b_n\}$ be the sequence of differences of the sequence $\{a_n\}$. When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k$$

2, 10, 24, 44, 70, 102, …

Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

8, 14, 20, 26, 32, …

$\therefore\ b_n=8+(n-1)\cdot 6=6n+2$

When $n \geqslant 2$, $a_n=a_1+\sum_{k=1}^{n-1}(6k+2)$

$$=2+6\cdot\frac{1}{2}(n-1)n+2(n-1)$$
$$=n(3n-1)\ \cdots ①$$

Since $a_1=2$, ① is also true when $n=1$.

$\therefore\ \mathbf{a_n=n(3n-1)}$

3, 4, 1, 10, −17, 64, …

Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

1, −3, 9, −27, 81, …

$\therefore\ b_n=1\cdot(-3)^{n-1}=(-3)^{n-1}$

When $n \geqslant 2$, $a_n=a_1+\sum_{k=1}^{n-1}(-3)^{k-1}$

$$=3+\frac{1-(-3)^{n-1}}{1-(-3)}$$
$$=\frac{13-(-3)^{n-1}}{4}\ \cdots ①$$

Since $a_1=3$, ① is also true when $n=1$.

$\therefore\ \mathbf{a_n=\frac{13-(-3)^{n-1}}{4}}$

Let S_n be the sum of the first n terms of the sequence $\{a_n\}$. The 1st term is $a_1 = S_1$.

When $n \geq 2$, $a_n = S_n - S_{n-1}$

$S_n=n^2+4n$

The 1st term a_1 is $a_1=S_1=1^2+4\cdot 1=5$.

When $n \geqslant 2$,

$$a_n=S_n-S_{n-1}$$
$$=(n^2+4n)-[(n-1)^2+4(n-1)]$$
$$=2n+3\ \cdots ①$$

Since $a_1=5$, ① is also true when $n=1$.

$\therefore\ \mathbf{a_n=2n+3}$

Variation: Each term in an arithmetic sequence $\{a_n\}$ multiplied by r^n or r^{n-1}

$S=1\cdot 1+2\cdot 3+3\cdot 3^2+4\cdot 3^3+\cdots+n\cdot 3^{n-1}\ \cdots ①$

Multiplying both sides of ① by 3,

$3S=1\cdot 3+2\cdot 3^2+3\cdot 3^3+\cdots+(n-1)\cdot 3^{n-1}+n\cdot 3^n\ \cdots ②$

①−②, $-2S=(1+3+3^2+3^3+\cdots+3^{n-1})-n\cdot 3^n$

$$=\frac{3^n-1}{3-1}-n\cdot 3^n$$
$$=-\frac{(2n-1)\cdot 3^n+1}{2}$$

$\therefore\ \mathbf{S=\frac{(2n-1)\cdot 3^n+1}{4}}$

Variation: Telescoping sum

$$S=\frac{3}{1\cdot 4}+\frac{3}{4\cdot 7}+\frac{3}{7\cdot 10}+\cdots+\frac{3}{(3n-2)(3n+1)}$$

Since $\frac{3}{(3k-2)(3k+1)}=\frac{1}{3k-2}-\frac{1}{3k+1}$

Therefore,

$$S=\left(\frac{1}{1}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\cdots+\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)$$
$$=1-\frac{1}{3n+1}=\mathbf{\frac{3n}{3n+1}}$$

N 41-50 : Recurrence Relations

The sequence $\{a_n\}$ defined by $a_{n+1} = a_n + d$ is an arithmetic sequence with common difference d.

The sequence $\{a_n\}$ defined by $a_{n+1} = ra_n$ is a geometric sequence with common ratio r.

Given sequence $\{a_n\}$ defined by $a_{n+1} = a_n + b_n$, then $a_n = a_1 + \sum_{k=1}^{n-1} b_k$ for $n \geq 2$.

$a_1=4,\ a_{n+1}=a_n+4n^3$

$a_{n+1}-a_n=4n^3$

Since the general term of the sequence of differences of $\{a_n\}$ is $4n^3$, when $n \geqslant 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}4k^3=4+4\left[\frac{1}{2}(n-1)n\right]^2 \quad \boxed{\sum_{k=1}^{n}k^3=\left[\frac{1}{2}n(n+1)\right]^2}$$
$$=n^4-2n^3+n^2+4\ \cdots ①$$

Since $a_1=4$, ① is also true when $n=1$.

$\therefore\ \mathbf{a_n=n^4-2n^3+n^2+4}$

The recurrence relation **$a_{n+1} = pa_n + q$** can be rearranged into $a_{n+1} - x = p(a_n - x)$ by using x which satisfies $x = px + q$. Letting $b_n = a_n - x$ we have that $\{b_n\}$ is a geometric sequence.

$a_1=4,\ a_{n+1}=2a_n+1$

Since $a_{n+1}=2a_n+1$, $a_{n+1}+1=2(a_n+1)$

Let $b_n=a_n+1$.

Replacing both a_{n+1} and a_n with x,
$x=2x+1$
$\therefore\ x=-1$

$$\begin{cases} b_{n+1}=2b_n & \cdots ① \\ b_1=a_1+1=5 & \cdots ② \end{cases}$$

From ① and ②, $b_n=5\cdot 2^{n-1}$ $\quad\therefore\ a_n=b_n-1=\mathbf{5\cdot 2^{n-1}-1}$

The recurrence relation $a_{n+2} + pa_{n+1} + qa_n = 0$ can be rearranged into $a_{n+2} - \alpha a_{n+1} = \beta(a_{n+1} - \alpha a_n)$ by using two solutions α and β of quadratic equation $x^2 + px + q = 0$. The sequence $\{a_{n+1} - \alpha a_n\}$ is a geometric sequence.

$a_1=2,\ a_2=1,\ a_{n+2}=a_{n+1}+6a_n$

Since $a_{n+2}=a_{n+1}+6a_n$, $a_{n+2}-a_{n+1}-6a_n=0 \quad \cdots①$

Rearranging ①,

$$\begin{cases} a_{n+2}+2a_{n+1}=3(a_{n+1}+2a_n) & \cdots② \\ a_{n+2}-3a_{n+1}=-2(a_{n+1}-3a_n) & \cdots③ \end{cases}$$

From ②, the sequence $\{a_{n+1}+2a_n\}$ is the geometric sequence with 1st term $a_2+2a_1=5$ and common ratio 3.

$\therefore\ a_{n+1}+2a_n=5\cdot 3^{n-1} \quad \cdots④$

From ③, the sequence $\{a_{n+1}-3a_n\}$ is the geometric sequence with 1st term $a_2-3a_1=-5$ and common ratio -2.

$\therefore\ a_{n+1}-3a_n=-5(-2)^{n-1} \quad \cdots⑤$

From ④ and ⑤, $\boldsymbol{a_n=3^{n-1}+(-2)^{n-1}}$

<u>Note:</u> Refer to the worksheets for more variations.

N 51-60 : Mathematical Induction

To prove that a proposition P is true for all natural numbers n by mathematical induction,

1) Prove that P is true when $n = 1$ (base number).

2) Assume that P is true when $n = k$.

3) Prove that P is also true when $n = k + 1$.

<u>Tips:</u> When proving the statement for $n = k + 1$, try to rearrange the expression so that it is possible to use the statement for $n = k$.

To prove an **inequality**, take the difference LHS – RHS and show that it is > 0 or < 0 when $n = k + 1$.

Refer to the worksheets for elaborated examples.

N 61-70 : Infinite Sequences

An infinite sequence may:

- **converge** $\quad \lim_{n\to\infty} a_n = \alpha$

- **diverge** $\quad \lim_{n\to\infty} a_n = \pm\infty$ or $\{a_n\}$ oscillates

<u>Properties of Limits of Sequences</u>

When the sequences $\{a_n\}$ and $\{b_n\}$ converge, where $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$,

1) $\lim_{n\to\infty} ka_n = k\alpha \qquad$ (k is a constant)

2) $\lim_{n\to\infty} a_n \pm b_n = \alpha \pm \beta$

3) $\lim_{n\to\infty} a_n b_n = \alpha\beta$

4) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\alpha}{\beta} \qquad (\beta \neq 0)$

There are several **techniques** to evaluate limits.

<u>Technique 1:</u> Divide the numerator and denominator by the term with the highest power in denominator

$$\lim_{n\to\infty}\frac{-2n^2(n+2)}{2n^3-5}=\lim_{n\to\infty}\frac{-2n^3-4n^2}{2n^3-5}=\lim_{n\to\infty}\frac{-2-\frac{4}{n}}{2-\frac{5}{n^3}}=-1$$

$$\lim_{n\to\infty}\frac{5+n+n^5}{n(4-3n^2)}=\lim_{n\to\infty}\frac{5+n+n^5}{4n-3n^3}=\lim_{n\to\infty}\frac{\frac{5}{n^3}+\frac{1}{n^2}+n^2}{\frac{4}{n^2}-3}=-\infty$$

$\lim_{n\to\infty}\left(\frac{5}{n^3}+\frac{1}{n^2}+n^2\right)=\infty$

$\lim_{n\to\infty}\left(\frac{4}{n^2}-3\right)=-3$

$$\lim_{n\to\infty}\frac{\sqrt{n^2+1}}{2n}=\lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n^2}}}{2}=\frac{1}{2}$$

$\frac{\sqrt{n^2+1}}{n}=\sqrt{\frac{n^2+1}{n^2}}=\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}$

$$\begin{aligned}\lim_{n\to\infty}[\log_3(2n^2-1)-\log_3(6n^2+2)]&=\lim_{n\to\infty}\log_3\frac{2n^2-1}{6n^2+2}\\&=\lim_{n\to\infty}\log_3\frac{2-\frac{1}{n^2}}{6+\frac{2}{n^2}}\\&=\log_3\frac{1}{3}=-1\end{aligned}$$

<u>Technique 2:</u> Factor out the dominant term

$$\lim_{n\to\infty}(n^2-3n)=\lim_{n\to\infty}n^2\left(1-\frac{3}{n}\right)=\infty$$

Taking out the term with the highest power

$$\lim_{n\to\infty}(2n^3-3n^2+4)=\lim_{n\to\infty}n^3\left(2-\frac{3}{n}+\frac{4}{n^3}\right)=\infty$$

$$\lim_{n\to\infty}(4n^2-2n^3)=\lim_{n\to\infty}n^3\left(\frac{4}{n}-2\right)=-\infty$$

<u>Technique 3:</u> Multiply the numerator and the denominator by conjugate surds

$$\begin{aligned}\lim_{n\to\infty}(\sqrt{n^2+3n}-n)&=\lim_{n\to\infty}\frac{(\sqrt{n^2+3n}-n)(\sqrt{n^2+3n}+n)}{\sqrt{n^2+3n}+n}\\&=\lim_{n\to\infty}\frac{3n}{\sqrt{n^2+3n}+n}\\&=\lim_{n\to\infty}\frac{3}{\sqrt{1+\frac{3}{n}}+1}\\&=\frac{3}{2}\end{aligned}$$

Considering $\sqrt{n^2+3n}-n=\frac{\sqrt{n^2+3n}-n}{1}$, then multiplying the numerator and the denominator by $\sqrt{n^2+3n}+n$

$$\begin{aligned}&\lim_{n\to\infty}\frac{2(\sqrt{n+5}-\sqrt{n+3})}{\sqrt{n+2}-\sqrt{n}}\\&=\lim_{n\to\infty}\frac{2(\sqrt{n+5}-\sqrt{n+3})(\sqrt{n+5}+\sqrt{n+3})(\sqrt{n+2}+\sqrt{n})}{(\sqrt{n+2}-\sqrt{n})(\sqrt{n+2}+\sqrt{n})(\sqrt{n+5}+\sqrt{n+3})}\\&=\lim_{n\to\infty}\frac{2(\sqrt{n+2}+\sqrt{n})}{\sqrt{n+5}+\sqrt{n+3}}\\&=\lim_{n\to\infty}\frac{2\left(\sqrt{1+\frac{2}{n}}+1\right)}{\sqrt{1+\frac{5}{n}}+\sqrt{1+\frac{3}{n}}}=2\end{aligned}$$

Technique 4: The Squeeze Theorem

For all n, when $a_n \le b_n$,

if $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$, then $\alpha \le \beta$

if $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} b_n = \infty$

For all n, when $a_n \le c_n \le b_n$,

if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \alpha$, then $\lim_{n\to\infty} c_n = \alpha$ (*)

(*) is known as the **Squeeze Theorem**.

$$\lim_{n\to\infty}\frac{1}{n}\sin\frac{n\pi}{2}$$

Since $-1 \leqslant \sin\frac{n\pi}{2} \leqslant 1$,

$$-\frac{1}{n} \leqslant \frac{1}{n}\sin\frac{n\pi}{2} \leqslant \frac{1}{n}$$

Then, since $\lim_{n\to\infty}\left(-\frac{1}{n}\right)=0$ and $\lim_{n\to\infty}\frac{1}{n}=0$,

$$\lim_{n\to\infty}\frac{1}{n}\sin\frac{n\pi}{2}=\mathbf{0}$$

$$\lim_{n\to\infty}\frac{n+1}{n^2}\cos n\theta \quad (\theta \text{ is a constant})$$

Since $-1 \leqslant \cos n\theta \leqslant 1$,

$$-\frac{n+1}{n^2} \leqslant \frac{n+1}{n^2}\cos n\theta \leqslant \frac{n+1}{n^2}$$

Then, since $\lim_{n\to\infty}\left(-\frac{n+1}{n^2}\right)=\lim_{n\to\infty}\left(-\frac{1}{n}-\frac{1}{n^2}\right)=0$ and

$$\lim_{n\to\infty}\frac{n+1}{n^2}=\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right)=0,$$

$$\lim_{n\to\infty}\frac{n+1}{n^2}\cos n\theta=\mathbf{0}$$

N 71-80 : Infinite Geometric Sequences

Limit of an Infinite Geometric Sequence $\{r_n\}$

When $r < 1$, $\lim_{n\to\infty} r^n = \infty$ ⋯ Diverges

When $r = 1$, $\lim_{n\to\infty} r^n = 1$ ⋯ Converges

When $|r| < 1$, $\lim_{n\to\infty} r^n = 0$ ⋯ Converges

When $r \le -1$, Oscillates ⋯ Diverges

Technique 1: Divide the numerator and the denominator by the term with the largest absolute value of the base in the denominator

$$\lim_{n\to\infty}\frac{(-2)^n+2\cdot 3^n}{3^n+1}=\lim_{n\to\infty}\frac{\left(-\frac{2}{3}\right)^n+2}{1+\left(\frac{1}{3}\right)^n}=2$$

$$\lim_{n\to\infty}\frac{-5^n+3^{n+1}}{3^n+4^n}=\lim_{n\to\infty}\frac{-\left(\frac{5}{4}\right)^n+3\left(\frac{3}{4}\right)^n}{\left(\frac{3}{4}\right)^n+1}=-\infty$$

Technique 2: Factor out the term with the largest absolute value of the base

$$\lim_{n\to\infty}(4^n-3^n)=\lim_{n\to\infty}4^n\left[1-\left(\frac{3}{4}\right)^n\right]=\infty$$

$$\lim_{n\to\infty}(3^n-5^n)=\lim_{n\to\infty}5^n\left[\left(\frac{3}{5}\right)^n-1\right]=-\infty$$

Variation 1: Consider the different cases of values of the geometric ratio

$$\lim_{n\to\infty}\frac{r^{2n+1}-1}{r^{2n}+1}$$

When $|r|<1$, $\lim_{n\to\infty}\frac{r^{2n+1}-1}{r^{2n}+1}=\frac{0-1}{0+1}=\mathbf{-1}$

When $r=1$, $\lim_{n\to\infty}\frac{r^{2n+1}-1}{r^{2n}+1}=\frac{1-1}{1+1}=\mathbf{0}$

When $r=-1$, $\lim_{n\to\infty}\frac{r^{2n+1}-1}{r^{2n}+1}=\frac{-1-1}{1+1}=\mathbf{-1}$

When $|r|>1$, $\lim_{n\to\infty}\frac{r^{2n+1}-1}{r^{2n}+1}=\lim_{n\to\infty}\frac{r-\left(\frac{1}{r}\right)^{2n}}{1+\left(\frac{1}{r}\right)^{2n}}=\frac{r-0}{1+0}=\mathbf{r}$

Variation 2: A geometric sequence converges when if and only if $-1 < r \le 1$.

Find the range of values of x for which the sequence $\left\{\left(\frac{x+1}{3}\right)^n\right\}$ converges. Then, state the limit values.

Since the common ratio is $\frac{x+1}{3}$, $-1 < \frac{x+1}{3} \leqslant 1$

$\therefore\ \mathbf{-4 < x \leqslant 2}$

Also, the limit values are,

when $-4<x<2$, $\lim_{n\to\infty}\left(\frac{x+1}{3}\right)^n=\mathbf{0}$ and

when $x=2$, $\lim_{n\to\infty}\left(\frac{x+1}{3}\right)^n=\mathbf{1}$

N 81-90 : Infinite Geometric Series

Given an infinite geometric series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$, the following is true.

When $a \ne 0$,

if $|r| < 1$, then the series converges to $\frac{a}{1-r}$

if $|r| \ge 1$, then the series diverges

When $a = 0$, the series converges to 0.

$$(\sqrt{3}-1)-2(2-\sqrt{3})+2(3\sqrt{3}-5)-4(7-4\sqrt{3})+\cdots$$

The 1st term is $\sqrt{3}-1$ and the common ratio is $-\sqrt{3}+1$.

Since $|-\sqrt{3}+1|<1$, the series **converges**.

$$\therefore\ S=\frac{\sqrt{3}-1}{1-(-\sqrt{3}+1)}=\frac{\sqrt{3}-1}{\sqrt{3}}=\mathbf{\frac{3-\sqrt{3}}{3}}$$

When $a \neq 0$, if $|r| < 1$, then the infinite geometric series converges.

Find the range of values of a real number x for which the following infinite geometric series converges. Then, find the sum S.

$2+2(x^2-3)+2(x^2-3)^2+\cdots$

Since the common ratio is x^2-3, $-1 < x^2-3 < 1$

So, $-1 < x^2-3$ $\cdots$① and also $x^2-3 < 1$ $\cdots$②

From ①, $x^2 > 2$; therefore, $x < -\sqrt{2}$, $\sqrt{2} < x$ $\cdots$③

From ②, $x^2 < 4$; therefore, $-2 < x < 2$ $\cdots$④

From ③ and ④, the range of values of x is

$$-2 < x < -\sqrt{2},\quad \sqrt{2} < x < 2$$

Also, $S=\dfrac{2}{1-(x^2-3)}=\dfrac{2}{4-x^2}$

Some other variations/ interesting applications:

Given the infinite geometric series whose sum is 9 and 2nd term is -4, find the 1st term a and the common ratio r.

Since the sum is 9, $a \neq 0$, $-1 < r < 1$ ← Since the sum is 9, the infinite geometric series converges.

$$\begin{cases} \dfrac{a}{1-r}=9 & \cdots ① \\ ar=-4 & \cdots ② \end{cases}$$

From ① and ②, ← From ①, $a=9(1-r)$ Substituting this into ②

$9(1-r)\cdot r=-4$

$9r^2-9r-4=0$

$(3r+1)(3r-4)=0 \qquad r=-\dfrac{1}{3},\ \dfrac{4}{3}$

Since $-1 < r < 1$, $r=-\dfrac{1}{3}$

From ②, $a=12$

$0.\dot{3}0\dot{6}=0.306+0.000306+0.000000306+\cdots$

$0.\dot{3}0\dot{6}$ is the infinite geometric series with 1st term 0.306 and common ratio 0.001. Since $|0.001| < 1$, it converges.

$$\therefore\ 0.\dot{3}0\dot{6}=\frac{0.306}{1-0.001}=\mathbf{\frac{34}{111}}$$

N 91-100 : Infinite Series

To determine whether an infinite series converges, first find the **partial sum** of the first n terms of the series, and then find the **limit** of the sum.

Key idea: Use standard tricks to obtain a telescoping sum, and then apply method of differences.

Technique 1: Multiply both the numerator and the denominator by the conjugate of the denominator

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$\frac{1}{\sqrt{n+1}+\sqrt{n}}=\frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}=\sqrt{n+1}-\sqrt{n}$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n=(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+\cdots+(\sqrt{n+1}-\sqrt{n})=-1+\sqrt{n+1}$$

$$\therefore\ \lim_{n\to\infty}S_n=\lim_{n\to\infty}(-1+\sqrt{n+1})=\infty$$

Thus, the series diverges.

Technique 2: Partial fraction decomposition

$$\frac{4}{1\cdot5}+\frac{4}{5\cdot9}+\frac{4}{9\cdot13}+\cdots+\frac{4}{(4n-3)(4n+1)}+\cdots$$

Let $\dfrac{4}{(4n-3)(4n+1)}=\dfrac{a}{4n-3}-\dfrac{b}{4n+1}$. $\quad\therefore\ a=1,\ b=1$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n=\left(\frac{1}{1}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)+\cdots+\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)=1-\frac{1}{4n+1}$$

$$\therefore\ \lim S_n=\lim\left(1-\frac{1}{4n+1}\right)=1$$

Thus, the series **converges** and the sum is **1**.

Technique 3: Consider partial sum of the first odd number of terms and the first even number of terms

$$\frac{1}{2}-\frac{2}{3}+\frac{2}{3}-\frac{3}{4}+\frac{3}{4}-\cdots-\frac{n}{n+1}+\frac{n}{n+1}-\frac{n+1}{n+2}+\cdots$$

Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n=2m-1$,

$$S_n=S_{2m-1}=\frac{1}{2}+\left(-\frac{2}{3}+\frac{2}{3}\right)+\left(-\frac{3}{4}+\frac{3}{4}\right)+\cdots+\left(-\frac{m}{m+1}+\frac{m}{m+1}\right)=\frac{1}{2}$$

$S_1=\dfrac{1}{2}$

$S_3=\dfrac{1}{2}+\left(-\dfrac{2}{3}+\dfrac{2}{3}\right)$

$S_5=\dfrac{1}{2}+\left(-\dfrac{2}{3}+\dfrac{2}{3}\right)+\left(-\dfrac{3}{4}+\dfrac{3}{4}\right)$

$\vdots$

$$\therefore\ \lim_{m\to\infty}S_{2m-1}=\frac{1}{2}$$

(ii) When $n=2m$,

$$S_n=S_{2m}=S_{2m-1}+\left(-\frac{m+1}{m+2}\right)$$

$S_{2m}=S_{2m-1}+$(the $2m^{\text{th}}$ term)

From (i), $\lim\limits_{m\to\infty}S_{2m-1}=\dfrac{1}{2}$

$$\therefore\ \lim_{m\to\infty}S_{2m}=\lim_{m\to\infty}\left(S_{2m-1}-\frac{1+\frac{1}{m}}{1+\frac{2}{m}}\right)=\frac{1}{2}$$

From (i) and (ii), $\lim\limits_{m\to\infty}S_{2m-1}\neq\lim\limits_{m\to\infty}S_{2m}$

Therefore, the series diverges. ← $\{S_n\}$ oscillates and diverges.

Properties of Infinite Series

When the infinite series $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ converge, where $\sum_{n=1}^{\infty}a_n=S$ and $\sum_{n=1}^{\infty}b_n=T$, the following properties are true:

$\sum_{n=1}^{\infty}ka_n=kS$ $\qquad$ (k is a constant)

$\sum_{n=1}^{\infty}(a_n\pm b_n)=S\pm T$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}-\frac{5}{4^n}\right)=\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}-\sum_{n=1}^{\infty}\frac{5}{4^n}=\frac{1}{1-\frac{1}{2}}-\frac{\frac{5}{4}}{1-\frac{1}{4}}=2-\frac{5}{3}=\frac{1}{3}$$

$$\sum_{n=1}^{\infty}\left(\frac{2}{3^n}-\frac{1}{6^{n-1}}\right)=\sum_{n=1}^{\infty}\frac{2}{3^n}-\sum_{n=1}^{\infty}\frac{1}{6^{n-1}}=\frac{\frac{2}{3}}{1-\frac{1}{3}}-\frac{1}{1-\frac{1}{6}}=1-\frac{6}{5}=-\frac{1}{5}$$

Important Theorem

If the infinite series $\sum_{n=1}^{\infty}a_n$ converges, then $\lim\limits_{n\to\infty}a_n=0$. If the sequence $\{a_n\}$ does not converge to 0, then the infinite series $\sum_{n=1}^{\infty}a_n$ diverges.

N 101-120 : Limits of Functions

Properties of Infinite Series

If $\lim\limits_{x\to a} f(x) = \alpha$ and $\lim\limits_{x\to a} g(x) = \beta$, then

$$\lim_{x\to a} kf(x) = k\alpha \qquad (k \text{ is a constant})$$

$$\lim_{x\to a}[f(x) \pm g(x)] = \alpha \pm \beta$$

$$\lim_{x\to a} f(x)g(x) = \alpha\beta$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\alpha}{\beta} \qquad (\beta \neq 0)$$

$$\lim_{x\to -2}\frac{x^3+8}{x^2-x-6}=\lim_{x\to -2}\frac{(x+2)(x^2-2x+4)}{(x+2)(x-3)}=\lim_{x\to -2}\frac{x^2-2x+4}{x-3}=-\frac{12}{5}$$

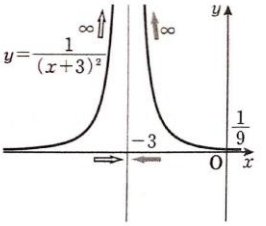
$$\lim_{x\to 1}\frac{x^2-4x+3}{x^3-1}=\lim_{x\to 1}\frac{(x-1)(x-3)}{(x-1)(x^2+x+1)}=\lim_{x\to 1}\frac{x-3}{x^2+x+1}=-\frac{2}{3}$$

To evaluate limits of rational functions (near the asymptote), we need to draw graphs.

$$\lim_{x\to -3}\frac{1}{(x+3)^2}$$

As $x\to -3$, $(x+3)^2\to 0$

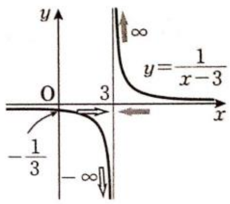
$$\therefore \lim_{x\to -3}\frac{1}{(x+3)^2}=\infty$$



$$\lim_{x\to 3}\frac{1}{x-3}$$

When $x>3$, as $x\to 3$, $\frac{1}{x-3}\to\infty$

When $x<3$, as $x\to 3$, $\frac{1}{x-3}\to-\infty$

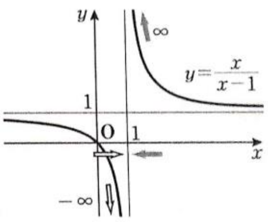


Definition: If $\lim\limits_{x\to a^+} f(x) \neq \lim\limits_{x\to a^-} f(x)$, then $\lim\limits_{x\to a} f(x)$ does not exist.

$$\lim_{x\to 1}\frac{x}{x-1}$$

$$\lim_{x\to 1^+}\frac{x}{x-1}=\infty,\quad \lim_{x\to 1^-}\frac{x}{x-1}=-\infty$$

Therefore, $\lim\limits_{x\to 1}\frac{x}{x-1}$ **does not exist.**



Techniques for evaluating limits of functions as x approaches infinity are similar to that of sequences (check N61–80).

$$\lim_{x\to\infty}\frac{2x^2-x-6}{3x^2-2x-8}=\lim_{x\to\infty}\frac{2-\frac{1}{x}-\frac{6}{x^2}}{3-\frac{2}{x}-\frac{8}{x^2}}=\frac{2}{3}$$

$$\lim_{x\to\infty}\frac{\sqrt{x+1}+\sqrt{3x-1}}{\sqrt{x-1}}=\lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{x}}+\sqrt{3-\frac{1}{x}}}{\sqrt{1-\frac{1}{x}}}=1+\sqrt{3}$$

Dividing the numerator and the denominator by $\sqrt{x}$

$$\lim_{x\to\infty}(x^2-3x+2)=\lim_{x\to\infty}x^2\left(1-\frac{3}{x}+\frac{2}{x^2}\right)=\infty$$

Taking out the term with the highest power

$$\lim_{x\to\infty}\left(2x+1-\sqrt{4x^2+2x+1}\right)$$
$$=\lim_{x\to\infty}\frac{\left(2x+1-\sqrt{4x^2+2x+1}\right)\left(2x+1+\sqrt{4x^2+2x+1}\right)}{2x+1+\sqrt{4x^2+2x+1}}$$
$$=\lim_{x\to\infty}\frac{2x}{2x+1+\sqrt{4x^2+2x+1}}$$
$$=\lim_{x\to\infty}\frac{2}{2+\frac{1}{x}+\sqrt{4+\frac{2}{x}+\frac{1}{x^2}}}=\frac{1}{2}$$

$$\lim_{x\to\infty}(10^x-3^x)=\lim_{x\to\infty}10^x\left[1-\left(\frac{3}{10}\right)^x\right]=\infty$$

$$\lim_{x\to\infty}(2^x-5^x)=\lim_{x\to\infty}5^x\left[\left(\frac{2}{5}\right)^x-1\right]=-\infty$$

$$\lim_{x\to\infty}5^x=\infty \qquad \lim_{x\to\infty}\left[\left(\frac{2}{5}\right)^x-1\right]=-1$$

$$\lim_{x\to\infty}\frac{4^x-1}{4^x+1}=\lim_{x\to\infty}\frac{1-\left(\frac{1}{4}\right)^x}{1+\left(\frac{1}{4}\right)^x}=1$$

For limits that tend to negative infinity, we usually introduce a transformation of variable.

$$\lim_{x\to-\infty}\frac{4x^2-x+2}{2x-3}$$

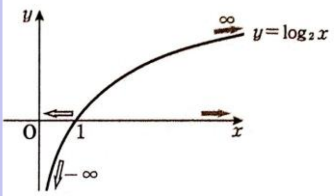
Let $x=-t$. As $x\to-\infty$, $t\to\infty$; therefore,

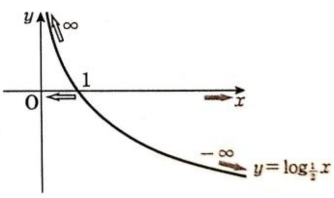
$$\lim_{x\to-\infty}\frac{4x^2-x+2}{2x-3}=\lim_{t\to\infty}\frac{4t^2+t+2}{-2t-3}=\lim_{t\to\infty}\frac{4t+1+\frac{2}{t}}{-2-\frac{3}{t}}=-\infty$$

Recall important properties of logarithms. (*Refer to the worksheets for more variation*)

$$\lim_{x\to\infty}\log_2 x=\infty \qquad \lim_{x\to\infty}\log_{\frac{1}{2}} x=-\infty$$

$$\lim_{x\to 0^+}\log_2 x=-\infty \qquad \lim_{x\to 0^+}\log_{\frac{1}{2}} x=\infty$$





Theorem: If $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \alpha < \infty$ and $\lim\limits_{x\to a} g(x) = 0$, then we must have $\lim\limits_{x\to a} f(x) = 0$.

N 121-130 : Limits of Trigonometric Functions

Main formulas: (can be used without proof)

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{\tan x}{x}=1$$

Important theorems:

(1) For all x close to a, when $f(x)\le h(x)\le g(x)$, if $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=\alpha$, then $\lim\limits_{x\to a} h(x)=\alpha$.

(2) If $\lim\limits_{x\to a}|f(x)|=0$, then $\lim\limits_{x\to a} f(x)=0$.

(1) is known as the **Squeeze Theorem**, (2) is known as the **Absolute Value Theorem**.

**Note that (2) is only true when the limit is zero.

$$\lim_{x\to0}\frac{2x}{\sin 3x}=\lim_{x\to0}\frac{3x}{\sin 3x}\cdot\frac{2}{3}=\lim_{x\to0}\frac{1}{\frac{\sin 3x}{3x}}\cdot\frac{2}{3}=\frac{1}{1}\cdot\frac{2}{3}=\frac{2}{3}$$

$$\lim_{x\to0}\frac{\sin 3x}{\tan 2x}=\lim_{x\to0}\frac{\sin 3x}{\frac{\sin 2x}{\cos 2x}}=\lim_{x\to0}\frac{\frac{\sin 3x}{3x}\cdot 3}{\frac{\sin 2x}{2x}\cdot\frac{2}{\cos 2x}}=\frac{1\cdot 3}{1\cdot\frac{2}{1}}=\frac{3}{2}$$

Sometimes, we need to multiply the numerator and the denominator by an identified conjugate

$$\lim_{x\to0}\frac{x\sin x}{1-\cos x}=\lim_{x\to0}\frac{x\sin x(1+\cos x)}{(1-\cos x)(1+\cos x)}=\lim_{x\to0}\frac{x\sin x(1+\cos x)}{\sin^2 x}=\lim_{x\to0}\frac{1+\cos x}{\frac{\sin x}{x}}=2$$

$$\lim_{x\to0}\frac{2\tan x}{\sqrt{1+x}-1}=\lim_{x\to0}\frac{2\sin x(\sqrt{1+x}+1)}{\cos x(\sqrt{1+x}-1)(\sqrt{1+x}+1)}=\lim_{x\to0}\frac{2\sin x(\sqrt{1+x}+1)}{x\cos x}=\lim_{x\to0}\frac{\sin x}{x}\cdot\frac{2(\sqrt{1+x}+1)}{\cos x}=4$$

Transformation of variable is sometimes necessary

$\lim_{x\to\infty} x\sin\frac{1}{x}$

Let $\frac{1}{x}=\theta$. As $x\to\infty$, $\theta\to0^+$

$\therefore \lim_{x\to\infty} x\sin\frac{1}{x}=\lim_{\theta\to0^+}\frac{1}{\theta}\sin\theta=\lim_{\theta\to0^+}\frac{\sin\theta}{\theta}=1$

$\lim_{x\to\frac{\pi}{4}}\frac{2\pi-8x}{\sin(4x-\pi)}$

Let $x-\frac{\pi}{4}=\theta$. $x=\theta+\frac{\pi}{4}$. As $x\to\frac{\pi}{4}$, $\theta\to0$

$\therefore \lim_{x\to\frac{\pi}{4}}\frac{2\pi-8x}{\sin(4x-\pi)}=\lim_{\theta\to0}\frac{-8\theta}{\sin 4\theta}=\lim_{\theta\to0}\frac{-2}{\frac{\sin 4\theta}{4\theta}}=-2$

Application of the Absolute Value Theorem

$\lim_{x\to\infty}\frac{\cos x}{x}$

Since $0\leqslant|\cos x|\leqslant1$, $0\leqslant\left|\frac{\cos x}{x}\right|=\left|\frac{1}{x}\right||\cos x|\leqslant\left|\frac{1}{x}\right|$

Then, since $\lim_{x\to\infty}\left|\frac{1}{x}\right|=0$, $\lim_{x\to\infty}\left|\frac{\cos x}{x}\right|=0$

$\therefore \lim_{x\to\infty}\frac{\cos x}{x}=0$

N 131-140 : Continuous & Discontinuous Functions

A function $f(x)$ is said to be continuous at $x=a$ if $f(x)$ satisfies the following three conditions:

(i) $\lim_{x\to a} f(x)$ exists.

(ii) $f(a)$ exists and is finite.

(iii) $\lim_{x\to a} f(x)=f(a)$ is true.

$f(x)=[x]$

(i) When $-1\leqslant x<0$, $[x]=-1$ ⟵ Examining if $\lim_{x\to0} f(x)$ exists

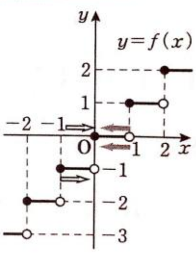
$\therefore \lim_{x\to0^-} f(x)=-1$

(ii) When $0\leqslant x<1$, $[x]=0$

$\therefore \lim_{x\to0^+} f(x)=0$

From (i) and (ii), $\lim_{x\to0} f(x)$ does not exist.

Therefore, $f(x)$ is discontinuous at $x=0$.



$$f(x)=\begin{cases}-ax^2-4a-1 & (x\leqslant-2)\\ -ax-9 & (x>-2)\end{cases}$$

If $f(x)$ is continuous for all x, then $f(x)$ must be continuous at $x=-2$. ⟵ $f(-2)=\lim_{x\to-2} f(x)$ needs to be satisfied.

$f(-2)=-a\cdot(-2)^2-4a-1=-8a-1$

$\lim_{x\to-2^+} f(x)=\lim_{x\to-2^+}(-ax-9)=2a-9$ ⟵ As $x\to-2^+$, $f(x)=-ax-9$

$\therefore -8a-1=2a-9$

$\therefore a=\frac{4}{5}$

Intermediate Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ and $f(a)\neq f(b)$, then for any $k\in\mathbb{R}$ which lies between $f(a)$ and $f(b)$, the equation $f(x)=k$ has at least one real solution in the open interval (a,b).

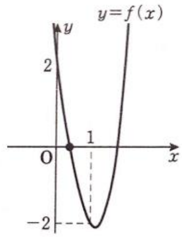
Prove that equation $x^4-5x+2=0$ has at least one real solution in the interval $0<x<1$.

Let $f(x)=x^4-5x+2$.

The function $f(x)$ is continuous on the $[0,1]$.

Also, $f(0)=2>0$, $f(1)=-2<0$

Therefore, from the Intermediate Value Theorem, $f(x)=0$ has at least one real solution in the interval $0<x<1$.



N 141-160 : Differentiation

$f(x)$ is said to be *differentiable* at $x=a$ if $f'(a)=\lim_{h\to0}\frac{f(a+h)-f(a)}{h}$ exists and is finite.

Properties of Derivatives

If $y=x^\alpha$, then $y'=\alpha x^{\alpha-1}$ (α is a real number)

If $y=kf(x)$, then $y'=kf'(x)$

If $y=f(x)\pm g(x)$, then $y'=f'(x)\pm g'(x)$

Product Rule

$$[f(x)g(x)]'=f'(x)g(x)+f(x)g'(x)$$

$y=(x^2-2)(x^3+3x-5)$

$y'=(x^2-2)'(x^3+3x-5)+(x^2-2)(x^3+3x-5)'$

$=2x(x^3+3x-5)+(x^2-2)(3x^2+3)$

$=5x^4+3x^2-10x-6$

$y=(3x+2)(x-1)(2x+5)=(3x^2-x-2)(2x+5)$

$y'=(6x-1)(2x+5)+(3x^2-x-2)\cdot2$

$=18x^2+26x-9$

Quotient Rule

$$\left[\frac{1}{g(x)}\right]'=-\frac{g'(x)}{[g(x)]^2}$$

$$\left[\frac{f(x)}{g(x)}\right]'=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

$$y=\frac{2x^2-3x+5}{x^2+2}$$

$$y'=\frac{(4x-3)(x^2+2)-(2x^2-3x+5)\cdot 2x}{(x^2+2)^2}=\frac{3x^2-2x-6}{(x^2+2)^2}$$

$$y=\frac{x^2-x+1}{x^2+x+1}$$

$$y'=\frac{(2x-1)(x^2+x+1)-(x^2-x+1)(2x+1)}{(x^2+x+1)^2}=\frac{2x^2-2}{(x^2+x+1)^2}$$

Chain Rule

$$[f(g(x))]'=f'(g(x))\cdot g'(x)$$

$$y=\frac{1}{(2-x)^3}=(2-x)^{-3} \qquad y=\left(\frac{x}{3x+1}\right)^4$$

$$y'=-3(2-x)^{-4}\cdot(-1) \qquad y'=4\left(\frac{x}{3x+1}\right)^3\cdot\frac{1\cdot(3x+1)-x\cdot 3}{(3x+1)^2}$$

$$=\frac{3}{(2-x)^4} \qquad =\frac{4x^3}{(3x+1)^5}$$

$$y=(x+\sqrt{x^2+1})^4=\left[x+(x^2+1)^{\frac{1}{2}}\right]^4$$

$$y'=4\left[x+(x^2+1)^{\frac{1}{2}}\right]^3\cdot\left[1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x\right]$$

$$=4(x+\sqrt{x^2+1})^3\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}$$

$$=\frac{4(x+\sqrt{x^2+1})^4}{\sqrt{x^2+1}}$$

$$\boxed{\left[x+(x^2+1)^{\frac{1}{2}}\right]'=1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot(x^2+1)'}$$

$$y=x\sqrt{1+x^2}=x(1+x^2)^{\frac{1}{2}}$$

$$y'=1\cdot(1+x^2)^{\frac{1}{2}}+x\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x$$

$$=\sqrt{1+x^2}+\frac{x^2}{\sqrt{1+x^2}}=\frac{1+2x^2}{\sqrt{1+x^2}}$$

$$y=\frac{x}{\sqrt{x^2+1}}=x(x^2+1)^{-\frac{1}{2}}$$

$$y'=1\cdot(x^2+1)^{-\frac{1}{2}}+x\cdot\left(-\frac{1}{2}\right)(x^2+1)^{-\frac{3}{2}}\cdot 2x$$

$$=\frac{1}{\sqrt{x^2+1}}-\frac{x^2}{(x^2+1)\sqrt{x^2+1}}=\frac{1}{(x^2+1)\sqrt{x^2+1}}$$

N 161-170: Differentiation Trigonometric Functions

Derivatives of Trigonometric Functions

$$(\sin x)'=\cos x, \qquad (\cos x)'=-\sin x,$$

$$(\tan x)'=\frac{1}{\cos^2 x}$$

Important Note: All rules & properties (Product Rule, Quotient Rule, Chain Rule) apply.

$$y=(3x-2)\sin x$$

$$y'=(3x-2)'\sin x+(3x-2)(\sin x)'$$

$$=3\sin x+(3x-2)\cos x$$

$$y=\cos(3-2x^2)$$

$$y'=-\sin(3-2x^2)\cdot(3-2x^2)'$$

$$=4x\sin(3-2x^2)$$

$$y=\cos^3(1-2x^2)$$

$$y'=3\cos^2(1-2x^2)\cdot[\cos(1-2x^2)]'$$

$$=3\cos^2(1-2x^2)\cdot[-\sin(1-2x^2)]\cdot(1-2x^2)'$$

$$=12x\cos^2(1-2x^2)\sin(1-2x^2)$$

$$y=\frac{\tan x}{\cos x}$$

$$y'=\frac{(\tan x)'\cos x-\tan x(\cos x)'}{\cos^2 x}$$

$$=\frac{\frac{1}{\cos^2 x}\cdot\cos x+\frac{\sin x}{\cos x}\cdot\sin x}{\cos^2 x}=\frac{1+\sin^2 x}{\cos^3 x}$$

$$y=\frac{1-\sin x}{1+\cos x}$$

$$y'=\frac{(1-\sin x)'(1+\cos x)-(1-\sin x)(1+\cos x)'}{(1+\cos x)^2}$$

$$=\frac{-\cos x(1+\cos x)+(1-\sin x)\sin x}{(1+\cos x)^2}$$

$$=\frac{-\cos x+\sin x-(\cos^2 x+\sin^2 x)}{(1+\cos x)^2}$$

$$=-\frac{\cos x-\sin x+1}{(1+\cos x)^2}$$

N 171-180 : Differentiation of Logarithmic & Exponential Functions

Derivatives of Logarithmic Functions

$$(\ln|x|)'=\frac{1}{x}, \qquad (\log_a|x|)'=\frac{1}{x\ln a}$$

$$y=\ln|x^2-3| \qquad y=\log_2|1-2x|$$

$$y'=\frac{(x^2-3)'}{x^2-3} \qquad y'=\frac{(1-2x)'}{(1-2x)\ln 2}$$

$$=\frac{2x}{x^2-3} \qquad =-\frac{2}{(1-2x)\ln 2}$$

$$y=\ln(x+\sqrt{x^2+1})$$

$$y'=\frac{(x+\sqrt{x^2+1})'}{x+\sqrt{x^2+1}}$$

$$=\frac{1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x}{x+\sqrt{x^2+1}}$$

$$=\frac{\sqrt{x^2+1}+x}{(x+\sqrt{x^2+1})\sqrt{x^2+1}}=\frac{1}{\sqrt{x^2+1}}$$

$$y=x(\ln x)^2$$

$$y'=(x)'(\ln x)^2+x[(\ln x)^2]'$$

$$=1\cdot(\ln x)^2+x\cdot 2\ln x\cdot(\ln x)'=(\ln x)^2+2\ln x$$

Useful Technique: Logarithmic Differentiation (appropriate when the function is a product/quotient of many factors)

$$y=\frac{(x+1)^3}{(x-2)^2(x+3)^4}$$

Taking the natural logarithm of the absolute values of both sides,

$$\ln|y|=\ln\left|\frac{(x+1)^3}{(x-2)^2(x+3)^4}\right|$$

$$=3\ln|x+1|-(2\ln|x-2|+4\ln|x+3|)$$ $\ln MN = \ln M + \ln N$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=\frac{3}{x+1}-\left(\frac{2}{x-2}+\frac{4}{x+3}\right)=-\frac{3x^2+x+16}{(x+1)(x-2)(x+3)}$$

$$\therefore\ y'=-\frac{3x^2+x+16}{(x+1)(x-2)(x+3)}\cdot\frac{(x+1)^3}{(x-2)^2(x+3)^4}$$

$$=-\frac{(3x^2+x+16)(x+1)^2}{(x-2)^3(x+3)^5}$$

Derivatives of Exponential Functions

$$(e^x)'=e^x,\qquad (a^x)'=a^x\ln a\ \ (a>0, a\neq 1)$$

$$y=5^{-x}$$

$$y'=5^{-x}\ln 5\cdot(-x)'$$

$$=-5^{-x}\ln 5$$

$$y=e^{\frac{1}{x}}$$

$$y'=e^{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)'=-\frac{e^{\frac{1}{x}}}{x^2}$$

$$y=xe^{-x^2}$$

$$y'=(x)'e^{-x^2}+x\left(e^{-x^2}\right)'$$

$$=1\cdot e^{-x^2}-2x^2e^{-x^2}$$

$$=(1-2x^2)e^{-x^2}$$

$$y=e^x\cos x$$

$$y'=(e^x)'\cos x+e^x(\cos x)'$$

$$=e^x\cos x-e^x\sin x$$

$$=e^x(\cos x-\sin x)$$

$$y=\frac{e^x-1}{e^x+1}$$

$$y'=\frac{(e^x-1)'(e^x+1)-(e^x-1)(e^x+1)'}{(e^x+1)^2}$$

$$=\frac{e^x(e^x+1)-(e^x-1)e^x}{(e^x+1)^2}=\frac{2e^x}{(e^x+1)^2}$$

N 181-190 : Differentiation of Various Functions & Higher Order Derivatives

Implicit differentiation when the function is not expressed explicitly in terms of x.

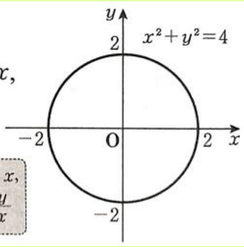
$$x^2+y^2=4$$

Differentiating both sides with respect to x,

$$2x+2y\cdot\frac{dy}{dx}=0$$

$$\therefore\ \frac{dy}{dx}=-\frac{x}{y}$$

Differentiating y^2 with respect to x, $\frac{d}{dx}y^2=\frac{d}{dy}y^2\cdot\frac{dy}{dx}=2y\cdot\frac{dy}{dx}$



Derivatives of Functions in Parametric Form

When $x=f(t)$ and $y=g(t)$,

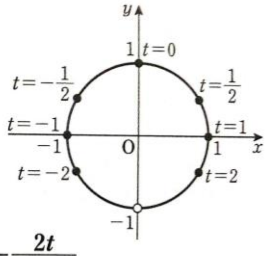
$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{g'(t)}{f'(t)}$$

$$x=\frac{2t}{1+t^2},\quad y=\frac{1-t^2}{1+t^2}$$

$$\frac{dx}{dt}=\frac{2(1+t^2)-2t\cdot 2t}{(1+t^2)^2}=\frac{2(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt}=\frac{-2t(1+t^2)-(1-t^2)\cdot 2t}{(1+t^2)^2}$$

$$=-\frac{4t}{(1+t^2)^2}\qquad \therefore\ \frac{dy}{dx}=-\frac{2t}{1-t^2}$$



The ***n*^th^ order derivative** of the function $f(x)$ is obtained by differentiating $f(x)$ for n times.

$$y=x^4+2x^3-x+1$$

$$y'=4x^3+6x^2-1$$

$$\therefore\ y''=12x^2+12x$$ Differentiating y'

$$y'''=24x+12$$ Differentiating y''

N 191-200 : Various Properties of Derivatives

Recall that $f(x)$ is differentiable at $x=a$ if $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists and is finite.

Theorem: If $f(x)$ is differentiable at $x=a$, then it is continuous at $x=a$.

Rolle's Theorem

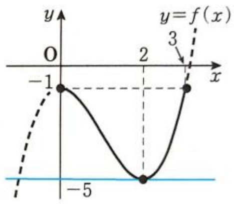
If the function $f(x)$ is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b) and $f(a)=f(b)$, then there exists at least one value c such that $f'(c)=0$ and $a<c<b$.

$$f(x)=x^3-3x^2-1\quad [0, 3]$$

$$f'(x)=3x^2-6x=3x(x-2)$$

$$f'(c)=3c(c-2)=0$$

Since $0<c<3$, $c=2$ — $c=0$ is not appropriate.



(Lagrange's) Mean Value Theorem

If the function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one value c with $a<c<b$ such that

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

$$f(x)=x^3-3x^2+2x\quad [0, 3]$$

$$\frac{f(3)-f(0)}{3-0}=\frac{6-0}{3}=2$$ $a=0,\ b=3$

$$f'(x)=3x^2-6x+2$$

From the Mean Value Theorem, there exists at least one value c such that $2=3c^2-6c+2$ and $0<c<3$.

$$3c(c-2)=0$$

Since $0<c<3$, $c=2$ — $c=0$ is not appropriate.