Programming Assignment #2 review

1. (Median-of-3 Partition) One way to improve the RANDOMIZED-QUICKSORT is to choose the pivot for partitioning more carefully than by picking a random element from the array. One common approach is to choose the pivot as the median of a set of 3 elements randomly selected from the array. Assume that all elements in the array are distinct. Please implement a Quicksort with Median-of-3 Partition in C or C++ and answer following questions in a separate paper.
(a) (review) What is the probability of getting an OK split if the pivot is chosen at random? Explain. (A split is "OK" if the smaller piece has at least *n*/4 elements.)

The split is OK if the pivot is an element of rank $n/4$ to $3n/4$. That is, half of the elements in the array are pivots that would yield an OK split. Therefore, if the pivot is chosen uniformly at random, the probability of getting an OK split is $1/2$.

(b) Roughly, what is the probability of getting an OK split with the new method? Explain.

Let E be the event that the split is OK when the pivot is chosen with the Median-of-3 Partition. First, we calculate the probability of the complement event $\bar{E}$, that is, the event that the split is not OK. The split is not OK if and only if (i) the median of three randomly chosen elements has rank $> n/4$ or (ii) it has rank $> 3n/4$. Case (i) happens when at least two out of three randomly chosen elements are of rank $< n/4$. Similarly, case (ii) happens when at least two out of three randomly chosen elements are of rank $> 3n/4$. Both (i) and (ii) have the same probability.
Case (i) happens if and only if (a) exactly 2 out of 3 random elements have rank $< n/4$ or (b) all 3 randomly chosen elements have rank $< n/4$. Each random element will have rank $< n/4$ with probability $1/4$. The probability that two fixed elements will have this rank and the remaining element will have rank $\geq n/4$ is $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{64}$. There are 3 ways to choose 2 elements out of 3. So, case (a) happens with probability $9/64$. Case (b) happens with probability $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$. Therefore, case (i) happens with probability $\frac{9}{64} + \frac{1}{64} = \frac{5}{32}$.
The probability of event $\bar{E}$ is twice the probability of case (i) (since case (ii) occurs with the same probability as case (i)). Therefore, $\Pr[\bar{E}] = 2 \times \frac{5}{32} = \frac{5}{16}$, and the probability of an OK split is $1 - \frac{5}{16} = \frac{11}{16}$.

(c) Let I be the indicator random variable for getting an OK split using the median-of-3 partition:

$$I = \begin{cases} 1 & \text{if the split is OK} \\ 0 & \text{otherwise} \end{cases}$$

What is the expectation of I?

$$E[I] = \Pr(I = 1) \cdot 1 + \Pr(I = 0) \cdot 0 = \Pr(I = 1) = \frac{11}{16}$$