LEVEL O

O 1-10 : Tangents & Normals

Equation of a Tangent

The equation of the tangent to the curve $y = f(x)$ at point $\big(a,\ f(a)\big)$ is

$$y - f(a) = f'(a)(x - a)$$

Equation of a Normal

The equation of the normal to the curve $y = f(x)$ at point $\big(a,\ f(a)\big)$ is

$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$

when $f'(a) \neq 0$, and

$$x = a \qquad \text{when } f'(a) = 0$$

Example:

$y=\dfrac{x}{2x+1}$, $\quad A\left(1, \dfrac{1}{3}\right)$

Let $f(x)=\dfrac{x}{2x+1}$.

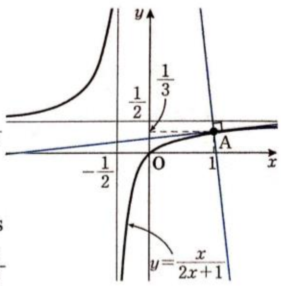
$f'(x)=\dfrac{1\cdot(2x+1)-x\cdot 2}{(2x+1)^2}=\dfrac{1}{(2x+1)^2}$

$f'(1)=\dfrac{1}{9}$

Therefore, **the equation of the tangent** is

$y-\dfrac{1}{3}=\dfrac{1}{9}(x-1) \qquad \therefore\ \boldsymbol{y=\dfrac{1}{9}x+\dfrac{2}{9}}$

The equation of the normal is $y-\dfrac{1}{3}=-9(x-1)$. $\therefore\ \boldsymbol{y=-9x+\dfrac{28}{3}}$



When the point of tangency is not specified, define an arbitrary point on the curve and solve for it.

Given that a line passing through point $(0, -3)$ is tangent to curve $y=x\ln x$, find the equation of the tangent and the coordinates of the tangent point.

Let $f(x)=x\ln x$.

$f'(x)=1\cdot\ln x+x\cdot\dfrac{1}{x}=\ln x+1$

Let the coordinates of the tangent point be $(a, a\ln a)$.

The equation of the tangent is

$y-a\ln a=(\ln a+1)(x-a)$

Since this line passes through $(0, -3)$,

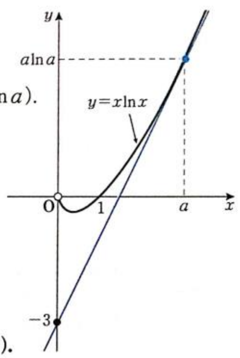
$-3-a\ln a=(\ln a+1)(0-a)$

$\therefore\ a=3$

Therefore, the equation of the tangent is

$\boldsymbol{y=(\ln 3+1)x-3}$

The coordinates of the tangent point are $\boldsymbol{(3, 3\ln 3)}$.



Sometimes, implicit differentiation is required.

$\dfrac{x^2}{8}+\dfrac{y^2}{2}=1$, $\quad A(2, 1)$

Differentiating both sides of $\dfrac{x^2}{8}+\dfrac{y^2}{2}=1$ with respect to x,

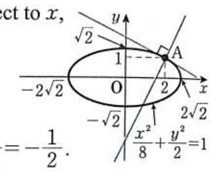
$\dfrac{2x}{8}+\dfrac{2y}{2}\cdot y'=0 \quad \Leftarrow \quad \boxed{\dfrac{d}{dx}y^2=\dfrac{d}{dy}y^2\cdot\dfrac{dy}{dx}=2yy'}$

Therefore, when $y\neq 0$, $y'=-\dfrac{x}{4y}$

Thus, the gradient of the tangent at point A is $-\dfrac{2}{4\cdot 1}=-\dfrac{1}{2}$.

Therefore, the equation of the tangent is $y-1=-\dfrac{1}{2}(x-2)$.

The equation of the normal is $y-1=2(x-2)$.



O 11-20 : Increasing/ Decreasing Functions & Relative Extreme Values

Given that a function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b),

$f'(x) > 0$ on $(a, b) \implies f(x)$ increases on $[a, b]$

$f'(x) < 0$ on $(a, b) \implies f(x)$ decreases on $[a, b]$

$f'(x) = 0$ on $(a, b) \implies f(x)$ is constant on $[a, b]$

Note: The converses are not necessarily true.

General tips for graphing a function

1) Determine the domain of the function.

E.g. $y = x - 2\sqrt{x}$, the domain is $x \geq 0$.

$y = \frac{x^2}{x+1}$, the domain is $x \neq 1$.

2) Determine the asymptotes (if any).

E.g. $y = 2x - \tan x$, the asymptotes are $x = \pm\frac{\pi}{2}$.

$y = \frac{x+1}{\sqrt{x^2+3}}$, the asymptotes are $y = \pm 1$.

Note: Graphs may cross a horizontal asymptote, but can never cross a vertical asymptote.

3) Find all stationary points (points with $y' = 0$) and create the variation table.

$y=\dfrac{x^2}{x+1}$

The domain is $x\neq -1$.

$y=x-1+\dfrac{1}{x+1}$

$y'=1-\dfrac{1}{(x+1)^2}=\dfrac{x(x+2)}{(x+1)^2}$

When $y'=0$, $x=-2,\ 0$

Creating the variation table,

x	$\cdots$	-2	$\cdots$	-1	$\cdots$	0	$\cdots$
y'	$+$	0	$-$		$-$	0	$+$
y	$\nearrow$	-4	$\searrow$		$\searrow$	0	$\nearrow$

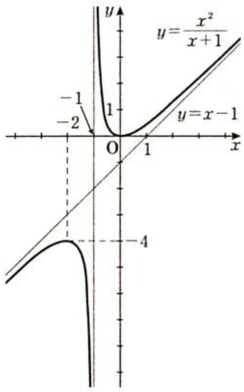
Therefore, **the relative maximum value is** $\boldsymbol{-4}$**, at** $\boldsymbol{x=-2}$ and

the relative minimum value is 0, at $\boldsymbol{x=0}$**.**

Also, $\lim\limits_{x\to -1^+} y=\infty$, $\lim\limits_{x\to -1^-} y=-\infty$,

$\lim\limits_{x\to\infty}[y-(x-1)]=0$ and $\lim\limits_{x\to-\infty}[y-(x-1)]=0$.

Thus, the asymptotes are $\boldsymbol{x=-1}$ and $\boldsymbol{y=x-1}$.



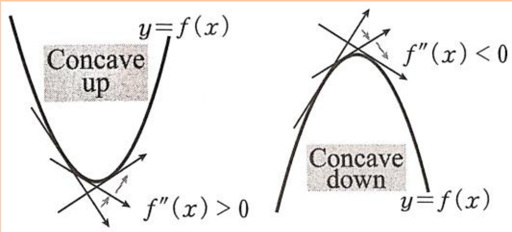
Note: If a differentiable function $f(x)$ has a relative extremum at $x = a$, then $f'(a) = 0$. However, the converse is not true. To determine if $f(x)$ has a relative extremum at $x = a$, we must look for a sign change of $f'(x)$ as x increases through a.

O 21-30 : Concavity of Curves

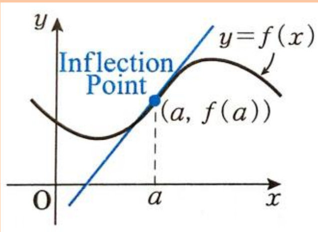
Given that a function $f(x)$ is twice differentiable on an interval, if

$f''(x) > 0 \Longrightarrow$ the curve $y = f(x)$ is concave up.

$f''(x) < 0 \Longrightarrow$ the curve $y = f(x)$ is concave down.



If point $(a,\ f(a))$ is an inflection point of the curve $y = f(x)$, then $f''(a) = 0$. However, the converse is not true. Point $(a,\ f(a))$ is an inflection point if the sign of $f''(x)$ changes as x increases through a and $f''(a) = 0$.



General Tips for graphing a function

Similar to O11-20, except that the variation table has four rows (x, y', y'', y) instead of three rows (x, y', y).

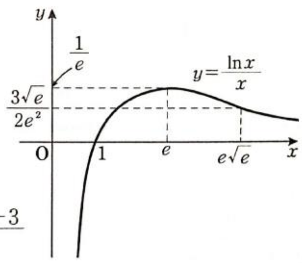
Example:

$y=\dfrac{\ln x}{x}$ $\left(\lim\limits_{x\to\infty}\dfrac{\ln x}{x}=0\right)$

The domain is $x>0$.

$$y'=\frac{\frac{1}{x}\cdot x-\ln x\cdot 1}{x^2}=\frac{1-\ln x}{x^2}$$

$$y''=\frac{-\frac{1}{x}\cdot x^2-(1-\ln x)\cdot 2x}{x^4}=\frac{2\ln x-3}{x^3}$$



when $y'=0$, $\ln x=1$, i.e. $x=e$; and

when $y''=0$, $\ln x=\frac{3}{2}$, i.e. $x=e\sqrt{e}$

Creating the variation table,

x	0	$\cdots$	e	$\cdots$	$e\sqrt{e}$	$\cdots$
y'	/	$+$	0	$-$	$-$	$-$
y''	/	$-$	$-$	$-$	0	$+$
y	/	↗	$\frac{1}{e}$	↘	$\frac{3\sqrt{e}}{2e^2}$	↘

Therefore, **the relative maximum value is** $\frac{1}{e}$, **at** $x=e$ and

there are no relative minimum values.

The inflection point is $\left(e\sqrt{e},\ \frac{3\sqrt{e}}{2e^2}\right)$. $\left[\frac{3\sqrt{e}}{2e^2}=\frac{3}{2e\sqrt{e}}\right]$

Also, since $\lim\limits_{x\to 0^+} y=-\infty$ and $\lim\limits_{x\to\infty} y=0$, the asymptotes are $x=0$, $y=0$.

Second Derivative Test for Relative Extrema

Given that $f''(x)$ is continuous on the interval including $x = a$, if

- $f'(a) = 0$ and $f''(a) > 0$, $f(a)$ represents the relative minimum value.
- $f'(a) = 0$ and $f''(a) < 0$, $f(a)$ represents the relative maximum value.

O 31-40 : Maxima & Minima

To find the (global) maximum and minimum values:

1) Create the variation table for the given domain.

2) Find the relative extreme values and the values on both ends of the interval, then compare the values.

Note: When the domain is all real numbers, we need to determine the end behaviour of the graph of $y = f(x)$, i.e. $\lim\limits_{x\to\infty} y$ and $\lim\limits_{x\to-\infty} y$.

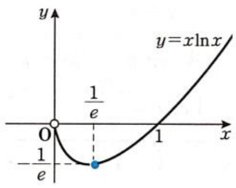
$y=x\ln x$ $\left(\lim\limits_{x\to 0^+} x\ln x=0\right)$

The domain is $x>0$.

$$y'=1\cdot\ln x+x\cdot\frac{1}{x}=\ln x+1$$

When $y'=0$ in $x>0$, $\ln x=-1$, i.e. $x=\frac{1}{e}$

x	0	$\cdots$	$\frac{1}{e}$	$\cdots$
y'	/	$-$	0	$+$
y	/	↘	$-\frac{1}{e}$	↗



Also, since $\lim\limits_{x\to 0^+} y=0$ and $\lim\limits_{x\to\infty} y=\infty$, from the variation table,

there are no maximum values, and

the minimum value is $-\frac{1}{e}$, **at** $x=\frac{1}{e}$.

Since the domain is $x>0$, determine $\lim\limits_{x\to 0^+} y$ and $\lim\limits_{x\to\infty} y$ as the values on both ends.

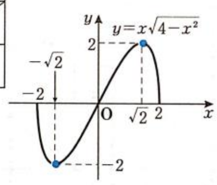
$y=x\sqrt{4-x^2}$

Since $4-x^2\geqslant 0$, the domain is $-2\leqslant x\leqslant 2$.

$$y'=1\cdot\sqrt{4-x^2}+x\cdot\frac{-x}{\sqrt{4-x^2}}=-\frac{2(x^2-2)}{\sqrt{4-x^2}}$$

When $y'=0$ in $-2<x<2$, $x=\pm\sqrt{2}$

x	-2	$\cdots$	$-\sqrt{2}$	$\cdots$	$\sqrt{2}$	$\cdots$	2
y'	/	$-$	0	$+$	0	$-$	/
y	0	↘	-2	↗	2	↘	0



From the variation table,

the maximum value is 2, at $x=\sqrt{2}$ and

the minimum value is -2, **at** $x=-\sqrt{2}$.

Refer to the worksheets for more variations.

If you have difficulty solving problems on O36-39, consult your instructor or a knowledgeable Maths tutor for additional guidance. You may also drop me a message on discord Peter Chang#4326.

O 41-50 : Various Applications of Differentiation

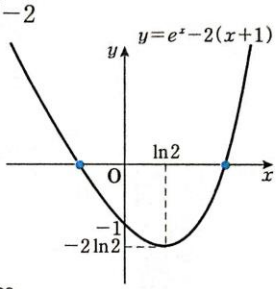
Using properties of derivatives, we can determine the number of real solutions of an equation.

$e^x-2(x+1)=0 \quad \left(\lim_{x\to\infty}[e^x-2(x+1)]=\infty\right)$

Let $f(x)=e^x-2(x+1)$. $f'(x)=e^x-2$

When $f'(x)=0$, $e^x=2$, i.e. $x=\ln 2$

x	$\cdots$	$\ln 2$	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	$-2\ln 2$	$\nearrow$



Also, $\lim_{x\to\infty} f(x)=\infty$ and $\lim_{x\to-\infty} f(x)=\infty$.

Therefore, the equation has **2 real solutions**.

Proving important inequalities

$e^x > 1+x+\frac{x^2}{2} \quad (x>0)$

Let $f(x)=e^x-\left(1+x+\frac{x^2}{2}\right)$.

$f'(x)=e^x-1-x$

$f''(x)=e^x-1$

Since $f'(x)>0$ cannot be confirmed with this expression, determine $f''(x)$.

When $x>0$, $e^x>1$; therefore, $f''(x)>0$

Thus, since $f'(x)$ increases in $x \geqslant 0$ and $f'(0)=0$, $f'(x)>0$

Therefore, since $f(x)$ increases in $x \geqslant 0$ and $f(0)=0$, $f(x)>0$

Thus, when $x>0$, $e^x>1+x+\frac{x^2}{2}$

Velocity and Acceleration

Let $x=f(t)$ be the coordinate of point P moving on a number line at time t. The velocity v and acceleration α of point P are

$$v=\frac{dx}{dt}=f'(t) \qquad \alpha=\frac{dv}{dt}=\frac{d^2x}{dt^2}=f''(t)$$

Linear Approximation I

aka 1st order Taylor Series

When the function $f(x)$ is differentiable at $x=a$ and the value of $|h|$ is approaching 0, then

$$f(a+h)\approx f(a)+f'(a)h$$

Linear Approximation II

aka 1st order Maclaurin Series

When the value of $|x|$ is approaching 0, then

$$f(x)\approx f(0)+f'(0)x$$

O 51-70 : Indefinite Integrals

Indefinite Integral of x^α

$$\int x^\alpha\,dx=\frac{1}{\alpha+1}x^{\alpha+1}+C \qquad (\alpha\neq -1)$$

$$\int x^{-1}\,dx=\int\frac{1}{x}dx=\ln|x|+C$$

Properties of Indefinite Integrals

$\int kf(x)\,dx=k\int f(x)\,dx \qquad$ (k is a constant)

$\int[f(x)\pm g(x)]\,dx=\int f(x)\,dx\pm\int g(x)\,dx$

Indefinite Integral of $f(ax+b)$

When $F'(x)=f(x)$, $a\neq 0$,

$$\int f(ax+b)\,dx=\frac{1}{a}F(ax+b)+C$$

Note: Sometimes, partial fraction decomposition of the integrand is necessary.

$$\int\sqrt{1-5x}\,dx=\int(1-5x)^{\frac{1}{2}}dx=-\frac{2}{15}(1-5x)^{\frac{3}{2}}+C$$
$$=-\frac{2}{15}(1-5x)\sqrt{1-5x}+C$$

$$\int\frac{(\sqrt{x}+2)^3}{x}dx=\int\frac{x\sqrt{x}+6x+12\sqrt{x}+8}{x}dx$$
$$=\int\left(x^{\frac{1}{2}}+6+12x^{-\frac{1}{2}}+\frac{8}{x}\right)dx$$
$$=\frac{2}{3}x^{\frac{3}{2}}+6x+24x^{\frac{1}{2}}+8\ln|x|+C$$
$$=\frac{2}{3}x\sqrt{x}+6x+24\sqrt{x}+8\ln x+C$$

$$\int\frac{x^2+x}{x+2}dx=\int\left(x-1+\frac{2}{x+2}\right)dx$$
$$=\frac{1}{2}x^2-x+2\ln|x+2|+C$$

$$\int\frac{3}{x^2-x-2}dx=\int\frac{3}{(x-2)(x+1)}dx$$
$$=\int\left(\frac{1}{x-2}-\frac{1}{x+1}\right)dx$$
$$=\ln|x-2|-\ln|x+1|+C$$
$$=\ln\left|\frac{x-2}{x+1}\right|+C$$

Indefinite Integral of Exponential Functions

$$\int e^x\,dx=e^x+C, \qquad \int a^x\,dx=\frac{a^x}{\ln a}+C$$

Indefinite Integral of Trigonometric Functions

$$\int\sin x\,dx=-\cos x+C \quad , \int\cos x\,dx=\sin x+C$$

$$\int\frac{1}{\cos^2 x}dx=\tan x+C,$$

$$\int\frac{1}{\sin^2 x}dx=-\frac{1}{\tan x}+C$$

Note: Recall all important trigonometric formulas and identities, e.g. Pythagorean Identity, Double angle identity, half angle identity etc.

$$\int \frac{e^{2x}-1}{e^x-1}\,dx=\int \frac{(e^x+1)(e^x-1)}{e^x-1}\,dx$$
$$=\int (e^x+1)\,dx = e^x+x+C$$

$$\int \frac{1+\cos^3 x}{\cos^2 x}\,dx=\int \left(\frac{1}{\cos^2 x}+\cos x\right)dx$$
$$=\tan x+\sin x+C$$

$$\int (\sin 5x+\cos 3x)\,dx=-\frac{1}{5}\cos 5x+\frac{1}{3}\sin 3x+C$$

$$\int (1-\cos x)^2\,dx=\int (1-2\cos x+\cos^2 x)\,dx$$
$$=\int \left[1-2\cos x+\frac{1}{2}(1+\cos 2x)\right]dx$$
$$=\int \left(\frac{3}{2}-2\cos x+\frac{1}{2}\cos 2x\right)dx$$
$$=\frac{3}{2}x-2\sin x+\frac{1}{4}\sin 2x+C$$

$$\int \sin^2 x\cos^3 x\,dx=\int \sin^2 x(1-\sin^2 x)\cos x\,dx$$

Let $\sin x=u$. $\cos x\,dx=du$

$$\therefore \int \sin^2 x\cos^3 x\,dx=\int u^2(1-u^2)\,du$$
$$=\int (u^2-u^4)\,du$$
$$=\frac{1}{3}u^3-\frac{1}{5}u^5+C=\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C$$

Useful Formula:

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C$$

$$\int \frac{4x^3}{x^4+1}\,dx=\int \frac{(x^4+1)'}{x^4+1}\,dx=\ln|x^4+1|+C=\ln(x^4+1)+C$$

$$\int \frac{x-1}{x^2-2x-3}\,dx=\int \frac{(x^2-2x-3)'}{x^2-2x-3}\cdot\frac{1}{2}\,dx=\frac{1}{2}\ln|x^2-2x-3|+C$$

O 71-80 : Integration by Substitution

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt \quad (x = g(t))$$

$$\int (x-2)\sqrt{3-x}\,dx$$

Let $\sqrt{3-x}=t$. Since $3-x=t^2$, $x=3-t^2$ $\therefore dx=-2t\,dt$

$$\therefore \int (x-2)\sqrt{3-x}\,dx=\int (1-t^2)t\cdot(-2t)\,dt$$
$$=2\int (t^4-t^2)\,dt$$
$$=2\left(\frac{1}{5}t^5-\frac{1}{3}t^3\right)+C$$
$$=\frac{2}{15}t^3(3t^2-5)+C$$
$$=\frac{2}{15}(4-3x)(3-x)\sqrt{3-x}+C$$

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \quad (g(x) = u)$$

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx$$

Let $\sqrt{x^2+1}=u$. Since $x^2+1=u^2$, $2x\,dx=2u\,du$

$\therefore x\,dx=u\,du$

$$\therefore \int \frac{x^3}{\sqrt{x^2+1}}\,dx=\int \frac{u^2-1}{u}\cdot u\,du$$
$$=\int (u^2-1)\,du$$
$$=\frac{1}{3}u^3-u+C$$
$$=\frac{1}{3}(x^2-2)\sqrt{x^2+1}+C$$

$$\int \frac{(\ln x)^2}{x}\,dx$$

Let $\ln x=u$. $\frac{1}{x}dx=du$

$$\therefore \int \frac{(\ln x)^2}{x}\,dx=\int u^2\,du=\frac{1}{3}u^3+C=\frac{1}{3}(\ln x)^3+C$$

O 81-90 : Integration by Parts

Integration by Parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

How to choose $f(x)$ in integration by parts

We choose $f(x)$ based on the LATE rule (1 being the first choice and 4 being the last choice)

1) **L** – Logarithmic function, e.g. $\ln x$
2) **A** – Algebraic expression, e.g. $x^2, \sqrt{x}$
3) **T** – Trigonometric function, e.g. $\sin x$
4) **E** – Exponential function, e.g. e^x

$$\int x^4\ln x\,dx=\int \left(\frac{1}{5}x^5\right)'\ln x\,dx$$

Using the LATE rule, let $f(x)=\ln x$, $g'(x)=x^4$

$$=\frac{1}{5}x^5\ln x-\int \frac{1}{5}x^5(\ln x)'\,dx$$
$$=\frac{1}{5}x^5\ln x-\frac{1}{5}\int x^4\,dx=\frac{1}{5}x^5\ln x-\frac{1}{25}x^5+C$$

$$\int x\sin x\,dx=\int x(-\cos x)'\,dx$$

Using the LATE rule, let $f(x)=x$, $g'(x)=\sin x$

$$=-x\cos x-\int (x)'(-\cos x)\,dx$$
$$=-x\cos x+\int \cos x\,dx=-x\cos x+\sin x+C$$

$$\int xe^{2x}\,dx=\int x\left(\frac{1}{2}e^{2x}\right)'dx$$

Using the LATE rule, let $f(x)=x$, $g'(x)=e^{2x}$

$$=\frac{1}{2}xe^{2x}-\int (x)'\cdot\frac{1}{2}e^{2x}\,dx$$
$$=\frac{1}{2}xe^{2x}-\frac{1}{2}\int e^{2x}\,dx=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$$

Note: In some cases, one might need to perform integration by parts twice (or multiple times).

$$\int(\ln x)^2dx=\int(x)'(\ln x)^2dx$$
$$=x(\ln x)^2-\int x[(\ln x)^2]'dx$$
$$=x(\ln x)^2-2\int\ln x\,dx$$
$$=x(\ln x)^2-2\int(x)'\ln x\,dx$$
$$=x(\ln x)^2-2\left[x\ln x-\int x(\ln x)'dx\right]$$
$$=x(\ln x)^2-2x\ln x+2\int dx$$
$$=\boldsymbol{x(\ln x)^2-2x\ln x+2x+C}$$

$$\boxed{\begin{aligned}&\int x[(\ln x)^2]'dx\\&=\int x\cdot 2\ln x\cdot\frac{1}{x}dx\\&=2\int\ln x\,dx\end{aligned}}$$

$$\int e^x\cos x\,dx=\int(e^x)'\cos x\,dx$$

Using the LATE rule
$f(x)=\cos x$, $g'(x)=e^x$

$$=e^x\cos x-\int e^x(\cos x)'dx$$
$$=e^x\cos x+\int e^x\sin x\,dx$$
$$=e^x\cos x+\int(e^x)'\sin x\,dx$$
$$=e^x\cos x+\left[e^x\sin x-\int e^x(\sin x)'dx\right]$$
$$=e^x\cos x+e^x\sin x-\int e^x\cos x\,dx$$
$$\therefore\ 2\int e^x\cos x\,dx=e^x(\cos x+\sin x)+C_1$$

Considering the constant of integration,

$$\int e^x\cos x\,dx=\frac{1}{2}e^x(\cos x+\sin x)+C$$

O 91-100 : Definite Integrals

If $F(x)$ is an indefinite integral of $f(x)$, then

$$\int_a^b f(x)\,dx=[F(x)]_a^b=F(b)-F(a)$$

$$\int_0^1\frac{dx}{\sqrt{x}+\sqrt{x+1}}=\int_0^1\frac{\sqrt{x}-\sqrt{x+1}}{(\sqrt{x}+\sqrt{x+1})(\sqrt{x}-\sqrt{x+1})}dx$$
$$=\int_0^1(\sqrt{x+1}-\sqrt{x})dx$$
$$=\left[\frac{2}{3}(x+1)^{\frac{3}{2}}-\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$$
$$=\left(\frac{2}{3}\cdot 2^{\frac{3}{2}}-\frac{2}{3}\cdot 1^{\frac{3}{2}}\right)-\frac{2}{3}\cdot 1^{\frac{3}{2}}=\boldsymbol{\frac{4}{3}(\sqrt{2}-1)}$$

$$\int_0^1\frac{dx}{(x+1)(x+2)}=\int_0^1\left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx$$

$$\boxed{\ln M-\ln N=\ln\frac{M}{N}}$$

$$=\Big[\ln|x+1|-\ln|x+2|\Big]_0^1$$
$$=\left[\ln\left|\frac{x+1}{x+2}\right|\right]_0^1=\ln\frac{2}{3}-\ln\frac{1}{2}=\ln\frac{4}{3}$$

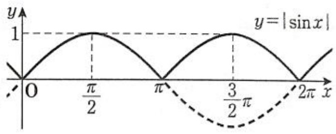
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin^2x\,dx=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1-\cos 2x)dx$$
$$=\frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$=\frac{1}{2}\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{1}{2}\right)\right]=\frac{\pi}{8}+\frac{1}{4}$$

Property of Definite Integrals I

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$$

$$\int_0^{2\pi}|\sin x|dx$$

$$|\sin x|=\begin{cases}\sin x & (0\leqslant x\leqslant\pi)\\-\sin x & (\pi\leqslant x\leqslant 2\pi)\end{cases}$$



$$\int_0^{2\pi}|\sin x|dx=\int_0^{\pi}\sin x\,dx+\int_{\pi}^{2\pi}(-\sin x)dx$$
$$=\Big[-\cos x\Big]_0^{\pi}+\Big[\cos x\Big]_{\pi}^{2\pi}$$
$$=[1-(-1)]+[1-(-1)]=4$$

$$\boxed{\begin{aligned}&\int_a^b f(x)dx\\&=\int_a^c f(x)dx+\int_c^b f(x)dx\end{aligned}}$$

O 101-110 : Integration by Substitution

When $x=g(t)$, if $a=g(\alpha)$ and $b=g(\beta)$, then

$$\int_a^b f(x)\,dx=\int_\alpha^\beta f\big(g(t)\big)g'(t)\,dt$$

$$\int_0^1\frac{x^3}{1+x^2}dx$$

x	$0\longrightarrow 1$
t	$1\longrightarrow 2$

Let $1+x^2=t$. Since $x^2=t-1$, $2x\,dx=dt$

$$\therefore\ \int_0^1\frac{x^3}{1+x^2}dx=\int_1^2\frac{t-1}{t}\cdot\frac{1}{2}dt$$
$$=\frac{1}{2}\int_1^2\left(1-\frac{1}{t}\right)dt$$
$$=\frac{1}{2}\Big[t-\ln|t|\Big]_1^2=\boldsymbol{\frac{1}{2}(1-\ln 2)}$$

Property of Definite Integrals II

$$\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx$$

$$\int_0^2 x\sqrt{4-x^2}\,dx$$

Let $\sqrt{4-x^2}=t$. Since $4-x^2=t^2$, $-2x\,dx=2t\,dt$ $\therefore x\,dx=-t\,dt$

$$\therefore\ \int_0^2 x\sqrt{4-x^2}\,dx=\int_2^0 t\cdot(-t)dt$$

x	$0\longrightarrow 2$
t	$2\longrightarrow 0$

$$=\int_0^2 t^2dt=\left[\frac{1}{3}t^3\right]_0^2=\boldsymbol{\frac{8}{3}}$$

Trigonometric Substitution:

Let $x=a\sin\theta$ for the definite integral of $\sqrt{a^2-x^2}$

Let $x=a\tan\theta$ for the definite integral of $\frac{1}{x^2+a^2}$

$$\int_0^2\frac{dx}{\sqrt{16-x^2}}$$

Let $x=4\sin\theta$. $dx=4\cos\theta\,d\theta$

When $0\leqslant\theta\leqslant\frac{\pi}{6}$, $\cos\theta>0$; therefore,

x	$0\longrightarrow 2$
θ	$0\longrightarrow\frac{\pi}{6}$

$$\sqrt{16-x^2}=\sqrt{16(1-\sin^2\theta)}=4\cos\theta$$

$$\therefore\int_0^2\frac{dx}{\sqrt{16-x^2}}=\int_0^{\frac{\pi}{6}}\frac{1}{4\cos\theta}\cdot 4\cos\theta\,d\theta=\int_0^{\frac{\pi}{6}}d\theta=\Big[\theta\Big]_0^{\frac{\pi}{6}}=\boldsymbol{\frac{\pi}{6}}$$

Integration of Even/Odd Functions

$f(x)$ is even function, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

$f(x)$ is odd function, $\int_{-a}^{a} f(x)\,dx = 0$

$$\int_{-e}^{e} xe^{x^2}dx$$

Let $f(x)=xe^{x^2}$. $f(-x)=-xe^{(-x)^2}=-xe^{x^2}=-f(x)$

$f(x)$ is an odd function.

$$\therefore \int_{-e}^{e} xe^{x^2}dx = \mathbf{0}$$

O 111-120 : Integration by Parts & Functions Represented by Definite Integrals

We use the LATE rule in integration by parts

$$\int_{e}^{2e} \ln x\,dx = \int_{e}^{2e} (x)'\ln x\,dx \quad \leftarrow \quad \boxed{\int \ln x\,dx = \int 1\cdot \ln x\,dx}$$

$$= \left[x\ln x\right]_{e}^{2e} - \int_{e}^{2e} x(\ln x)'dx \qquad \boxed{\begin{aligned}&\ln 2e\\ &=\ln 2+\ln e\\ &=\ln 2+1\end{aligned}}$$

$$= 2e\ln 2 + e - \int_{e}^{2e} dx$$

$$= 2e\ln 2 + e - \left[x\right]_{e}^{2e} = \mathbf{2e\ln 2}$$

$$\int_{-1}^{1} x^2 e^{1-x}dx = \int_{-1}^{1} x^2(-e^{1-x})'dx$$

$$= \left[-x^2e^{1-x}\right]_{-1}^{1} - \int_{-1}^{1}(x^2)'(-e^{1-x})dx$$

$$= -1+e^2+2\int_{-1}^{1} xe^{1-x}dx$$

$$= -1+e^2+2\int_{-1}^{1} x(-e^{1-x})'dx$$

$$= -1+e^2+2\left\{\left[-xe^{1-x}\right]_{-1}^{1} - \int_{-1}^{1}(x)'(-e^{1-x})dx\right\}$$

$$= -e^2-3+2\int_{-1}^{1} e^{1-x}dx$$

$$= -e^2-3+2\left[-e^{1-x}\right]_{-1}^{1} = \mathbf{e^2-5}$$

Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_{a}^{x} f(t)\,dt = f(x)$$

$$F(x) = \int_{0}^{x}(x-t)\cos t\,dt$$

x is not affected by the variable of integration t and can therefore be taken to the front of $\int$.

$$F(x) = x\int_{0}^{x}\cos t\,dt - \int_{0}^{x} t\cos t\,dt$$

$f(x)=x,\ g(x)=\int_0^x \cos t\,dt$

$[f(x)g(x)]'=f'(x)g(x)+f(x)g'(x)$

$$F'(x) = 1\cdot\int_{0}^{x}\cos t\,dt + x\left(\frac{d}{dx}\int_{0}^{x}\cos t\,dt\right) - x\cos x$$

$$= \int_{0}^{x}\cos t\,dt + x\cos x - x\cos x \qquad \boxed{\frac{d}{dx}\int_{a}^{x} f(t)dt = f(x)}$$

$$= \left[\sin t\right]_{0}^{x} \qquad \boxed{\frac{d}{dx}\int_{a}^{x} f(t)dt = f(x)}$$

$$= \sin x$$

Property of Definite Integrals III

$$\int_{a}^{a} f(x)\,dx = 0$$

O 121-130 : Integration by Quadrature & Proof of Inequalities

If $f(x)$ is continuous on the interval $[a, b]$,

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \int_{a}^{b} f(x)\,dx$$

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$

Important result:

$$\lim_{\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_{0}^{1} f(x)\,dx$$

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}$$

$$= \int_{0}^{1}\frac{dx}{1+x} = \left[\ln|1+x|\right]_{0}^{1} = \mathbf{\ln 2}$$

When proving inequalities, we need to use the following two results:

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} < 2\sqrt{n}-1$$

Let k be a natural number. When $k \leqslant x \leqslant k+1$, $\frac{1}{\sqrt{k+1}} \leqslant \frac{1}{\sqrt{x}}$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_{k}^{k+1}\frac{dx}{\sqrt{k+1}} < \int_{k}^{k+1}\frac{dx}{\sqrt{x}}$$

$$\therefore \frac{1}{\sqrt{k+1}} < \int_{k}^{k+1}\frac{dx}{\sqrt{x}} \quad \cdots①$$

Substituting $k=1,\ 2,\ 3,\ \ldots,\ n-1$ into ① and adding up the terms on each side, when $n \geqslant 2$,

$$\sum_{k=1}^{n-1}\frac{1}{\sqrt{k+1}} < \sum_{k=1}^{n-1}\int_{k}^{k+1}\frac{dx}{\sqrt{x}}$$

$$\text{LHS} = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{n}}$$

$$\text{RHS} = \int_{1}^{2}\frac{dx}{\sqrt{x}}+\int_{2}^{3}\frac{dx}{\sqrt{x}}+\int_{3}^{4}\frac{dx}{\sqrt{x}}+\cdots+\int_{n-1}^{n}\frac{dx}{\sqrt{x}}$$

$$= \int_{1}^{n}\frac{dx}{\sqrt{x}}$$

$$= \left[2x^{\frac{1}{2}}\right]_{1}^{n} = 2\sqrt{n}-2$$

$$\therefore \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{n}} < 2\sqrt{n}-2$$

Adding 1 to both sides,

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} < 2\sqrt{n}-1$$

① On the interval $[a, b]$, if $f(x) \geq g(x)$, then

$$\int_{a}^{b} f(x)\,dx \geq \int_{a}^{b} g(x)\,dx$$

Equality only when $f(x) = g(x)$ for all values of x

② If $a_n \geq b_n$ for $n = 1, 2, \cdots, N$, then

$$\sum_{n=1}^{N} a_n \geq \sum_{n=1}^{N} b_n$$

Equality only when $a_n = b_n$ for $n \geq 1$

Cauchy-Schwarz Inequality

$$\left[\int_a^b f(x)g(x)\,dx\right]^2 \leq \left(\int_a^b [f(x)]^2\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right)$$

O 131-140 : Areas

Given area S enclosed by the curve $y = f(x)$, the x-axis and two lines $x = a$ and $x = b$ $(a < b)$,

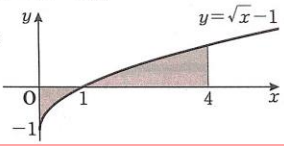
- When $f(x) \geq 0$ on $[a, b]$, $S = \int_a^b f(x)\,dx$
- When $f(x) \leq 0$ on $[a, b]$, $S = -\int_a^b f(x)\,dx$

$y=\sqrt{x}-1,\quad x=0,\quad x=4$

Since $y \leqslant 0$ in $0 \leqslant x \leqslant 1$ and $y \geqslant 0$ in $1 \leqslant x \leqslant 4$,

$$S=-\int_0^1(\sqrt{x}-1)dx+\int_1^4(\sqrt{x}-1)dx$$

$$=-\left[\frac{2}{3}x^{\frac{3}{2}}-x\right]_0^1+\left[\frac{2}{3}x^{\frac{3}{2}}-x\right]_1^4=2$$



On the interval $[a, b]$, when $f(x) \geq g(x)$, the area between the curves $y = f(x)$ and $y = g(x)$

$$S = \int_a^b [f(x) - g(x)]\,dx$$

Find the area S enclosed by the following two curves.

$f(x)=\frac{1}{8}x^2,\quad g(x)=\sqrt{x}$

Since $\frac{1}{8}x^2=\sqrt{x}$, the x-coordinates of the points of intersection are

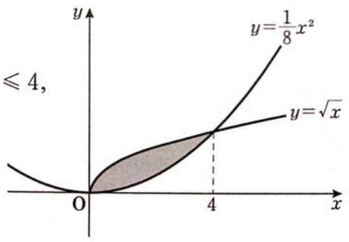
$x^4-64x=0$

$\therefore\ x=0,\ 4$

Since $f(x) \leqslant g(x)$ in $0 \leqslant x \leqslant 4$,

$$S=\int_0^4\left(\sqrt{x}-\frac{1}{8}x^2\right)dx$$

$$=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{24}x^3\right]_0^4=\frac{8}{3}$$



Given area S enclosed by the curve $x = g(y)$, the y-axis and two lines $y = c$ and $y = d$ $(c < d)$,

- When $g(y) \geq 0$ on $[c, d]$, $S = \int_c^d g(y)\,dy$

Find the area S enclosed by two curves $x=4y-y^2$ and $y=\sqrt{3x}$.

Since $y=\sqrt{3x}$, $x=\frac{1}{3}y^2$

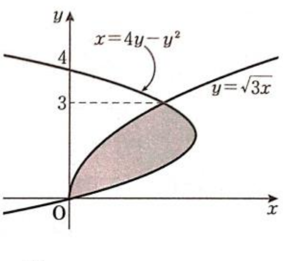
When $4y-y^2=\frac{1}{3}y^2$,

$\therefore\ y=0,\ 3$

Since $4y-y^2 \geqslant \frac{1}{3}y^2$ in $0 \leqslant y \leqslant 3$,

$$S=\int_0^3\left[(4y-y^2)-\frac{1}{3}y^2\right]dy$$

$$=\int_0^3\left(4y-\frac{4}{3}y^2\right)dy=\left[2y^2-\frac{4}{9}y^3\right]_0^3=6$$



Sometimes, curves are expressed in parametric form

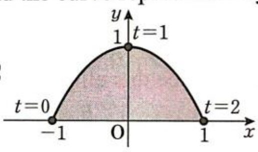
Find the area enclosed by the x-axis and the curve represented by $x=t-1,\quad y=2t-t^2$ in $0 \leqslant t \leqslant 2$.

When $y=0$, since $t(2-t)=0$, $t=0,\ 2$

Also, $y \geqslant 0$ in $0 \leqslant t \leqslant 2$.

Then, since $x=t-1$, $dx=dt$

Therefore, let S be the area to be found.



x	$-1 \longrightarrow 1$
t	$0 \longrightarrow 2$

$$S=\int_{-1}^1 y\,dx=\int_0^2(2t-t^2)dt=\left[t^2-\frac{1}{3}t^3\right]_0^2=\frac{4}{3}$$

O 141-150 : Volumes

Let $S(x)$ be the cross-sectional area of a solid cut by the plane perpendicular to the x-axis and intersecting the x-axis at x, then the volume V in $a \leq x \leq b$ is

$$V = \int_a^b S(x)\,dx$$

Volume of Revolution about the x-axis

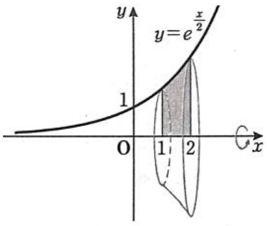
$$V = \pi\int_a^b [f(x)]^2\,dx = \pi\int_a^b y^2\,dx$$

Given a solid formed by rotating the area enclosed by curve $y=e^{\frac{x}{2}}$, the x-axis and two lines $x=1$ and $x=2$ once about the x-axis, find the volume V of this solid.

$$V=\pi\int_1^2 e^x\,dx$$

$$=\pi\left[e^x\right]_1^2$$

$$=\pi e(e-1)$$



Volume of Revolution about the y-axis

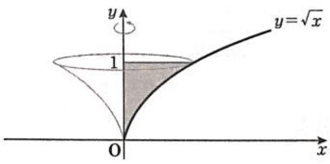
$$V = \pi\int_c^d [g(y)]^2\,dy = \pi\int_c^d x^2\,dy$$

Given a solid formed by rotating the area enclosed by curve $y=\sqrt{x}$, the y-axis and line $y=1$ once about the y-axis, find the volume V of this solid.

Since $x=y^2$,

$$V=\pi\int_0^1 y^4\,dy$$

$$=\pi\left[\frac{1}{5}y^5\right]_0^1=\frac{\pi}{5}$$



Given a solid formed by rotating the area enclosed by curve $y=\sqrt[3]{x}$ and line $y=x$ once about the x-axis, find the volume V of this solid. $(x \geqslant 0)$

When $\sqrt[3]{x}=x$, $x^3-x=0$

$x(x+1)(x-1)=0$

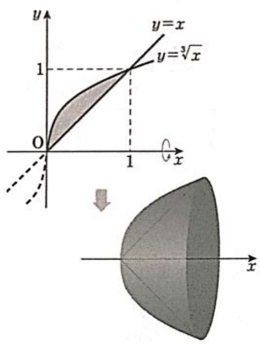
Since $x \geqslant 0$, $x=0,\ 1$

Since $\sqrt[3]{x} \geqslant x \geqslant 0$ in $0 \leqslant x \leqslant 1$,

$$V=\pi\int_0^1(\sqrt[3]{x})^2dx-\pi\int_0^1 x^2\,dx$$

$$=\pi\int_0^1\left(x^{\frac{2}{3}}-x^2\right)dx$$

$$=\pi\left[\frac{3}{5}x^{\frac{5}{3}}-\frac{1}{3}x^3\right]_0^1=\frac{4}{15}\pi$$



Refer to the worksheets for more variations.

O 151-160 : Length of a Curve, Velocity & Distance

Length of a Curve

The length L of the curve $x = f(t), y = g(t)$ $(a \le t \le b)$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

If instead we have $y = f(x)$ $(a \le x \le b)$, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

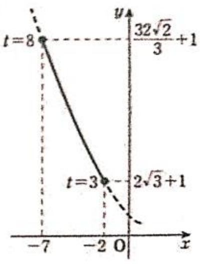
Given that the coordinates of point P(x, y) moving on the plane at time t are $x = -t+1$ and $y = \frac{2}{3}t\sqrt{t}+1$, find the distance l travelled by point P from $t=3$ to $t=8$.

$\frac{dx}{dt} = -1,\ \frac{dy}{dt} = \sqrt{t}$

$\therefore\ l = \int_3^8 \sqrt{(-1)^2 + (\sqrt{t})^2}\, dt$

$= \int_3^8 \sqrt{1+t}\, dt$

$= \left[\frac{2}{3}(1+t)^{\frac{3}{2}}\right]_3^8 = \frac{38}{3}$

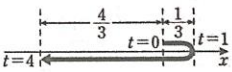


Let $x = f(t)$ be the coordinate of point P moving on a number line at time *t*, and let *v* be its velocity. Therefore, the displacement ***s*** and the distance ***l*** of point P from *t* = *a* to *t* = *b* are given as

$$s = \int_a^b v\, dt \quad , \quad l = \int_a^b |v|\, dt$$

Given that the velocity v of point P moving on a number line at time t is $1-\sqrt{t}$, find the displacement s of point P and the distance l travelled by point P from $t=0$ to $t=4$.

$s = \int_0^4 (1-\sqrt{t})dt = \left[t - \frac{2}{3}t^{\frac{3}{2}}\right]_0^4 = -\frac{4}{3}$



Also, $1-\sqrt{t} \geqslant 0$ when $0 \leqslant t \leqslant 1$

$1-\sqrt{t} \leqslant 0$ when $1 \leqslant t \leqslant 4$

$\therefore\ l = \int_0^1 (1-\sqrt{t})dt + \int_1^4 [-(1-\sqrt{t})]dt$

$= \left[t - \frac{2}{3}t^{\frac{3}{2}}\right]_0^1 + \left[-t + \frac{2}{3}t^{\frac{3}{2}}\right]_1^4 = 2$

Refer to the worksheets for more applications.

O 161-170 : Differential Equations

We use method of separating variables if the differential equation has form $f(y)\frac{dy}{dx} = g(x)$.

$\frac{dy}{dx} = y\cos x \quad \cdots①$

(i) It is obvious that constant function $y=0$ is a solution.

(ii) When $y \neq 0$, rearranging ①,

$\frac{1}{y}\cdot\frac{dy}{dx} = \cos x \qquad \int \frac{dy}{y} = \int \cos x\, dx$

$\therefore\ \ln|y| = \sin x + C_1$

$\therefore\ y = \pm e^{\sin x + C_1} = \pm e^{C_1}\cdot e^{\sin x}$

Let $\pm e^{C_1} = C$. $y = Ce^{\sin x}$ (C is an arbitrary constant, $C \neq 0$)

When $C=0$ in (ii), $y=0$, which is the same as (i).

$\therefore\ y = Ce^{\sin x}$ **(C is an arbitrary constant)**

If an initial condition is given, solve for the arbitrary constant in the general solution

$\sqrt{1+x}\frac{dy}{dx} = \sqrt{1+y} \quad \cdots①$ [When $x=3$, $y=8$]

Constant functions $x=-1$ and $y=-1$ do not satisfy the given condition.

When $x \neq -1$ and $y \neq -1$, rearranging ①,

$\frac{1}{\sqrt{1+y}}\cdot\frac{dy}{dx} = \frac{1}{\sqrt{1+x}} \qquad \int \frac{dy}{\sqrt{1+y}} = \int \frac{dx}{\sqrt{1+x}}$

$\therefore\ 2\sqrt{1+y} = 2\sqrt{1+x} + C$

When $x=3$, $y=8$; therefore, $C=2$

$\therefore\ \sqrt{1+y} = \sqrt{1+x} + 1$

Solving using substitution is a clever trick

$\frac{dy}{dx} = \frac{1-x-y}{x+y} \quad \cdots①$

Differentiating both sides of $x+y=u$ with respect to x,

$1 + \frac{dy}{dx} = \frac{du}{dx} \quad \cdots②$

From ① and ②, $\frac{du}{dx} = 1 + \frac{1-u}{u} = \frac{1}{u}$, i.e. $u\frac{du}{dx} = 1$

$\int u\, du = \int dx \qquad \therefore\ \frac{1}{2}u^2 = x + C_1$

$\therefore\ (x+y)^2 = 2x + 2C_1$

Let $2C_1 = C$.

$(x+y)^2 = 2x + C$ **(C is an arbitrary constant)**

O 171-180 : Applications of Differential Equations

This particular set includes applications of differential equations in Natural/ Social Science. The following provides explanation on a few differential equations from O176, 177

__O176a:__ $L\frac{dx}{dt}$ is the voltage drop due to an inductor, Rx is the voltage drop due to a resistor (from Ohm's Law). The differential equation $L\frac{dx}{dt} + Rx = V$ is derived from one of the Kirchhoff's Laws.

__O176b:__ $mg - kv$ is the resultant force acting on the raindrop (in the direction towards the ground). The differential equation $m\frac{dv}{dt} = mg - kv$ is a result of Newton's second law of motion.

__O177a:__ $A\frac{dx}{dt}$ is the instantaneous rate of change in water volume, and that is equal to the area of the exit hole *a* multiplied by the rate of water outflow $\sqrt{2gx}$ (from Torricelli's Law).

O 181-200 : Applications of Calculus

This particular set includes **challenging problems** that require higher order thinking skills. Make sure to go through the problems (and solutions) *carefully* with your instructor.

If you have difficulty solving them, you may drop me a message on discord Peter Chang#4326.