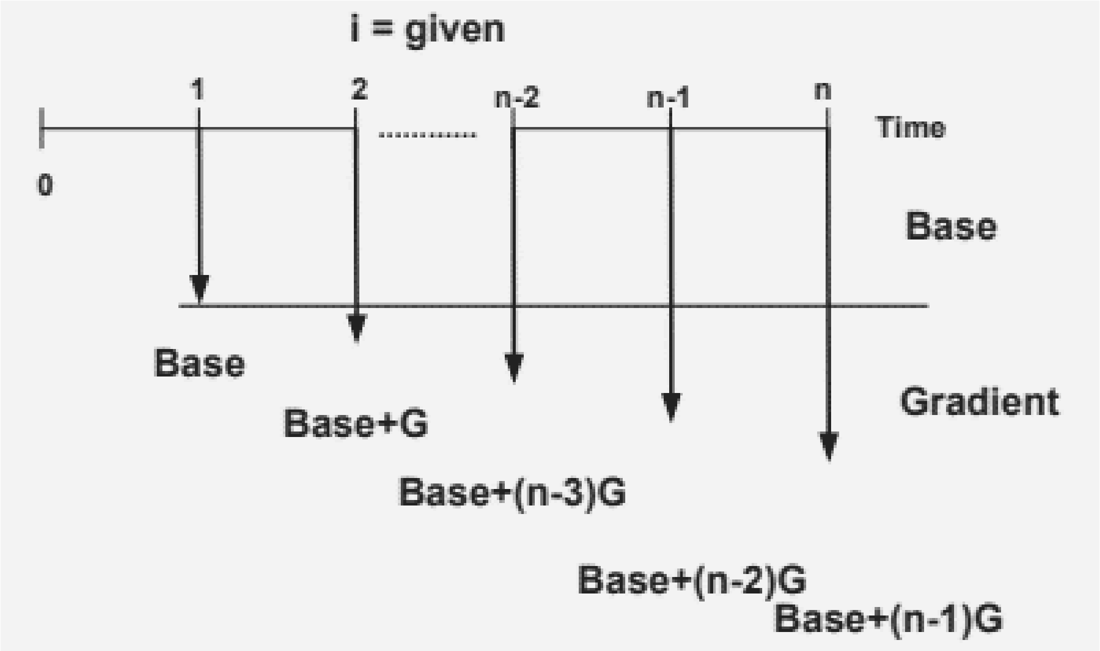
- **Course Title:** Engineering Cost Analysis & Economy (ENGR 222)
- **Session:** Fall 2024
- **Instructor:** Sudipta Chowdhury (chowdhurys@marshall.edu)
- **Class Time: TR 9.30-10.45 AM**
- **Office hours: TR 11.00 AM-12.30 PM**

Arithmetic Gradient/ Geometric Gradient Series Factors

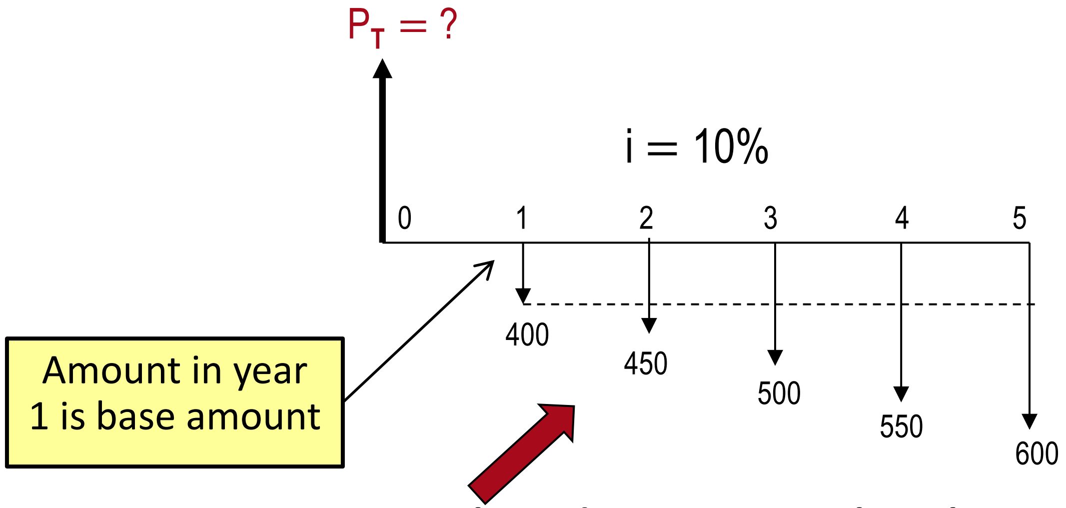
Arithmetic Gradient

- An arithmetic gradient is a cash flow that either increases or decreases by a constant amount.
- The cash flow, changes by the same amount each period.
- The amount of decrease or the increase is the gradient or G. Therefore, it is composed of base amount and gradient part.



The cash flow in period n (CFn) is given as:
CFn = base amount +(n − 1)G

- Note that the gradient begins between years 1 and 2.
- This is because cash flow in year 1 is usually not equal to G and is handled separately as a *base amount*



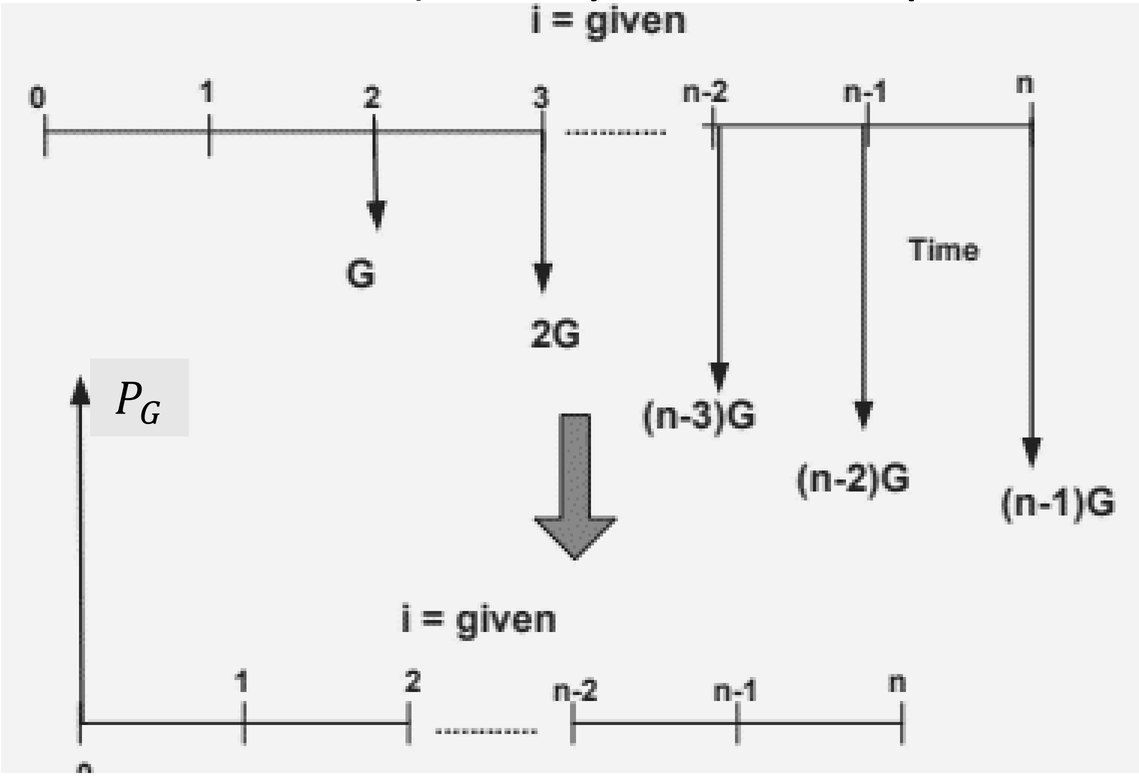
This diagram = this *base amount* plus this gradient

Present worth of gradient series only (P_G)

$$P_G = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]$$

- The $\frac{1}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]$ is the arithmetic-gradient present worth factor or P/G factor that converts the arithmetic gradient (without base amount) for n years to the present worth at year 0.
- The standard notation is (P/G,i,n)

Total present worth,
$P_T = P_A \pm P_G$



Example 1. A local university has initiated a logo-licensing program with the clothier Holister, Inc. Estimated revenues are $80,000 for the first year with uniform increases to a total of $200,000 by the end of year 9. Determine the arithmetic gradient and construct a cash flow diagram that identifies the base amount and the gradient series.

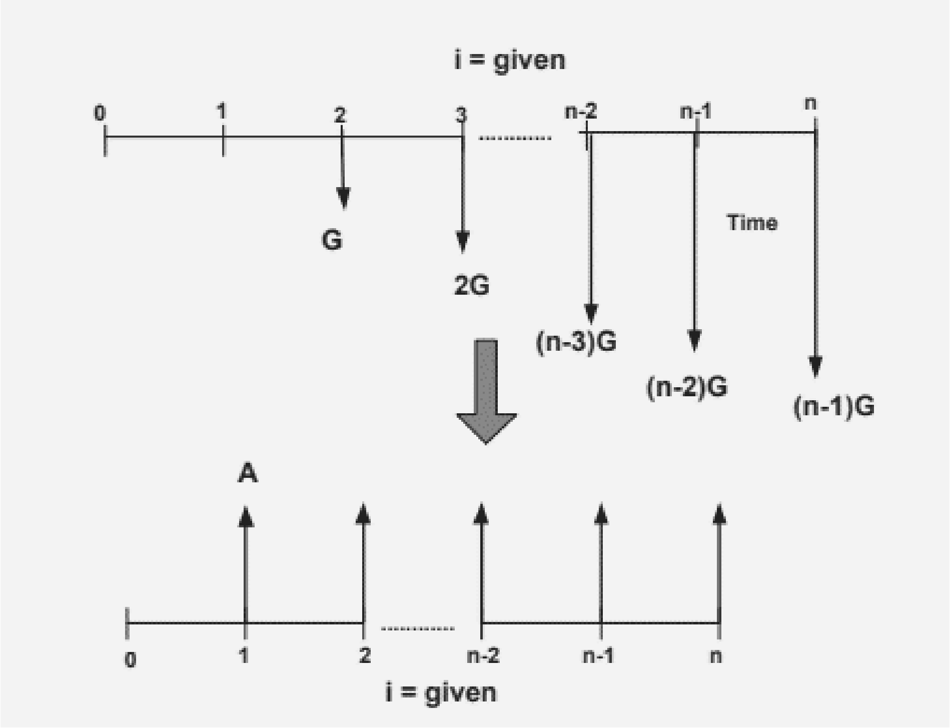
Example 2. Neighboring parishes in Louisiana have agreed to pool road tax resources already designated for bridge refurbishment. At a recent meeting, the engineers estimated that a total of $500,000 will be deposited at the end of next year into an account for the repair of old and safety-questionable bridges throughout the area. Further, they estimate that the deposits will increase by $100,000 per year for only 9 years thereafter, then cease. Determine the equivalent present worth if public funds earn at a rate of 5% per year.

5%	Compound Interest Factors								5%
	Single Payment		Uniform Payment Series				Arithmetic Gradient		
	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Compound Amount Factor	Present Worth Factor	Gradient Uniform Series	Gradient Present Worth	
n	Find *F* Given *P* *F/P*	Find *P* Given *F* *P/F*	Find *A* Given *F* *A/F*	Find *A* Given *P* *A/P*	Find *F* Given *A* *F/A*	Find *P* Given *A* *P/A*	Find *A* Given *G* *A/G*	Find *P* Given *G* *P/G*	*n*
1	1.050	.9524	1.0000	1.0500	1.000	0.952	0	0	1
2	1.102	.9070	.4878	.5378	2.050	1.859	0.488	0.907	2
3	1.158	.8638	.3172	.3672	3.152	2.723	0.967	2.635	3
4	1.216	.8227	.2320	.2820	4.310	3.546	1.439	5.103	4
5	1.276	.7835	.1810	.2310	5.526	4.329	1.902	8.237	5
6	1.340	.7462	.1470	.1970	6.802	5.076	2.358	11.968	6
7	1.407	.7107	.1228	.1728	8.142	5.786	2.805	16.232	7
8	1.477	.6768	.1047	.1547	9.549	6.463	3.244	20.970	8
9	1.551	.6446	.0907	.1407	11.027	7.108	3.676	26.127	9
10	1.629	.6139	.0795	.1295	12.578	7.722	4.099	31.652	10

The equivalent uniform annual series A for an arithmetic gradient G is found by

$$A_G = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$$

- The $\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$ term is the arithmetic-gradient uniform factor or A/G factor that converts the arithmetic gradient (without base amount) for n years to the annual worths.
- The standard notation is (A/G,i,n)



$Total\ Annual\ worth,$
$A_T = A_A \pm A_G$

Example 3. Neighboring parishes in Louisiana have agreed to pool road tax resources already designated for bridge refurbishment. At a recent meeting, the engineers estimated that a total of $500,000 will be deposited at the end of next year into an account for the repair of old and safety-questionable bridges throughout the area. Further, they estimate that the deposits will increase by $100,000 per year for only 9 years thereafter, then cease. Determine the equivalent annual worth if public funds earn at a rate of 5% per year.

5% Compound Interest Factors 5%

	Single Payment		Uniform Payment Series				Arithmetic Gradient		
n	Compound Amount Factor Find F Given P F/P	Present Worth Factor Find P Given F P/F	Sinking Fund Factor Find A Given F A/F	Capital Recovery Factor Find A Given P A/P	Compound Amount Factor Find F Given A F/A	Present Worth Factor Find P Given A P/A	Gradient Uniform Series Find A Given G A/G	Gradient Present Worth Find P Given G P/G	n
1	1.050	.9524	1.0000	1.0500	1.000	0.952	0	0	1
2	1.102	.9070	.4878	.5378	2.050	1.859	0.488	0.907	2
3	1.158	.8638	.3172	.3672	3.152	2.723	0.967	2.635	3
4	1.216	.8227	.2320	.2820	4.310	3.546	1.439	5.103	4
5	1.276	.7835	.1810	.2310	5.526	4.329	1.902	8.237	5
6	1.340	.7462	.1470	.1970	6.802	5.076	2.358	11.968	6
7	1.407	.7107	.1228	.1728	8.142	5.786	2.805	16.232	7
8	1.477	.6768	.1047	.1547	9.549	6.463	3.244	20.970	8
9	1.551	.6446	.0907	.1407	11.027	7.108	3.676	26.127	9
10	1.629	.6139	.0795	.1295	12.578	7.722	4.099	31.652	10

Example 4. The announcement of the HBNA steel plant states that the $200 million (M) investment is planned for 2020. Most large investment commitments are actually spread out over several years as the plant is constructed and production is initiated. Assume the amount planned for 2021 is $100 M with constant decreases of $25 M each year thereafter (2021 through 2024). As before, assume the time value of money for investment capital is 10% per year to answer the following question using tabulated factors.

Given the planned investment series, what is the equivalent annual amount that will be invested from 2021 to 2024?

10% Compound Interest Factors 1(

	Single Payment		Uniform Payment Series				Arithmetic Gradient	
n	Compound Amount Factor Find *F* Given *P* *F/P*	Present Worth Factor Find *P* Given *F* *P/F*	Sinking Fund Factor Find *A* Given *F* *A/F*	Capital Recovery Factor Find *A* Given *P* *A/P*	Compound Amount Factor Find *F* Given *A* *F/A*	Present Worth Factor Find *P* Given *A* *P/A*	Gradient Uniform Series Find *A* Given *G* *A/G*	Gradient Present Worth Find *P* Given *G* *P/G*
1	1.100	.9091	1.0000	1.1000	1.000	0.909	0	0
2	1.210	.8264	.4762	.5762	2.100	1.736	0.476	0.826
3	1.331	.7513	.3021	.4021	3.310	2.487	0.937	2.329
4	1.464	.6830	.2155	.3155	4.641	3.170	1.381	4.378

An F/G factor (arithmetic-gradient future worth factor) can be found by

$$F_G = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i} - n\right]$$

The standard notation is (F/G,i,n)

$$\textit{Total future worth,}$$
$$F_T = F_A \pm F_G$$

Example 5. Determine the total future worth when A=100, G=25, n=5, and i=7%.

$F_T = F_A + F_G =$A(F/A, i%, n)+G(F/G, i%, n)

= 100*5.751+25* $\frac{1}{i}\left[\frac{(1+i)^n - 1}{i} - n\right]$

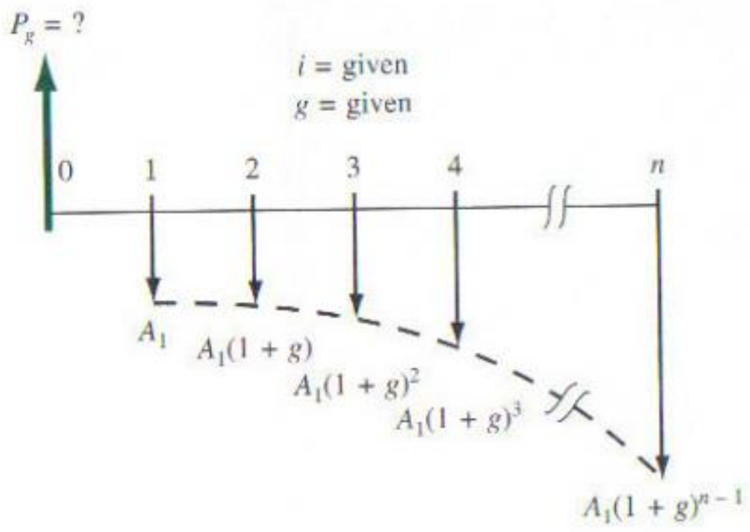
=575.1+268.12

=843.22

7%	Compound Interest Factors								7%
	Single Payment		Uniform Payment Series				Arithmetic Gradient		
n	Compound Amount Factor Find F Given P F/P	Present Worth Factor Find P Given F P/F	Sinking Fund Factor Find A Given F A/F	Capital Recovery Factor Find A Given P A/P	Compound Amount Factor Find F Given A F/A	Present Worth Factor Find P Given A P/A	Gradient Uniform Series Find A Given G A/G	Gradient Present Worth Find P Given G P/G	n
1	1.070	.9346	1.0000	1.0700	1.000	0.935	0	0	1
2	1.145	.8734	.4831	.5531	2.070	1.808	0.483	0.873	2
3	1.225	.8163	.3111	.3811	3.215	2.624	0.955	2.506	3
4	1.311	.7629	.2252	.2952	4.440	3.387	1.416	4.795	4
5	1.403	.7130	.1739	.2439	5.751	4.100	1.865	7.647	5

Geometric Gradient

- It is common for cash flow series, such as operating costs, construction costs, and revenues to increase or decrease from period to period by a constant percentage.
- The uniform rate of change defines a geometric gradient series of cash flows:



Geometric Gradient

- The uniform change is called the rate of change.

  g = constant rate of change, in decimal form, by which cash flow values increase or decrease from one period to the next. The gradient g can be + or -

  $\boldsymbol{A_1}$= initial cash flow in year 1 of the geometric series

  $\boldsymbol{P_g}$ = present worth of the entire geometric gradient series, including the initial amount A_1

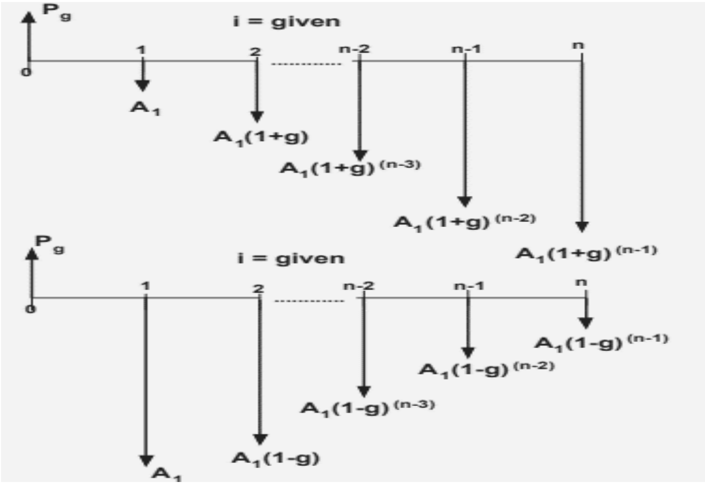
Geometric Gradient

- To compute a present amount from a geometric gradient series use:

$$P_g = A_1\left[\frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}\right] \qquad g\neq i$$

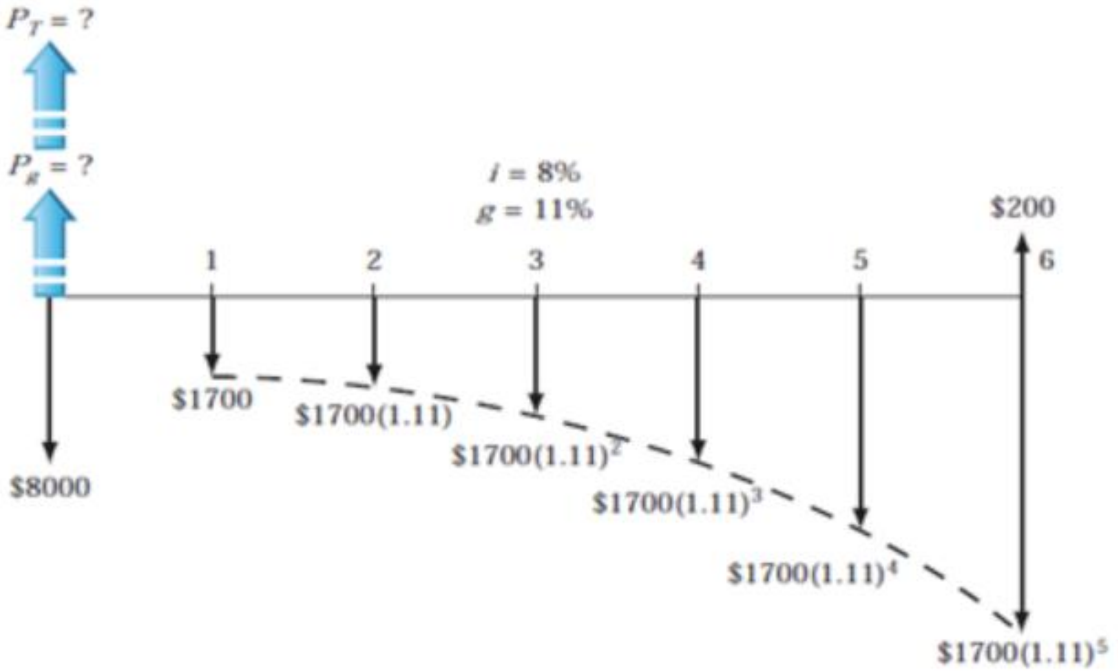
$$P_g = \left[\frac{nA_1}{1+i}\right] \qquad g = i$$

The term in the brackets is called the geometric-gradient-series present worth factor



- It is easier to determine the P_g amount and then multiply by A/P and F/P factors

Example 6. A coal-fired power plant has upgraded an emission control valve. The modification costs only $8000 and is expected to last 6 years with a $200 salvage value. The maintenance cost is expected to be high at $1700 the first year, increasing by 11% per year thereafter. Assuming interest rate to be 8%, determine the equivalent present worth of the modification and maintenance cost



8% Compound Interest Factors 8%

	Single Payment		Uniform Payment Series				Arithmetic Gradient		
	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Compound Amount Factor	Present Worth Factor	Gradient Uniform Series	Gradient Present Worth	
n	Find F Given P F/P	Find P Given F P/F	Find A Given F A/F	Find A Given P A/P	Find F Given A F/A	Find P Given A P/A	Find A Given G A/G	Find P Given G P/G	n
1	1.080	.9259	1.0000	1.0800	1.000	0.926	0	0	1
2	1.166	.8573	.4808	.5608	2.080	1.783	0.481	0.857	2
3	1.260	.7938	.3080	.3880	3.246	2.577	0.949	2.445	3
4	1.360	.7350	.2219	.3019	4.506	3.312	1.404	4.650	4
5	1.469	.6806	.1705	.2505	5.867	3.993	1.846	7.372	5
6	1.587	.6302	.1363	.2163	7.336	4.623	2.276	10.523	6

Example 7. A civil engineer planning for her retirement places 10% of her salary each year into a high-technology stock fund. If her salary this year (end of year 1) is $160,000 and she expects her salary to increase by 3% each year, what will be the future worth of her retirement fund after 15 years provided it earns 7% per year?

7%	Compound Interest Factors							
	Single Payment		Uniform Payment Series				Arithmetic Gradient	
n	Compound Amount Factor Find F Given P F/P	Present Worth Factor Find P Given F P/F	Sinking Fund Factor Find A Given F A/F	Capital Recovery Factor Find A Given P A/P	Compound Amount Factor Find F Given A F/A	Present Worth Factor Find P Given A P/A	Gradient Uniform Series Find A Given G A/G	Gradient Present Worth Find P Given G P/G
1	1.070	.9346	1.0000	1.0700	1.000	0.935	0	0
2	1.145	.8734	.4831	.5531	2.070	1.808	0.483	0.873
3	1.225	.8163	.3111	.3811	3.215	2.624	0.955	2.506
4	1.311	.7629	.2252	.2952	4.440	3.387	1.416	4.795
5	1.403	.7130	.1739	.2439	5.751	4.100	1.865	7.647
6	1.501	.6663	.1398	.2098	7.153	4.767	2.303	10.978
7	1.606	.6227	.1156	.1856	8.654	5.389	2.730	14.715
8	1.718	.5820	.0975	.1675	10.260	5.971	3.147	18.789
9	1.838	.5439	.0835	.1535	11.978	6.515	3.552	23.140
10	1.967	.5083	.0724	.1424	13.816	7.024	3.946	27.716
11	2.105	.4751	.0634	.1334	15.784	7.499	4.330	32.467
12	2.252	.4440	.0559	.1259	17.888	7.943	4.703	37.351
13	2.410	.4150	.0497	.1197	20.141	8.358	5.065	42.330
14	2.579	.3878	.0443	.1143	22.551	8.745	5.417	47.372
15	2.759	.3624	.0398	.1098	25.129	9.108	5.758	52.446

Example 8. To improve crack detection in military aircraft, the Air Force combined ultrasonic inspection procedures with laser heating to identify fatigue cracks. Early detection of cracks may reduce repair costs by as much as $200,000 per year. If the savings start at the end of year 1 and increase by 3% each year through year 5, calculate the present worth of these savings at an interest rate of 10% per year.

Unknown Interest Rate i

Unknown interest rate problems involve solving for i, given n and 2 other values (P, F, or A)

(Usually requires a trial and error solution or interpolation in interest tables)

Procedure: Set up equation with all symbols involved and solve for i

Your grandmother deposited $10,000 in an investment account on the day you were born to help pay the tuition when you go to college. If the account was worth $50,000 seventeen years after she made the deposit, what was the rate of return on the account?

(a) 5% (b) 8% (c) 9% (d) 12%

Solution: Can use either the P/F or F/P factor. Using A/P:

$$50{,}000 = 10{,}000(\mathrm{F/P}, \mathrm{i}\%, 17)$$
$$(\mathrm{F/P}, \mathrm{i}\%, 17) = 5$$

From F/P column at n =17 in the interest tables, i is between 9% and 10%

9% Compound Interest Factors

	Single Payment		Uniform Payment Series		
n	Compound Amount Factor Find *F* Given *P* F/P	Present Worth Factor Find *P* Given *F* P/F	Sinking Fund Factor Find *A* Given *F* A/F	Capital Recovery Factor Find *A* Given *P* A/P	Compound Amount Factor Find *F* Given *A* F/A
1	1.090	.9174	1.0000	1.0900	1.000
2	1.188	.8417	.4785	.5685	2.090
3	1.295	.7722	.3051	.3951	3.278
4	1.412	.7084	.2187	.3087	4.573
5	1.539	.6499	.1671	.2571	5.985
6	1.677	.5963	.1329	.2229	7.523
7	1.828	.5470	.1087	.1987	9.200
8	1.993	.5019	.0907	.1807	11.028
9	2.172	.4604	.0768	.1668	13.021
10	2.367	.4224	.0658	.1558	15.193
11	2.580	.3875	.0569	.1469	17.560
12	2.813	.3555	.0497	.1397	20.141
13	3.066	.3262	.0436	.1336	22.953
14	3.342	.2992	.0384	.1284	26.019
15	3.642	.2745	.0341	.1241	29.361
16	3.970	.2519	.0303	.1203	33.003
17	4.328	.2311	.0270	.1170	36.974

10% Compound

	Single Payment		Uniform P	
n	Compound Amount Factor Find *F* Given *P* F/P	Present Worth Factor Find *P* Given *F* P/F	Sinking Fund Factor Find *A* Given *F* A/F	Capital Recovery Factor Find *A* Given *P* A/P
1	1.100	.9091	1.0000	1.1000
2	1.210	.8264	.4762	.5762
3	1.331	.7513	.3021	.4021
4	1.464	.6830	.2155	.3155
5	1.611	.6209	.1638	.2638
6	1.772	.5645	.1296	.2296
7	1.949	.5132	.1054	.2054
8	2.144	.4665	.0874	.1874
9	2.358	.4241	.0736	.1736
10	2.594	.3855	.0627	.1627
11	2.853	.3505	.0540	.1540
12	3.138	.3186	.0468	.1468
13	3.452	.2897	.0408	.1408
14	3.797	.2633	.0357	.1357
15	4.177	.2394	.0315	.1315
16	4.595	.2176	.0278	.1278
17	5.054	.1978	.0247	.1247

Unknown Recovery Period n

Unknown recovery period problems involve solving for n, given i and 2 other values (P, F, or A)

(Like interest rate problems, they usually require a trial & error solution or interpolation in interest tables)

Procedure: Set up equation with all symbols involved and solve for n

A contractor purchased equipment for \$60,000 that provided income of \$8,000 per year. At an interest rate of 10% per year, the length of time required to recover the investment was closest to:

(a) 10 years (b) 12 years (c) 15 years (d) 18 years

Solution: Can use either the P/A or A/P factor. Using A/P:

$$60{,}000(A/P, 10\%, n) = 8{,}000$$

$$(A/P, 10\%, n) = 0.13333$$

From A/P column in $i = 10\%$ interest tables, n is between 14 and 15 years

10% Compound Interest Factors

	Single Payment		Uniform Payment Series		
n	Compound Amount Factor Find F Given P F/P	Present Worth Factor Find P Given F P/F	Sinking Fund Factor Find A Given F A/F	Capital Recovery Factor Find A Given P A/P	Compound Amount Factor Find F Given A F/A
1	1.100	.9091	1.0000	1.1000	1.000
2	1.210	.8264	.4762	.5762	2.100
3	1.331	.7513	.3021	.4021	3.310
4	1.464	.6830	.2155	.3155	4.641
5	1.611	.6209	.1638	.2638	6.105
6	1.772	.5645	.1296	.2296	7.716
7	1.949	.5132	.1054	.2054	9.487
8	2.144	.4665	.0874	.1874	11.436
9	2.358	.4241	.0736	.1736	13.579
10	2.594	.3855	.0627	.1627	15.937
11	2.853	.3505	.0540	.1540	18.531
12	3.138	.3186	.0468	.1468	21.384
13	3.452	.2897	.0408	.1408	24.523
14	3.797	.2633	.0357	.1357	27.975
15	4.177	.2394	.0315	.1315	31.772

QUESTIONS?