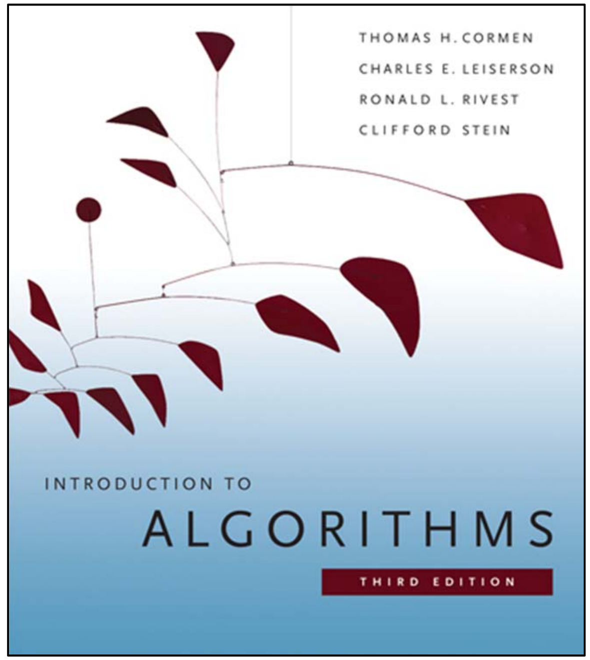
# 6.006

# *Introduction to Algorithms*




# Lecture 2: Peak Finding

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# Today

- Peak finding *(new problem)*
  - 1D algorithms
  - 2D algorithms
- Divide & conquer *(new technique)*

# Finding Water…

## IN SPACE

- You are Geordi LaForge
- Trapped on alien mountain range
- Need to find a pool where water accumulates
- Can teleport, but can't see

http://en.wikipedia.org/wiki/File:GeordiLaForge.jpg

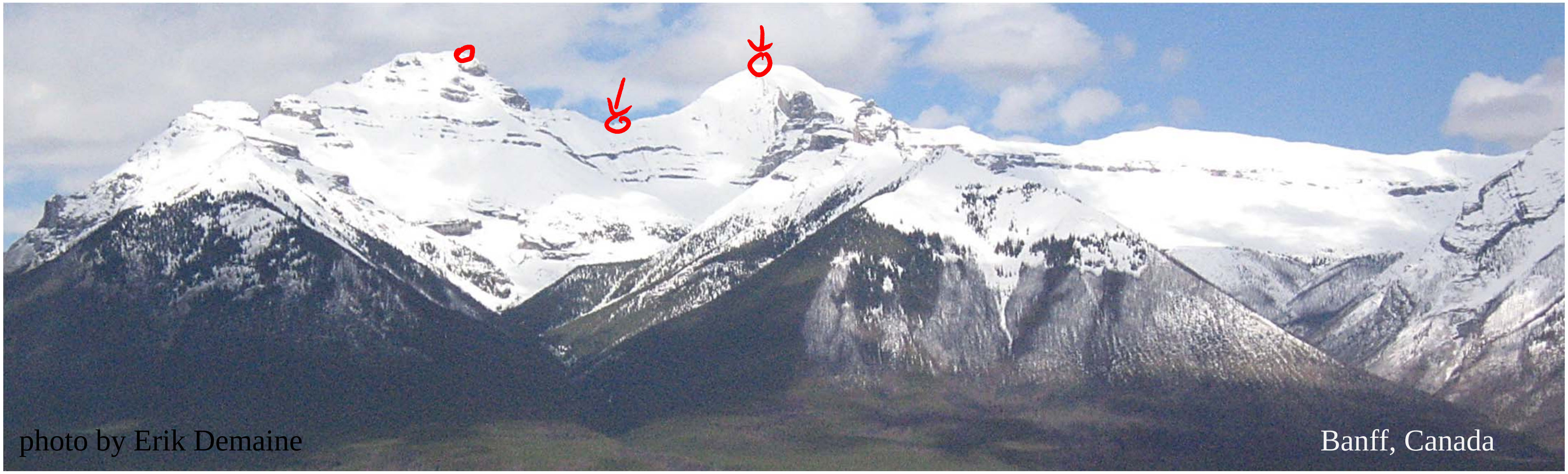
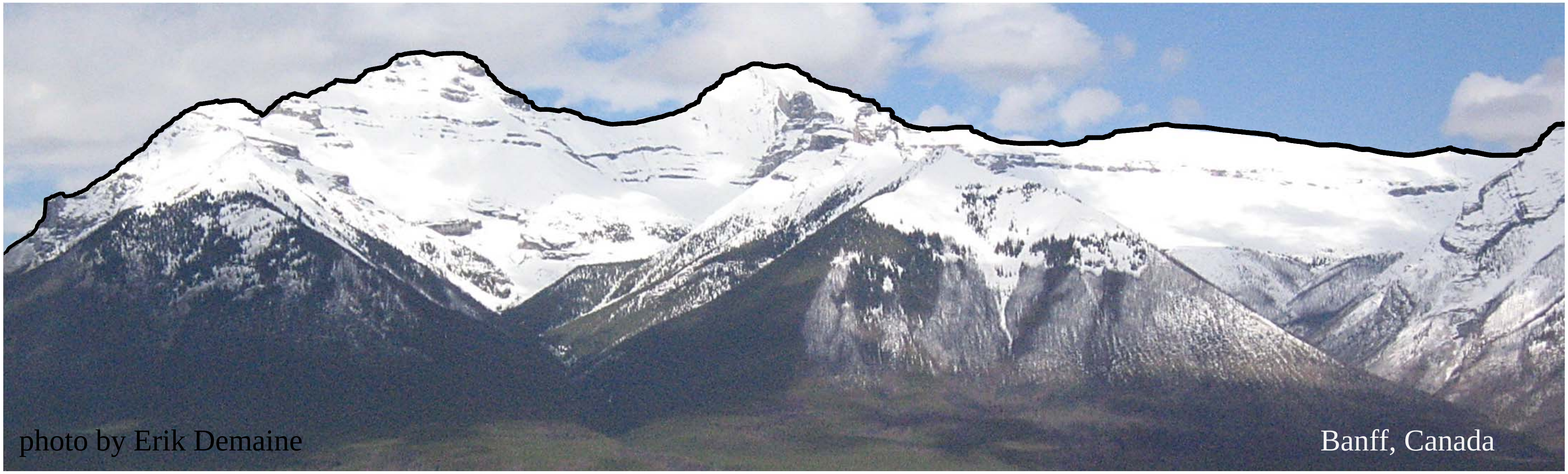


photo by Erik Demaine

Banff, Canada

# Finding Water…
## IN SPACE

- Problem: Find a local minimum or maximum in a terrain by sampling



# 1D Peak Finding

- Given an array $A[0..n-1]$:

| $A$: | $-\infty$ | 1 | 2 | 6 | 5 | 3 | 7 | 4 | $-\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | |

- $A[i]$ is a **peak** if it is not smaller than its neighbor(s):

$$A[i-1] \le A[i] \ge A[i+1]$$

  where we imagine

$$A[-1] = A[n] = -\infty$$

- <u>Goal:</u> Find *any* peak

# "Brute Force" Algorithm

- Test all elements for peakyness

for $i$ in range$(n)$:
  if $A[i-1] \leq A[i] \geq A[i+1]$:
    return $i$

$O(1)$ $O(n)$

$A$:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 6 | 5 | 3 | 7 | 4 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |

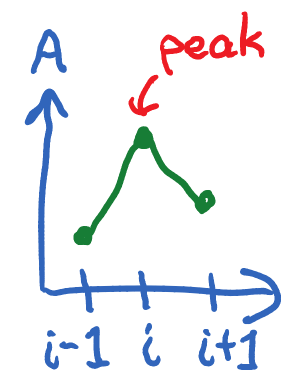
# Algorithm 1½

- max$(A)$
  - Global maximum is a local maximum

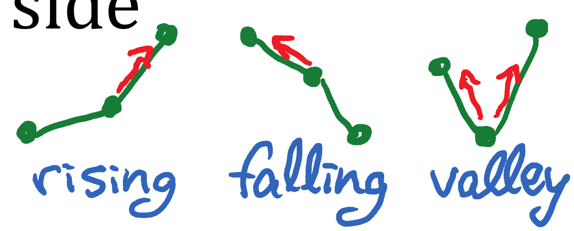
$m = 0$
for $i$ in range$(1, n)$:
    if $A[i] > A[m]$: $\Theta(1)$ $\Theta(n)$
        $m = i$
return $m$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $A$: | 1 | 2 | 6 | 5 | 3 | 7 | 4 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

# Cleverer Idea




- Look at any element $A[i]$ and its neighbors $A[i-1]$ & $A[i+1]$
  - If peak: return $i$
  - Otherwise: locally rising on some side
    - Must be a peak in that direction
    - So can throw away rest of array, leaving $A[:i]$ or $A[i+1:]$




| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $A$: | 1 | 2 | 6 | 5 | 3 | 7 | 4 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

# Where to Sample?

- Want to minimize the worst-case remaining elements in array
  - Balance $A[:i]$ of length $i$ with $A[i+1:]$ of length $n-i-1$
  - $i = n-i-1$
  - $i = (n-1)/2$: **middle element**
  - Reduce $n$ to $(n-1)/2$

$A$:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 6 | 5 | 3 | 7 | 4 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |

# Algorithm

```
peak1d(A, i, j):
    m = ⌊(i + j)/2⌋
    if A[m − 1] ≤ A[m] ≥ A[m + 1]:
        return m
    elif A[m − 1]  >  A[m]:
        return peak1d(A, i, m − 1)
    elif A[m]  <  A[m + 1]:
        return peak1d(A, m + 1, j)
```

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $A$: | 1 | 2 | 6 | 5 | 3 | 7 | 4 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

# Divide & Conquer

- General design technique:

1. **Divide** input into part(s)
2. **Conquer** each part recursively
3. **Combine** result(s) to solve original problem

- 1D peak:

1. One half
2. Recurse
3. Return

# Divide & Conquer Analysis

- **Recurrence** for time $T(n)$ taken by problem size $n$

1. **Divide** input into part(s):
   $n_1, n_2, \ldots, n_k$
2. **Conquer** each part recursively
3. **Combine** result(s) to solve original problem

$T(n) =$

divide cost +

$T(n_1) + T(n_2) + \cdots + T(n_k)$

+ combine cost

# 1D Peak Finding Analysis

- Divide problem into 1 problem of size $\sim \frac{n}{2}$
- Divide cost: $O(1)$
- Combine cost: $O(1)$
- Recurrence:

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

# Solving Recurrence

don't use O(1) notation to keep track of constant

$$T(n) = T\left(\frac{n}{2}\right) + c$$

$$T(n) = T\left(\frac{n}{4}\right) + c + c$$

$$T(n) = T\left(\frac{n}{8}\right) + c + c + c$$

$$T(n) = T\left(\frac{n}{2^k}\right) + c\,k$$

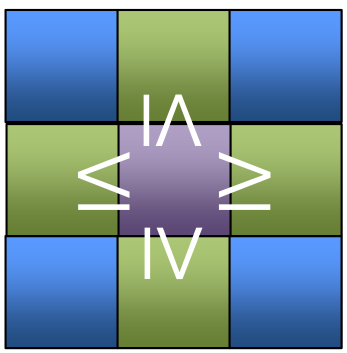
$$T(n) = T\left(\frac{\cancel{n}}{\cancel{2^{\lg n}}}\right) + c \lg n$$

$$T(n) = T(1) + c \lg n$$

$$T(n) = \Theta(\lg n)$$

# 2D Peak Finding

- Given $n \times n$ matrix of numbers
- Want an entry not smaller than its (up to) 4 neighbors:




| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 3 | 5 | 2 | 4 | 9 | 8 |
| 7 | 2 | 5 | 1 | 4 | 0 | 3 |
| 9 | 8 | 9 | 3 | 2 | 4 | 8 |
| 7 | 6 | 3 | 1 | 3 | 2 | 3 |
| 9 | 0 | 6 | 0 | 4 | 6 | 4 |
| 8 | 9 | 8 | 0 | 5 | 3 | 0 |
| 2 | 1 | 2 | 1 | 1 | 1 | 1 |

# Divide & Conquer #0

- Looking at center element doesn't split the problem into pieces…

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 3 | 5 | 2 | 4 | 9 | 8 |
| 7 | 2 | 5 | 1 | 4 | 0 | 3 |
| 9 | 8 | 9 | 3 | 2 | 4 | 8 |
| 7 | 6 | 3 | 1 | 3 | 2 | 3 |
| 9 | 0 | 6 | 0 | 4 | 6 | 4 |
| 8 | 9 | 8 | 0 | 5 | 3 | 0 |
| 2 | 1 | 2 | 1 | 1 | 1 | 1 |

# Divide & Conquer #½

- Consider max element in each column
- 1D algorithm would solve max array in $O(\lg n)$ time
- But $\Theta(n^2)$ time to compute max array

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 3 | 5 | 2 | 4 | 9 | 8 |
| 7 | 2 | 5 | 1 | 4 | 0 | 3 |
| 9 | 8 | 9 | 3 | 2 | 4 | 8 |
| 7 | 6 | 3 | 1 | 3 | 2 | 3 |
| 9 | 0 | 6 | 0 | 4 | 6 | 4 |
| 8 | 9 | 8 | 0 | 5 | 3 | 0 |
| 2 | 1 | 2 | 1 | 1 | 1 | 1 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 9 | 9 | 3 | 5 | 9 | 8 |

# Divide & Conquer #1

- Look at center column
- Find global max within
- If peak: return it
- Else:
  - Larger left/right neighbor
  - Larger max in that column
  - Recurse in left/right half
- <u>Base case:</u> 1 column
  - Return global max within

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 3 | 5 | 2 | 4 | 9 | 8 |
| 7 | 2 | 5 | 1 | 4 | 0 | 3 |
| 9 | 8 | 9 | 3 | 2 | 4 | 8 |
| 7 | 6 | 3 | 1 | 3 | 2 | 3 |
| 9 | 0 | 6 | 0 | 4 | 6 | 4 |
| 8 | 9 | 8 | 0 | 5 | 3 | 0 |
| 2 | 1 | 2 | 1 | 1 | 1 | 1 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 9 | 9 | 3 | 5 | 9 | 8 |

# Analysis #1

- $O(n)$ time to find max in column
- $O(\lg n)$ iterations (like binary search)
- $O(n \lg n)$ time total

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 3 | 5 | 2 | 4 | 9 | 8 |
| 7 | 2 | 5 | 1 | 4 | 0 | 3 |
| 9 | 8 | 9 | 3 | 2 | 4 | 8 |
| 7 | 6 | 3 | 1 | 3 | 2 | 3 |
| 9 | 0 | 6 | 0 | 4 | 6 | 4 |
| 8 | 9 | 8 | 0 | 5 | 3 | 0 |
| 2 | 1 | 2 | 1 | 1 | 1 | 1 |

- Can we do better?

# Divide & Conquer #2

- Look at boundary, center row, and center column (**window**)
- Find global max within
- If it's a peak: return it
- Else:
  - Find larger neighbor
  - Can't be in window
  - Recurse in quadrant, including green boundary

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 9 | 3 | 5 | 2 | 4 | 9 | 8 | 0 |
| 0 | 7 | 2 | 5 | 1 | 4 | 0 | 3 | 0 |
| 0 | 9 | 8 | 9 | 3 | 2 | 4 | 8 | 0 |
| 0 | 7 | 6 | 3 | 1 | 3 | 2 | 3 | 0 |
| 0 | 9 | 0 | 6 | 0 | 4 | 6 | 4 | 0 |
| 0 | 8 | 9 | 8 | 0 | 5 | 3 | 0 | 0 |
| 0 | 2 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# Correctness

- <u>Lemma:</u> If you enter a quadrant, it contains a peak of the overall array [climb up]
- <u>Invariant:</u> Maximum element of window never decreases as we descend in recursion
- <u>Theorem:</u> Peak in visited quadrant is also peak in overall array

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 9 | 3 | 5 | 2 | 4 | 9 | 8 | 0 |
| 0 | 7 | 2 | 5 | 1 | 4 | 0 | 3 | 0 |
| 0 | 9 | 8 | 9 | 3 | 2 | 4 | 8 | 0 |
| 0 | 7 | 6 | 3 | 1 | 3 | 2 | 3 | 0 |
| 0 | 9 | 0 | 6 | 0 | 4 | 6 | 4 | 0 |
| 0 | 8 | 9 | 8 | 0 | 5 | 3 | 0 | 0 |
| 0 | 2 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

→ proofs in recitation

# Analysis #2

- Reduce $n \times n$ matrix to $\sim \frac{n}{2} \times \frac{n}{2}$ submatrix in $O(n)$ time ($|\text{window}|$)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 9 | 3 | 5 | 2 | 4 | 9 | 8 | 0 |
| 0 | 7 | 2 | 5 | 1 | 4 | 0 | 3 | 0 |
| 0 | 9 | 8 | 9 | 3 | 2 | 4 | 8 | 0 |
| 0 | 7 | 6 | 3 | 1 | 3 | 2 | 3 | 0 |
| 0 | 9 | 0 | 6 | 0 | 4 | 6 | 4 | 0 |
| 0 | 8 | 9 | 8 | 0 | 5 | 3 | 0 | 0 |
| 0 | 2 | 1 | 2 | 1 | 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$$T(n) = T\left(\frac{n}{2}\right) + c\,n$$

$$T(n) = T\left(\frac{n}{4}\right) + c\frac{n}{2} + c\,n$$

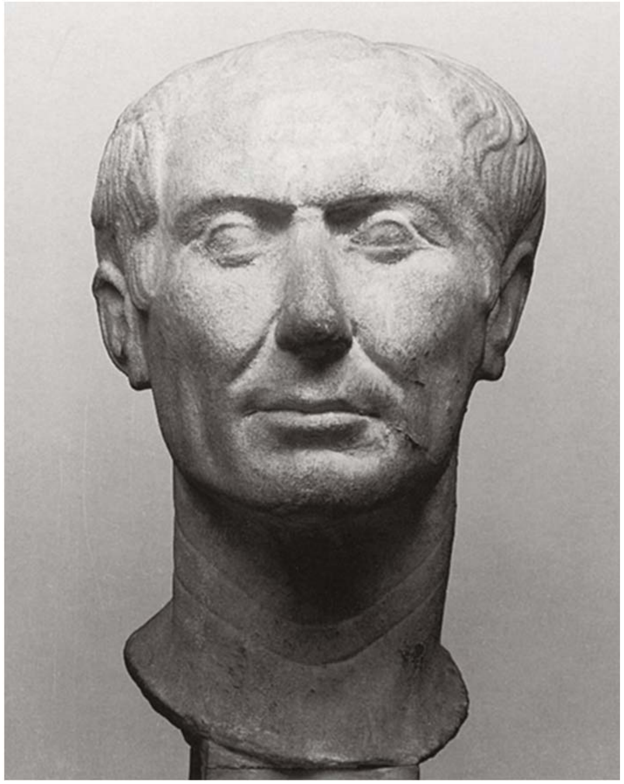
$$T(n) = T\left(\frac{n}{8}\right) + c\frac{n}{4} + c\frac{n}{2} + c\,n$$

$\Theta(n)$

$$T(n) = T(1) + c\left(1 + 2 + 4 + \cdots + \frac{n}{4} + \frac{n}{2} + n\right)$$

# Divide & Conquer Wrapup

- Leads to surprisingly efficient algorithms
- Not terribly general, but still quite useful
- We'll use it again in
  – Module 4 (sorting)
  – Module 8 (geometry)



http://en.wikipedia.org/wiki/File:CaesarTusculum.jpg