LEVEL K

K 1-10 : Review of Linear Functions

Any linear function can be expressed in the form $y = mx + b$, where m is the gradient of the line, b is the y-intercept.

Given a line passes through (x_1, y_1) and (x_2, y_2), the gradient of the line can be found by:

$$gradient = \frac{y_2 - y_1}{x_2 - x_1}$$

Line $x = k$ is a straight line **parallel to the y-axis**.

Line $y = k$ is a straight line **parallel to the x-axis**.

K 11-30 : Quadratic Functions & Graphs

Given a quadratic function of the form $y = a(x - p)^2 + q$, where $a \neq 0$

- The axis of symmetry is $x = p$
- The vertex is (p, q)

How to complete the square,

$$y = 3x^2 + 7x + 1$$
$$= 3\left(x^2 + \frac{7}{3}x\right) + 1$$
$$= 3\left[\left(x + \frac{7}{6}\right)^2 - \frac{49}{36}\right] + 1$$
$$= 3\left(x + \frac{7}{6}\right)^2 - 3 \times \frac{49}{36} + 1$$
$$= 3\left(x + \frac{7}{6}\right)^2 - \frac{37}{12}$$

The graph of $y = a(x - p)^2 + q$ is a translation of $y = ax^2$, p units along the x-axis and q units along the y-axis. The vertex is (p, q)

The graph of $y = 2(x + 4)^2 - 5$ is a translation of $y = 2x^2$, -4 unit(s) along the x-axis, -5 unit(s) along the y-axis.
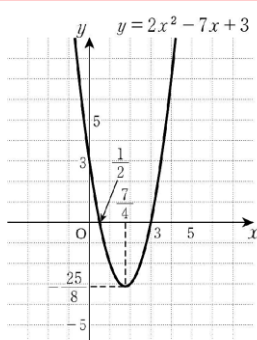
To draw the graph of a quadratic function, find the (1) vertex, (2) y-intercept and (3) x-intercept

$y = 2x^2 - 7x + 3$

① The vertex is $\left(\frac{7}{4}, -\frac{25}{8}\right)$.

② If $x = 0$, then $y = 3$.
  Therefore, the y-intercept is $(0, 3)$.

③ If $y = 0$, then $x = \frac{1}{2}$, 3.
  Therefore, the x-intercepts are $\left(\frac{1}{2}, 0\right)$ and $(3, 0)$.

$y = 2x^2 - 7x + 3$

K 31-40 : Equations of Quadratic Functions

Form 1: Vertex form

$y = a(x - p)^2 + q$, where (p, q) is the vertex of the parabola

If $a > 0$, the parabola opens upward.

If $a < 0$, the parabola opens downward.

Form 2: x-intercept form

$y = a(x - \alpha)(x - \beta)$, where α and β are the zeros / x-intercepts of the quadratic function.

The parabola crosses the x-axis at $(\alpha, 0)$ and $(\beta, 0)$

Form 3: General form

$y = ax^2 + bx + c$, where c is the y-intercept

To determine the equation of a quadratic function, we first identify the type of information given:

Type 1: Vertex & one point

The vertex is $(2, 1)$, and the parabola passes through point $(0, 5)$.

[Sol] Since the vertex is $(2, 1)$,
let $y = a(x - 2)^2 + 1$
Because the parabola passes through $(0, 5)$,

$5 = 4a + 1$

Substituting $x = 0$ and $y = 5$ into the equation above.

Type 2: x-intercepts & one point

The parabola intersects the x-axis at points $(1, 0)$ and $(3, 0)$, and intersects the y-axis at point $(0, 6)$.

[Sol] Since the parabola intersects the x-axis at $(1, 0)$ and $(3, 0)$, let $y = a(x - 1)(x - 3)$

Since the parabola intersects the y-axis at $(0, 6)$,

$6 = 3a \qquad a = 2$

Type 3: Three points

The parabola passes through three points $(-1, 0), (2, 3)$ and $(3, -4)$.

[Sol] Let $y = ax^2 + bx + c$

Since the parabola passes through $(-1, 0), (2, 3)$ and $(3, -4)$,

$$\begin{cases} a - b + c = 0 & \cdots ① \\ 4a + 2b + c = 3 & \cdots ② \\ 9a + 3b + c = -4 & \cdots ③ \end{cases}$$

K 41-70 : Maxima & Minima of Quadratic Functions

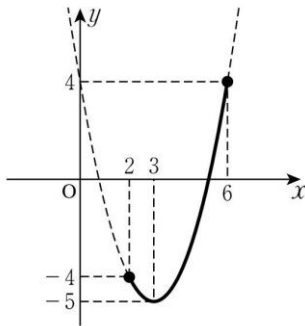Given a quadratic function $y = ax^2 + bx + c$

When $a > 0$, the function has a **minimum** value at the vertex. When $a < 0$, the function has a **maximum** value at the vertex.

To find the minimum/maximum value:

1) Draw the graph of the function

2) Highlight only the part of the curve which lies within the domain

3) Range: Minimum value $\leq y \leq$ Maximum value

$$y = x^2 - 6x + 4 \quad \text{when } 2 \leqslant x \leqslant 6$$
$$= (x-3)^2 - 5$$

Maximum value: 4 , at $x = 6$

Minimum value: -5 , at $x = 3$

Range: $-5 \leqslant y \leqslant 4$

How to find the minimum/maximum value (when the domain is arbitrary, K51-60):

1) Prepare two rulers, pretend that they are the lower boundary and the upper boundary of the domain.

2) Place the rulers on the graph. Gradually move both rulers <u>from left to right</u>.

3) Observe how the maximum and minimum value changes as the ruler passes through certain "critical points". E.g. when ruler $x=a+4$ is to the left of $x=5$, the minimum value of the function is f($a+4$), but when it moves to the right of $x=5$, the minimum value becomes f(5).

(when the function is arbitrary, K61-70):

1) As (1) above.

2) Place the rulers on the graph. Move both rulers <u>from right to left</u>. In that sense, the parabola is moving from left to right.

3) As (3) above.

Visualise the method:

Pretend this is $x = a$ Pretend this is $x = a+4$

Maximum here

Minimum here

5

x
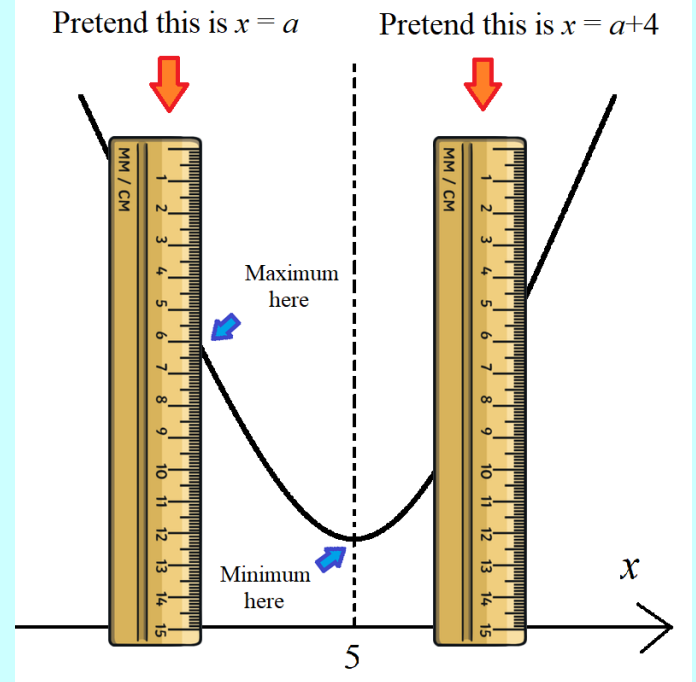
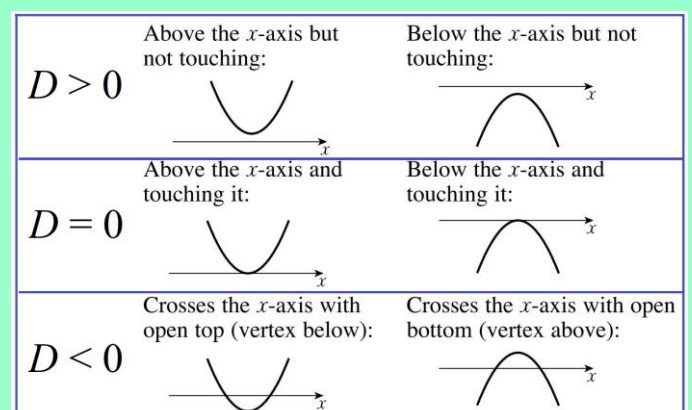**K 71-80** : Quadratic Functions & Equations

The number of real roots of $ax^2 + bx + c = 0$ is the same as the number of common points of $y = ax^2 + bx + c$ and the **x-axis**.

Given a quadratic function $y = ax^2 + bx + c$, and define $D = b^2 - 4ac$.

- 2 common points $\Leftrightarrow$ $D > 0$
- 1 common point $\Leftrightarrow$ $D = 0$
- 0 common points $\Leftrightarrow$ $D < 0$
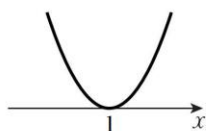
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Similarly, the number of real solutions of the simultaneous equations $\begin{cases} y = ax^2 + bx + c \\ y = mx + n \end{cases}$ is the same as the number of common points of $y = ax^2 + bx + c$ and $y = mx + n$.

*** The quadratic equation would be $ax^2 + (b - m)x + c - n = 0$

K 81-90 : Quadratic Functions & Inequalities

To solve a quadratic inequality, first find the x-intercept(s) of the quadratic function.

$x^2 - 2x - 15 > 0$

[Sol] $(x+3)(x-5) > 0$

From the sketch,
$x < -3, \ x > 5$

$x^2 - 6x + 4 < 0$

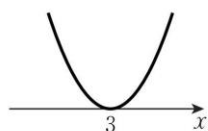[Sol] $x^2 - 6x + 4 = 0$
$x = 3 \pm \sqrt{5}$

From the sketch,
$3 - \sqrt{5} < x < 3 + \sqrt{5}$

$x^2 - 6x + 9 > 0$
[Sol] $y = x^2 - 6x + 9$
$= (x-3)^2$

From the sketch,
$y > 0$ for all values of x,
except when $x = 3$.
$x < 3, x > 3 \quad (x \neq 3)$

$x^2 - 2x + 1 < 0$
[Sol] $y = x^2 - 2x + 1$
$= (x-1)^2$

From the sketch above,
when $y < 0$ there are no
values of x.
No solution

In some cases, the quadratic function does not touch /cross the x-axis ($D < 0$).

$x^2 - 3x + 4 > 0$

[Sol] Calculating the discriminant,
D, of $x^2 - 3x + 4 = 0$,
$D = 9 - 16 = -7 < 0$

From the sketch,
$y > 0$ for all values of x.
All real numbers

$x^2 - 3x + 4 \geq 0$

[Sol] From the sketch,
All real numbers

$x^2 - 3x + 4 < 0$

[Sol] From the sketch,
when $y < 0$ there are no
values of x.
No solution

Summary: (assume $a > 0$)

The sign of D	$D>0$	$D=0$	$D<0$
The graph of $y = ax^2 + bx + c$			
Solution of:			
$ax^2 + bx + c > 0$	$x < \alpha, \ x > \beta$	$x < \alpha, \ x > \alpha$	All real numbers
$ax^2 + bx + c \geq 0$	$x \leq \alpha, \ x \geq \beta$	All real numbers	All real numbers
$ax^2 + bx + c < 0$	$\alpha < x < \beta$	No solution	No solution
$ax^2 + bx + c \leq 0$	$\alpha \leq x \leq \beta$	$x = \alpha$	No solution

K 91-100 : Solutions of Quadratic Equations

Given a quadratic equation $ax^2 + bx + c = 0 \ (a > 0)$ with two solutions α and β $(\alpha < \beta)$, we can determine the signs of α and β by determining the signs of D, $-\dfrac{b}{2a}$ and y-intercept, $f(0)$.

（ⅰ） When $D > 0$, $-\dfrac{b}{2a} > 0, f(0) > 0$, then $\alpha > 0, \beta > 0$.

（ⅱ） When $D > 0$, $-\dfrac{b}{2a} < 0, f(0) > 0$, then $\alpha < 0, \beta < 0$.

（ⅲ） When $f(0) < 0$, then $\alpha < 0, \beta > 0$.

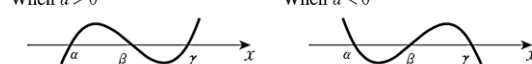Given quadratic function $f(x) = ax^2 + bx + c$ in the domain $h < x < k$,

$f(h) \cdot f(k) < 0 \Rightarrow$ there is exactly one solution between h and k

A general theorem (Intermediate Value Theorem) will be revisited in level N.
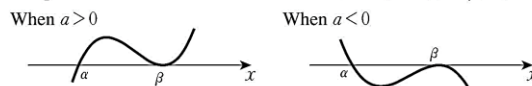
K 101-110 : Higher Degree Functions

A function that can be written as $y = ax^3 + bx^2 + cx + d$ is called a cubic function.
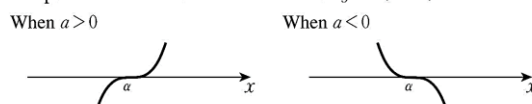
• The possible sketches of the cubic function $y = a(x-\alpha)(x-\beta)(x-\gamma)$ $(\alpha < \beta < \gamma)$ are:
When $a > 0$ When $a < 0$

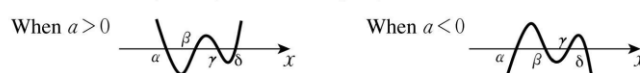• The possible sketches of the cubic function $y = a(x-\alpha)(x-\beta)^2$ $(\alpha < \beta)$ are:
When $a > 0$ When $a < 0$

• The possible sketches of the cubic function $y = a(x-\alpha)^3$ are:
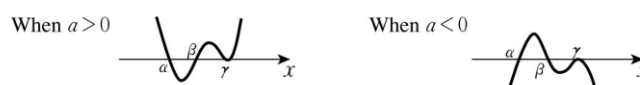When $a > 0$ When $a < 0$

A function that can be written as $y = ax^4 + bx^3 + cx^2 + dx + e$ is called a quartic function.

• The possible sketches of the quartic function $y = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ $(\alpha < \beta < \gamma < \delta)$ are:
When $a > 0$ When $a < 0$

• The possible sketches of the quartic function $y = a(x-\alpha)(x-\beta)(x-\gamma)^2$ $(\alpha < \beta < \gamma)$ are:
When $a > 0$ When $a < 0$
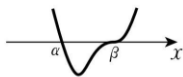

• The possible sketches of the quartic function $y = a(x-\alpha)^2(x-\beta)^2$ $(\alpha < \beta)$ are:
When $a > 0$ When $a < 0$

- The possible sketches of the quartic function
$y = a(x-\alpha)(x-\beta)^3$ $(\alpha < \beta)$ are:
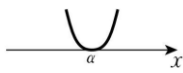
When $a > 0$
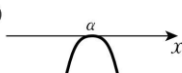

When $a < 0$

- The possible sketches of the quartic function $y = a(x-\alpha)^4$ are:
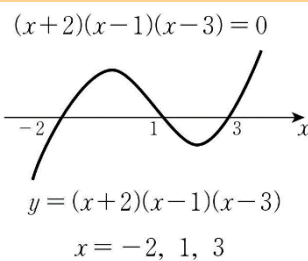
When $a > 0$

When $a < 0$

Extra Notes:

The graph of $y = a(x-p)^3 + q$ is a translation of $y = ax^3$, p units along the x-axis and q units along the y-axis.

Similarly, the graph of $y = b(x-p)^4 + q$ is a translation of $y = bx^4$, p units along the x-axis and q units along the y-axis.

K 111-120 : Higher Degree Equations & Inequalities

To solve higher degree equations, we factorise the polynomial expression and equate to zero.

$$(x+2)(x-1)(x-3) = 0$$

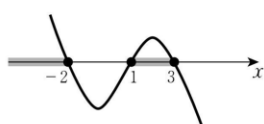$$y = (x+2)(x-1)(x-3)$$
$$x = -2, \ 1, \ 3$$

To solve higher degree inequalities, sketch the graph of the function and determine the range of values of x which satisfies the inequality.

$(x+2)(x-1)(3-x) \geqslant 0$

[Sol] $-(x+2)(x-1)(x-3) \geqslant 0$

Sketching
$y = -(x+2)(x-1)(x-3)$,

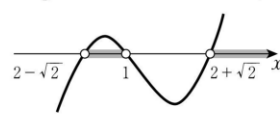Therefore, $x \leqslant -2, \ 1 \leqslant x \leqslant 3$

$(x-1)(x^2-4x+2) > 0$

[Sol] Solving $x^2-4x+2 = 0$,
$$x = 2 \pm \sqrt{2}$$
Sketching
$y = (x-1)(x^2-4x+2)$,
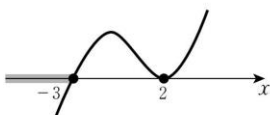

Therefore,
$2-\sqrt{2} < x < 1, \ x > 2+\sqrt{2}$

$(x+3)(x-2)^2 \leqslant 0$

[Sol] Sketching
$y = (x+3)(x-2)^2$,

Therefore,
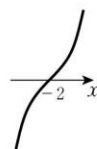$x \leqslant -3, \ x = 2$

$(x+2)(x^2+3x+5) > 0$

[Sol] Let D be the discriminant of the equation $x^2+3x+5 = 0$.
As $D = 9-20 = -11 < 0$,
so $x^2+3x+5 > 0$ for all x.
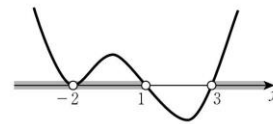Therefore,
$x+2 > 0$
Thus,
$x > -2$

$(x+2)^2(x-1)(x-3) > 0$

[Sol] Sketching
$y = (x+2)^2(x-1)(x-3)$,

Therefore,
$x < -2, \ -2 < x < 1, \ x > 3$

$(x+2)(x-1)(x^2+3x+3) < 0$

[Sol]
Calculating the discriminant, D, of $x^2+3x+3 = 0$,
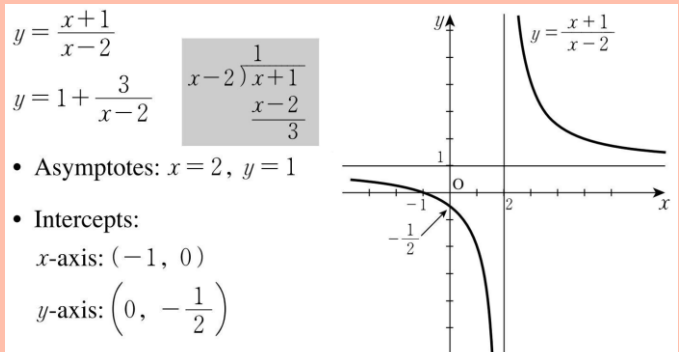$$D = 9-12 = -3 < 0.$$
So $x^2+3x+3 > 0$ for all x.
Therefore,
$$(x+2)(x-1) < 0$$
Thus, $-2 < x < 1$

Note: One may need to apply the factor theorem.

K 121-140 : Graphs of Fractional Functions

The asymptotes of the graph of $\boxed{y = \dfrac{k}{x-p} + q}$ $(k \neq 0)$ are $x = p$ and $y = q$.

$y = \dfrac{x+1}{x-2}$

$y = 1 + \dfrac{3}{x-2}$

$$\begin{array}{r} 1 \\ x-2 \overline{)\, x+1} \\ \underline{x-2} \\ 3 \end{array}$$

- Asymptotes: $x = 2$, $y = 1$
- Intercepts:
  x-axis: $(-1, \ 0)$
  y-axis: $\left(0, \ -\dfrac{1}{2}\right)$

Extra Notes:

Given a fractional function $y = \dfrac{k}{x-p} + q$ $(k \neq 0)$, the graph's asymptotes are $x = p$ and $y = q$. This graph has been translated from $y = \dfrac{k}{x}$, p units along the x-axis and q units along the y-axis.

If a domain is specified, highlight only the part of the graph which lies within the domain.

$f(x) = \dfrac{2x-5}{x-3}$ $(2 \leqslant x \leqslant 6, \text{ but with } x \neq 3)$
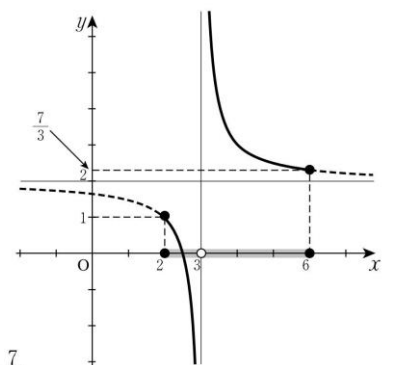
[Sol] $f(x) = 2 + \dfrac{1}{x-3}$

- Asymptotes:
  $x = 3$, $y = 2$
- Intercepts:
  x-axis: $\left(\dfrac{5}{2}, \ 0\right)$
  y-axis: $\left(0, \ \dfrac{5}{3}\right)$
- Range:
  $f(2) = 1$, $f(6) = \dfrac{7}{3}$

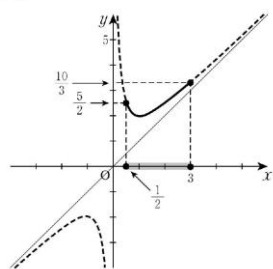From the graph, $f(x) \leqslant 1$, $f(x) \geqslant \dfrac{7}{3}$

The asymptotes of the graph of $\boxed{y = ax + \dfrac{b}{x}}$ are $x = 0$ and $y = ax$ (oblique asymptote).

Given a fractional function $y = x - p + \dfrac{k}{x-p} + q$.

The graph has been translated from $y = x + \dfrac{k}{x}$, p units along the x-axis and q units along the y-axis.

Given the fractional function $f(x) = x + \dfrac{1}{x}$ $\left(\dfrac{1}{2} \leqslant x \leqslant 3\right)$, draw the graph and find the maximum value.

[Sol] $f\left(\dfrac{1}{2}\right) = \dfrac{1}{2} + \dfrac{1}{\frac{1}{2}} = \dfrac{5}{2}$

$f(3) = 3 + \dfrac{1}{3} = \dfrac{10}{3}$

From the graph:

Maximum value: $f(3) = \dfrac{10}{3}$

K 141-150 : Fractional Equations & Inequalities

To solve fractional equations,

$\dfrac{x+4}{x+2} = \dfrac{x}{3}$

[Sol] Multiplying both sides by $3(x+2)$,

$3(x+4) = x(x+2)$ (where $x \neq -2$)

$x^2 - x - 12 = 0$

$(x-4)(x+3) = 0$

$x = 4, -3$

We must always check each solution and reject any that make a denominator in the equation zero.

$\dfrac{1}{x-1} - \dfrac{2}{x+1} = \dfrac{x}{x-1}$

[Sol] Multiplying both sides of the equation by $(x-1)(x+1)$,

$x+1 - 2(x-1) = x(x+1)$ (where $x \neq 1, x \neq -1$)

$x^2 + 2x - 3 = 0$

$x = -3, 1$

$x = 1$ is invalid. ☜🔲 ▨ Since the denominator is 0 when $x = 1$.

Therefore, $x = -3$

To solve a fractional inequality,

Method 1: Graphical method

$\dfrac{(x+1)(x-2)}{x} \leqslant 0$

[Sol] Rewriting, $x - 1 - \dfrac{2}{x} \leqslant 0$,

$x - 1 \leqslant \dfrac{2}{x}$

Solving $x - 1 = \dfrac{2}{x}$,

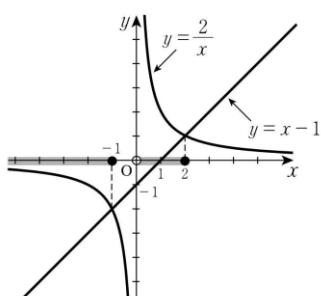$x(x-1) = 2$ (where $x \neq 0$)

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

From the graph,

$x \leqslant -1, \ 0 < x \leqslant 2$

Method 2: Multiply the square of the denominator

$\dfrac{5}{x+2} \geqslant \dfrac{2}{x-1}$

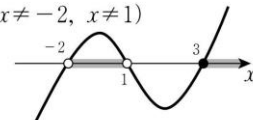[Sol] $\dfrac{5}{x+2} - \dfrac{2}{x-1} \geqslant 0 \ \Rightarrow \ \dfrac{5(x-1) - 2(x+2)}{(x+2)(x-1)} \geqslant 0$

$\dfrac{3(x-3)}{(x+2)(x-1)} \geqslant 0$

Multiplying both sides by $(x+2)^2(x-1)^2$,

$3(x-3)(x+2)(x-1) \geqslant 0$ (where $x \neq -2, \ x \neq 1$)

From the graph,

$-2 < x < 1, \ x \geqslant 3$

K 151-160 : Graphs of Irrational Functions

- Given an irrational function of the form $y = \sqrt{k(x-p)} + q$, the domain and range of the function are as follows:
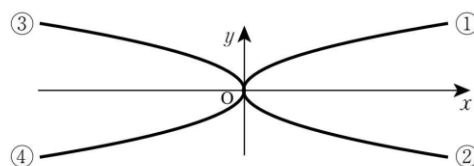 When $k > 0$, Domain: $x \geqslant p$, Range: $y \geqslant q$
  When $k < 0$, Domain: $x \leqslant p$, Range: $y \geqslant q$

- An irrational function of the form $y = \sqrt{k(x-p)} + q$ is a translation of $y = \sqrt{kx}$, p units along the x-axis, and q units along the y-axis.

- The graphs of some common irrational functions are shown below.
  ① $y = \sqrt{x}$, ② $y = -\sqrt{x}$, ③ $y = \sqrt{-x}$, ④ $y = -\sqrt{-x}$

K 161-170 : Irrational Equations & Inequalities

When solving irrational equation, check if solutions correspond to common points on the graph.

$\sqrt{4-x} = x - 2$
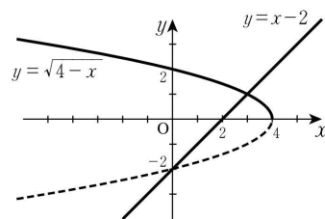
[Sol] Squaring both sides,

$4 - x = (x-2)^2$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, \ 3$

From the graph, $x = 0$ is an extraneous solution.

Therefore, $x = 3$

Alternatively, substitute the solution into the original equation and check if LHS = RHS.

Solve the irrational equation $\sqrt{5-2x} = -x + 1$.

[Sol] Squaring both sides,

$5 - 2x = (-x+1)^2 \ \Rightarrow \ x^2 = 4 \ \Rightarrow \ x = \pm 2$

Substituting into the original equation,

(i) When $x = 2$, LHS $= 1$, RHS $= -1$
        Therefore, $x = 2$ is an extraneous solution.

(ii) When $x = -2$, LHS $= 3$, RHS $= 3$

From (i) and (ii), $x = -2$

To solve irrational inequalities, first solve the equation, then look for range of values of x that satisfy the inequality from the **graph**.

$\sqrt{2x+1} > x-1$

[Sol] Let $\sqrt{2x+1} = x-1$ ☞ First, solve the equation.
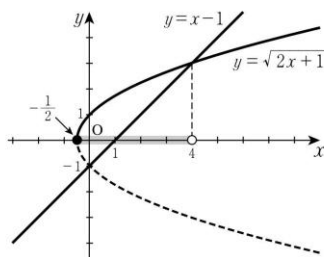
Squaring both sides,

$2x+1 = (x-1)^2$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, 4$

From the graph, $x = 0$ is an extraneous solution.

Thus, $x = 4$

Therefore, $-\dfrac{1}{2} \leq x < 4$ ☞ For these values of x, the graph of $y = \sqrt{2x+1}$ lies above the graph of $y = x-1$.

K 171-190 : Exponential Functions and Graphs

Properties of exponents:

Laws of Exponents I

When $a \neq 0$, $b \neq 0$,

$a^0 = 1$ $\qquad a^m \times a^n = a^{m+n} \qquad (a^m)^n = a^{mn}$

$a^{-n} = \dfrac{1}{a^n} \qquad a^m \div a^n = a^{m-n} \qquad (ab)^m = a^m b^m$

Laws of Exponents II

When $a > 0$, $b > 0$ and m, n and p are positive integers:

$\sqrt[n]{a^m} = (\sqrt[n]{a})^m \qquad \sqrt[mp]{a^{np}} = \sqrt[m]{a^n} \qquad \sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$

$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab} \qquad \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

Laws of Exponents III

When $a > 0$, m is an integer, and n is a positive integer,

$a^{\frac{m}{n}} = \sqrt[n]{a^m} \qquad (a^{\frac{1}{n}} = \sqrt[n]{a})$
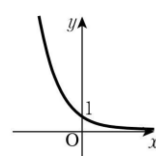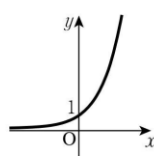
The graph of an exponential function:

When $a > 0$ and $a \neq 1$, a function of the form $y = a^x$ is called an exponential function of x, where a is the base.

When $0 < a < 1$ When $a > 1$

The graph of $y = a^x$ passes through point $(0, 1)$, and the x-axis is the asymptote.

Transformation effects:

$y = \left(\dfrac{1}{a}\right)^x$ or $y = a^{-x}$ (where $a > 0$ and $a \neq 1$) is a **reflection** of $y = a^x$ with respect to the **y-axis**.

$y = -a^x$ is a **reflection** of $y = a^x$ with respect to the **x-axis**.

The graph of $y = a^{x-p} + q$ (where $a > 0$ and $a \neq 1$) is a translation of $y = a^x$, p units along the x-axis and q units along the y-axis.

Since an exponential function is monotonic,

Given the exponential function $y = a^x$

★ $a > 1$, then as x increases, y increases. Therefore, $p < q$ if and only if $a^p < a^q$.

★ $0 < a < 1$, then as x increases, y decreases. Therefore, $p < q$ if and only if $a^p > a^q$.

Given the exponential functions $y = a^x$ and $y = b^x$, when $a > 0$, $b > 0$ and $p > 0$, $a < b$ if and only if $a^p < b^p$.

K 191-200 : Exponential Equations & Inequalities

When solving equations, use a common base.

$8^x = 2^{x+1}$ $\qquad\qquad 9^x = 3\sqrt{3}$

[Sol] $2^{3x} = 2^{x+1}$ $\qquad$ [Sol] $3^{2x} = 3^{\frac{3}{2}}$

$3x = x+1 \qquad\qquad\qquad 2x = \dfrac{3}{2}$

$2x = 1$

$x = \dfrac{1}{2} \qquad\qquad\qquad\quad x = \dfrac{3}{4}$

$2^{2x+1} + 3 \cdot 2^x - 2 = 0$

[Sol]

$2(2^x)^2 + 3 \cdot 2^x - 2 = 0 \qquad$ Since $X > 0$, $X = -2$ is an extraneous solution.

Let $2^x = X$ (where $X > 0$)

$2X^2 + 3X - 2 = 0 \qquad\qquad$ Therefore, $X = \dfrac{1}{2}$

$X = \dfrac{1}{2}, -2 \qquad\qquad\qquad\quad x = -1$

When solving inequalities, we switch the inequality sign if the base is < 1, leave the inequality sign unchanged if the base is > 1.

$9^x < 27 \qquad\qquad \left(\dfrac{1}{4}\right)^x < \dfrac{1}{8}$

[Sol] $3^{2x} < 3^3 \qquad$ [Sol] $\left(\dfrac{1}{2}\right)^{2x} < \left(\dfrac{1}{2}\right)^3$

$2x < 3 \qquad\qquad\qquad 2x > 3$

$x < \dfrac{3}{2} \qquad\qquad\qquad x > \dfrac{3}{2}$

$2^{2x+1} + 3 \cdot 2^x - 2 < 0$

[Sol]

$2(2^x)^2 + 3 \cdot 2^x - 2 < 0 \qquad$ Since $X > 0$,

Let $2^x = X$ (where $X > 0$) $\qquad 0 < X < \dfrac{1}{2}$

$2X^2 + 3X - 2 < 0$

$(X+2)(2X-1) < 0 \qquad\qquad 0 < 2^x < \dfrac{1}{2}$

$-2 < X < \dfrac{1}{2} \qquad\qquad\qquad x < -1$