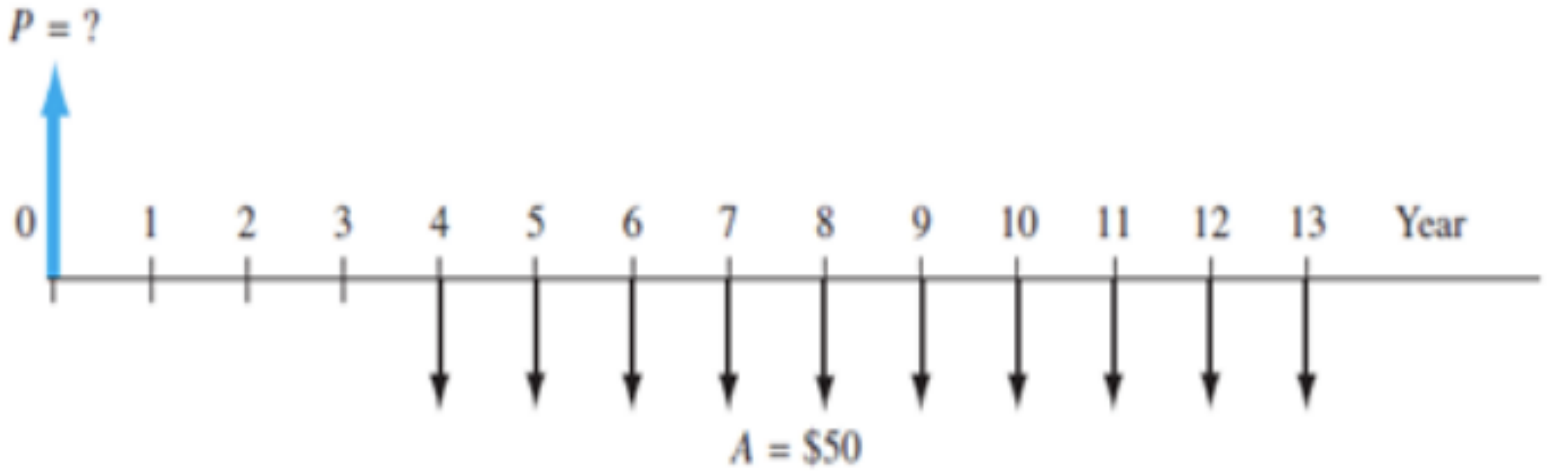
- **Course Title:** Engineering Cost Analysis & Economy (ENGR 222)
- **Session:** Spring 2024
- **Instructor:** Sudipta Chowdhury (chowdhurys@marshall.edu)
- **Class Time: TR 9.30 AM-10.45 AM**
- **Office hours: TR 11.00 AM-12.30 PM**

Shifted Cash Flow Series

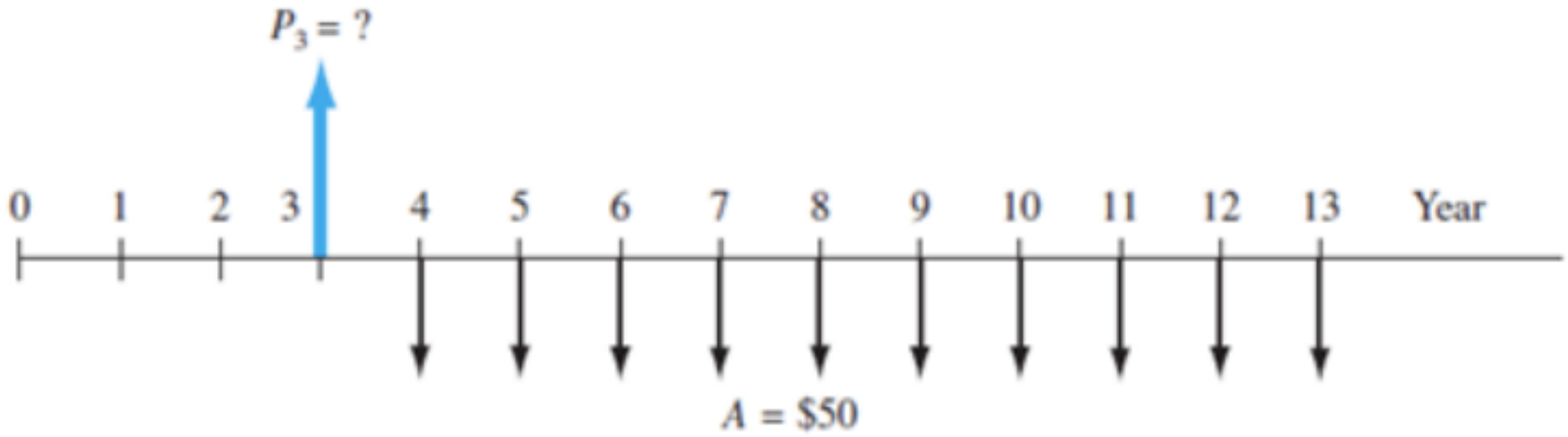
- When a uniform series begins at a time other than at the end of period 1, it is called a **shifted series**
- The cash flow diagram below is an example of a shifted series



A uniform series that is shifted

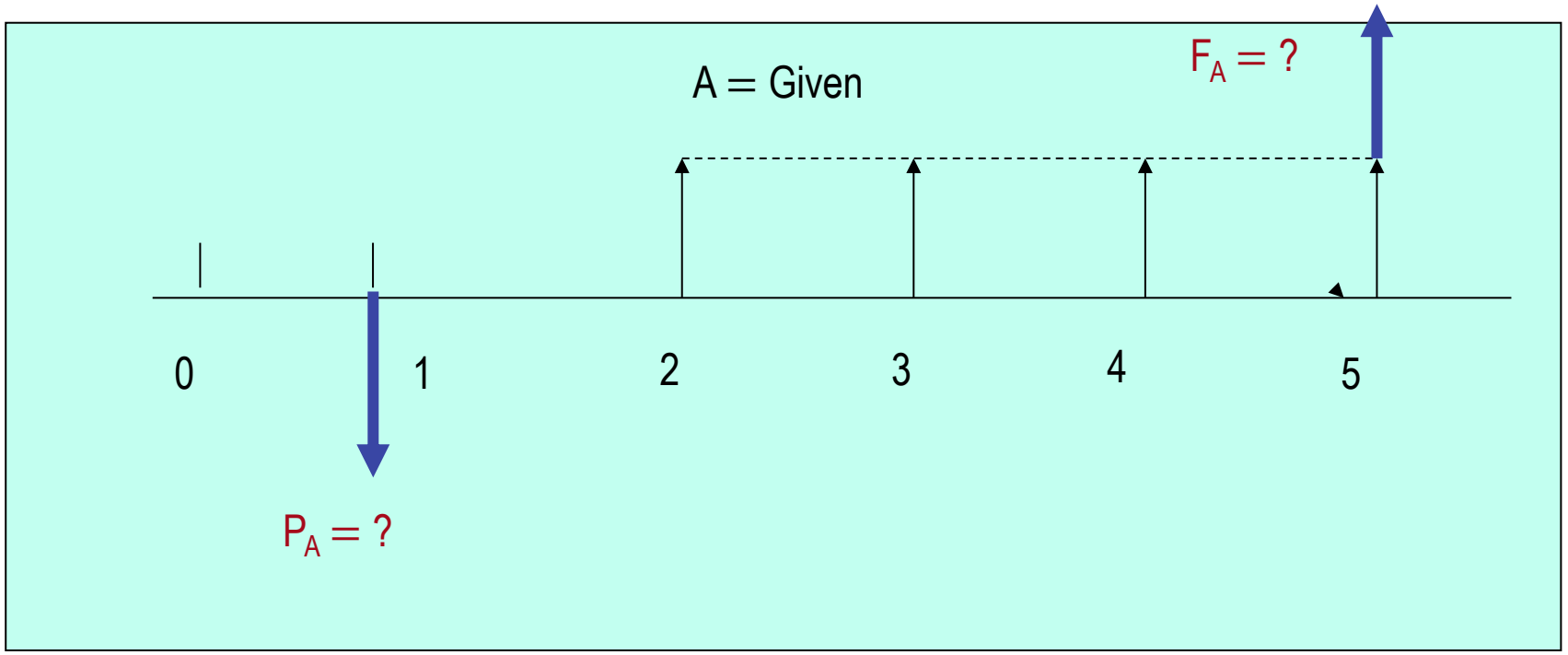
Key point!

- It is important to remember that when using the **P/A factor**, the 'P' value is always located **one year prior** to the first 'A' value because that 's how the equation was derived.
- For example, if a uniform series of payments extended from year 4 through year 13. the P/A factor would yield a 'P' value in vear 3. not vear 4!



Key point!

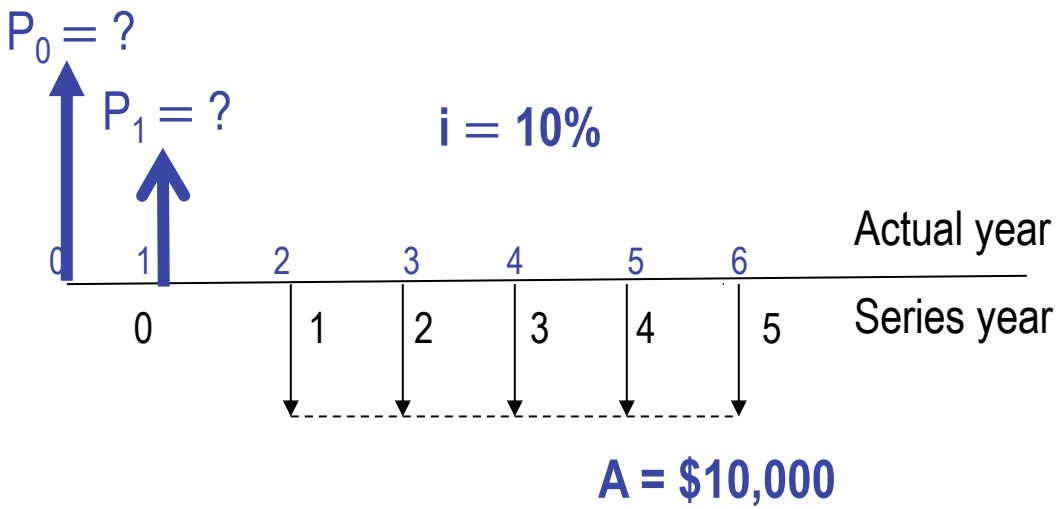
- The future worth is always located in **the same period** as the last uniform series amount when using the F/A factor.
- The following figure shows both cases:



Example 1

The present worth of the cash flow shown below at i = 10% is:

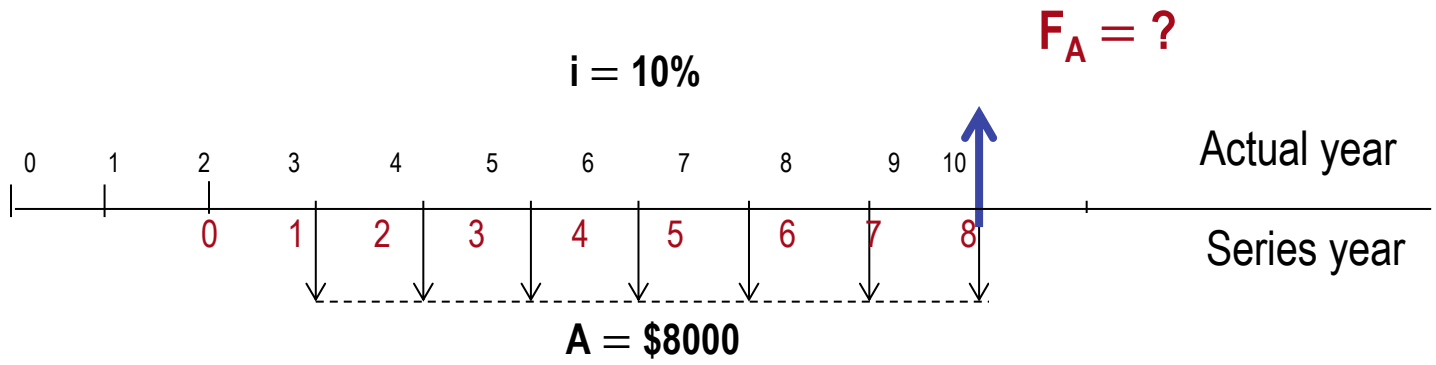
(a) $25,304 (b) $29,562 (c) $34,462 (d) $37,908



10%	Compound Interest Factors								10%
	Single Payment		Uniform Payment Series				Arithmetic Gradient		
n	Compound Amount Factor Find F Given P F/P	Present Worth Factor Find P Given F P/F	Sinking Fund Factor Find A Given F A/F	Capital Recovery Factor Find A Given P A/P	Compound Amount Factor Find F Given A F/A	Present Worth Factor Find P Given A P/A	Gradient Uniform Series Find A Given G A/G	Gradient Present Worth Find P Given G P/G	n
1	1.100	.9091	1.0000	1.1000	1.000	0.909	0	0	1
2	1.210	.8264	.4762	.5762	2.100	1.736	0.476	0.826	2
3	1.331	.7513	.3021	.4021	3.310	2.487	0.937	2.329	3
4	1.464	.6830	.2155	.3155	4.641	3.170	1.381	4.378	4
5	1.611	.6209	.1638	.2638	6.105	3.791	1.810	6.862	5
6	1.772	.5645	.1296	.2296	7.716	4.355	2.224	9.684	6
7	1.949	.5132	.1054	.2054	9.487	4.868	2.622	12.763	7
8	2.144	.4665	.0874	.1874	11.436	5.335	3.004	16.029	8
9	2.358	.4241	.0736	.1736	13.579	5.759	3.372	19.421	9
10	2.594	.3855	.0627	.1627	15.937	6.145	3.725	22.891	10

Example 2

How much money would be available in year 10 if $8000 is deposited each year in years 3 through 10 at an interest rate of 10% per year?



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	Single Payment		Uniform Payment Series				Arithmetic Gradient		
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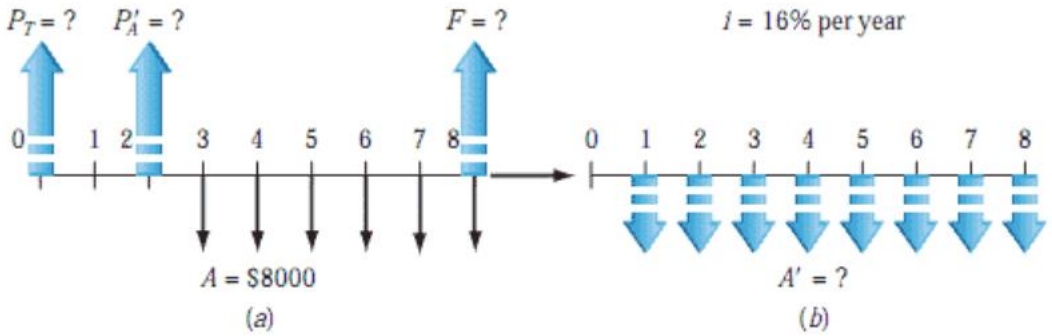
Example 3

Recalibration of sensitive measuring devices costs $8000 per year. If the machine will be recalibrated for each of 6 years starting 3 years after purchase, calculate the 8-year equivalent uniform series at 16% per year.



16% TABLE 20 Discrete Cash Flow: Compound Interest Factors 16%

	Single Payments		Uniform Series Payments				Arithmetic Gradients	
n	Compound Amount F/P	Present Worth P/F	Sinking Fund A/F	Compound Amount F/A	Capital Recovery A/P	Present Worth P/A	Gradient Present Worth P/G	Gradient Uniform Series A/G
1	1.1600	0.8621	1.00000	1.0000	1.16000	0.8621		
2	1.3456	0.7432	0.46296	2.1600	0.62296	1.6052	0.7432	0.4630
3	1.5609	0.6407	0.28526	3.5056	0.44526	2.2459	2.0245	0.9014
4	1.8106	0.5523	0.19738	5.0665	0.35738	2.7982	3.6814	1.3156
5	2.1003	0.4761	0.14541	6.8771	0.30541	3.2743	5.5858	1.7060
6	2.4364	0.4104	0.11139	8.9775	0.27139	3.6847	7.6380	2.0729
7	2.8262	0.3538	0.08761	11.4139	0.24761	4.0386	9.7610	2.4169
8	3.2784	0.3050	0.07022	14.2401	0.23022	4.3436	11.8962	2.7388
9	3.8030	0.2630	0.05708	17.5185	0.21708	4.6065	13.9998	3.0391
10	4.4114	0.2267	0.04690	21.3215	0.20690	4.8332	16.0399	3.3187
11	5.1173	0.1954	0.03886	25.7329	0.19886	5.0286	17.9941	3.5783
12	5.9360	0.1685	0.03241	30.8502	0.19241	5.1971	19.8472	3.8189
13	6.8858	0.1452	0.02718	36.7862	0.18718	5.3423	21.5899	4.0413
14	7.9875	0.1252	0.02290	43.6720	0.18290	5.4675	23.2175	4.2464
15	9.2655	0.1079	0.01936	51.6595	0.17936	5.5755	24.7284	4.4352
16	10.7480	0.0930	0.01641	60.9250	0.17641	5.6685	26.1241	4.6086

What happens if there is a randomly placed single amount?

Example 4

A design-build-operate engineering company in Texas that owns a sizable amount of land plans to lease the drilling rights to a mining and exploration company. The contract calls for the mining company to pay $20,000 per year for 20 years beginning 3 years from now (i.e., beginning at the end of year 3 and continuing through year 22) plus $10,000 six years from now and $15,000 sixteen years from now. Determine the five equivalent values listed below at 16% per year.

1. Total present worth P in year 0
2. Future worth F in year 22
3. Annual series over all 22 years

16%	End-of-Period Compound Interest Factors								16%
	Single Payment		Uniform Payment Series				Arithmetic Gradient		
N	Compound Amount Factor F/P	Present Worth Factor P/F	Capital Recovery Factor A/P	Present Worth Factor P/A	Sinking Fund Factor A/F	Compound Amount Factor F/A	Present Worth Factor P/G	Uniform Payment Factor A/G	N
1	1.160	.8621	1.1600	.862	1.0000	1.000	0	0	1
2	1.346	.7432	.6230	1.605	.4630	2.160	.743	.463	2
3	1.561	.6407	.4453	2.246	.2853	3.506	2.024	.901	3
4	1.811	.5523	.3574	2.798	.1974	5.066	3.681	1.316	4
5	2.100	.4761	.3054	3.274	.1454	6.877	5.586	1.706	5
6	2.436	.4104	.2714	3.685	.1114	8.977	7.638	2.073	6
7	2.826	.3538	.2476	4.039	.0876	11.414	9.761	2.417	7
8	3.278	.3050	.2302	4.344	.0702	14.240	11.896	2.739	8
9	3.803	.2630	.2171	4.607	.0571	17.519	14.000	3.039	9
10	4.411	.2267	.2069	4.833	.0469	21.321	16.040	3.319	10
11	5.117	.1954	.1989	5.029	.0389	25.733	17.994	3.578	11
12	5.936	.1685	.1924	5.197	.0324	30.850	19.847	3.819	12
13	6.886	.1452	.1872	5.342	.0272	36.786	21.590	4.041	13
14	7.988	.1252	.1829	5.468	.0229	43.672	23.217	4.246	14
15	9.266	.1079	.1794	5.575	.0194	51.660	24.728	4.435	15
16	10.748	.0930	.1764	5.668	.0164	60.925	26.124	4.609	16
17	12.468	.0802	.1740	5.749	.0140	71.673	27.407	4.768	17
18	14.463	.0691	.1719	5.818	.0119	84.141	28.583	4.913	18
19	16.777	.0596	.1701	5.877	.0101	98.603	29.656	5.046	19
20	19.461	.0514	.1687	5.929	.00867	115.4	30.632	5.167	20
21	22.574	.0443	.1674	5.973	.00742	134.8	31.518	5.277	21
22	26.186	.0382	.1664	6.011	.00635	157.4	32.320	5.377	22

Shifted Arithmetic Gradients

Shifted gradient begins at a time other than between periods 1 and 2

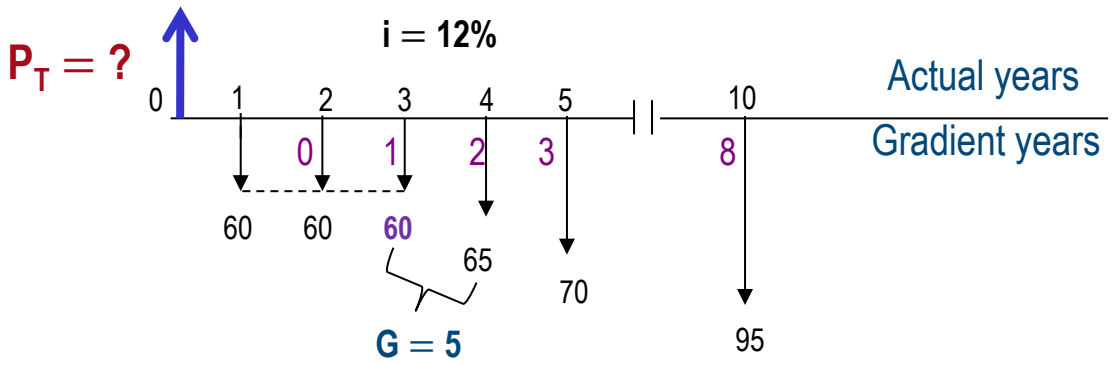
Present worth P_G is located 2 periods before gradient starts

Must use multiple factors to find P_T in actual year 0

To find equivalent A series, find P_T at actual time 0 and apply (A/P,i,n)

Example 5

John Deere expects the cost of a tractor part to increase by $5 per year beginning 4 years from now. If the cost in years 1-3 is $60, determine the *present worth in year 0* of the cost through year 10 at an interest rate of 12% per year.



12%	Compound Interest Factors								12%
	Single Payment		Uniform Payment Series				Arithmetic Gradient		
n	Compound Amount Factor Find *F* Given *P* *F/P*	Present Worth Factor Find *P* Given *F* *P/F*	Sinking Fund Factor Find *A* Given *F* *A/F*	Capital Recovery Factor Find *A* Given *P* *A/P*	Compound Amount Factor Find *F* Given *A* *F/A*	Present Worth Factor Find *P* Given *A* *P/A*	Gradient Uniform Series Find *A* Given *G* *A/G*	Gradient Present Worth Find *P* Given *G* *P/G*	n
1	1.120	.8929	1.0000	1.1200	1.000	0.893	0	0	**1**
2	1.254	.7972	.4717	.5917	2.120	1.690	0.472	0.797	**2**
3	1.405	.7118	.2963	.4163	3.374	2.402	0.925	2.221	**3**
4	1.574	.6355	.2092	.3292	4.779	3.037	1.359	4.127	**4**
5	1.762	.5674	.1574	.2774	6.353	3.605	1.775	6.397	**5**
6	1.974	.5066	.1232	.2432	8.115	4.111	2.172	8.930	**6**
7	2.211	.4523	.0991	.2191	10.089	4.564	2.551	11.644	**7**
8	2.476	.4039	.0813	.2013	12.300	4.968	2.913	14.471	**8**
9	2.773	.3606	.0677	.1877	14.776	5.328	3.257	17.356	**9**
10	3.106	.3220	.0570	.1770	17.549	5.650	3.585	20.254	**10**

Shifted Geometric Gradients

Shifted gradient begins at a time other than between periods 1 and 2

Equation yields P_g for *all* cash flows (base amount A_1 is included)

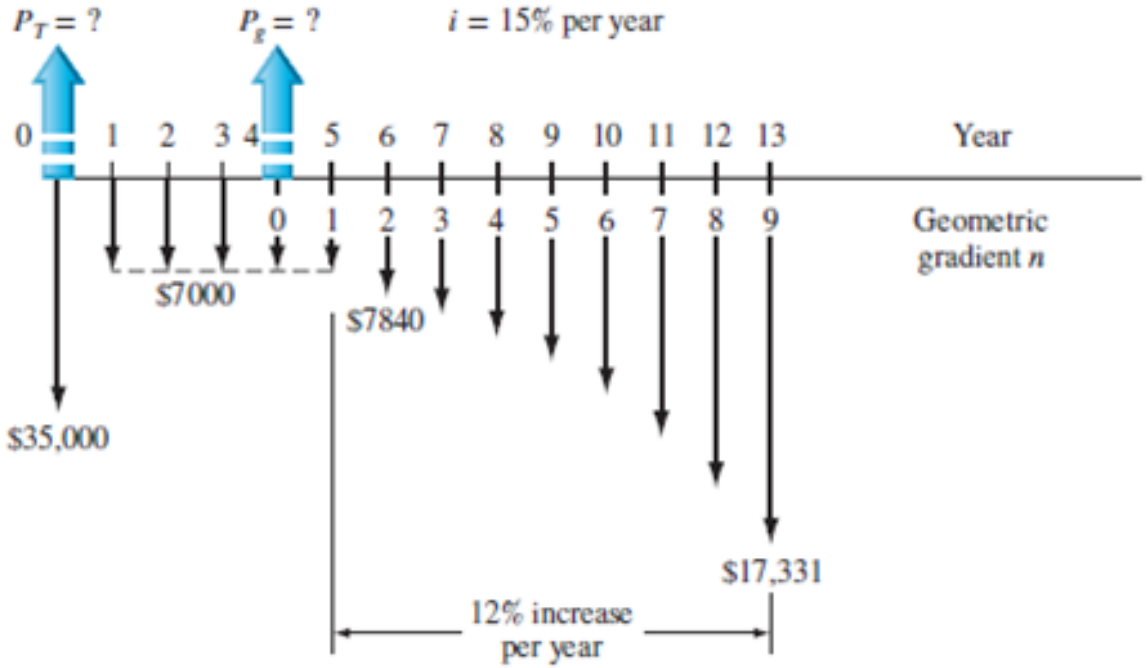
$$P_g = A_1 \left[\frac{1-(\frac{1+g}{1+i})^n}{i-g} \right] \qquad g \neq i$$

For negative gradient, change signs on both g values

There are no tables for geometric gradient factors

Example 6

Weirton Steel signed a 5-year contract to purchase water treatment chemicals from a local distributor for $7000 per year. When the contract ends, the cost of the chemicals is expected to increase by 12% per year for the next 8 years. If an initial investment in storage tanks is $35,000, determine the equivalent present worth in year 0 of all of the cash flows at i = 15% per year.



15%	Compound Interest Factors								15%
	Single Payment		Uniform Payment Series				Arithmetic Gradient		
n	Compound Amount Factor Find *F* Given *P* *F/P*	Present Worth Factor Find *P* Given *F* *P/F*	Sinking Fund Factor Find *A* Given *F* *A/F*	Capital Recovery Factor Find *A* Given *P* *A/P*	Compound Amount Factor Find *F* Given *A* *F/A*	Present Worth Factor Find *P* Given *A* *P/A*	Gradient Uniform Series Find *A* Given *G* *A/G*	Gradient Present Worth Find *P* Given *G* *P/G*	*n*
1	1.150	.8696	1.0000	1.1500	1.000	0.870	0	0	1
2	1.322	.7561	.4651	.6151	2.150	1.626	0.465	0.756	2
3	1.521	.6575	.2880	.4380	3.472	2.283	0.907	2.071	3
4	1.749	.5718	.2003	.3503	4.993	2.855	1.326	3.786	4
5	2.011	.4972	.1483	.2983	6.742	3.352	1.723	5.775	5
6	2.313	.4323	.1142	.2642	8.754	3.784	2.097	7.937	6
7	2.660	.3759	.0904	.2404	11.067	4.160	2.450	10.192	7
8	3.059	.3269	.0729	.2229	13.727	4.487	2.781	12.481	8
9	3.518	.2843	.0596	.2096	16.786	4.772	3.092	14.755	9
10	4.046	.2472	.0493	.1993	20.304	5.019	3.383	16.979	10

Negative Shifted Gradients

For negative arithmetic gradients, change sign on G term from + to −

General equation for determining P: **P = present worth of base amount − P_G**

Changed from + to −

For negative geometric gradients, change signs on both g values

Changed from + to −

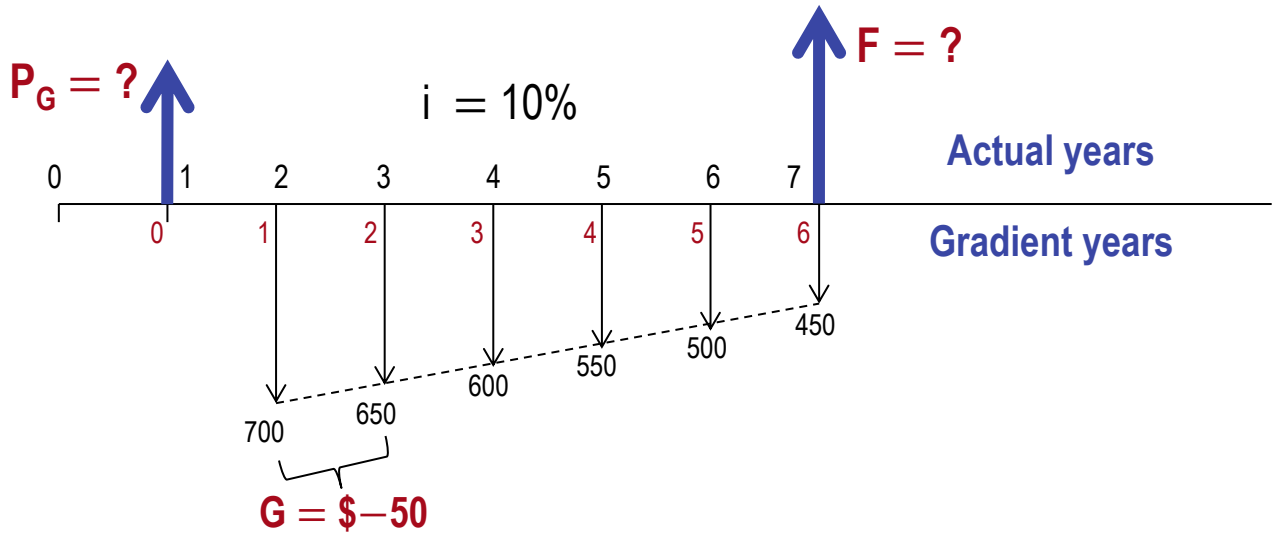
$$P_g = A1\{1 - [(1 \mp g)/(1 + i)]^n/(i + g)\}$$

Changed from − to +

All other procedures are the same as for positive gradients

Example 7

For the cash flows shown, find the future worth in year 7 at i = 10% per year



QUESTIONS?