# Enter title here

## Enter subtitle here

Enter subsubtitle here

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## Enter section title here

### Enter subsection title here

#### Enter subsubsection title here

Enter text here. Enter TraditionalForm input for evaluation in a separate cell below:

$$\int x\, dx + \sqrt{z}$$

$$\frac{x^2}{2} + \sqrt{z}$$

- Enter bulleted item text here.

  Enter item paragraph text here.

  - Enter subitem text here.

    Enter item paragraph text here.

    - Enter subitem text here.

      Enter item paragraph text here.

Enter text here. Enter formula for display in a separate cell below:

$$\int x\, dx + \sqrt{z}$$

Enter text here. Enter an inline formula like this: $2 + 2$.

**1.** Enter numbered item text here.

Enter item paragraph text here.

**1.1.** Enter numbered subitem text here.

Enter item paragraph text here.

**1.1.1.** Enter subitem text here.

Enter item paragraph text here.

Enter text here. Enter formula for numbered display in a separate cell below:

$$\int x \, dx + \sqrt{z} \tag{1}$$

Enter text here. Enter Wolfram Language program code below.

In[∘]:=
```
fun[x_]:=1
```

Enter text here. Enter non-Wolfram Language program code below.

```
DLLEXPORT int fun(WolframLibraryData libData, mreal A1, mreal *Res)
{
 mreal R0_0;
 mreal R0_1;
 R0_0 = A1;
 R0_1 = R0_0 * R0_0;
 *Res = R0_1;
 funStructCompile->WolframLibraryData_cleanUp(libData, 1);
 return 0;
}
```

# #1

* **5.** Is the following proposition true or false? Justify your conclusion.

For each nonnegative integer $n$, $8^n \mid (4n)!$.

The first nonnegative integer is 0.

The statement is true for 0.

$8^0 \mid (4\times 0)!$

In[∘]:= $8^0$

Out[∘]= 1

In[∘]:= $4\times 0$

Out[∘]= 0

In[∘]:= $(4\times 0)!$

Out[∘]= 1

$8^0$ is 1 and 4 times 0 is 0 and 0! is 1. We have 1 divides 1, which is true.

We can assume the statement is true for i so we have P(i). The next question is will P(i+1) will be true? P is the proposition that 8^n divides (4n)!.

$8^i \mid (4\,i)!$

$8^{i+1} \mid (4\,(i+1))!$

$8^{i+1} \mid (4\,i+4)!$

$8 \times 8^{i} \mid (4\,i + 4)!$

$8 \times 8^{i} \mid (4\,i)!\,(4\,i + 1)\,(4\,i + 2)\,(4\,i + 3)\,(4\,i + 4)$

We assumed $8^{i} \mid (4\,i)!$ from which we know that $\exists_{b,b\in\mathbb{Z}}\ (4\,i)!\,b = 8^{i}$

We showed that $8^{i+1} \mid (4\,(i + 1))!$ is the same as $8 \times 8^{i} \mid (4\,i)!\,(4\,i + 1)\,(4\,i + 2)\,(4\,i + 3)\,(4\,i + 4)$, which would mean $\exists_{c,c\in\mathbb{Z}}\ (4\,i)!\,(4\,i + 1)\,(4\,i + 2)\,(4\,i + 3)\,(4\,i + 4)\,c = 8 \times 8^{i}$ if it were true.

We can divide the two equalities. We can cancel (4i)! when dividing the larger factorial by the smaller factorial to simplify $\frac{(4\,i)!\,(4\,i+1)\,(4\,i+2)\,(4\,i+3)\,(4\,i+4)}{(4\,i)!}$ to $(1 + 4\,i)\,(2 + 4\,i)\,(3 + 4\,i)\,(4 + 4\,i)$. We can cancel $8^{i}$ as well to simplify $\frac{8\times 8^{i}}{8^{i}}$ to 8. After dividing $(4\,i)!\,(4\,i + 1)\,(4\,i + 2)\,(4\,i + 3)\,(4\,i + 4)\,c = 8 \times 8^{i}$ by $(4\,i)!\,b = 8^{i}$ we have $(1 + 4\,i)\,(2 + 4\,i)\,(3 + 4\,i)\,(4 + 4\,i)$c/b==8. We can clear the denominator to get $(1 + 4\,i)\,(2 + 4\,i)\,(3 + 4\,i)\,(4 + 4\,i)$c==8b. We know that when we expand the left we get $24 + 200\,i + 560\,i^{2} + 640\,i^{3} + 256\,i^{4}$. I used a Groebner basis to factor this to $8\left(3 + 25\,i + 70\,i^{2} + 80\,i^{3} + 32\,i^{4}\right)$, which means that the left side $(1 + 4\,i)\,(2 + 4\,i)\,(3 + 4\,i)\,(4 + 4\,i)$c is divisible by 8 and the statement P(i+1) is true. That means the proposition is true for all non-negative integers from 0 to infinity.

In[ ]:=
```
Expand[(4 i + 1) (4 i + 2) (4 i + 3) (4 i + 4)]
```
Out[ ]=

$24 + 200\,i + 560\,i^{2} + 640\,i^{3} + 256\,i^{4}$

In[ ]:=
```
GroebnerBasis[Expand[(4 i + 1) (4 i + 2) (4 i + 3) (4 i + 4)], i]
```
Out[ ]=

$\{3 + 25\,i + 70\,i^{2} + 80\,i^{3} + 32\,i^{4}\}$

In[ ]:=

$$\frac{(4\,i)!\,(4\,i + 1)\,(4\,i + 2)\,(4\,i + 3)\,(4\,i + 4)}{(4\,i)!}$$

Out[ ]=

$(1 + 4\,i)\,(2 + 4\,i)\,(3 + 4\,i)\,(4 + 4\,i)$

In[ ]:=

$$\frac{8 \times 8^{i}}{8^{i}}$$

Out[ ]=

8

In[ ]:=

$$\frac{(4\,i)!}{(4\,i)!\,(4\,i + 1)\,(4\,i + 2)\,(4\,i + 3)\,(4\,i + 4)}$$

Out[ ]=

$$\frac{1}{(1 + 4\,i)\,(2 + 4\,i)\,(3 + 4\,i)\,(4 + 4\,i)}$$

In[∘]:=
$$\frac{(4\,i)\,!\,b == 8^{i}}{(4\,i)\,!\,(4\,i+1)\,(4\,i+2)\,(4\,i+3)\,(4\,i+4) == 8\times 8^{i}}\ \text{// Simplify}$$

Out[∘]=
$$\frac{8^{i} == b\,(4\,i)\,!}{8\,(1+i)\,(1+2\,i)\,(1+4\,i)\,(3+4\,i)\,(4\,i)\,! == 8^{1+i}}$$

In[∘]:=
$$\frac{(4\,(i+1))\,!}{(4\,i)\,!}$$

Out[∘]=
$$\frac{(4\,(1+i))\,!}{(4\,i)\,!}$$

In[∘]:=
$$\text{FullSimplify}\left[\frac{(4\,(1+i))\,!}{(4\,i)\,!},\ \text{Assumptions}\rightarrow i\in\mathbb{Z}_{\geq 0}\right]$$

Out[∘]=
$$\frac{\text{Gamma}[5+4\,i]}{(4\,i)\,!}$$

In[∘]:=
```
FullSimplify[((4 (i + 1)) !) == ((4 i + 4) !) == ((4 i) ! (4 i + 1) (4 i + 2) (4 i + 3) (4 i + 4))]
```

Out[∘]=
```
True
```

In[∘]:=
```
Table[Divisible[(4 i) !, 8^i], {i, 0, 10}]
```

Out[∘]=
```
{True, True, True, True, True, True, True, True, True, True, True}
```

In[∘]:=
```
AllTrue[Range[0, 10], (i ↦ Divisible[(4 i) !, 8^i])]
```

Out[∘]=
```
True
```

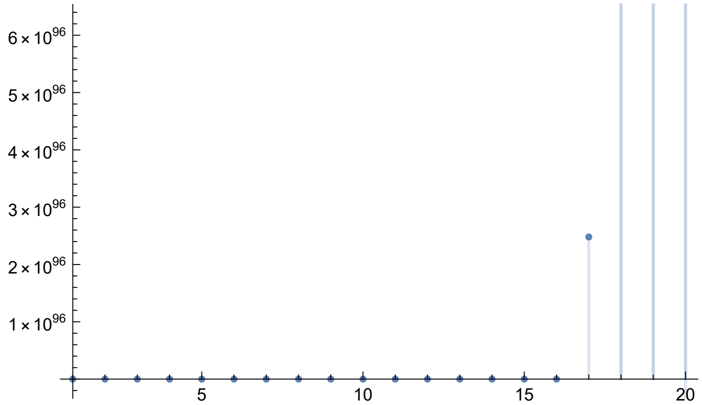
In[∘]:=
```
(4 i) !
```

Out[∘]=
```
(4 i) !
```

In[∘]:=
```
FunctionExpand[(4 i) !]
```

Out[∘]=
```
Gamma[1 + 4 i]
```

In[∘]:=
```
DiscretePlot[Gamma[1 + 4 i], {i, 1, 20}]
```
Out[∘]=



In[∘]:=
```
FunctionDomain[(4 i) !, {}]
```
Out[∘]=

True

---

# #2

**6.** Let $y = \ln x$.

4.2. Other Forms of Mathematical Induction 197

**(a)** Determine $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, and $\frac{d^4y}{dx^4}$.

**(b)** Let $n$ be a natural number. Formulate a conjecture for a formula for $\frac{d^n y}{dx^n}$. Then use mathematical induction to prove your conjecture.

In[∘]:=
```
Table[D[Log[x], {x, i}], {i, 1, 4}]
```
Out[∘]=

$$\left\{\frac{1}{x}, -\frac{1}{x^2}, \frac{2}{x^3}, -\frac{6}{x^4}\right\}$$

If y is function that maps y to ln(x) aka $y \longmapsto \log(x)$ we then have $\frac{dy}{dx} = \frac{1}{x}$, $\frac{dy}{dx} = \frac{-1}{x^2}$, $\frac{dy}{dx} = \frac{2}{x^3}$, and $\frac{dy}{dx} = \frac{-6}{x^4}$.

Let n be a natural number. Formulate a conjecture for a formula for $\frac{d^n y}{dx^n}$. Then use mathematical induction to prove your conjecture.

I have a table for the first 7 cases:

In[∘]:=
```
Table[D[Log[x], {x, i}], {i, 1, 7}]
```
Out[∘]=

$$\left\{\frac{1}{x}, -\frac{1}{x^2}, \frac{2}{x^3}, -\frac{6}{x^4}, \frac{24}{x^5}, -\frac{120}{x^6}, \frac{720}{x^7}\right\}$$

Get the numerators:

In[∘]:=
```
Numerator[Table[D[Log[x], {x, i}], {i, 1, 7}]]
```
Out[∘]=
```
{1, -1, 2, -6, 24, -120, 720}
```

Find a sequence function:

In[∘]:=
```
FindSequenceFunction[Numerator[Table[D[Log[x], {x, i}], {i, 1, 7}]], i]
```
Out[∘]=

$-(-1)^i$ Pochhammer[1, -1 + i]

Find a sequence function directly.

In[∘]:=
```
FindSequenceFunction[Table[D[Log[x], {x, i}], {i, 1, 7}], i]
```
Out[∘]=

$-\left(-\frac{1}{x}\right)^i$ Pochhammer[1, -1 + i]

The Pochhammer function is also known as the rising factorial, ascending factorial, rising sequential product and upper factorial. The Pochhammer function is defined as

$$x^{(n)} = x^{\overline{n}} = (\overbrace{x(x+1)(x+2)\ldots(x+n-1)}^{n\text{ factors}}) = \prod_{k=1}^{n}(x+k-1) = \prod_{k=0}^{n-1}(x+k)$$

My conjecture is that the i-th derivative is:

$$-\left(-\frac{1}{x}\right)^i \prod_{k=1}^{i-1}(1+k-1)$$

## proving the conjecture by induction

We start with the P(1). P(1) is the first derivative of log(x), which is $\frac{1}{x}$.

Let's see if $-\left(-\frac{1}{x}\right)^i \prod_{k=1}^{i-1}(1+k-1)$ matches this.

In[∘]:=

$$-\left(\frac{-1}{x}\right)^1 \prod_{k=1}^{1-1}(1+k-1)$$

Out[∘]=

$$\frac{1}{x}$$

Good. We have checked the basis case. We can step onto the letter. Next we show you can keep moving up the ladder once you're on the ladder.

Assume P(1) to P(i) is true.

Divide P(i+1) by P(i).

In[∘]:=

$$\frac{-\left(\frac{-1}{x}\right)^{(i+1)} \prod_{k=1}^{(i+1)-1} (1+k-1)}{-\left(\frac{-1}{x}\right)^{i} \prod_{k=1}^{i-1} (1+k-1)} \text{ // FullSimplify // TraditionalForm}$$

Out[∘]//TraditionalForm=

$$-\frac{i}{x}$$

So we take the previous expression i and multiply by -1/x.

In[∘]:=

```
Table[D[Log[x], {x, i}], {i, 1, 7}]
```

Out[∘]=

$$\left\{\frac{1}{x}, -\frac{1}{x^2}, \frac{2}{x^3}, -\frac{6}{x^4}, \frac{24}{x^5}, -\frac{120}{x^6}, \frac{720}{x^7}\right\}$$

In[∘]:=

$$D\left[-\left(\frac{-1}{x}\right)^{i} \prod_{k=1}^{i-1} (1+k-1), x\right] \text{ // Simplify // TraditionalForm}$$

Out[∘]//TraditionalForm=

$$\frac{i\,(i-1)!\left(-\frac{1}{x}\right)^{i}}{x}$$

In[∘]:=

$$D\left[-\left(\frac{-1}{x}\right)^{i} \prod_{k=1}^{i-1} (1+k-1), x\right] == -\left(\frac{-1}{x}\right)^{i+1} \prod_{k=1}^{i+1-1} (1+k-1) \text{ // FullSimplify}$$

Out[∘]=

True

## Pochhammer work

$-\left(\frac{-1}{x}\right)^{i} \prod_{k=1}^{i-1} (x+k-1)$

In[∘]:=

$$\text{Table}\left[-\left(\frac{-1}{x}\right)^{i} \prod_{k=1}^{i-1} (i+k-1), \{i, 7\}\right]$$

Out[∘]=

$$\left\{\frac{1}{x}, -\frac{2}{x^2}, \frac{12}{x^3}, -\frac{120}{x^4}, \frac{1680}{x^5}, -\frac{30\,240}{x^6}, \frac{665\,280}{x^7}\right\}$$

In[∘]:=

```
Table[D[Log[x], {x, i}], {i, 1, 7}]
```

Out[∘]=

$$\left\{\frac{1}{x}, -\frac{1}{x^2}, \frac{2}{x^3}, -\frac{6}{x^4}, \frac{24}{x^5}, -\frac{120}{x^6}, \frac{720}{x^7}\right\}$$

In[∘]:= Table[FullSimplify[Pochhammer[1, -1 + i] - $\prod_{k=1}^{i-1}$ (x + k - 1)], {i, 10}]

Out[∘]=

$\{0,\ 1-x,\ 2-x\,(1+x),\ 6-x\,(1+x)\,(2+x),\ 24-x\,(1+x)\,(2+x)\,(3+x),$
$120-x\,(1+x)\,(2+x)\,(3+x)\,(4+x),\ 720-x\,(1+x)\,(2+x)\,(3+x)\,(4+x)\,(5+x),$
$5040-x\,(1+x)\,(2+x)\,(3+x)\,(4+x)\,(5+x)\,(6+x),$
$40320-x\,(1+x)\,(2+x)\,(3+x)\,(4+x)\,(5+x)\,(6+x)\,(7+x),$
$362880-x\,(1+x)\,(2+x)\,(3+x)\,(4+x)\,(5+x)\,(6+x)\,(7+x)\,(8+x)\}$

In[∘]:= AsymptoticProduct[x + k - 1, {k, 1, i - 1}, i → ∞]

Out[∘]=

$$\frac{e^{-i}\, i^{-\frac{3}{2}+i+x}\,\sqrt{2\pi}}{\text{Gamma}[x]}$$

In[∘]:= DiscreteAsymptotic[$\prod_{k=1}^{i-1}$ (x + k - 1), i → ∞]

Out[∘]=

$$\frac{e^{-i}\, i^{-\frac{3}{2}+i+x}\,\sqrt{2\pi}}{\text{Gamma}[x]}$$

In[∘]:= Pochhammer[1, -1 + i]

Out[∘]=

Pochhammer[1, -1 + i]

In[∘]:= Pochhammer[1, -1 + i] // TraditionalForm

Out[∘]//TraditionalForm=

$(1)_{i-1}$

In[∘]:= Pochhammer[a, n] // TraditionalForm

Out[∘]//TraditionalForm=

$(a)_n$

In[∘]:= Pochhammer[a, i - 1] // TraditionalForm

Out[∘]//TraditionalForm=

$(a)_{i-1}$

In[∘]:= Pochhammer[1, i - 1] // TraditionalForm

Out[∘]//TraditionalForm=

$(1)_{i-1}$

$$\prod_{k=1}^{n}(x+k-1)$$

In[ ]:= $\prod_{k=1}^{i-1}(1+k-1)$

Out[ ]= $(-1+i)!$

In[ ]:= FullSimplify$\left[\prod_{k=1}^{i-1}(1+k-1) - \text{Pochhammer}[1, i-1]\right]$

Out[ ]= 0

In[ ]:= FullSimplify$\left[\prod_{k=1}^{i-1}(1+k-1) == \text{Pochhammer}[1, i-1]\right]$

Out[ ]= True

In[ ]:= `FindSequenceFunction[Table[D[Log[x], {x, i}], {i, 1, 7}], i]`

Out[ ]= $-\left(-\frac{1}{x}\right)^{i}$ Pochhammer[1, $-1+i$]

$-\left(-\frac{1}{x}\right)^{i} \prod_{k=1}^{i-1}(1+k-1)$