Exam II
MTH 300
Fall 2022

Instructions:
Complete each of the following exercises, showing the work you did to arrive at your conclusion and writing a conclusion statement.
For reference materials, you may use only:
- Your very bright and creative brains.
- Your notes from this class.
- My notes from this class.
- Definitions from your textbook, and Theorems 5.18, 5.20; 5.30; 5.31

Your completed Exam is due on Sunday, Nov. 20th at 11:59 p.m.

I didn't use Google, Wikipedia, Wolfram Alpha, Mathematica, Wolfram Math World or other resources on the test.
I also didn't use my calculator or other mathematical texts. I just used the resources above.

Exercise I: [10pts]

A) State The Principle of Mathematical Induction.

Let T be an inductive set. A set $\mathbb{T} \subset \mathbb{Z}$ is called an inductive set iff for each integer $k \in \mathbb{T}$, if $k \in \mathbb{T}$ then $(k+1) \in \mathbb{T}$.

If $\mathbb{T} \subset \mathbb{N}$ such that

{1} $1 \in \mathbb{T}$

and

{2} For all $k \in \mathbb{N}$, if $k \in \mathbb{T}$ then $k+1 \in \mathbb{T}$.

Then we can say $\mathbb{T} = \mathbb{N}$

B) Describe the difference between The Principle of Mathematical Induction and The Extended Principle of Mathematical Induction.

The general form of a statement that we will prove using induction is $(\forall n \in \mathbb{N})(P(n))$ where $P(n)$ is a predicate statement. This means ~~that~~ $P(n)$ is a mathematical expression such that when a value of n is substituted into the expression you have a mathematical statement (logical statement about mathematics).

Principle of Mathematical Induction

Step 1 prove $P(1)$ is true.

Step 2: Inductive step: For all $k \in \mathbb{N}$, if ~~[illegible]~~ $P(k)$ is true, then $P(k+1)$ is true.

The Extended Principle of Mathematical Induction

Let M be an integer. If T is a subset of $\mathbb{Z}$ such that

{a} $M \in T$, and

{b} For every $k \in \mathbb{Z}$ with $k \geq M$, if $k \in T$, then $(k+1) \in T$.

Then T contains all integers greater than or equal to M.

The difference is in the principle of mathematical induction you prove the basis step $P(1)$ first but in the extended principle of mathematical induction you prove ~~the~~ $P(M)$ first.

Exercise II: Prove exactly 3 of the propositions from the following list:

[30 pts]

A) Prop: For all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

B) Prop:
Suppose D is an operation taking functions defined on $\mathbb{R}$ to functions defined on $\mathbb{R}$.
Assume that $D(x)=1$ and that

$$D(f(x)g(x)) = D(f(x))\cdot g(x) + f(x)\cdot D(g(x))$$

for all functions $f=f(x)$ and $g=g(x)$.

Then,

$$D(x^n) = nx^{n-1}$$

C) Prop:
For all positive integers, n,

$$\frac{1}{1^2} + \frac{2}{2^2} + \frac{3}{3^2} + \dots + \frac{n}{n^2} \leq 2 - \frac{1}{n}$$

D) Prop:
Let x be a real number with $x>0$.
For every $n \in \mathbb{N}$, $n \geq 2$,

$$(1+x)^n > 1+nx.$$

$\boxed{A}$ Letter of your choice, from page 3

Proposition: For all $n \in \mathbb{N}$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

our result over on the left of $\frac{2k^3+9k^2+13k+6}{6}$ matches our result on the bottom of $\frac{2k^3+9k^2+13k+6}{6}$

Therefore the proposition is true.

Proof: ~~Step~~ (1) step 1 Prove $p(1)$ is true.
by Induction

$p(1)$: For $1 \in \mathbb{N}$, $\sum_{k=1}^{1} k^2 = \frac{n(n+1)(2n+1)}{6}$

$$\frac{2k^3+9k^2+13k+6}{6}$$

$$\frac{(2k^3+3k^2+k)}{6} + \frac{6k^2+12k+6}{6}$$

$$\cancel{\frac{2k^3+4k^2+5k^2+3k+1}{6}}$$

$$1^2 = \frac{1(1+1)(2(1)+1)}{6}$$

$$1 = \frac{1 \times 2 \times (2+1)}{6}$$

$$1 = \frac{1 \times 2 \times 3}{6}$$

$$1 = \frac{6}{6}$$

$$1 = 1$$

Step 2 Inductive step

For all $k \in \mathbb{N}$, if $p(k)$ is true, then $P(k+1)$ is true.

$\cancel{\frac{2k^3+4k^2+k+k^2+2k+1}{6}}$ We Let $p(k)$ be true. I will use i as the indexing letter.

That is,

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6} = \frac{k(k+1)(2k+1)}{6}$$

$$\frac{k(2k^2+3k+1)}{6} + \frac{6(k^2+2k+1)}{6}$$

$$\frac{k(k+1)(2k+1)}{6} + \frac{k^2+2k+1}{6}$$

Then

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + \sum_{i=k+1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$= \frac{(k+1)(k+2)(2k+2+1)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \cancel{\frac{i(i+1)(2i+1)}{6}}$$

$$\frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$\frac{(k^2+3k+2)(2k+3)}{6} = \frac{2k^3+6k^2+4k+3k^2+9k+6}{6} = \frac{(2k^3+9k^2+13k+6)}{6}$$

C Prop:

For all positive integers, n,

$$\frac{1}{1^2}+\frac{2}{2^2}+\frac{3}{3^2}+\dots+\frac{n}{n^2} \le 2-\frac{1}{n}.$$

I will disprove this proposition that (I use p instead of n)

$$\sum_{i=1}^{p} \frac{i}{i^2} = \sum_{i=1}^{p} \frac{1}{i} \le 2-\frac{1}{p}$$

P(5) would be $\sum_{i=1}^{5} \frac{1}{i} \le 2-\frac{1}{n}$. P(5) is false.

$$\sum_{i=1}^{5} \frac{1}{i} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$$
$$= \frac{60+30+20+15+12}{60}$$
$$= \frac{137}{60}$$

137 is not less than or equal to $2-\frac{1}{5} = \frac{10-2}{5} = \frac{8}{5}$

so the proposition does not hold for all positive integers p (or n).

we know $\frac{137}{60}$ is bigger than $2=\frac{120}{60}$, and 2 is bigger than $2-\frac{1}{5}$, so the counter example disproves the proposition.

1. Prop:

Let x be a real number with $x > 0$.

For every $n \in \mathbb{N}$, $n \geq 2$

$(1+x)^n > 1+nx$

I will prove for every $p \in \mathbb{N}$, $p \geq 2$

$(1+x)^p > 1+px$.

~~we will do~~ ~~Basis step~~ we will do the basis step first.

$\cancel{(1+x)^2 > 1+(p-2)x}$ we assume $x > 0$.

Then $x^2 > 0$. Then $x^2 + 2x + 1 > 1 + 2x$.

Then $(x+1)^2 > 1+2x$ and we have proved the statement is true for $p=2$.

Inductive step

Assume $(1+x)^p > 1+px$. Assume for contradiction

$(1+x)^{p+1} \leq 1+(p+1)x$ or $(1+x)^{p+1} \leq 1+px+x$

we have $1+px < (1+x)^p$ which we add

$$(1+x)^{p+1} \leq 1+px+x$$
$$+(1+px) \quad + (1+x)^p$$
$$1+px+(1+x)^{p+1} \leq 1+px+x+(1+x)^p$$
$$(1+x)^{p+1} \leq x+(1+x)^p$$
$$(x+1)(1+x)^p \leq x+(1+x)^p$$
$$x(1+x)^p + (1+x)^p \leq x + (1+x)^p$$
$$x(1+x)^p \leq x$$
$$(1+x)^p \leq 1$$

~~This~~ The equality only holds when $1 \leq 1$ or $1=1$ or $(1+x)^p < 1$. We would have to make $x \geq 0$ to make this true but we assumed $x > 0$. We have a contradiction so $(1+x)^{p+1} > 1+(p+1)x$. We have finished the inductive step. This completes the proof by contradiction and induction.

Exercise III: [10 pts]

A) For a set $A \subseteq U$, define the Power Set of A, $\mathcal{P}(A)$

Definition: If A is a subset of a universal set U, then the set whose members are all the subsets of A is called the power set of A. We denote the power set of A by $P(A)$. Symbollically, we write $P(A) = \{X \subseteq U \mid X \subseteq A\}$

That is, $X \in P(A)$ if and only if $X \subseteq A$.

B) Create sets A and B, $A, B \subseteq \mathbb{R}$, with at least 3 elements in each set and $A \cap B \neq \phi$.

Determine, for your sets, if

$$\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$$

use
$p = 1$
$q = 2$
$r = 3$
$s = 4$
$t = 5$
$v = 6$

$A = \{p, q, r\}$

$B = \{s, t, v, r\}$

~~$A \cap B = \{\}$~~ $A \cap B = \{r\}$

\# of elements $2^3 = 8$ $P(A) = \{\phi, \{p\}, \{q\}, \{r\}, \{p, r\}, \{q, r\}, \{p, q\}, \{p, q, r\}\}$

$2^4 = 16$ $P(B) = \{\{s\}, \{t\}, \{v\}, \{r\}, \{s, t\}, \{s, v\}, \{s, r\},$
$\{t, v\}, \{t, r\}, \{v, r\}, \{s, t, v\}, \{s, t, r\}, \{s, v, r\},$
$\{t, v, r\}, \{s, t, v, r\}, \phi\}$

$P(A) \cap P(B) = \{\phi, \{r\}\}$

$P(A \cap B) = P(\{r\}) = \{\phi, \{r\}\}$

$(P(A \cap B) = \{\phi, \{r\}\}) \subseteq (P(A) \cap P(B) = \{\phi, \{r\}\})$

True

c) Create sets A and B, $A, B \subseteq \mathbb{R}$, with at least 3 elements in each set and $A \cap B = \emptyset$. $\in$ not $\neq$

Determine for your sets, if

$$\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$$

$A = \{1, 2, 3\}$

$B = \{4, 5, 6\}$

$A \cap B = \emptyset$

$P(A \cap B) = \{\emptyset\}$

$P(A) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \emptyset\}$

$P(B) = \{\{4\}, \{5\}, \{6\}, \{4,5\}, \{4,6\}, \{5,6\}, \{4,5,6\}, \emptyset\}$

$P(A) \cap P(B) = \{\emptyset\}$

$(P(A \cap B) = \{\emptyset\}) \subseteq (P(A) \cap P(B) = \{\emptyset\})$

The statement is true.

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D) Prove or Disprove the following Proposition:

Prop:
Let $A, B \subseteq U$.

$\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$

Prove

$A \cap B = \{p \mid p \in A \text{ and } p \in B\}$

$P(A \cap B) = \{q \mid \cancel{q \subseteq A \cap B}\ q \subseteq (A \cap B)\}$

$P(A) = \{r \mid r \subseteq A\}$

$P(B) = \{s \mid s \subseteq B\}$

$P(A) \cap P(B) = \{t \mid t \subseteq A \text{ and } t \subseteq B$

$P(A \cap B) = \{t \mid t \subseteq A \text{ and } t \subseteq B\}$

$P(A \cap B) = P(A) \cap P(B)$ because

$\{t \mid t \subseteq A \text{ and } t \subseteq B\} = \{t \mid t \subseteq A \text{ and } t \subseteq B\}$

Exercise IV: Prove exactly 3 of the propositions from the following list:

A) Prove by chasing elements.

Prop: Let A and B be subsets of $\mathcal{U}$.

$$A = (A - B) \cup (A \cap B).$$

B) Prop:

Let A and B be subsets of $\mathcal{U}$.

Then $(A - B) \cup (B - A) = (A \cap B)' - (A \cup B)'$

(Hint: Use some nice Thms!)

C) Prop:

Let A and B be sets.

$A \times B$ and $B \times A$ are disjoint iff A and B are disjoint.

D) Prop:

Let Λ be a nonempty indexing set an $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$ be a family of set.

Then
$$B - \left(\bigcap_{\alpha \in \Lambda} A_\alpha\right) = \bigcup_{\alpha \in \Lambda} (B - A_\alpha).$$

(Hint: Use some nice Thms!)

$\boxed{A}$ Letter of your choice

Prove by chasing elements:

Prop: Let A and B be subsets of V.

$$A = (A-B) \cup (A \cap B)$$

I will prove
Let C and D be subsets of the universal set S.

$$C = (C-D) \cup (C \cap D)$$

Let $p \in C$. Then $p \notin C^c$.

~~Since $p \in C$, $p \in (C-D)$.~~

Since $p \in$

Either $p \in D$ or $p \notin D$.

If $p \in D$, $p \in C \cap D$ because $p \in C$ and $p \in D$.

If $p \notin D$, $p \in C-D$ because $p \in C$ and $p \notin D$. Then $p \in (C-D) \cup (C \cap D)$.

Therefore $C \subseteq (C-D) \cup (C \cap D)$

Let $q \in (C-D) \cup (C \cap D)$.

Then $q \in (C-D)$ for sure or $q \in (C \cap D)$ for sure.

If $q \in C-D$, $q \in C$.

If $q \in C \cap D$, $q \in C$.

Therefore $q \in C$. Therefore $C \supseteq (C-D) \cup (C \cap D)$.

Since $C \subseteq (C-D) \cup (C \cap D)$ and $C \supseteq (C-D) \cup (C \cap D)$,

$C = (C-D) \cup (C \cap D)$ This completes the proof. ∎

I wanted to use letters that aren't vowels and I thought it would confusing to use B for A so I used C D and S instead of A B and V.

D) Prop:
Let Λ be a nonempty indexing set on $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$ be a family of sets.

Then $B - \bigcap_{\alpha \in \Lambda} A_\alpha = \bigcup_{\alpha \in \Lambda} B - A_\alpha$

Hint: use some nice theorems!

Proof: $B - \bigcap_{\alpha \in \Lambda} A_\alpha = B \cap \left(\bigcap_{\alpha \in \Lambda} A_\alpha\right)^c$

De Morgan's law from Theorem 5.30 allows us to transform the indexed intersection's complement into the union of the complement of indexed sets.

$$B \cap \left(\bigcap_{\alpha \in \Lambda} A_\alpha\right)^c = B \cap \left(\bigcup_{\alpha \in \Lambda} A_\alpha^c\right)$$

The distributive law allows us to simplify this further.

$$B \cap \left(\bigcup_{\alpha \in \Lambda} A_\alpha^c\right) = \bigcup_{\alpha \in \Lambda} B \cap A_\alpha^c$$

We use Theorem 5.20 to transform $B \cap A_\alpha^c$ into $B - A_\alpha$ to get

$$\bigcup_{\alpha \in \Lambda} B \cap A_\alpha^c = \bigcup_{\alpha \in \Lambda} B - A_\alpha$$

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(B) Proposition

Let A and B be subsets of U.

Then $(A-B) \cup (B-A) = (A \cap B)^C - (A \cup B)^C$

We will transform the right hand side into

$A^C \cup B^C - A^C \cap B^C$

into

$(A^C \cup B^C) \cap (A^C \cap B^C)^C$

into

$(A^C \cup B^C) \cap ((A^C)^C \cup (B^C)^C)$

into

$(A^C \cup B^C) \cap (A \cup B)$

into

$(A \cap B)^C \cap (A \cup B)$

into

$(A \cup B) \cap (A \cap B)^C$

into

$A \cup B - A \cap B$

We now prove that

$A \cup B = (A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)$.

~~There are 3 possi~~

First, $A \cup B \subseteq (A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)$

Let $p \in A \cup B$.

If $p \in A$ and $p \notin B$, $p \in A \cap B^C$, then $p \in (A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)$.

If $p \in A$ and $p \in B$, $p \in A \cap B$, so then $p \in (A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)$.

If $p \notin A$ and $p \in B$, then, $p \in B \cap A^C$. ~~then the~~ Since $p \in B \cap A^C$, $p \in (A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)$.

The only case we haven't considered is $p \notin A$ and $p \notin B$, but we don't need to consider this because we assumed $p \in A \cup B$.

We proved $A \cup B \subseteq (A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)$. We now prove $A \cup B \supseteq (A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)$.

~~If~~ Let $p \in (A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)$. There are 3 cases.

1) ~~$p \in A$ and $p \in B$. When $p \in A$ and~~ $p \in A \cap B^C$. When $p \in A \cap B^C$, $p \in A$ and $p \notin B$. Since $p \in A$, ~~and~~ $p \in A \cup B$.

2) $p \in B \cap A^C$. When $p \in B \cap A^C$, $p \in B$ and $p \notin A$. Since $p \in B$ ~~and~~, $p \in A \cup B$.

3) $p \in A \cap B$. When $p \in A \cap B$, $p \in A$ and $p \in B$. Since $p \in A$ ~~and~~, $p \in A \cup B$.

This is a proof that $A \cup B \supseteq (A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)$. We can combine this proof with our proof that $A \cup B \subseteq (A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)$ to conclude $A \cup B = (A \cap B^C) \cup (B \cap A^C) \cup (A \cap B)$.

We will now transform $A - B \cup B - A$ into $(A \cap B^C) \cup (B \cap A^C)$. ~~Later~~ We will now add the set $A \cap B$ and take it away so we have $A \cap B^C \cup (B \cap A^C) \cup (A \cap B) - A \cap B$. We use our earlier proof to transform this into $A \cup B - A \cap B$, which ~~matches~~ is identical to the right hand side. This completes the proof. ■