

Abstract

Expressing neutrino oscillation probabilities in matter can be done either approximately with very good precision, or exactly with complicated expressions. Exact solutions require solving for the eigenvalues and, in turn, the eigenvectors. While there is no shortcut for the exact expressions for the eigenvalues, given those the eigenvectors can be determined in a straightforward fashion. Finally, I will show that while CPV in matter appears to extremely complicated, it is much simpler than expected both exactly and after approximations.

Recent Results in Neutrino Oscillation Theory

Peter B. Denton

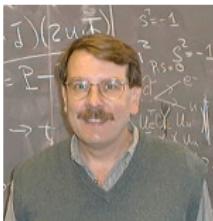
UMass Amherst Seminar

November 15, 2019

BROOKHAVEN
NATIONAL LABORATORY



Analytic Oscillation Probability Collaborators



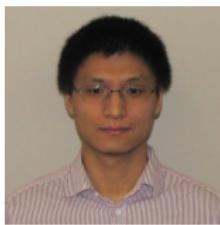
Stephen Parke



Hisakazu Minakata



Gabriela Barenboim



Xining Zhang



Christoph Ternes



Terrence Tao

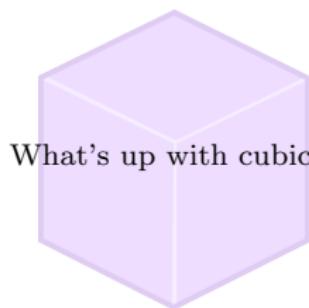
[1604.08167](#), [1806.01277](#), [1808.09453](#),
[1902.00517](#), [1902.07185](#), [1907.02534](#)
[1908.03795](#), [1909.02009](#)

github.com/PeterDenton/Nu-Pert

github.com/PeterDenton/Nu-Pert-Compare

Path

1. **Statement** of oscillation question
2. Get the eigen**values**
3. Get the eigen**vectors**
4. Useful **approximations**
5. **CP** violation in matter



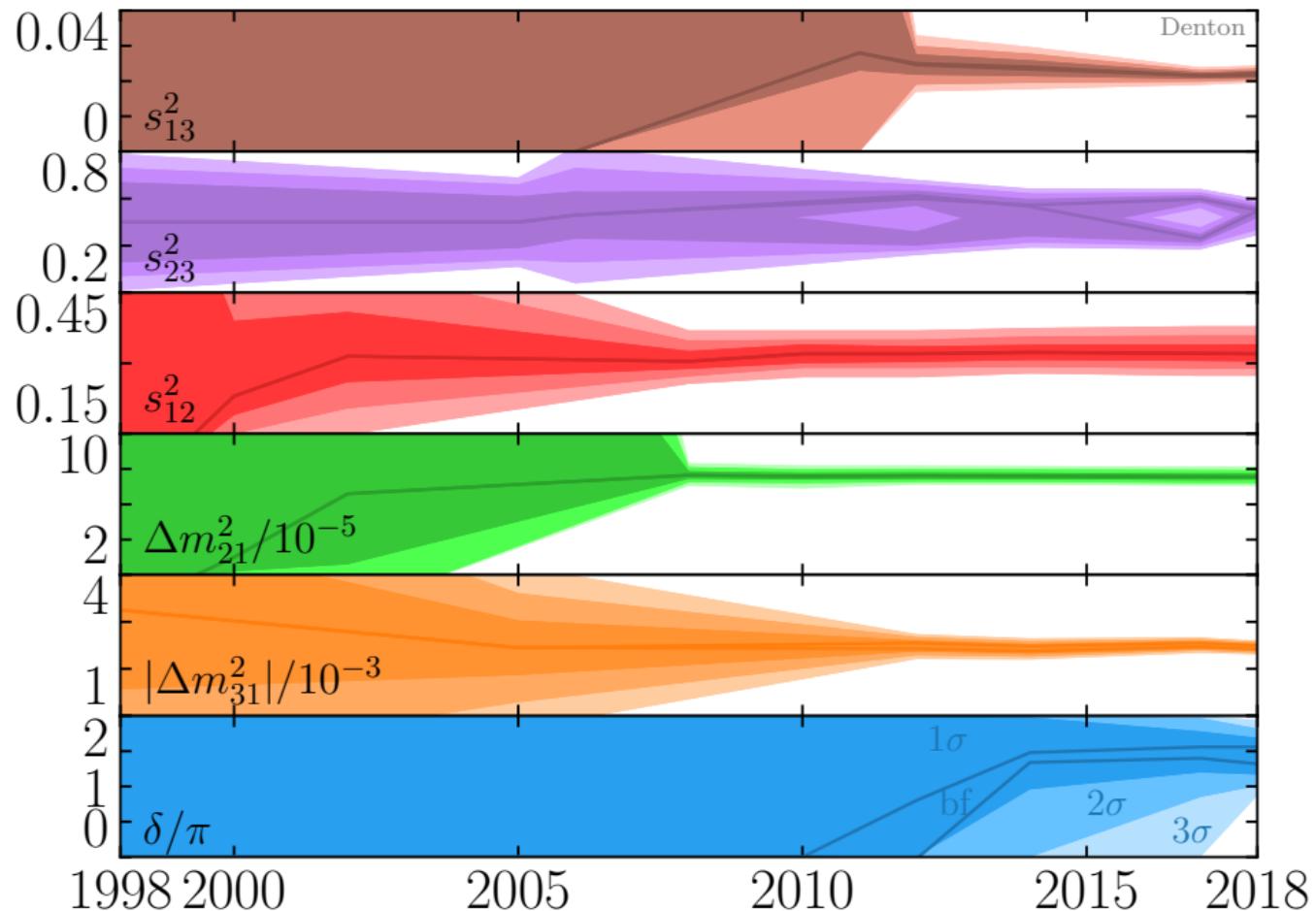
Statement of oscillation question

Experiment to Oscillation Parameters

Six oscillation parameters: θ_{12} , θ_{13} , θ_{23} , δ , Δm_{21}^2 , Δm_{31}^2

- ▶ Atmospheric ν_μ disappearance $\rightarrow \sin 2\theta_{23}$, $|\Delta m_{31}^2|$
SuperK, IMB, IceCube
- ▶ Solar ν_e disappearance $\rightarrow \pm \cos 2\theta_{12}$, $\pm \Delta m_{21}^2$
SNO, Borexino, SuperK
- ▶ Reactor ν_e disappearance:
 - ▶ LBL $\rightarrow \sin 2\theta_{12}$ and $|\Delta m_{21}^2|$
KamLand
 - ▶ Future LBL $\rightarrow \pm \Delta m_{31}^2$
JUNO
 - ▶ MBL $\rightarrow \theta_{13}$, $|\Delta m_{31}^2|$
Daya Bay, RENO, Double Chooz
- ▶ Accelerator LBL ν_e appearance: $\pm \Delta m_{31}^2$, $\pm \cos 2\theta_{23}$, θ_{13} , δ
T2K, NOvA, T2HK, DUNE

Denton



The Big Question

What is $P(\nu_\mu \rightarrow \nu_e)$?

$$P({}^{\langle}\overline{\nu}_{\mu} \rightarrow {}^{\langle}\overline{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \quad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}} \mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23} \sin \Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23}) \sin \Delta_{21}$$

$$\Delta_{ij} = \Delta m^2 {}_{ij} L / 4E$$

The Big Question

What is $P(\nu_\mu \rightarrow \nu_e)$?

$$P({}^{\langle}\overline{\nu}_{\mu} \rightarrow {}^{\langle}\overline{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \quad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}} \mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23} \sin \Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23}) \sin \Delta_{21}$$

$$\Delta_{ij} = \Delta m^2 {}_{ij} L / 4E$$

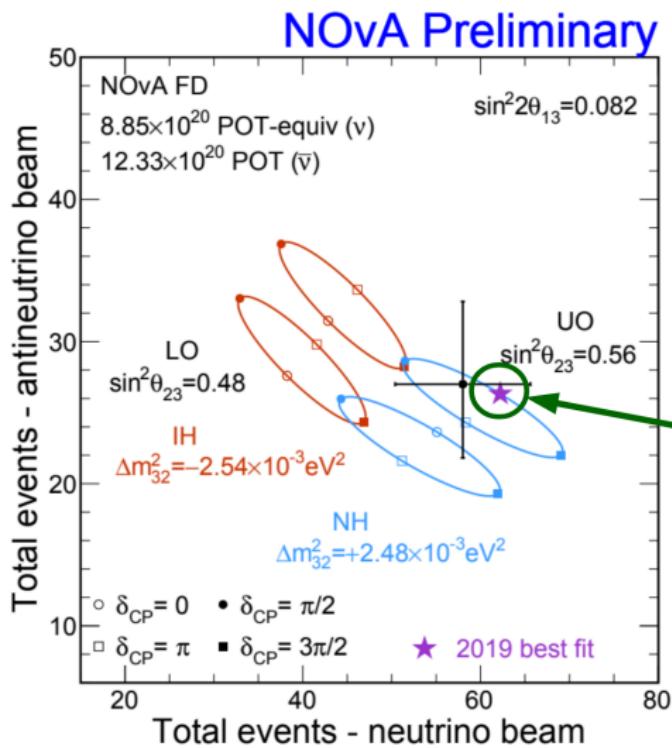
. . . in matter?

Now: NOvA, T2K, MINOS, . . .

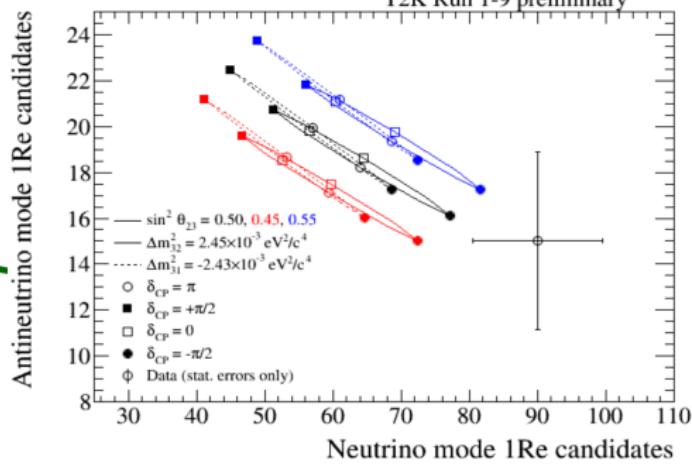
Upcoming: DUNE, T2HK, . . .

Second maximum: T2HKK? ESSnuSB? . . .
nuSTORM?

Current Best Long Baseline Bievent Plots



810 km



Analytic Oscillation Probabilities in Matter

- ☒ Solar: $P_{ee} \simeq \sin^2 \theta_\odot$

Approx: S. Mikheev, A. Smirnov [Nuovo Cim. C9 \(1986\) 17-26](#)

Exact: S. Parke [PRL 57 \(1986\) 2322](#)

- ☒ Long-baseline: All three flavors

Exact: H. Zaglauer, K. Schwarzer [Z.Phys. C40 \(1988\) 273](#)

Approx: PBD, H. Minakata, S. Parke, [1604.08167](#)

- ☒ ν_e disappearance (neutrino factory):

$$\Delta\widehat{m^2}_{ee} = \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - \Delta m^2_{21} c^2_{12})$$

PBD, S. Parke, [1808.09453](#)

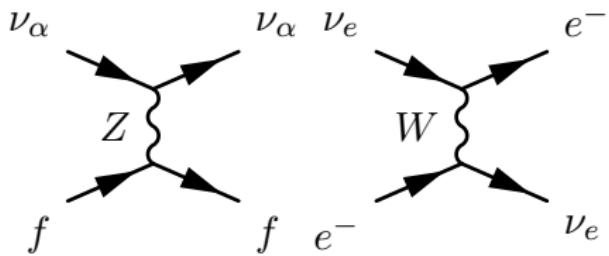
- ☐ Atmospheric

Matter Effects Matter

Call Schrödinger equation's eigenvalues m_i^2 and eigenvectors U_i .

$$\mathcal{A}(\nu_\alpha \rightarrow \nu_\beta) = \sum_{i=1}^3 U_{\alpha i}^* U_{\beta i} e^{-im_i^2 L/2E} \quad P = |\mathcal{A}|^2$$

In matter ν 's propagate in a new basis that depends on $a \propto \rho E$.



L. Wolfenstein PRD 17 (1978)

Eigenvalues: $m_i^2 \rightarrow \widehat{m^2}_i(a)$

Eigenvectors are given by $\theta_{ij} \rightarrow \widehat{\theta}_{ij}(a)$ \Leftarrow Unitarity

Hamiltonian Dynamics

$$H = \frac{1}{2E} \left[U \begin{pmatrix} 0 & & \\ & \Delta m_{21}^2 & \\ & & \Delta m_{31}^2 \end{pmatrix} U^\dagger + \begin{pmatrix} a & 0 & \\ & 0 & \\ & & 0 \end{pmatrix} \right]$$
$$a = 2\sqrt{2}G_F N_e E$$

$$U = \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}$$

Find eigenvalues and eigenvectors:

$$H = \frac{1}{2E} \widehat{U} \begin{pmatrix} 0 & & \\ & \widehat{\Delta m_{21}^2} & \\ & & \widehat{\Delta m_{31}^2} \end{pmatrix} \widehat{U}^\dagger$$

J. Kopp [physics/0610206](#)

Computationally works, but we can do better than a **black box**...
Analytic expression?

Get the eigen**values**

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation: eigen**values**

G. Cardano *Ars Magna* 1545

V. Barger, et al. PRD 22 (1980) 2718

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

Then write down eigen**vectors** (mixing angles)

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

K. Kimura, A. Takamura, H. Yokomakura [hep-ph/0205295](#)

PBD, S. Parke, X. Zhang [1907.02534](#)

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation: eigen**values**

G. Cardano *Ars Magna* 1545

V. Barger, et al. PRD 22 (1980) 2718

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

Then write down eigen**vectors** (mixing angles)

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

K. Kimura, A. Takamura, H. Yokomakura [hep-ph/0205295](#)

PBD, S. Parke, X. Zhang [1907.02534](#)

“Unfortunately, the algebra is rather impenetrable.”

V. Barger, et al.

The Cubic

Math history aside

Linear: $ax + b = 0$



Quadratic: $ax^2 + bx + c = 0$



Cubic: $ax^3 + bx^2 + cx + d = 0$



Quartic: $ax^4 + bx^3 + cx^2 + dx + e = 0$



Qunitic+: $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$



Abel-Ruffini theorem, 1824

Eigenvalues Analytically: The Exact Solution

The cubic solution (in neutrino terms)

$$\widehat{m^2}_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3BS} - \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widehat{m^2}_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3BS} + \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widehat{m^2}_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2 - 3BS}$$

$$A = \Delta m_{21}^2 + \Delta m_{31}^2 + a$$

$$B = \Delta m_{21}^2 \Delta m_{31}^2 + a [c_{13}^2 \Delta m_{31}^2 + (c_{12}^2 c_{13}^2 + s_{13}^2) \Delta m_{21}^2]$$

$$C = a \Delta m_{21}^2 \Delta m_{31}^2 c_{12}^2 c_{13}^2$$

$$S = \cos \left\{ \frac{1}{3} \cos^{-1} \left[\frac{2A^3 - 9AB + 27C}{2(A^2 - 3B)^{3/2}} \right] \right\}$$

Get the eigen**vectors**

Values and Vectors

Probability amplitude:

$$\mathcal{A}_{\alpha\beta} = \sum_i \hat{U}_{\alpha i}^* \hat{U}_{\beta i} e^{-im^2_i L/2E}$$

- ▶ Eigen**values** give the frequencies of the oscillations
Where should DUNE be?

- ▶ Eigen**vectors** give the amplitudes of the oscillations
How many events will DUNE see?

Exact Neutrino Oscillations in Matter: Mixing Angles

$$s_{\widehat{12}}^2 = \frac{-[(\widehat{m^2}_2)^2 - \alpha \widehat{m^2}_2 + \beta] \Delta \widehat{m^2}_{31}}{[(\widehat{m^2}_1)^2 - \alpha \widehat{m^2}_1 + \beta] \Delta \widehat{m^2}_{32} - [(\widehat{m^2}_2)^2 - \alpha \widehat{m^2}_2 + \beta] \Delta \widehat{m^2}_{31}}$$

$$s_{\widehat{13}}^2 = \frac{(\widehat{m^2}_3)^2 - \alpha \widehat{m^2}_3 + \beta}{\Delta \widehat{m^2}_{31} \Delta \widehat{m^2}_{32}}$$

$$s_{\widehat{23}}^2 = \frac{s_{23}^2 E^2 + c_{23}^2 F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\widehat{\delta}} = \frac{c_{23}s_{23}(e^{-i\delta}E^2 - e^{i\delta}F^2) + (c_{23}^2 - s_{23}^2)EF}{\sqrt{(s_{23}^2 E^2 + c_{23}^2 F^2 + 2EFc_{23}s_{23}c_\delta)(c_{23}^2 E^2 + s_{23}^2 F^2 - 2EFc_{23}s_{23}c_\delta)}}$$

$$\alpha = c_{13}^2 \Delta m_{31}^2 + (c_{12}^2 c_{13}^2 + s_{13}^2) \Delta m_{21}^2, \quad \beta = c_{12}^2 c_{13}^2 \Delta m_{21}^2 \Delta m_{31}^2$$

$$E = c_{13}s_{13} \left[(\widehat{m^2}_3 - \Delta m_{21}^2) \Delta m_{31}^2 - s_{12}^2 (\widehat{m^2}_3 - \Delta m_{31}^2) \Delta m_{21}^2 \right]$$

$$F = c_{12}s_{12}c_{13} (\widehat{m^2}_3 - \Delta m_{31}^2) \Delta m_{21}^2$$

H. Zaglauer, K. Schwarzer [Z.Phys. C40 \(1988\) 273](#)

Too “Impenetrable”: Approximations

- ▶ Small matter potential: $a/\Delta m^2$

Y. Li, Y. Wang, Z-z. Xing [1605.00900](#)

I. Martinez-Soler, H. Minakata [1904.07853](#)

A. Khan, H. Nunokawa, S. Parke [1910.12900](#)

- ▶ $s_{13} \sim 0.14$, $s_{13}^2 \sim 0.02$

A. Cervera, et al. [hep-ph/0002108](#)

H. Minakata [0910.5545](#)

K. Asano, H. Minakata [1103.4387](#)

- ▶ $\Delta m_{21}^2/\Delta m_{31}^2 \sim 0.03$

J. Arafune, J. Sato, [hep-ph/9607437](#)

A. Cervera, et al. [hep-ph/0002108](#)

M. Freund, [hep-ph/0103300](#)

E. Akhmedov, et al. [hep-ph/0402175](#)

S. Agarwalla, Y. Kao, T. Takeuchi [1302.6773](#)

H. Minakata, S. Parke [1505.01826](#)

PBD, H. Minakata, S. Parke [1604.08167](#)

(See G. Barenboim, **PBD**, S. Parke, C. Ternes [1902.00517](#) for a review)



Eigenvalues to Eigenvectors

KTY pushed calculating the eigenvectors from the eigenvalues.

K. Kimura, A. Takamura, H. Yokomakura [hep-ph/0205295](#)

$$\hat{U}_{\alpha i} \hat{U}_{\beta i}^* = \frac{\hat{p}_{\alpha\beta} \widehat{m^2}_i + \hat{q}_{\alpha\beta} - \delta_{\alpha\beta} \widehat{m^2}_i (\widehat{m^2}_j + \widehat{m^2}_k)}{\Delta \widehat{m^2}_{ji} \Delta \widehat{m^2}_{ki}}$$

$$\hat{p}_{\alpha\beta} = (2E)H_{\alpha\beta}$$

$$\hat{q}_{\alpha\beta} = \sum_{i < j} \widehat{m^2}_i \widehat{m^2}_j \hat{U}_{\alpha k} \hat{U}_{\beta k}^* \text{ for } k \neq i, j$$

valid for $\alpha \neq \beta$.

Wanted to preserve phase information for $\hat{\delta}$.

Eigenvalues: the Rosetta Stone

We realized:

$$|\widehat{U}_{\alpha i}|^2 = \frac{(\widehat{m^2}_i - \xi_\alpha)(\widehat{m^2}_i - \chi_\alpha)}{\Delta \widehat{m^2}_{ij} \Delta \widehat{m^2}_{ik}}$$

PBD, S. Parke, X. Zhang [1907.02534](#)

where ξ_α and χ_α are the submatrix eigenvalues of H_α

$$H = \begin{pmatrix} H_{\alpha\alpha} & H_{\alpha\beta} & H_{\alpha\gamma} \\ H_{\beta\alpha} & H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\alpha} & H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix} \quad \rightarrow \quad H_\alpha = \begin{pmatrix} H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix}$$

e.g.

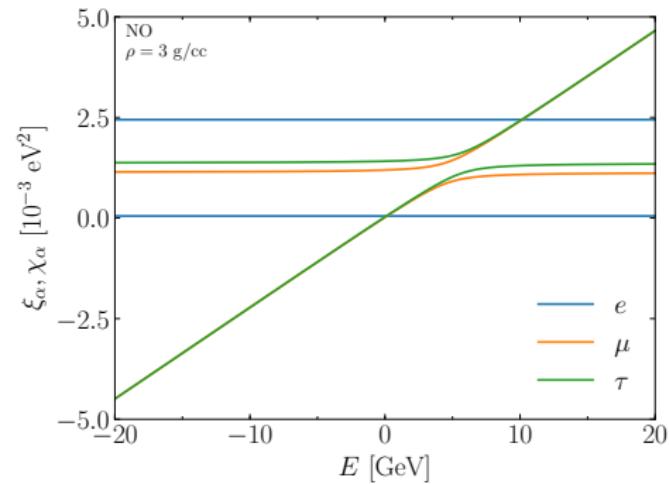
$$\xi_e + \chi_e = \Delta m_{21}^2 + \Delta m_{ee}^2 c_{13}^2$$

$$\xi_e \chi_e = \Delta m_{21}^2 [\Delta m_{ee}^2 c_{13}^2 c_{12}^2 + \Delta m_{21}^2 (s_{12}^2 c_{12}^2 - s_{13}^2 s_{12}^2 c_{12}^2)]$$

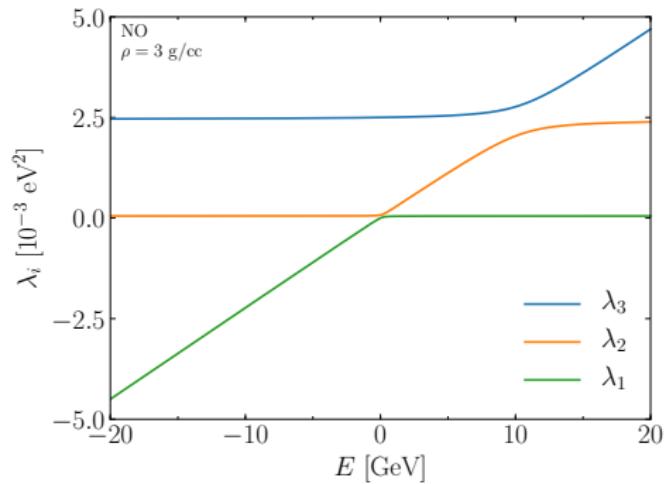
$$\Delta m_{ee}^2 = c_{12}^2 \Delta m_{31}^2 + s_{12}^2 \Delta m_{32}^2$$

H. Nunokawa, S. Parke, R. Z. Funchal [hep-ph/0503283](#)

Submatrix Eigenvalues



Submatrix Eigenvalues



Eigenvalues

Eigenvalues: the Rosetta Stone

$$s_{\widehat{13}}^2 = |\widehat{U}_{e3}|^2 = \frac{(\widehat{m^2}_3 - \xi_e)(\widehat{m^2}_3 - \chi_e)}{\Delta \widehat{m^2}_{31} \Delta \widehat{m^2}_{32}}$$

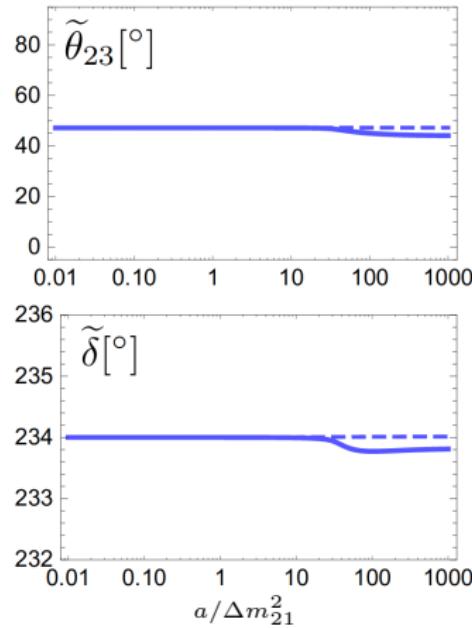
$$s_{\widehat{12}}^2 c_{\widehat{13}}^2 = |\widehat{U}_{e2}|^2 = -\frac{(\widehat{m^2}_2 - \xi_e)(\widehat{m^2}_2 - \chi_e)}{\Delta \widehat{m^2}_{32} \Delta \widehat{m^2}_{21}}$$

$$s_{\widehat{23}}^2 c_{\widehat{13}}^2 = |\widehat{U}_{\mu 3}|^2 = \frac{(\widehat{m^2}_3 - \xi_\mu)(\widehat{m^2}_3 - \chi_\mu)}{\Delta \widehat{m^2}_{31} \Delta \widehat{m^2}_{32}}$$

What about $\widehat{\delta}$?

CPV From Rosetta

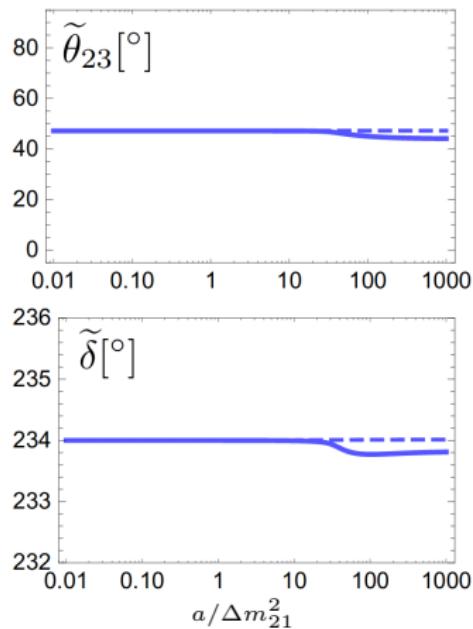
$\hat{\delta}$ nearly constant, but have to get it right



Z-z. Xing, S. Zhou
Y-L. Zhou [1802.00990](#)

CPV From Rosetta

$\hat{\delta}$ nearly constant, but have to get it right



Toshev identity:

$$\sin \hat{\delta} = \frac{\sin 2\theta_{23}}{\sin 2\hat{\theta}_{23}} \sin \delta$$

Z-z. Xing, S. Zhou
Y-L. Zhou [1802.00990](#)

S. Toshev [MPL A6 \(1991\) 455](#)

Get the sign of $\cos \hat{\delta}$ from e.g. $|\hat{U}_{\mu 1}|^2$.

In General

Two flavor:

$$|\widehat{U}_{\alpha i}|^2 = \frac{\widehat{m^2}_i - \xi_\alpha}{\Delta \widehat{m^2}_{ij}}$$

leads to

$$\begin{aligned} \sin^2 \widehat{\theta} &= |\widehat{U}_{e2}|^2 = \frac{\widehat{m^2}_2 - \xi_e}{\widehat{m^2}_2 - \widehat{m^2}_1} \\ &= \frac{1}{2} \left(1 - \frac{\Delta m^2 \cos 2\theta - a}{\sqrt{(\Delta m^2 \cos 2\theta - a)^2 + (\Delta m^2 \sin 2\theta)^2}} \right) \end{aligned}$$

In General

Two flavor:

$$|\widehat{U}_{\alpha i}|^2 = \frac{\widehat{m^2}_i - \xi_\alpha}{\Delta \widehat{m^2}_{ij}}$$

leads to

$$\begin{aligned} \sin^2 \widehat{\theta} &= |\widehat{U}_{e2}|^2 = \frac{\widehat{m^2}_2 - \xi_e}{\widehat{m^2}_2 - \widehat{m^2}_1} \\ &= \frac{1}{2} \left(1 - \frac{\Delta m^2 \cos 2\theta - a}{\sqrt{(\Delta m^2 \cos 2\theta - a)^2 + (\Delta m^2 \sin 2\theta)^2}} \right) \end{aligned}$$

Numerically checked for $N = 4, 5$.

True for all N ?

A Cheery Firehose

1. Terry posted on the same question with a different answer
2. We emailed our result, < 2 hours later:
 - ▶ “Very nice identity!”
 - ▶ 3 distinct proofs
3. 6d later, we’ve sorted 1 proof, send a draft, < 1 hr later:
 - ▶ Agrees to a paper
 - ▶ Adds a corollary
 - ▶ Adds several new observation
4. Barely processed that, another email $< \frac{1}{2}$ day later
 - ▶ A more general proof
5. We sent confirmation that the v_i are normed
6. < 1 day showed how that followed from proof 4

A Cheery Firehose

1. Terry posted on the same question with a different answer
2. We emailed our result, < 2 hours later:
 - ▶ “Very nice identity!”
 - ▶ 3 distinct proofs
3. 6d later, we’ve sorted 1 proof, send a draft, < 1 hr later:
 - ▶ Agrees to a paper
 - ▶ Adds a corollary
 - ▶ Adds several new observation
4. Barely processed that, another email $< \frac{1}{2}$ day later
 - ▶ A more general proof
5. We sent confirmation that the v_i are normed
6. < 1 day showed how that followed from proof 4

“He’s famously like a cheery firehose of mathematics
Guess he’s power-washing you today”

EIGENVECTORS FROM EIGENVALUES

PETER B. DENTON, STEPHEN J. PARKE, TERENCE TAO, AND XINING ZHANG

ABSTRACT. We present a new method of succinctly determining eigenvectors from eigenvalues. Specifically, we relate the norm squared of the elements of eigenvectors to the eigenvalues and the submatrix eigenvalues.

$$|v_{i,j}|^2 = \frac{\prod_{k=1}^{n-1} (\lambda_i - \xi_{j,k})}{\prod_{k=1; k \neq i}^n (\lambda_i - \lambda_k)}$$

Proofs

1. From previous result with $n - 1$ subvectors using derivatives

L. Erdos, B. Schlein, H-T. Yau [0711.1730](#)

T. Tao, V. Vu [0906.0510](#)

2. Geometric formulation with exterior algebra
3. Using determinants and a Cauchy-Binet variant
4. Adjugate matrices

Can get off-diagonal elements, thus CP phase

5. Cramer's rule
6. Two other mathematicians provided other proofs
7. Another mathematician generalized it to all square matrices

[https://terrytao.wordpress.com/2019/08/13/
eigenvectors-from-eigenvalues/](https://terrytao.wordpress.com/2019/08/13/eigenvectors-from-eigenvalues/)

Same result previously appeared

R. Thompson, P. McEnteggert [Lin. Alg. App. 1 211-243 \(1968\)](#)

Useful **approximations**

Focus On Eigenvalues

Now eigenvectors are easy enough given eigenvalues

Had previously derived approximations to both

PBD, H. Minakata, S. Parke [1604.08167](#)

Use approximate eigenvalues in “Rosetta” formula

Rotations Are Key

Two techniques to improve precision of an approximate system:

1. Perturbation theory

- ▶ Straightforward procedure to continue ad infinitum
- ▶ Each step is more complicated than the previous
- ▶ Careful to avoid level crossings

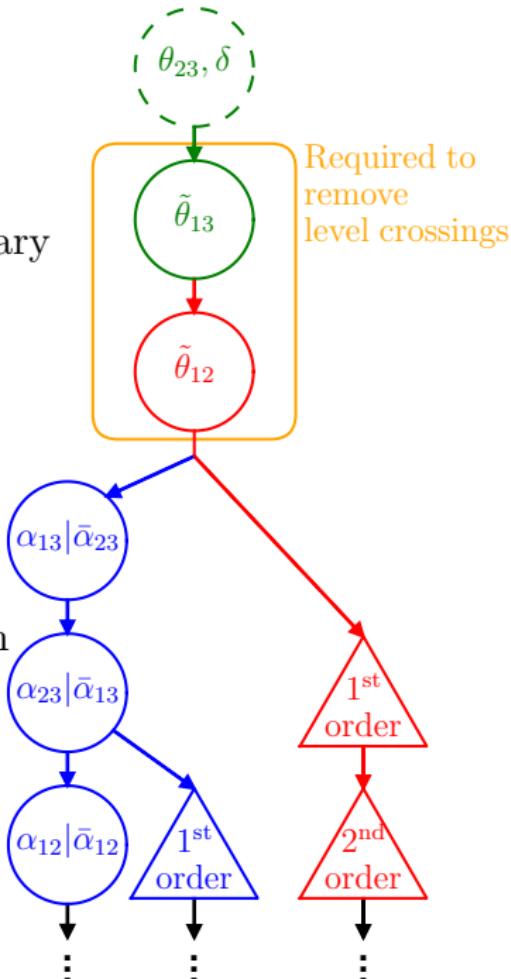
2. Rotations

- ▶ Removes level crossings
- ▶ Each step is as complicated as the last
- ▶ Can improve the precision arbitrarily
- ▶ Order matters: care must be taken

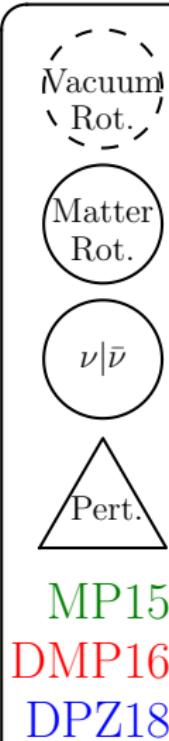
We employ a hybrid approach

Roadmap

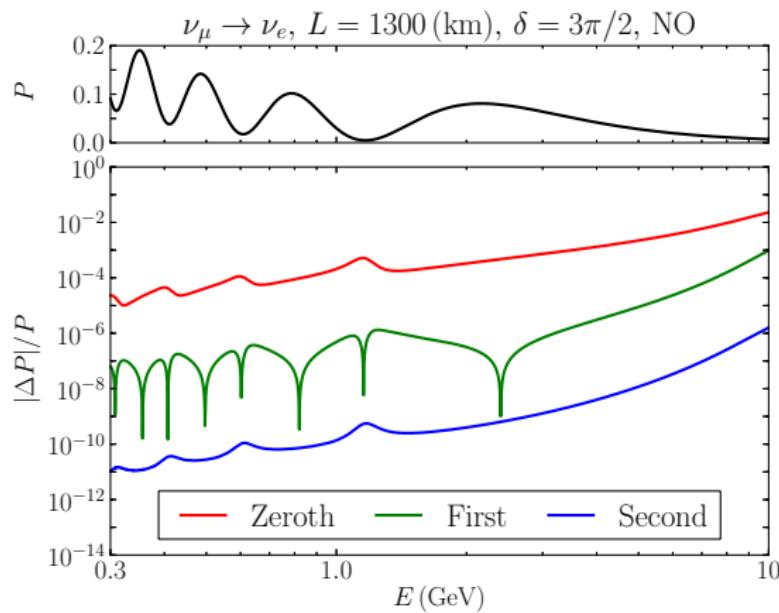
- ▶ Two rotations are necessary
- ▶ Order is lucky



- ▶ Further precision through perturbation theory or rotations

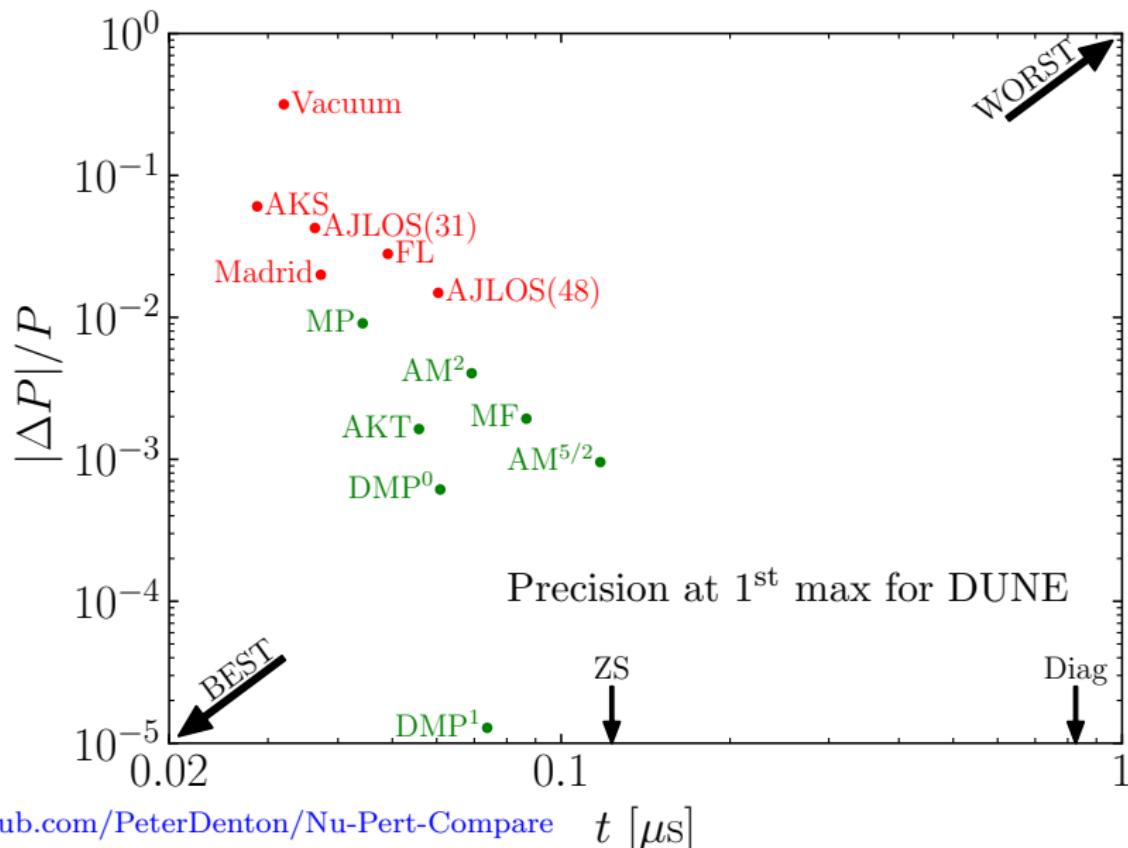


Precision



DUNE: NO, $\delta = 3\pi/2$	First min	First max
$P(\nu_\mu \rightarrow \nu_e)$	0.0047	0.081
E (GeV)	1.2	2.2
$\frac{ \Delta P }{P}$	Zeroth	5×10^{-4}
$\frac{ \Delta P }{P}$	First	3×10^{-7}
	Second	6×10^{-10}
		4×10^{-4}
		2×10^{-7}
		5×10^{-10}

Speed \propto Simplicity



DMP Eigenvalues: Perturbative Odd Orders

After two matter rotations

$$H = \frac{1}{2E} \begin{pmatrix} \widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3 \end{pmatrix} + \epsilon'(a) \frac{\Delta m_{ee}^2}{2E} \begin{pmatrix} 0 & 0 & -s_{12} \\ 0 & 0 & c_{12} \\ -s_{12} & c_{12} & 0 \end{pmatrix}$$

$$\epsilon' \lesssim 1\%, \epsilon'(a=0) = 0$$

Corrections to eigenvalues:

$$(\widetilde{m^2}_i)^{(1)} = (2E)H_{ii}^1 = 0$$

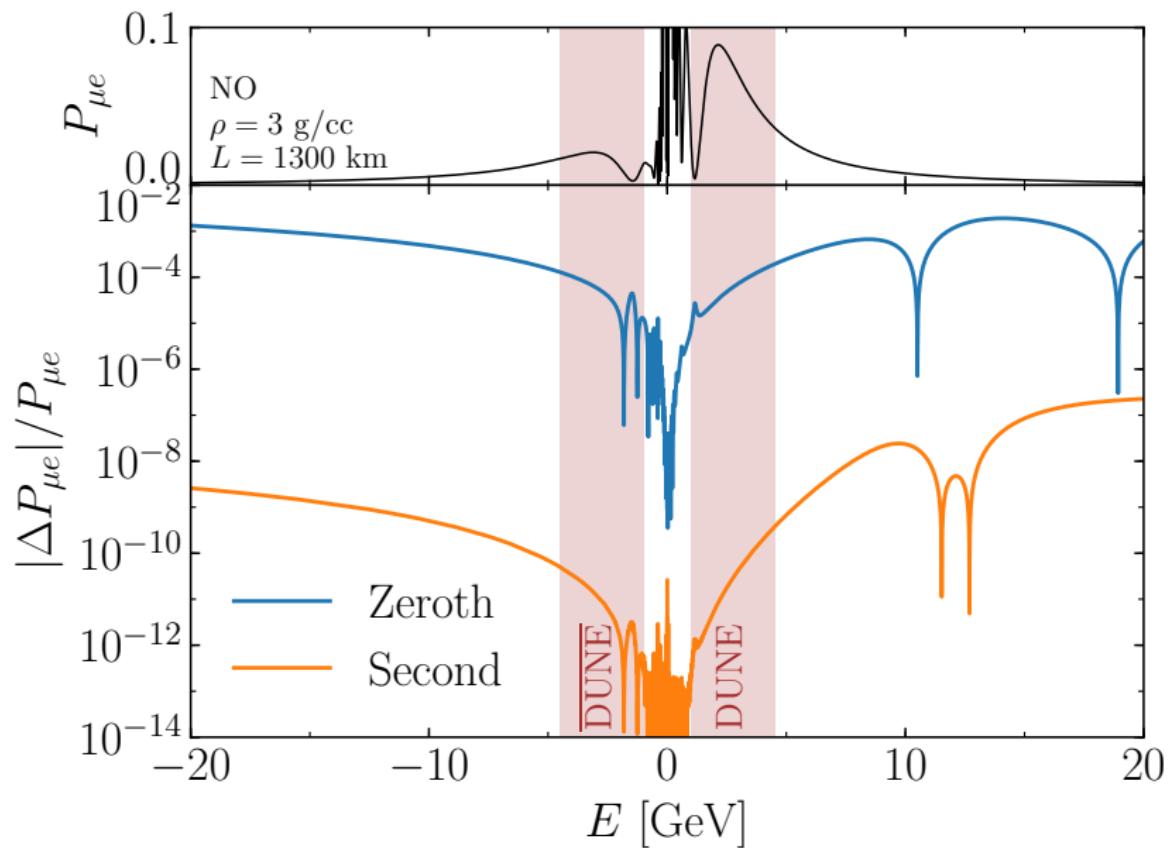
In fact, *all* odd orders vanish

X. Zhang, PBD, S. Parke [1907.02534](#)

Smallness parameter is actually $(\epsilon')^2 \sim 10^{-5}$!

Can be done for any 3×3 or 4×4 but not higher in general

DMP Eigenvalues + Rosetta



Lots of Rotations

In a 3×3 always one zero off diagonal element:

$$H_1 = \begin{pmatrix} 0 & 0 & \epsilon^a x \\ 0 & 0 & \epsilon^b y \\ \epsilon^a x^* & \epsilon^b y^* & 0 \end{pmatrix} \quad \begin{array}{l} \epsilon \ll 1 \\ x, y = \mathcal{O}(1) \\ 0 < a \leq b \end{array}$$

Rotate away the 1-3 term:

$$H'_1 = \begin{pmatrix} 0 & \epsilon^{a+b} x' & 0 \\ \epsilon^{a+b} x'^* & 0 & \epsilon^b y' \\ 0 & \epsilon^b y'^* & 0 \end{pmatrix}$$

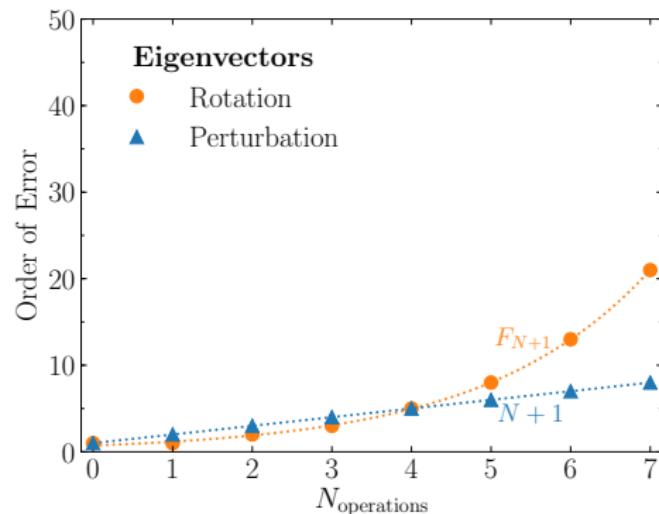
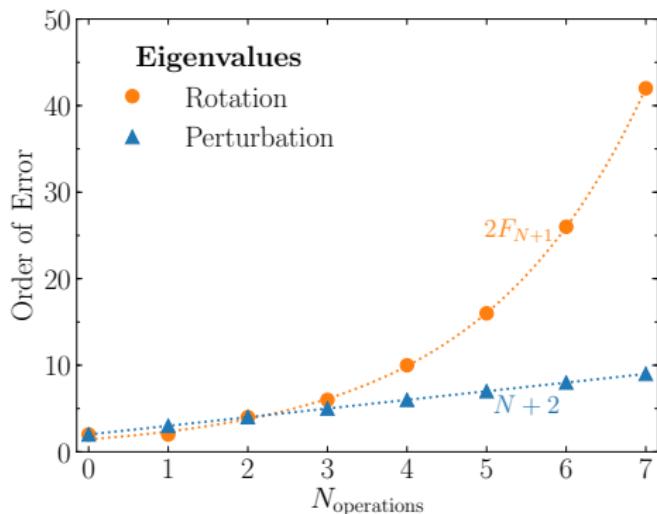
Keep on rotating large terms



Fibonacci

Continuing with rotations the error shrinks rapidly.

Take $a = b = 1$ (neutrino oscillation case)

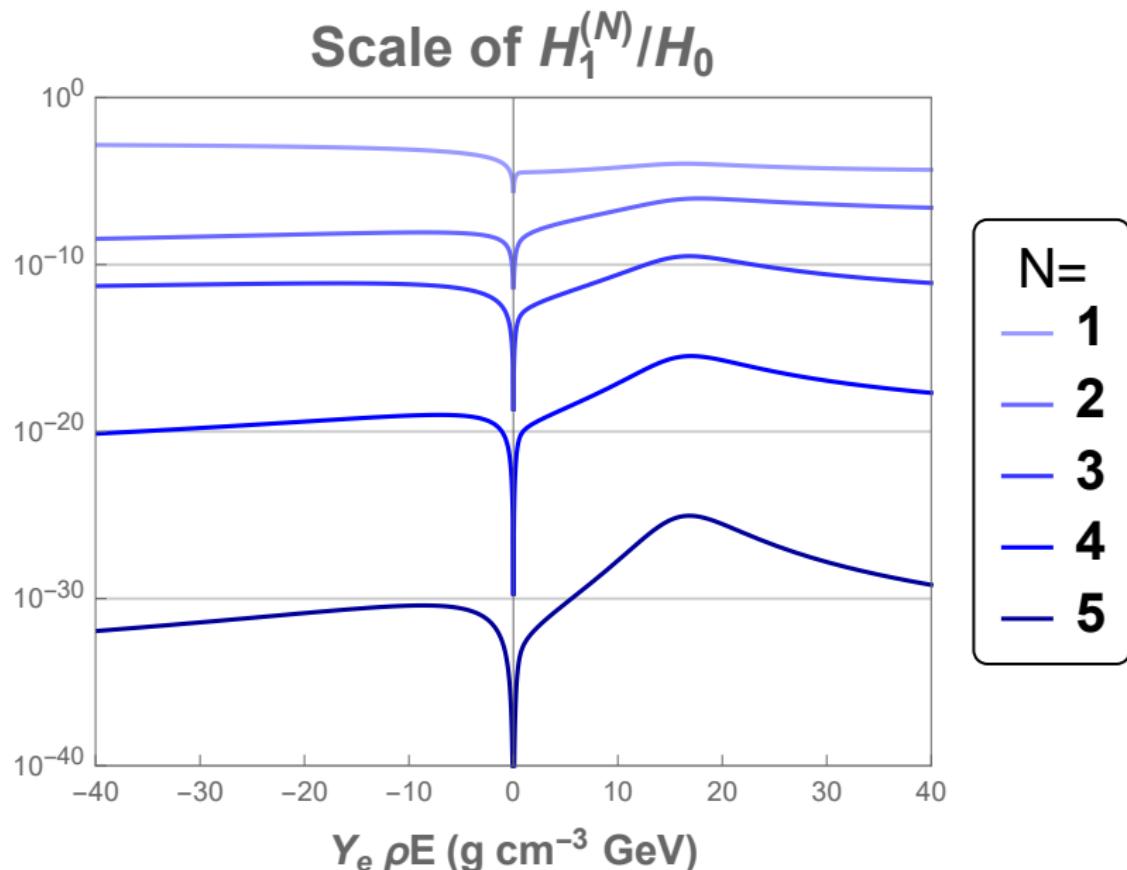


$$F_0 \equiv 0, \quad F_1 \equiv 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n > 1$$

$$\lim_{n \rightarrow \infty} F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n$$

X. Zhang, PBD, S. Parke [1909.02009](#)

Exponential (Fibonacci) Improvement



CP violation in matter

The CPV Term in Matter

The amount of CPV is

$$P_{\alpha\beta} - \bar{P}_{\alpha\beta} = \pm 16J \sin \Delta_{21} \sin \Delta_{31} \sin \Delta_{32} \quad \alpha \neq \beta$$

where the Jarlskog is

$$J \equiv \Im[U_{\alpha i} U_{\beta j} U_{\alpha j}^* U_{\beta i}^*] \quad \alpha \neq \beta, i \neq j$$

$$J = c_{12}s_{12}c_{13}^2s_{13}c_{23}s_{23} \sin \delta$$



C. Jarlskog [PRL 55 \(1985\)](#)

The exact term in matter is known to be

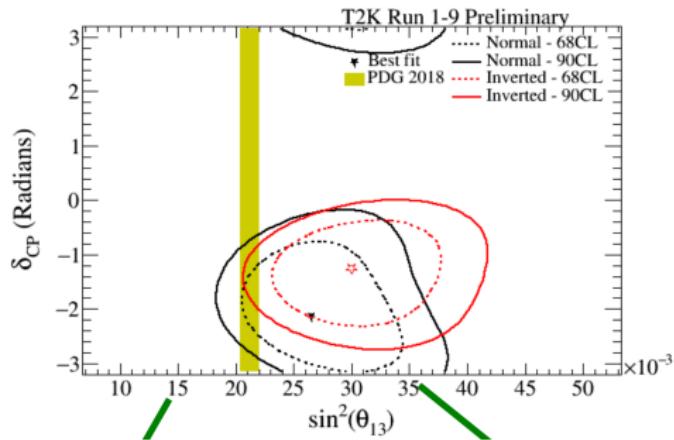
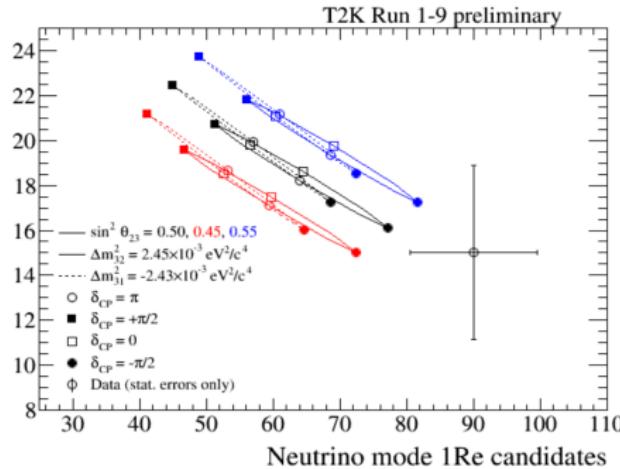
$$\frac{\hat{J}}{J} = \frac{\Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2}{\Delta \hat{m}_{21}^2 \Delta \hat{m}_{31}^2 \Delta \hat{m}_{32}^2}$$

V. Naumov [IJMP 1992](#)

P. Harrison, W. Scott [hep-ph/9912435](#)

CPV Tension at T2K

Antineutrino mode 1Re candidates



$$J = c_{12}s_{12}c_{13}^2s_{13}c_{23}s_{23} \sin \delta$$

CPV in Matter

CPV in matter can be written sans $\cos(\frac{1}{3} \cos^{-1}(\dots))$ term.

$$\frac{\widehat{J}}{J} = \frac{\Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2}{\widehat{\Delta m^2}_{21} \widehat{\Delta m^2}_{31} \widehat{\Delta m^2}_{32}}$$

$$\left(\widehat{\Delta m^2}_{21} \widehat{\Delta m^2}_{31} \widehat{\Delta m^2}_{32} \right)^2 = (A^2 - 4B)(B^2 - 4AC) + (2AB - 27C)C$$

$$A \equiv \sum_j \widehat{m^2}_j = \Delta m_{31}^2 + \Delta m_{21}^2 + a$$

$$B \equiv \sum_{j>k} \widehat{m^2}_j \widehat{m^2}_k = \Delta m_{31}^2 \Delta m_{21}^2 + a(\Delta m_{ee}^2 c_{13}^2 + \Delta m_{21}^2)$$

$$C \equiv \prod_j \widehat{m^2}_j = a \Delta m_{31}^2 \Delta m_{21}^2 c_{13}^2 c_{12}^2$$

This is the only oscillation quantity in matter that can be written exactly without $\cos(\frac{1}{3} \cos^{-1}(\dots))$!

H. Yokomakura, K. Kimura, A. Takamura [hep-ph/0009141](https://arxiv.org/abs/hep-ph/0009141)

CPV Factorizes

Thus \hat{J}^{-2} is fourth order in matter potential:
only two matter corrections are needed.

$$\frac{\hat{J}}{J} = \frac{1}{|1 - (a/\alpha_1)e^{i2\theta_1}| |1 - (a/\alpha_2)e^{i2\theta_2}|}$$

CPV Factorizes

Thus \hat{J}^{-2} is fourth order in matter potential:
only two matter corrections are needed.

$$\frac{\hat{J}}{J} = \frac{1}{|1 - (a/\alpha_1)e^{i2\theta_1}| |1 - (a/\alpha_2)e^{i2\theta_2}|}$$

CPV in matter can be well approximated:

$$\frac{\hat{J}}{J} \approx \frac{1}{|1 - (a/\Delta m_{ee}^2)e^{i2\theta_{13}}| |1 - (\textcolor{orange}{c}_{13}^2 a/\Delta m_{21}^2)e^{i2\theta_{12}}|}$$

PBD, Parke [1902.07185](#)

See also X. Wang, S. Zhou [1901.10882](#)

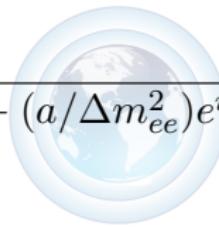
Precise at the < 0.04% level!

CPV Factorizes

Thus \hat{J}^{-2} is fourth order in matter potential:
only two matter corrections are needed.

$$\frac{\hat{J}}{J} = \frac{1}{|1 - (a/\alpha_1)e^{i2\theta_1}| |1 - (a/\alpha_2)e^{i2\theta_2}|}$$

CPV in matter can be well approximated:

$$\frac{\hat{J}}{J} \approx \frac{1}{|1 - (a/\Delta m_{ee}^2)e^{i2\theta_{13}}| |1 - (\textcolor{orange}{c_{13}^2} a/\Delta m_{21}^2)e^{i2\theta_{12}}|}$$


PBD, Parke [1902.07185](#)

See also X. Wang, S. Zhou [1901.10882](#)

Precise at the < 0.04% level!

CPV Factorizes Part II

- ▶ Option 1: Use NHS identity and $\Delta\widehat{m^2}$'s
- ▶ Option 2: Use the angles?

$$\frac{\widehat{J}}{J} = \frac{s_{12}\widehat{c_{12}}s_{13}\widehat{c_{13}^2}s_{23}\widehat{c_{23}}\sin\widehat{\delta}}{s_{12}c_{12}s_{13}\widehat{c_{13}^2}s_{23}c_{23}\sin\delta}$$

Toshev: θ_{23} , δ :

S. Toshev [MPL A6 \(1991\) 455](#)

$$\frac{\widehat{J}}{J} = \frac{s_{12}\widehat{c_{12}}s_{13}\widehat{c_{13}^2}}{s_{12}c_{12}s_{13}\widehat{c_{13}^2}}$$

How to split up?

CPV Factorizes Part II

- ▶ Option 1: Use NHS identity and $\Delta\widehat{m^2}$'s
- ▶ Option 2: Use the angles?

$$\frac{\widehat{J}}{J} = \frac{s_{12}\widehat{c_{12}}s_{13}\widehat{c_{13}^2}s_{23}\widehat{c_{23}}\sin\widehat{\delta}}{s_{12}c_{12}s_{13}\widehat{c_{13}^2}s_{23}c_{23}\sin\delta}$$

Toshev: θ_{23} , δ :

S. Toshev [MPL A6 \(1991\) 455](#)

$$\frac{\widehat{J}}{J} = \frac{s_{12}\widehat{c_{12}}s_{13}\widehat{c_{13}^2}}{s_{12}c_{12}s_{13}\widehat{c_{13}^2}}$$

How to split up?

From DMP:

[PBD](#), H. Minakata, S. Parke [1604.08167](#)

$$\frac{1}{|1 - (a/\Delta m_{ee}^2)e^{i2\theta_{13}}|} \approx \frac{s_{13}\widehat{c_{13}}}{s_{13}c_{13}}$$

Hopefully:

$$\frac{1}{|1 - (c_{13}^2 a/\Delta m_{21}^2)e^{i2\theta_{12}}|} \approx \frac{s_{12}\widehat{c_{12}}\widehat{c_{13}}}{s_{12}c_{12}c_{13}}$$

CPV Factorizes Part II

- ▶ Option 1: Use NHS identity and $\Delta\widehat{m^2}$'s
- ▶ Option 2: Use the angles?

$$\frac{\widehat{J}}{J} = \frac{s_{12}\widehat{c_{12}}s_{13}\widehat{c_{13}}^2 s_{23}\widehat{c_{23}} \sin \widehat{\delta}}{s_{12}c_{12}s_{13}\widehat{c_{13}}^2 s_{23}c_{23} \sin \delta}$$

Toshev: θ_{23} , δ :

S. Toshev [MPL A6 \(1991\) 455](#)

$$\frac{\widehat{J}}{J} = \frac{s_{12}\widehat{c_{12}}s_{13}\widehat{c_{13}}^2}{s_{12}c_{12}s_{13}\widehat{c_{13}}^2} \quad \text{How to split up?}$$

From DMP:

[PBD](#), H. Minakata, S. Parke [1604.08167](#)

$$\frac{1}{|1 - (a/\Delta m_{ee}^2)e^{i2\theta_{13}}|} \approx \frac{s_{13}\widehat{c_{13}}}{s_{13}c_{13}} \quad \sim 0.4\%$$

Hopefully:

$$\frac{1}{|1 - (c_{13}^2 a/\Delta m_{21}^2)e^{i2\theta_{12}}|} \approx \frac{s_{12}\widehat{c_{12}}c_{13}\widehat{c_{13}}}{s_{12}c_{12}c_{13}} \quad \sim 0.4\%$$

Key Cancellation

Expect s_{13}^2 or $\Delta m_{21}^2/\Delta m_{ee}^2 \sim 2 - 3\%$ precision

The atmospheric term:

$$\Delta m_{31}^2 \quad \Delta m_{ee}^2 \quad \Delta m_{32}^2$$

Solar correction:

$$1 \quad c_{13}^2 \quad \cos 2\theta_{13}$$

Key Cancellation

Expect s_{13}^2 or $\Delta m_{21}^2/\Delta m_{ee}^2 \sim 2 - 3\%$ precision

The atmospheric term:

$$\begin{array}{ccc} \Delta m_{31}^2 & \Delta m_{ee}^2 & \Delta m_{32}^2 \\ \text{X} & \checkmark & \text{X} \end{array}$$

Solar correction:

$$\begin{array}{ccc} 1 & c_{13}^2 & \cos 2\theta_{13} \\ \text{X} & \checkmark & \text{X} \end{array}$$

Key Cancellation

Expect s_{13}^2 or $\Delta m_{21}^2/\Delta m_{ee}^2 \sim 2 - 3\%$ precision

The atmospheric term:

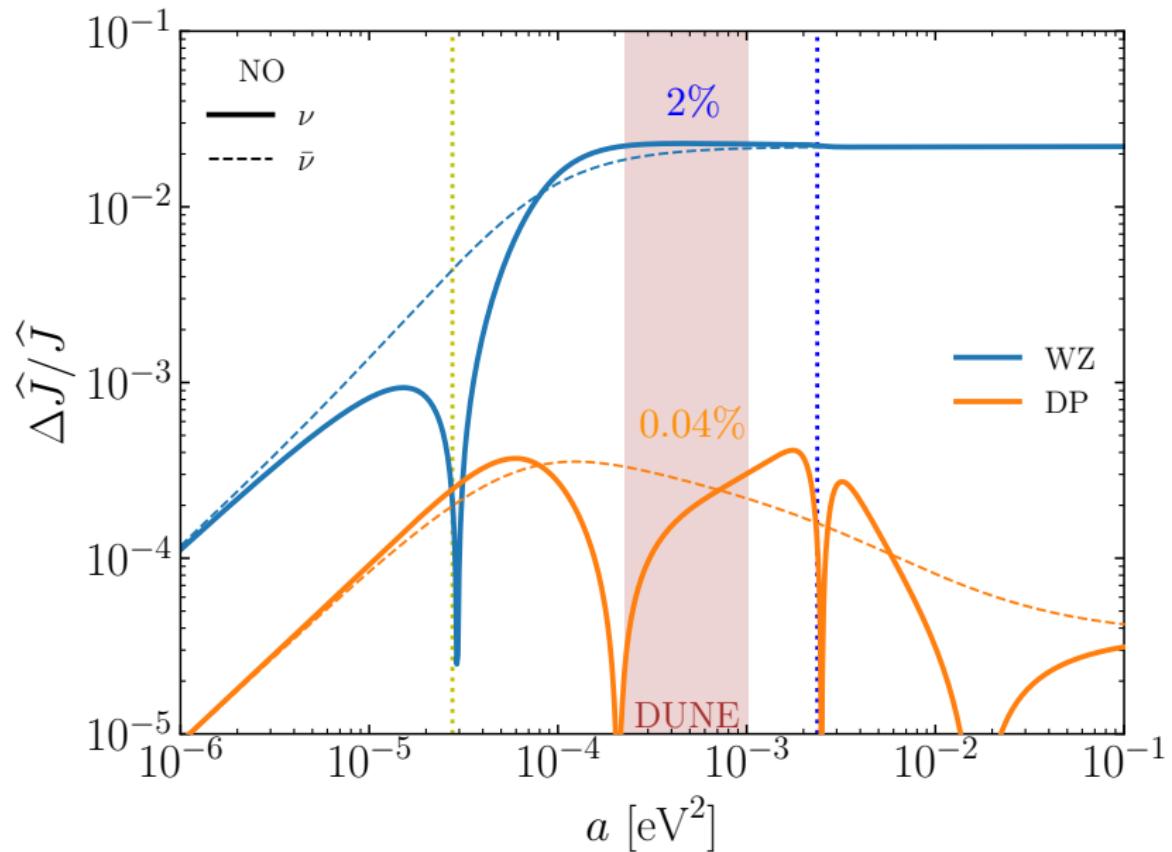
$$\begin{array}{ccc} \Delta m_{31}^2 & \Delta m_{ee}^2 & \Delta m_{32}^2 \\ \text{X} & \checkmark & \text{X} \end{array}$$

Solar correction:

$$\begin{array}{ccc} 1 & c_{13}^2 & \cos 2\theta_{13} \\ \text{X} & \checkmark & \text{X} \end{array}$$

$$\frac{\Delta \hat{J}}{\hat{J}} \sim \mathcal{O}\left(s_{13}^2 \frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right) + \mathcal{O}\left[\left(\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right)^2\right] \sim 0.04\%$$

CPV In Matter Approximation Precision



New Physics

DUNE and T2HK will unprecedented capabilities to test the three-neutrino oscillation picture

Extend DMP to new physics progress report:

- Sterile

S. Parke, X. Zhang [1905.01356](#)

- NSI
- Neutrino decay
- Decoherence
- ...

Given Rosetta, extensions should be considerably simpler

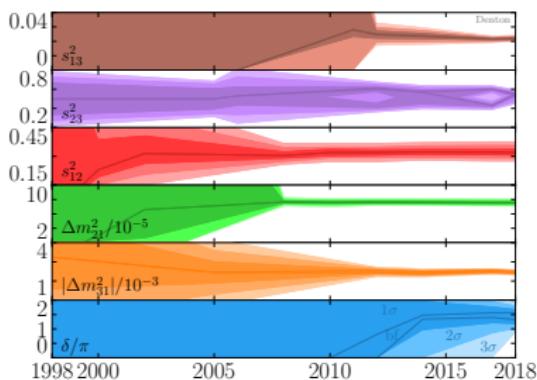
Key Points

- ▶ Understanding probabilities in matter is key to current/future LBL
- ▶ Long-baseline oscillations are fundamentally three-flavor
- ▶ Approximate eigenvalues are key
- ▶ Eigenvectors follow from eigenvalues
- ▶ Exact and approximate CPV in matter are simpler than expected

Thanks!

Backups

References



SK [hep-ex/9807003](#)

M. Gonzalez-Garcia, et al. [hep-ph/0009350](#)

M. Maltoni, et al. [hep-ph/0207227](#)

SK [hep-ex/0501064](#)

SK [hep-ex/0604011](#)

T. Schwetz, M. Tortola, J. Valle [0808.2016](#)

M. Gonzalez-Garcia, M. Maltoni, J. Salvado [1001.4524](#)

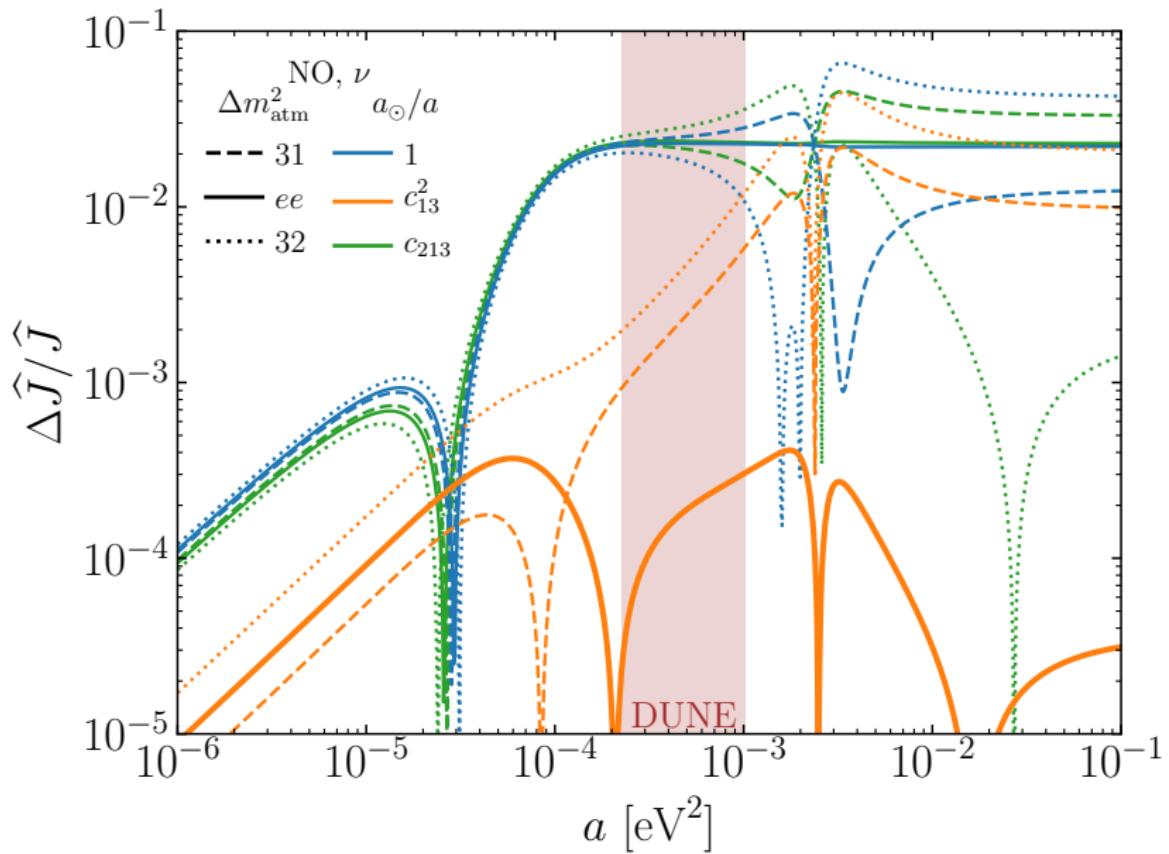
T2K [1106.2822](#)

D. Forero, M. Tortola, J. Valle [1205.4018](#)

D. Forero, M. Tortola, J. Valle [1405.7540](#)

P. de Salas, et al. [1708.01186](#)

Factorization Conditions



Proper Expansions

Parameter x is an expansion parameter iff

$$\lim_{x \rightarrow 0} P_{\text{approx}}(x) = P_{\text{exact}}(x = 0)$$

	ϵ	s_{13}	$a/\Delta m_{31}^2$	
Madrid(like)	✗	✗	✗	Cervera+ hep-ph/0002108
AKT	✓	✓	✓	Agarwalla+ 1302.6773
MP	✓	✗	✗	Minakata, Parke 1505.01826
DMP	✓	✓	✓	PBD+ 1604.08167
AKS	✗	✗	✗	Arafune+ hep-ph/9703351
MF	✓	✗	✗	Freund hep-ph/0103300
AJLOS(48)	✓	✗	✗	Akhmedov+ hep-ph/0402175
AM	✗	✗	✗	Asano, Minakata 1103.4387

$$\epsilon \simeq \frac{\Delta m_{21}^2}{\Delta m_{ee}^2}$$

The Cubic

Math history aside

1. Ancients (20-16C BC)

Babylonians, Greeks, Chinese, Indians, Egyptians:
thought about cubics, calculated cube roots

$$x^3 = a$$

2. Chinese Wang Xiaotong (7C AD):

numerically solved 25 general cubics

3. Persian Omar Khayyam (11C AD):

realized there are multiple solutions

The Cubic

Math history aside: The Italian Job (16C AD)

4. Scipione del **Ferro**:

Secret solution, nearly all (didn't know negative numbers)

$$x^3 + mx = n$$

5. Antonio **Fiore**: Ferro's student, from just before his death
6. Niccol **Tartaglia**: Claimed a solution, was challenged by Fiore
7. Gerolamo **Cardano**: Gets Tartaglia's (winner) solution, promises to keep it secret. Later publishes Ferro's solution via Fiore
8. Tartaglia challenges Cardano who denies it. Cardano's student **Ferrari** accepted, Tartaglia lost along with prestige and income

Quartic was (nearly) solved around the same time by Ferrari,
before the cubic solution was published