Abstract

I will develop a perturbative framework for neutrino oscillations in uniform matter density so that the resulting oscillation probabilities are accurate for the complete matter potential versus baseline divided by neutrino energy plane. The expansion parameter used is related to the ratio of the solar to the atmospheric Δm^2 scales but with a unique choice of the atmospheric Δm^2, Δm^2_{ee}, such that certain first-order effects are taken into account in the zeroth-order Hamiltonian. I will also show how the first and second order results improve the precision by approximately two or more orders of magnitude per perturbative order. I will show recent work on comparing the effect of additional rotations versus perturbations. I will then compare this expressions with the others on the market. Finally, I will explore how Δm^2_{ee} is modified in matter.

Neutrino Oscillation Probabilities in Matter

Peter B. Denton

Virginia Tech

February 13, 2019

with S. Parke

X. Zhang, C. Ternes, H. Minakata, G. Barenboim.

1604.08167, 1806.01277, 1808.09453, 1902.00517

github.com/PeterDenton/Nu-Pert

github.com/PeterDenton/Nu-Pert-Compare

BROOKHAVEN NATIONAL LABORATORY

Neutrino Oscillation Parameters Status

Six parameters:

1. $\theta_{13} = (8.6 \pm 0.1)^\circ$
2. $\theta_{12} = (33.8 \pm 0.8)^\circ$
3. $\Delta m^2_{21} = (7.4 \pm 0.2) \times 10^{-5}\ \mathrm{eV}^2$
4. $\theta_{23} \sim 45^\circ$ (octant)
5. $|\Delta m^2_{31}| = (2.52 \pm 0.03) \times 10^{-3}\ \mathrm{eV}^2$ (mass ordering)
6. $\delta = ???$

NuFIT, `1811.05487`

PMNS order allows for easy measurement of θ_{13} and θ_{12}.

θ_{23} and δ_{CP} require full three-flavor description.

Analytic Oscillation Probabilities in Matter

- Solar: $P_{ee}=\sin^2\theta_\odot$

  Approx: S. Mikheev, A. Smirnov, Nuovo Cim. C9 (1986) 17-26

  Exact: S. Parke, PRL 57 (1986) 2322

- Long-baseline: All three flavors

  Exact: H. Zaglauer, K. Schwarzer, Z.Phys. C40 (1988) 273

  Approx: PBD, H. Minakata, S. Parke, `1604.08167`

- ν_e disappearance (neutrino factory):
  $\Delta\widehat{m^2}_{ee}=\widehat{m^2}_3-(\widehat{m^2}_1+\widehat{m^2}_2-\Delta m^2_{21}c^2_{12})$

  PBD, S. Parke, `1808.09453`

- Atmospheric?

The Several Billion Dollar Question

What is $P(\nu_\mu \to \nu_e)$?

$$P(\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \qquad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}}\mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23}\sin\Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23})\sin\Delta_{21}$$

$$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$$

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$$\Delta_{ij} = \Delta m^2{}_{ij}L/4E$$

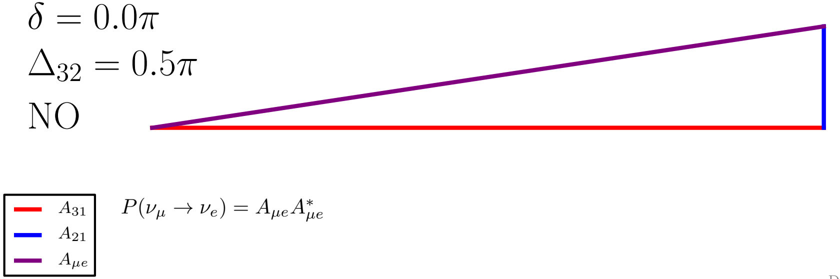
...in matter?

Now: NOvA, T2K, MINOS, ...

Upcoming: DUNE, T2HK, ...

Second maximum: T2HKK? ESSnuSB? ...

Neutrino oscillations in vacuum: appearance



$P(\nu_\mu \to \nu_e) = A_{\mu e} A_{\mu e}^*$

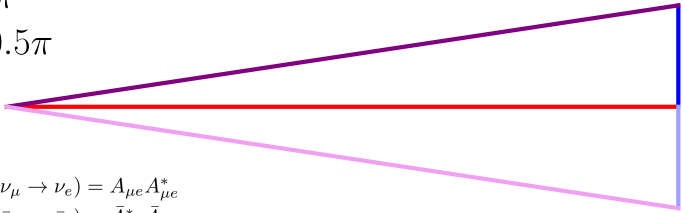
Denton & Parke

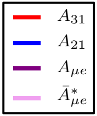
Neutrino oscillations in vacuum: appearance

$\delta = 0.0\pi$

$\Delta_{32} = 0.5\pi$

NO





$P(\nu_\mu \to \nu_e) = A_{\mu e} A^*_{\mu e}$

$P(\bar{\nu}_\mu \to \bar{\nu}_e) = \bar{A}^*_{\mu e} \bar{A}_{\mu e}$

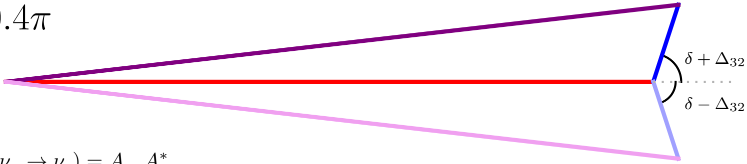
Denton & Parke

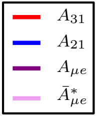
Neutrino oscillations in vacuum: appearance

$\delta = 0.0\pi$

$\Delta_{32} = 0.4\pi$

NO





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Denton & Parke

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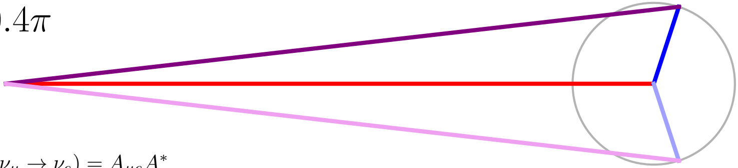


red	A_{31}
blue	A_{21}
purple	$A_{\mu e}$
pink	$\bar{A}^*_{\mu e}$

$$P(\nu_\mu \to \nu_e) = A_{\mu e} A^*_{\mu e}$$
$$P(\bar{\nu}_\mu \to \bar{\nu}_e) = \bar{A}^*_{\mu e} \bar{A}_{\mu e}$$

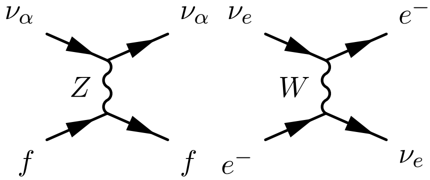
Denton & Parke

Matter Effects Matter

Call Schrödinger equation's eigenvalues m_i^2 and eigenvectors U_i.

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_i U^*_{\alpha i} U_{\beta i} e^{-i m_i^2 L/2E} \qquad P = |\mathcal{A}|^2$$

In matter ν's propagate in a new basis that depends on $a \propto \rho E$.



L. Wolfenstein, PRD 17 (1978)

Eigenvalues: $m_i^2 \to \widehat{m^2}_i(a)$

Eigenvectors are given by $\theta_{ij} \to \widehat{\theta}_{ij}(a) \qquad \Leftarrow \qquad$ Unitarity

Variable Matter Density

We assume ρ is constant. Is this okay?

If ρ varies only slowly, we can set ρ to the average:

$$\rho = \bar{\rho} = \frac{1}{L}\int_0^L \rho(x)dx$$

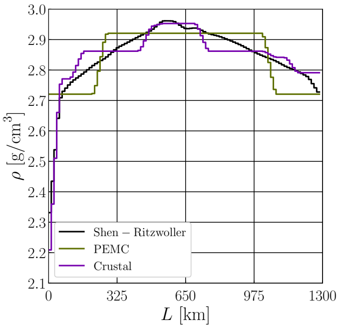
ρ doesn't vary "too much" when

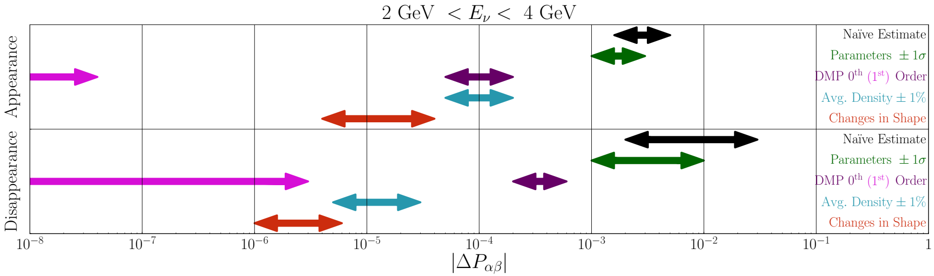
$$|\dot{\hat{\theta}}| \ll \left|\frac{\Delta\widehat{m^2}}{2E}\right|$$

True for DUNE?

Variable Matter Density

This is a fine approximation at DUNE:





K. Kelly, S. Parke, 1802.06784

Hamiltonian Dynamics

$$H = \frac{1}{2E}\left[U\begin{pmatrix}0 & & \\ & \Delta m^2_{21} & \\ & & \Delta m^2_{31}\end{pmatrix}U^\dagger + \begin{pmatrix}a & & \\ & 0 & \\ & & 0\end{pmatrix}\right]$$

$a = 2\sqrt{2}G_F N_e E$

Find eigenvalues and eigenvectors:

$$H = \frac{1}{2E}\widehat{U}\begin{pmatrix}0 & & \\ & \Delta\widehat{m^2}_{21} & \\ & & \Delta\widehat{m^2}_{31}\end{pmatrix}\widehat{U}^\dagger$$

Computationally works, but we can do better than a $\boxed{\textbf{black box}}$...

Analytic expression?

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation.

$$\widehat{m^2}_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2-3B}\,S - \frac{\sqrt{3}}{3}\sqrt{A^2-3B}\sqrt{1-S^2}$$

$$\widehat{m^2}_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2-3B}\,S + \frac{\sqrt{3}}{3}\sqrt{A^2-3B}\sqrt{1-S^2}$$

$$\widehat{m^2}_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2-3B}\,S$$

$$A = \Delta m^2_{21} + \Delta m^2_{31} + a$$

$$B = \Delta m^2_{21}\Delta m^2_{31} + a\left[c^2_{13}\Delta m^2_{31} + (c^2_{12}c^2_{13} + s^2_{13})\Delta m^2_{21}\right]$$

$$C = a\Delta m^2_{21}\Delta m^2_{31}c^2_{12}c^2_{13}$$

$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3-9AB+27C}{2\left(A^2-3B\right)^{3/2}}\right]\right\}$$

H. Zaglauer, K. Schwarzer, Z. Phys. C Particles and Fields, 40 (1988)

Traded one $\boxed{\textbf{black box}}$ for another...

We're physicists so . . .

Perturbation theory

A Tale of Two Tools

Split the Hamiltonian into:

- Large, diagonal part (H_0)
- Small, *off-diagonal* part (H_1)

- Improves precision at zeroth order
- Naturally leads to using $\Delta m^2_{ee} \equiv c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32}$

H. Nunokawa, S. Parke, R. Zukanovich, hep-ph/0503283

1. **Rotations**:
   - A two-flavor rotation only requires solving a quadratic
   - Diagonalize away the big off-diagonal terms
   - Follows the order of the PMNS matrix
2. **Perturbative expansion**:
   - Smallness parameter is $|\epsilon'| \leq 0.015$
   - Correct eigenvalues ($\widetilde{m^2}_i$) and eigenvectors ($\widetilde{\theta}_{ij}$)
   - Eigenvalues already include 1^{st} order corrections at 0^{th} order
   - Can improve the precision to arbitrary order

"What is Δm^2_{ee}?"

$$\Delta m^2_{ee} = c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32} = \Delta m^2_{31} - s^2_{12}\Delta m^2_{21}$$

H. Nunokawa, S. Parke, R. Funchal, hep-ph/0503283

S. Parke, 1601.07464

Additional expressions for $\Delta m^2_{\mu\mu}, \Delta m^2_{\tau\tau}$

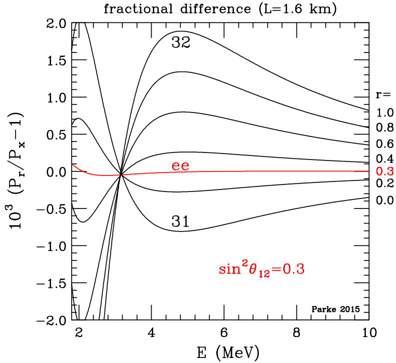
Useful definitions:

- ν_e weighted average of atmospheric splittings:

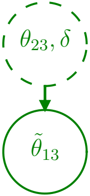
$$m_3^2 - \frac{|U_{e1}|^2 m_1^2 + |U_{e2}|^2 m_2^2}{|U_{e1}|^2 + |U_{e2}|^2}$$

- Measured by reactor experiments with smallest L/E error
- Simple form:
  $P_{ee} \approx 1 - \sin^2 2\theta_{13} \sin^2 \Delta_{ee}$

$$\Delta_{ij} = \Delta m^2{}_{ij} L/4E$$

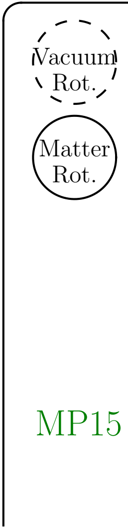


Atmospheric Resonance

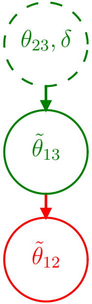


1. $U_{23}(\theta_{23}, \delta)$ commutes with matter potential
2. Largest off-diagonal term: $s_{13}c_{13}\Delta m^2_{ee}$ in the 1-3 position

- Eigenvalues still cross at the solar resonance:
  - No perturbation theory there
- Smallness parameter:
  - After U_{23}: $s_{13}c_{13} = 0.15$
  - After U_{13}: $s_{12}c_{12}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.015$



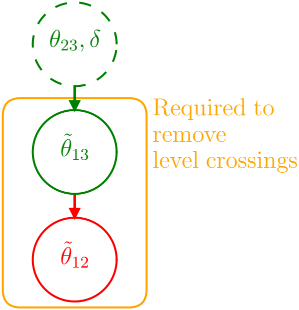
Solar Resonance



3. Largest off-diagonal term:
   $s_{12}c_{12}c_{\widetilde{\theta}_{13}-\theta_{13}}\Delta m^2_{21}$ in the 1-2 position
   - Largest except for ν's above the atmospheric resonance

- $|\epsilon'| < 0.015$, zero in vacuum
- Perturbation theory valid everywhere now
- Rotation order matches PMNS
- Take vacuum expressions, replace θ_{13}, θ_{12}, and Δm^2_{ij}
- Extremely precise $|\Delta P/P| < 10^{-3}$

Solar Resonance



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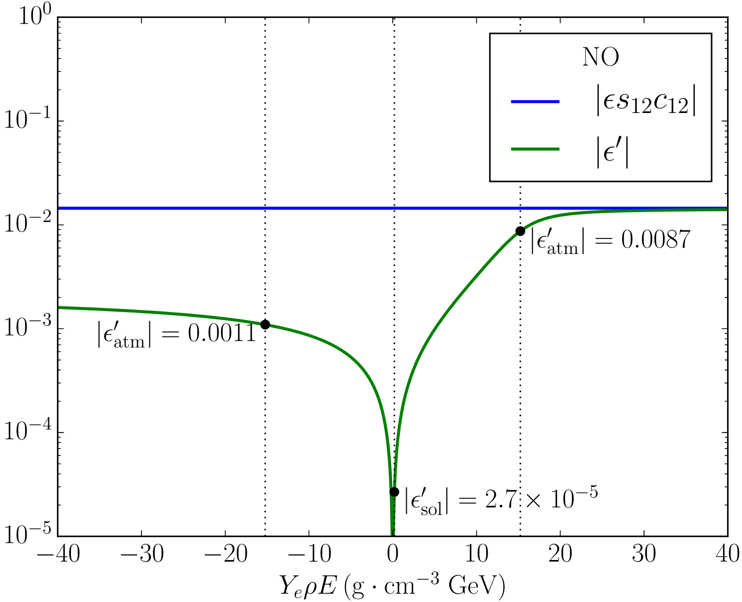
Expansion Parameter



Probability in Matter: DMP 0$^{\text{th}}$

Matter expression $\Rightarrow$ Vacuum expression

$$\widetilde{P}_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\Delta\widetilde{m^2}_{21}, \Delta\widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

Probability in Matter: DMP 0^{th}

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Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m^2_{ee} \cos 2\theta_{13} - a}{\Delta\widetilde{m^2}_{ee}}$$

$$\Delta\widetilde{m^2}_{ee} = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

Probability in Matter: DMP 0$^{\text{th}}$

Matter expression $\Rightarrow$ Vacuum expression

$$\widetilde{P}_{\alpha\beta}(\Delta m^2_{21}, \Delta m^2_{31}, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\Delta\widetilde{m^2}_{21}, \Delta\widetilde{m^2}_{31}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

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$$\Delta\widetilde{m^2}_{ee} = \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

$$\cos 2\widetilde{\theta}_{12} = \frac{\Delta m^2_{21}\cos 2\theta_{12} - a_{12}}{\Delta\widetilde{m^2}_{21}}, \qquad a_{12} = (a + \Delta m^2_{ee} - \Delta\widetilde{m^2}_{ee})/2$$

$$\Delta\widetilde{m^2}_{21} = \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a_{12}/\Delta m^2_{21})^2 + \cos^2(\widetilde{\theta}_{13} - \theta_{13})\sin^2 2\theta_{12}}$$

Probability in Matter: DMP 0^{th}

Matter expression $\Rightarrow$ Vacuum expression

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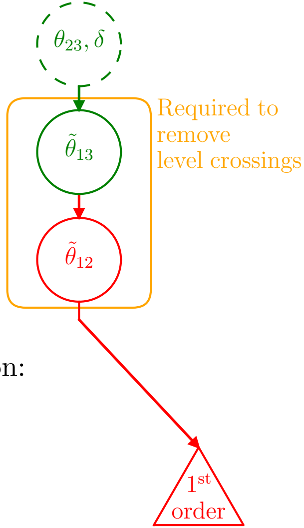
$$\Delta\widetilde{m^2}_{21} = \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a_{12}/\Delta m^2_{21})^2 + \cos^2(\widetilde{\theta}_{13} - \theta_{13})\sin^2 2\theta_{12}}$$

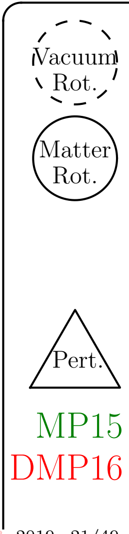
$$\Delta\widetilde{m^2}_{31} = \Delta m^2_{31} + \frac{1}{4}a + \frac{1}{2}(\Delta\widetilde{m^2}_{21} - \Delta m^2_{21}) + \frac{3}{4}(\Delta\widetilde{m^2}_{ee} - \Delta m^2_{ee})$$

Improve with Perturbation



4. $\gtrsim 2$ orders of magnitude of improvement in precision: $|\Delta P/P| < 10^{-6}$

- Eigenvalues need no correction
- Compact form utilizes a $\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2$, $\widetilde\theta_{12} \leftrightarrow \widetilde\theta_{12} \pm \pi/2$ symmetry



$\widetilde{m^2}_{1,2} - \widetilde{\theta}_{12}$ Symmetry

From the shape of $U_{12}(\widetilde{\theta}_{12})$, it is clear that the probabilities are invariant under a simultaneous interchange of

$$\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2 \,, \qquad \text{and} \qquad \widetilde{\theta}_{12} \to \widetilde{\theta}_{12} \pm \frac{\pi}{2} \,.$$

Since only even powers of $\widetilde{\theta}_{12}$ trig functions $c^2_{\widetilde{12}}, s^2_{\widetilde{12}}, c_{\widetilde{12}} s_{\widetilde{12}}, \cos(2\widetilde{\theta}_{12}), \sin(2\widetilde{\theta}_{12})$ appear in the probabilities, the sign degeneracy is irrelevant.

More usefully, we can write that the probabilities are invariant under the simultaneous interchange of

$$\widetilde{m^2}_1 \leftrightarrow \widetilde{m^2}_2 \,, \qquad c^2_{\widetilde{12}} \leftrightarrow s^2_{\widetilde{12}} \,, \qquad \text{and} \qquad c_{\widetilde{12}} s_{\widetilde{12}} \to -c_{\widetilde{12}} s_{\widetilde{12}} \,.$$

This interchange constrains the $\sin^2 \Delta_{21}$ term,
and the $\sin^2 \Delta_{32}$ term easily follows from the $\sin^2 \Delta_{31}$ term.

General Form of the First Order Coefficients

$$P_{\alpha\beta} = \delta_{\alpha\beta} + 4C_{21}^{\alpha\beta}\sin^2\Delta_{21} + 4C_{31}^{\alpha\beta}\sin^2\Delta_{31} + 4C_{32}^{\alpha\beta}\sin^2\Delta_{32} + 8D^{\alpha\beta}\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32}$$

Can reduce 8 expressions down to 3:

$$(C_{21}^{\alpha\beta})^{(1)} = \epsilon'\Delta m_{ee}^2\left(\frac{F_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} + \frac{F_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}}\right)$$

$$(C_{31}^{\alpha\beta})^{(1)} = \epsilon'\Delta m_{ee}^2\left(\frac{F_1^{\alpha\beta}+G_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} - \frac{F_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}}\right)$$

$$(C_{32}^{\alpha\beta})^{(1)} = \epsilon'\Delta m_{ee}^2\left(-\frac{F_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} + \frac{F_2^{\alpha\beta}+G_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}}\right)$$

$$(D^{\alpha\beta})^{(1)} = \epsilon'\Delta m_{ee}^2\left(\frac{K_1^{\alpha\beta}}{\widetilde{\Delta m^2}_{31}} - \frac{K_2^{\alpha\beta}}{\widetilde{\Delta m^2}_{32}}\right)$$

$$K_1^{\alpha\beta} = \mp s_{23}c_{23}c_{\widetilde{13}}s_{\widetilde{12}}^2(c_{\widetilde{13}}^2c_{\widetilde{12}}^2 - s_{\widetilde{13}}^2)s_\delta\,,\quad \alpha\neq\beta$$

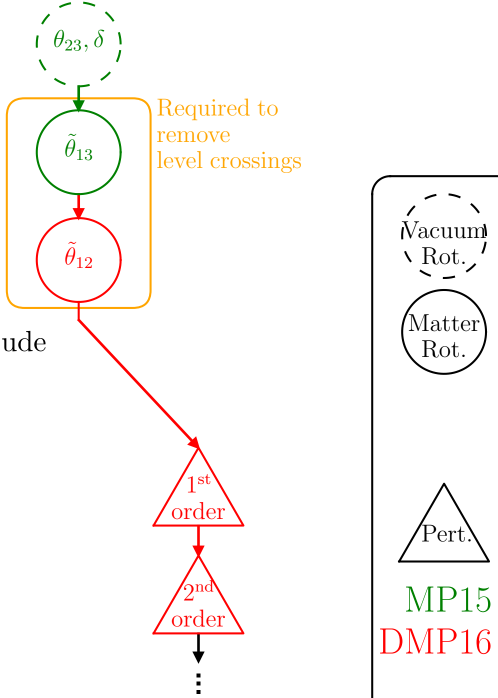
First Order Coefficients

$\nu_\alpha \to \nu_\beta$	$F_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$-2c^3_{\widetilde{13}}s_{\widetilde{13}}s^3_{\widetilde{12}}c_{\widetilde{12}}$
$\nu_\mu \to \nu_e$	$c_{\widetilde{13}}s^2_{\widetilde{12}}[s_{\widetilde{13}}s_{\widetilde{12}}c_{\widetilde{12}}(c^2_{23}+c_{2\widetilde{13}}s^2_{23})$ $-s_{23}c_{23}(s^2_{\widetilde{13}}s^2_{\widetilde{12}}+c_{2\widetilde{13}}c^2_{\widetilde{12}})c_\delta]$
$\nu_\mu \to \nu_\mu$	$2c_{\widetilde{13}}s_{\widetilde{12}}(s^2_{23}s_{\widetilde{13}}c_{\widetilde{12}}+s_{23}c_{23}s_{\widetilde{12}}c_\delta)\times$ $(c^2_{23}c^2_{\widetilde{12}}-2s_{23}c_{23}s_{\widetilde{13}}s_{\widetilde{12}}c_{\widetilde{12}}c_\delta+s^2_{23}s^2_{\widetilde{13}}s^2_{\widetilde{12}})$

$\nu_\alpha \to \nu_\beta$	$G_1^{\alpha\beta}$
$\nu_e \to \nu_e$	$2s_{\widetilde{13}}c_{\widetilde{13}}s_{\widetilde{12}}c_{\widetilde{12}}c_{2\widetilde{13}}$
$\nu_\mu \to \nu_e$	$-2s_{\widetilde{13}}c_{\widetilde{13}}s_{\widetilde{12}}(s^2_{23}c_{2\widetilde{13}}c_{\widetilde{12}}-s_{23}c_{23}s_{\widetilde{13}}s_{\widetilde{12}}c_\delta)$
$\nu_\mu \to \nu_\mu$	$-2c_{\widetilde{13}}s_{\widetilde{12}}(s^2_{23}s_{\widetilde{13}}c_{\widetilde{12}}+s_{23}c_{23}s_{\widetilde{12}}c_\delta)$ $\times(1-2c^2_{\widetilde{13}}s^2_{23})$

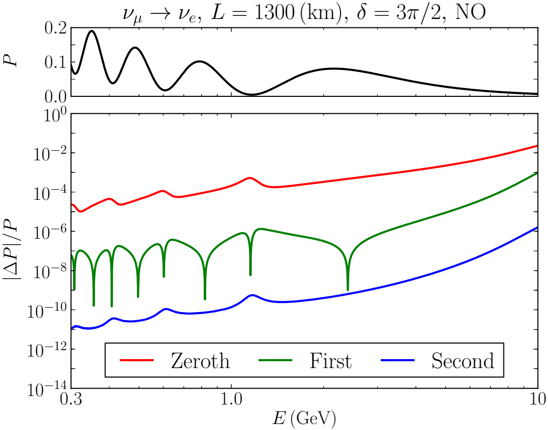
Three channels gives them all with unitarity!

Higher Orders



5. $\gtrsim 2$ more orders of magnitude of improvement per order: $|\Delta P/P| < 10^{-9}$, ...

Precision



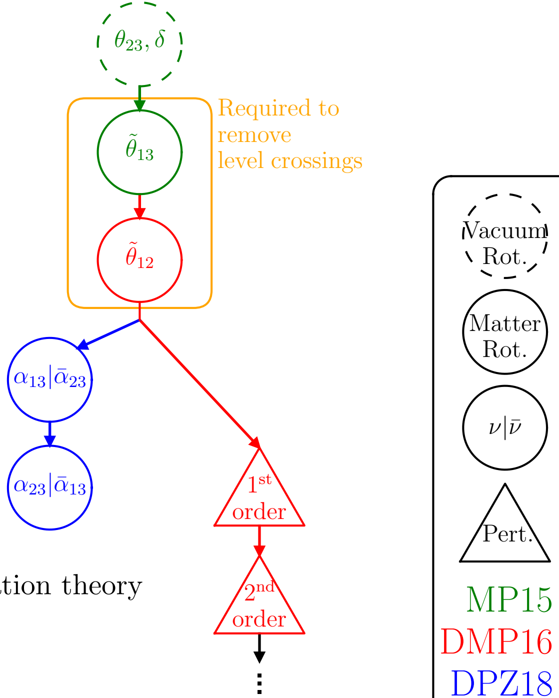
DUNE: NO, $\delta = 3\pi/2$		First min	First max
$P(\nu_\mu \to \nu_e)$		0.0047	0.081
E (GeV)		1.2	2.2
$\frac{\lvert\Delta P\rvert}{P}$	Zeroth	5×10^{-4}	4×10^{-4}
	First	3×10^{-7}	2×10^{-7}
	Second	6×10^{-10}	5×10^{-10}

More Rotations

Instead continue to diagonalize large terms

4. 1-3 sector for ν's
   2-3 sector for $\bar{\nu}$'s
5. Then opposite

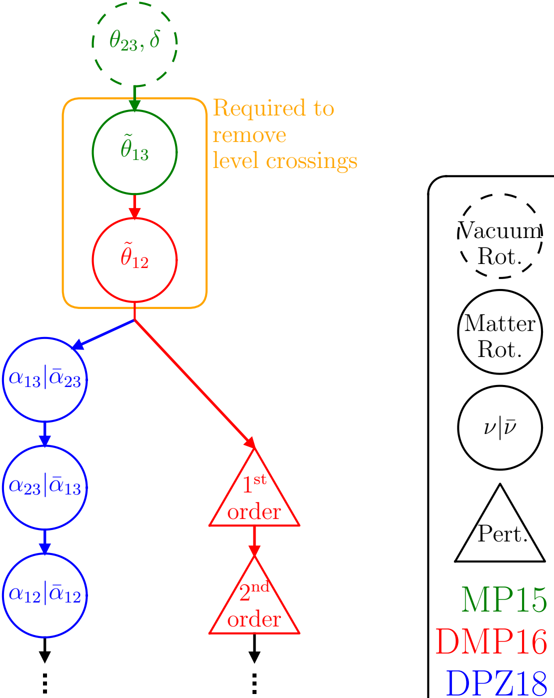
- 2 additional rotations
  $\equiv$ 1 order of perturbation theory



Even More Rotations

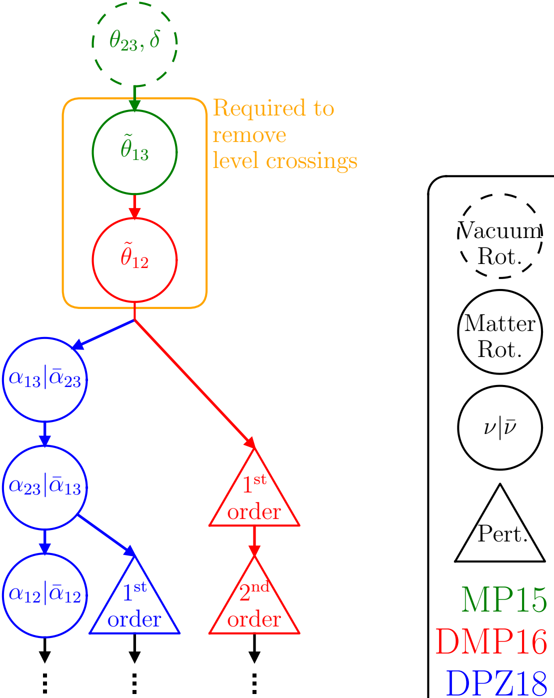
6. 1-2 sector for either $\nu/\bar{\nu}$

- 3 additional rotations $\equiv$ 2 orders of pert. th.



More Options

6. Perturbation theory after 2 additional rotations



- 2 additional rotations
  + 1 order of pert. th.
  ≡ 2 orders of pert. th.

Angles in Matter

Angles receive corrections at first order:

$$\widetilde{\theta}_{12}^{(1)} = \epsilon' \Delta m^2_{ee} s_{\widetilde{12}} c_{\widetilde{12}} \left(\frac{1}{\widetilde{\Delta m^2}_{32}} - \frac{1}{\widetilde{\Delta m^2}_{31}} \right)$$

$$\widetilde{\theta}_{13}^{(1)} = -\epsilon' \Delta m^2_{ee} \frac{s_{\widetilde{13}}}{c_{\widetilde{13}}} \left(\frac{s^2_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} + \frac{c^2_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} \right)$$

$$\widetilde{\theta}_{23}^{(1)} = \epsilon' \Delta m^2_{ee} \frac{c_\delta}{c_{\widetilde{13}}} \left(\frac{s^2_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} + \frac{c^2_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} \right)$$

$$\widetilde{\delta}^{(1)} = -\epsilon' \Delta m^2_{ee} \frac{2 c_{2\widetilde{23}} s_\delta}{s_{2\widetilde{23}} c_{\widetilde{13}}} \left(\frac{s^2_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} + \frac{c^2_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} \right)$$

Second order: see paper

Verifying the CPV Term in Matter

The amount of CPV is

$$J \sin\Delta_{21} \sin\Delta_{31} \sin\Delta_{32}$$

where the Jarlskog is

$$J = 8c_{12}s_{12}c_{13}^2 s_{13}c_{23}s_{23}s_\delta$$

C. Jarlskog: PRL 55 (1985)

The exact term in matter is known to be

$$\frac{\widehat{J}}{J} = \frac{\Delta m^2_{21}\Delta m^2_{31}\Delta m^2_{32}}{\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

V. Naumov, Int. J. Mod. Phys. 1992

P. Harrison, W. Scott, hep-ph/9912435

Our expression reproduces this order by order in ϵ' for all channels.

Is DMP the best?

Is DMP the best?

yes

Other Expressions

We were not the first to examine this problem.

- Madrid: drop $\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ and s_{13} terms; ∼sum of two amplitudes

$$P_{\mu e} = 4s^2_{23}s^2_{13}c^2_{13}\left(\frac{\Delta m^2_{31}}{b}\right)^2 \sin^2\Delta_b + 4c^2_{23}s^2_{12}c^2_{12}\left(\frac{\Delta m^2_{21}}{a}\right)^2 \sin^2\Delta_a$$
$$+ 8J_r\frac{\Delta m^2_{21}}{a}\frac{\Delta m^2_{31}}{b}\sin\Delta_a\sin\Delta_b\cos\left(\delta+\Delta_{31}\right)\,, \quad b = a - \Delta m^2_{31}$$

A. Cervera, et al., hep-ph/0002108

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A. Cervera, et al., hep-ph/0002108

E. Akhmedov, et al., hep-ph/0402175

A. Friedland, C. Lunardini, hep-ph/0606101

H. Nunokawa, S. Parke, J. Valle, 0710.0554

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A. Cervera, et al., hep-ph/0002108

E. Akhmedov, et al., hep-ph/0402175

A. Friedland, C. Lunardini, hep-ph/0606101

H. Nunokawa, S. Parke, J. Valle, 0710.0554

- AKT: from mass basis rotated 12 then 23 converted into 13
  - Δm^2_{ee} appears all over the expressions

S. Agarwalla, Y. Kao, T. Takeuchi, 1302.6773

Other Expressions

- AM: Powers of $s_{13}^2 \simeq \frac{\Delta m_{21}^2}{\Delta m_{31}^2}$ through the 5/2 order

  K. Asano, H. Minakata, 1103.4387

- Various other expressions

  J. Arafune, M. Koike, J. Sato, hep-ph/9703351

  M. Freund, hep-ph/0103300

  E. Akhmedov, et al., hep-ph/0402175

Others...

Which is best?

Other Expressions

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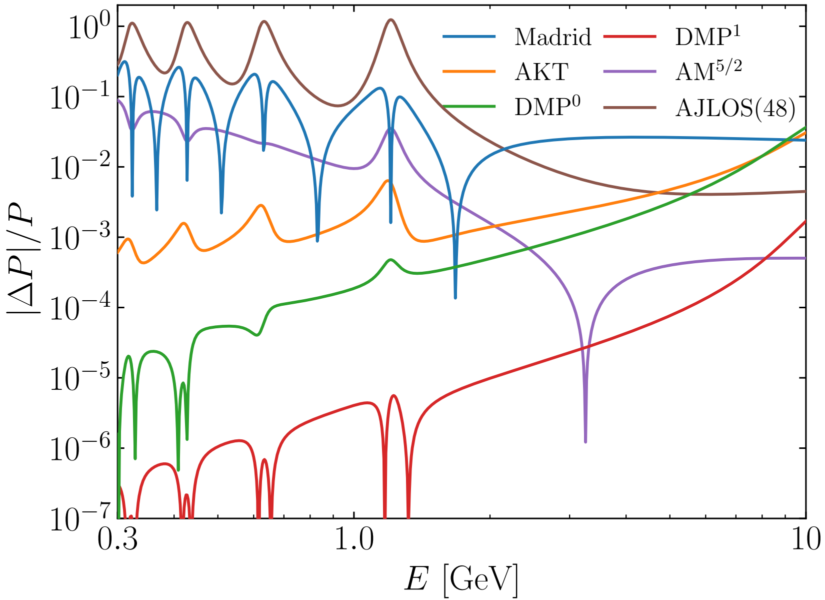
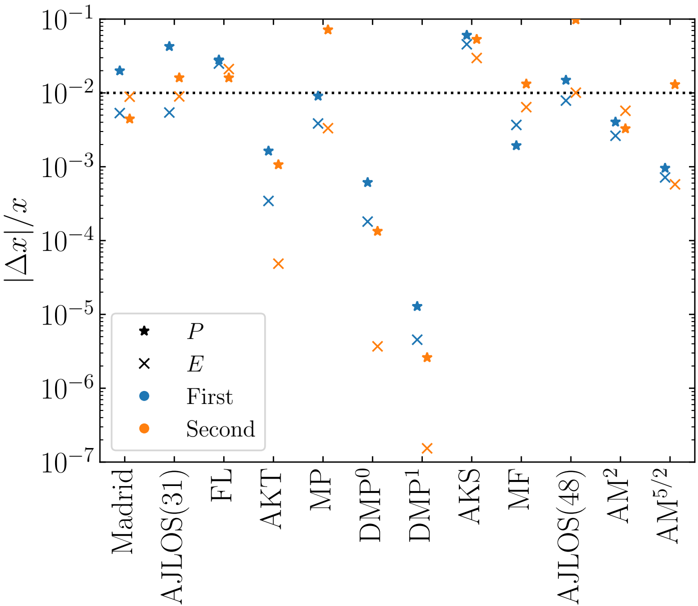
Others...

Which is best?
What does "best" mean?

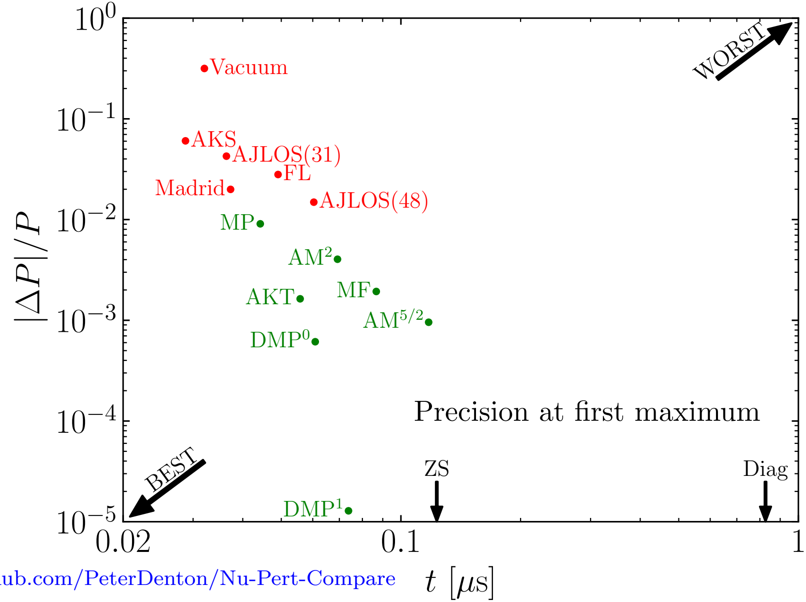
Comparative Precision ($L = 1300$ km)



Comparative Precision: At the Peaks



Speed ∝̃ Simplicity



github.com/PeterDenton/Nu-Pert-Compare

Proper Expansions

Parameter x is an *expansion parameter* iff

$$\lim_{x\to 0} P_{\text{approx}}(x) = P_{\text{exact}}(x=0)$$

	ϵ	s_{13}	$a/\Delta m^2_{31}$	
Madrid(like)	✗	✗	✗	Cervera+, hep-ph/0002108
AKT	✓	✓	✓	Agarwalla+, 1302.6773
MP	✓	✗	✗	Minakata, Parke, 1505.01826
DMP	✓	✓	✓	PBD+, 1604.08167
AKS	✗	✗	✗	Arafune+, hep-ph/9703351
MF	✓	✗	✗	Freund, hep-ph/0103300
AJLOS(48)	✓	✗	✗	Akhmedov+, hep-ph/0402175
AM	✗	✗	✗	Asano, Minakata, 1103.4387

Comparative Review

- Many expressions in the literature (12 considered)
- Most are not at the 1% level
- Most are not exact in vacuum
- Changing the basis to remove level crossings seems best
  - AKT, (MP), DMP
  - Δm^2_{ee} naturally appears (regardless of the name)
- The order of rotations matters:
  - Constant 23 rotation, then in matter: 13, 12
- First order DMP corrections are quite simple

The Effective Δm^2_{ee} in Matter

$$\Delta m^2_{ee} = c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32}$$

Δm^2_{ee} is an important quantity for understanding oscillations:

- Optimal expression for reactor experiments

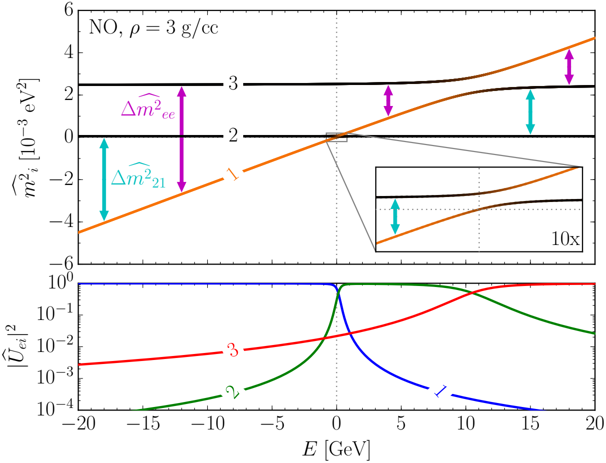
  H. Nunokawa, S. Parke, R. Zukanovich, hep-ph/0503283

  S. Parke, 1601.07464

- Shows up naturally in DMP on long-baseline matter effect

How does Δm^2_{ee} evolve in matter?

Asymptotic Evolution of $\Delta\widehat{m^2}_{ee}$



$$\Delta\widehat{m^2}_{ee} = \begin{cases} \widehat{m^2}_3 - \widehat{m^2}_1 & E \to -\infty \\ \widehat{m^2}_3 - \widehat{m^2}_2 & E \to +\infty \end{cases}$$

Intermediate Evolution of $\Delta\widehat{m^2}_{ee}$

$$\Delta\widehat{m^2}_{ee} = \begin{cases} \widehat{m^2}_3 - \widehat{m^2}_1 & E \to -\infty \\ \widehat{m^2}_3 - \widehat{m^2}_2 & E \to +\infty \end{cases}$$

Since $\widehat{m^2}_2(E \to -\infty) = \widehat{m^2}_1(E \to +\infty) =$ constant,
call $m_0^2 \equiv \Delta m^2_{21} c^2_{12}$

Now we can define

$$\Delta\widehat{m^2}_{ee} \equiv \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - m_0^2)$$

$$\Delta\widehat{m^2}_{ee} - \Delta m^2_{ee} = (\widehat{m^2}_3 - m_3^2) - (\widehat{m^2}_1 - m_1^2) - (\widehat{m^2}_2 - m_2^2)$$

Easy to see that $\Delta\widehat{m^2}_{ee}(E = 0) = \Delta m^2_{ee}$

Relationship to vacuum expression?

Relationship to Vacuum Expression

In vacuum we can equivalently write:

$$\Delta m^2_{ee} = \begin{cases} c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32} \\ m^2_3 - (m^2_1 + m^2_2 - m^2_0) \end{cases}$$

Elevate everything to matter equivalent,
except m^2_0 which we know we want to be a constant.

$$\Delta\widehat{m^2}_{ee} = \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - m^2_0)$$

$$\Delta\widehat{m^2}_{EE} = c^2_{\widehat{12}}\Delta\widehat{m^2}_{31} + s^2_{\widehat{12}}\Delta\widehat{m^2}_{32}$$

The difference between these similar formulas:

$$\Delta_{Ee} = \widehat{m^2}_1 + c^2_{\widehat{12}}\Delta\widehat{m^2}_{21} - c^2_{12}\Delta m^2_{21}$$

Use DMP!

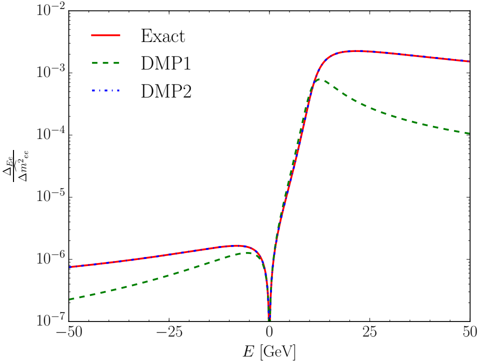
At zeroth order: $\Delta_{Ee}^{(0)} = 0$

At first order, only correction is to $\widetilde{\theta}_{12}$,

PBD, S. Parke, X. Zhang, 1806.01277

$$\Delta_{Ee}^{(1)} = t_{\widetilde{13}} s_{12}^2 c_{12}^2 \sin 2\theta_{13} a \frac{(\Delta m_{21}^2)^2}{\widetilde{\Delta m^2}_{32} \widetilde{\Delta m^2}_{31}}$$

At second order eigenvalues are also corrected, $\Delta_{Ee}^{(2)} = \cdots$



Error is quantified with DMP^2:

- First order isn't enough, ...
  - Second is
- Exact in vacuum

A Third Option

Avoid $\cos(\frac{1}{3}\cos^{-1}\cdots)$, use DMP:

$$\widetilde{\Delta m^2}_{ee,\mathrm{DMP}} \equiv \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

Which is best?

What does *best* mean?

Using one $\Delta m^2 \Rightarrow$ using a two-flavor picture:

$$P_{ee} \approx 1 - \sin^2 2\widehat{\theta}_{13} \sin^2 \frac{\Delta\widehat{m^2}_{ee} L}{4E}$$

$\Rightarrow$ want the first minimum correct

Take exact expression at $dP/dL = 0$ for a given E, then

$$\frac{\Delta\widehat{m^2}_{ee} L}{4E} = \frac{\pi}{2}$$

What does *best* mean?

Using one $\Delta m^2 \Rightarrow$ using a two-flavor picture:

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$\Rightarrow$ want the first minimum correct

Take exact expression at $dP/dL = 0$ for a given E, then

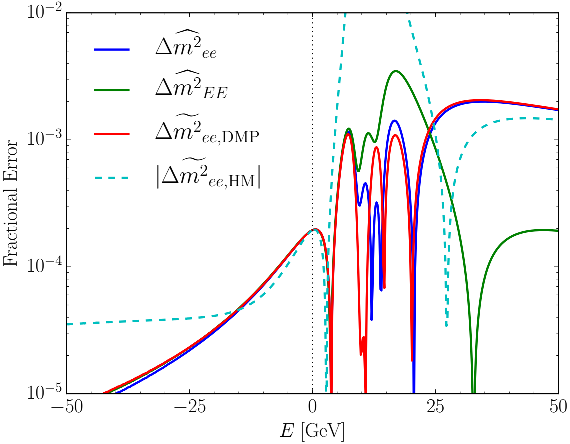
$$\frac{\Delta\widehat{m^2}_{ee} L}{4E} = \frac{\pi}{2}$$

Could do $dP/dE = 0$ for a given L,

H. Minakata, 1702.03332

but L/E show up together except where $a = a(E)$ appears,
and both $\widehat{\theta}_{13}$, $\Delta\widehat{m^2}_{ee}$ are complicated functions of a.

Comparison of Two-Flavor Precision



The $\Delta\widehat{m^2}_{ee}$ expression also leads to a simple rewriting of the eigenvalues.

HM: H. Minakata, 1702.03332
21 term in probability not included

The winner is: $\Delta\widehat{m^2}_{ee} \equiv \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - m_0^2)$!

Precision is better than 0.06%

Depth of Oscillations

The depth of the minimum is well-described by

$$\sin^2 2\widehat{\theta_{13}} \approx \sin^2 2\theta_{13}\left(\frac{\Delta m^2_{ee}}{\widehat{\Delta m^2}_{ee}}\right)^2$$

$$\approx \frac{\sin^2 2\theta_{13}}{(\cos^2 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

Using DMP

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$$\sin^2 2\widehat{\theta}_{13} \approx \sin^2 2\theta_{13}\left(\frac{\Delta m^2_{ee}}{\Delta\widehat{m^2}_{ee}}\right)^2$$

$$\approx \frac{\sin^2 2\theta_{13}}{(\cos^2 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

Using DMP

The disappearance probability in matter is well described by

$$P_{ee} \approx 1 - \sin^2 2\theta_{13}\left(\frac{\Delta m^2_{ee}}{\Delta\widehat{m^2}_{ee}}\right)^2 \sin^2\frac{\Delta\widehat{m^2}_{ee}L}{4E}$$

$$\Delta\widehat{m^2}_{ee} \equiv \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2) - [m^2_3 - (m^2_1 + m^2_2)] + \Delta m^2_{ee}$$

Key Points

- Include 1^{st} order corrections in 0^{th} order eigenvalues (Δm^2_{ee})
- Rotate **large terms first** $\Rightarrow$ PMNS order, removes level crossings
- 0^{th} order probabilities: **same structure as vacuum** probabilities
- 0^{th} order: **accurate** enough for current & future experiments
- Δm^2_{ee} is the optimal Δm^2_{atm} in both reactor and long-baseline
- DMP is the most precise while just as simple

Backups

Neutrino Oscillations in Vacuum: Disappearance

It is easy to calculate the *exact* disappearance expression in vacuum:

$$P(\nu_\alpha \to \nu_\alpha) = 1 - 4\sum_{i<j} |U_{\alpha i}|^2 |U_{\alpha j}|^2 \sin^2 \Delta_{ji}$$

For the electron case this expression is simple:

$$\begin{aligned} P(\nu_e \to \nu_e) = 1 & \\ & - 4c_{12}^2 s_{12}^2 c_{13}^4 \sin^2 \Delta_{21} \\ & - 4c_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{31} \\ & - 4s_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{32} \end{aligned}$$

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$
$$\Delta m_{ij}^2 = m_i^2 - m_j^2$$

A Simple Solution

For two-flavor oscillations:

$$P(\nu_\alpha \to \nu_\beta) = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

- Solar: θ_{21}, Δm^2_{21}
- Reactor: θ_{13}, Δm^2_{ee}

Alternative Solutions: Example

$$P_0 = \sin^2\theta_{23}\frac{\sin^2 2\theta_{13}}{\hat{C}^2}\sin^2(\hat{\Delta}\hat{C}), \tag{36a}$$

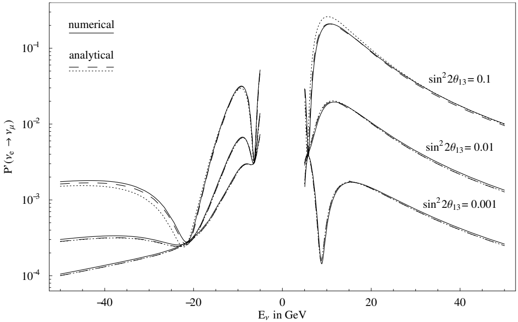
$$P_{\sin\delta} = \frac{1}{2}\alpha\frac{\sin\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2}\sin(\hat{C}\hat{\Delta}) \times\{\cos(\hat{C}\hat{\Delta}) - \cos((1+\hat{A})\hat{\Delta})\}, \tag{36b}$$

$$P_{\cos\delta} = \frac{1}{2}\alpha\frac{\cos\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2}\sin(\hat{C}\hat{\Delta}) \times\{\sin((1+\hat{A})\hat{\Delta}) \mp \sin(\hat{C}\hat{\Delta})\}, \tag{36c}$$

$$P_1 = -\alpha\frac{1-\hat{A}\cos 2\theta_{13}}{\hat{C}^3}\sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\theta_{23}\hat{\Delta} \times\sin(2\hat{\Delta}\hat{C}) + \alpha\frac{2\hat{A}(-\hat{A}+\cos 2\theta_{13})}{\hat{C}^4} \times\sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\theta_{23}\sin^2(\hat{\Delta}\hat{C}), \tag{36d}$$

$$P_2 = \alpha\frac{\mp 1+\hat{C}\pm\hat{A}\cos 2\theta_{13}}{2\hat{C}^2\hat{A}\cos^2\theta_{13}}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13} \times\sin 2\theta_{23}\sin^2(\hat{\Delta}\hat{C}), \tag{36e}$$

$$P_3 = \alpha^2\frac{2\hat{C}\cos^2\theta_{23}\sin^2 2\theta_{12}}{\hat{A}^2\cos^2\theta_{13}(\mp\hat{A}+\hat{C}\pm\cos 2\theta_{13})} \times\sin^2\left(\frac{1}{2}(1+\hat{A}\mp\hat{C})\hat{\Delta}\right). \tag{36f}$$



M. Freund, hep-ph/0103300

Our Methodology

- Start with $\epsilon = \frac{\Delta m^2_{21}}{\Delta m^2_{ee}} = 0.03$
- Perform one fixed and two variable rotations: (θ_{23}, δ), $\widetilde{\theta}_{13}$, $\widetilde{\theta}_{12}$
- Write the probabilities with simple L/E dependence:

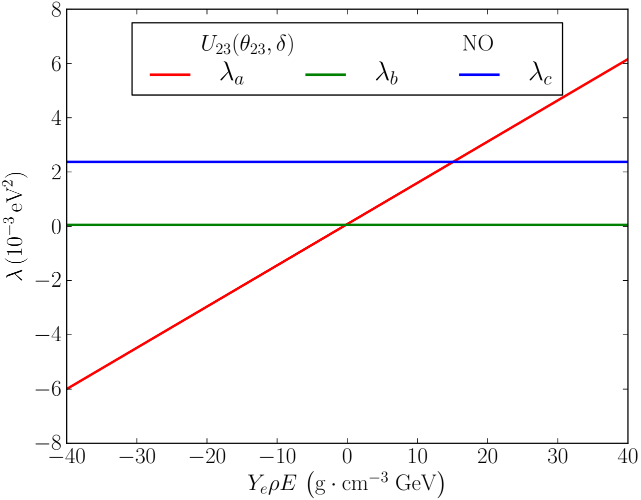
$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \sum_{i<j} \Re\left[U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}\right]\sin^2\Delta_{ij}$$
$$+ 8\Im\left[U_{\alpha 1}U^*_{\beta 2}U^*_{\alpha 2}U_{\beta 1}\right]\sin\Delta_{32}\sin\Delta_{31}\sin\Delta_{21}$$

C. Jarlskog: PRL 55 (1985)

Nonvanishing Wronskian $\Rightarrow$ fewest number of L/E functions

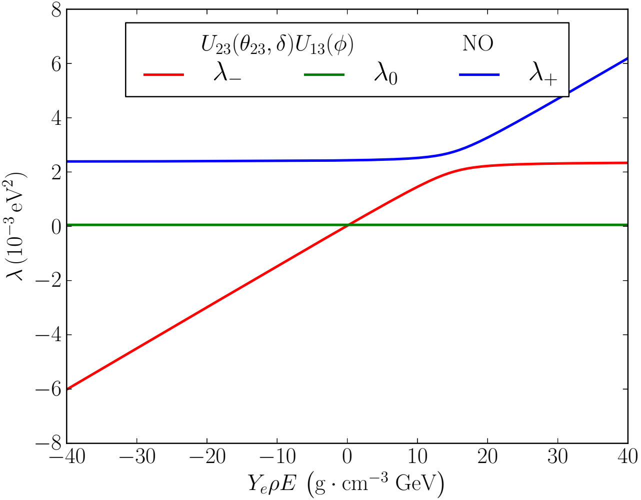
Clear that the CPV term is $\mathcal{O}[(L/E)^3]$ not $\mathcal{O}[(L/E)^1]$

Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_a = a + (s_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2\,,\ \widetilde{m^2}_b = \epsilon c_{12}^2 \Delta m_{ee}^2\,,\ \widetilde{m^2}_c = (c_{13}^2 + \epsilon s_{12}^2)\Delta m_{ee}^2$$

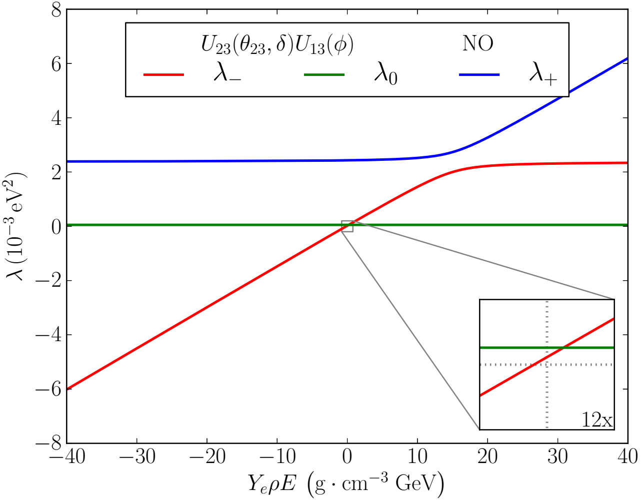
Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_{\mp} = \frac{1}{2}\left[(\widetilde{m^2}_a + \widetilde{m^2}_c) \mp \mathrm{sgn}(\Delta m^2_{ee})\sqrt{(\widetilde{m^2}_c - \widetilde{m^2}_a)^2 + (2 s_{13} c_{13} \Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_0 = \widetilde{m^2}_b$$

Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m^2}_{\mp} = \frac{1}{2}\left[(\widetilde{m^2}_a + \widetilde{m^2}_c) \mp \mathrm{sgn}(\Delta m^2_{ee})\sqrt{(\widetilde{m^2}_c - \widetilde{m^2}_a)^2 + (2s_{13}c_{13}\Delta m^2_{ee})^2}\right]$$

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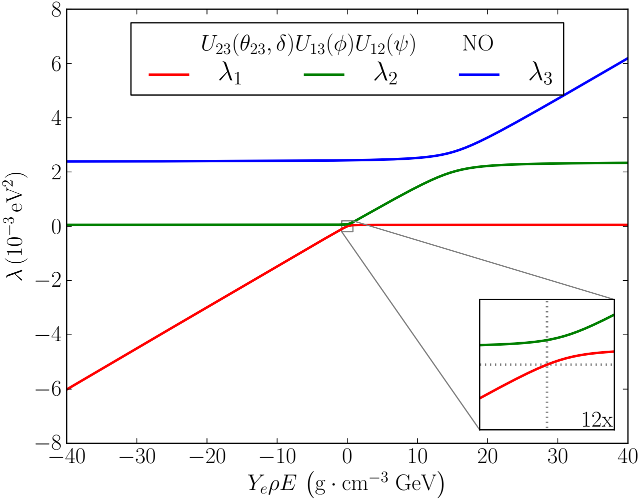
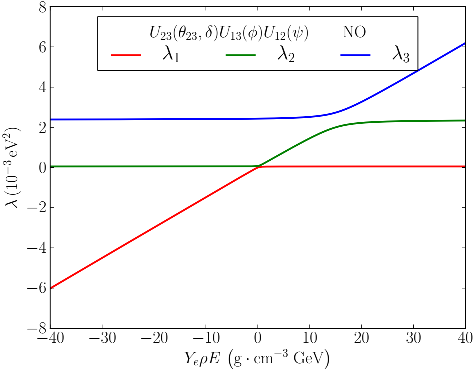
Eigenvalues in Matter: Two Rotations are Needed

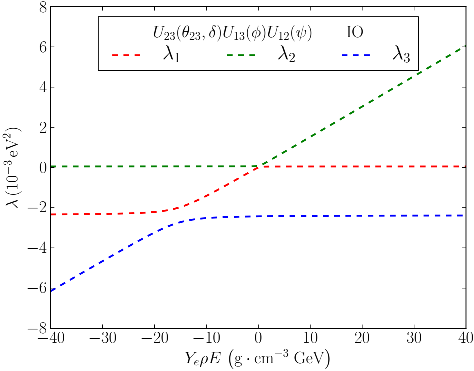


$$\widetilde{m^2}_{1,2} = \frac{1}{2}\left[(\widetilde{m^2}_0 + \widetilde{m^2}_-) \mp \sqrt{(\widetilde{m^2}_0 - \widetilde{m^2}_-)^2 + (2\epsilon c_{(\widetilde{\theta}_{13}-\theta_{13})} c_{12} s_{12} \Delta m^2_{ee})^2}\right]$$

$$\widetilde{m^2}_3 = \widetilde{m^2}_+$$

Eigenvalues in Matter: Mass Ordering





NO

$$\widetilde{m^2}_1 < \widetilde{m^2}_2 < \widetilde{m^2}_3$$

IO

$$\widetilde{m^2}_3 < \widetilde{m^2}_1 < \widetilde{m^2}_2$$

$1+2$ Rotations

1. Perform a constant $U_{23}(\theta_{23},\delta)$ rotation
   - U_{23} commutes with the matter potential
   - Resultant Hamiltonian is real
   - 'Expansion parameter' is $c_{13}s_{13}=0.15$ at this point
2. Diagonalize the diagonal and $\mathcal{O}(\epsilon^0)$ off-diagonal terms with $U_{13}(\widetilde{\theta}_{13})$
   - $\widetilde{\theta}_{13}(a=0)=\theta_{13}$
   - Expansion parameter is $c_{12}s_{12}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}}=0.015$

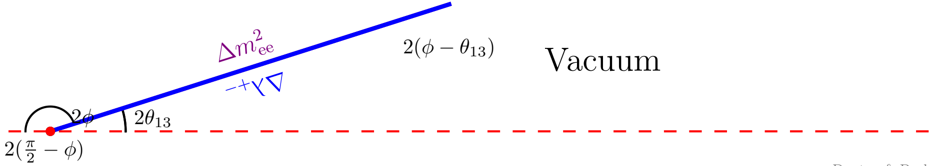
H. Minakata, S. Parke, 1505.01826

3. Diagonalize the terms non-zero in vacuum with $U_{12}(\widetilde{\theta}_{12})$
   - $\widetilde{\theta}_{12}(a=0)=\theta_{12}$
   - Expansion parameter is now $\epsilon'=c_{12}s_{12}s_{(\widetilde{\theta}_{13}-\theta_{13})}\frac{\Delta m^2_{21}}{\Delta m^2_{ee}}<0.015$
   - $\epsilon'(a=0)=0$

31 Triangle

$$\frac{a}{\Delta m^2_{ee}\cos 2\theta_{13}} = 0.0$$

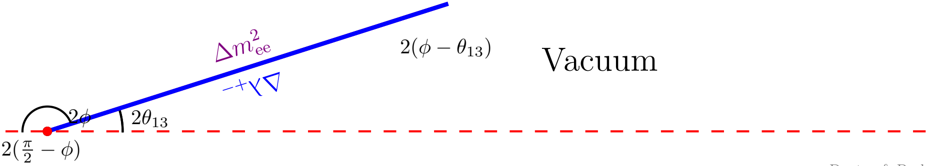
Vacuum



Denton & Parke

31 Triangle

$$\frac{a}{\Delta m^2_{ee}\cos 2\theta_{13}} = 0.0$$

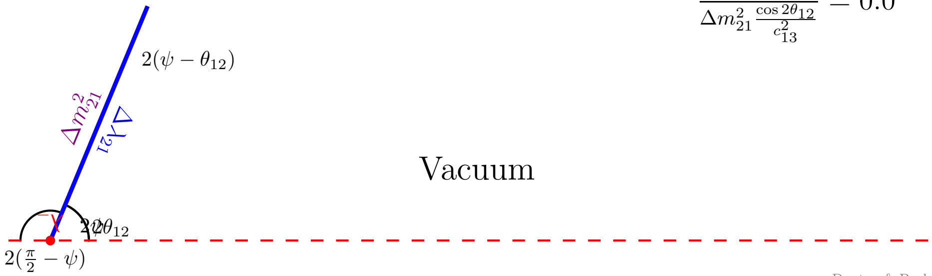


Vacuum

Denton & Parke

21 Triangle

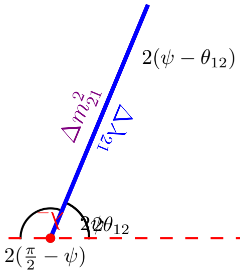
$$\frac{a}{\Delta m^2_{21}\frac{\cos 2\theta_{12}}{c^2_{13}}} = 0.0$$



Vacuum

Denton & Parke

21 Triangle



$$\frac{a}{\Delta m^2_{21}\frac{\cos 2\theta_{12}}{c^2_{13}}}=0.0$$

Vacuum

Denton & Parke

Exact Neutrino Oscillations in Matter: Mixing Angles

$$s^2_{\widehat{12}} = \frac{-\left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widetilde{m^2}_{31}}{\left[(\widehat{m^2}_1)^2 - \alpha\widehat{m^2}_1 + \beta\right]\Delta\widetilde{m^2}_{32} - \left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widetilde{m^2}_{31}}$$

$$s^2_{\widehat{13}} = \frac{(\widehat{m^2}_3)^2 - \alpha\widehat{m^2}_3 + \beta}{\Delta\widetilde{m^2}_{31}\Delta\widetilde{m^2}_{32}}$$

$$s^2_{\widehat{23}} = \frac{s^2_{23}E^2 + c^2_{23}F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\widehat{\delta}} = \frac{c^2_{23}s^2_{23}\left(e^{-i\delta}E^2 - e^{i\delta}F^2\right) + \left(c^2_{23} - s^2_{23}\right)EF}{\sqrt{\left(s^2_{23}E^2 + c^2_{23}F^2 + 2EFc_{23}s_{23}c_\delta\right)\left(c^2_{23}E^2 + s^2_{23}F^2 - 2EFc_{23}s_{23}c_\delta\right)}}$$

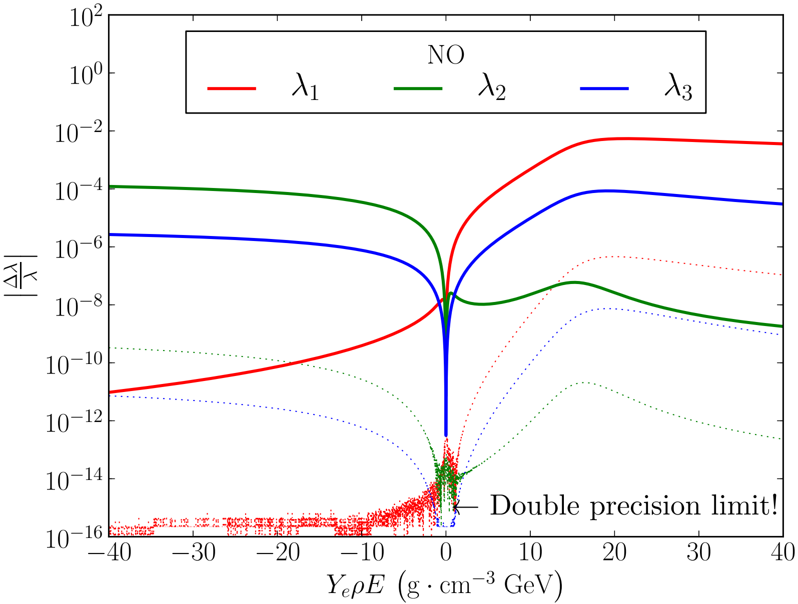
$$\alpha = c^2_{13}\Delta m^2_{31} + \left(c^2_{12}c^2_{13} + s^2_{13}\right)\Delta m^2_{21}\,,\ \ \beta = c^2_{12}c^2_{13}\Delta m^2_{21}\Delta m^2_{31}$$

$$E = c_{13}s_{13}\left[\left(\widehat{m^2}_3 - \Delta m^2_{21}\right)\Delta m^2_{31} - s^2_{12}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}\right]$$

$$F = c_{12}s_{12}c_{13}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}$$

H. Zaglauer, K. Schwarzer, Z. Phys. C Particles and Fields, 40 (1988)

Eigenvalues: Precision



Hamiltonians

After a constant (θ_{23},δ) rotation, $2E\tilde{H}=$

$$\begin{pmatrix}\widetilde{m^2}_a & & s_{13}c_{13}\Delta m^2_{ee}\\ & \widetilde{m^2}_b & \\ s_{13}c_{13}\Delta m^2_{ee} & & \widetilde{m^2}_c\end{pmatrix}+\epsilon s_{12}c_{12}\Delta m^2_{ee}\begin{pmatrix} & c_{13} & \\ c_{13} & & -s_{13}\\ & -s_{13} & \end{pmatrix}$$

After a $U_{13}(\widetilde{\theta}_{13})$ rotation, $2E\hat{H}=$

$$\begin{pmatrix}\widetilde{m^2}_- & & \\ & \widetilde{m^2}_0 & \\ & & \widetilde{m^2}_+\end{pmatrix}+\epsilon c_{12}s_{12}\Delta m^2_{ee}\begin{pmatrix} & c_{(\widetilde{\theta}_{13}-\theta_{13})} & \\ c_{(\widetilde{\theta}_{13}-\theta_{13})} & & s_{(\widetilde{\theta}_{13}-\theta_{13})}\\ & s_{(\widetilde{\theta}_{13}-\theta_{13})} & \end{pmatrix}$$

After a $U_{12}(\widetilde{\theta}_{12})$ rotation, $2E\check{H}=$

$$\begin{pmatrix}\widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3\end{pmatrix}+\epsilon s_{(\widetilde{\theta}_{13}-\theta_{13})}s_{12}c_{12}\Delta m^2_{ee}\begin{pmatrix} & & -s_{\widetilde{12}}\\ & & c_{\widetilde{12}}\\ -s_{\widetilde{12}} & c_{\widetilde{12}} & \end{pmatrix}$$

Perturbative Expansion

Hamiltonian: $\check{H} = \check{H}_0 + \check{H}_1$

$$\check{H}_0 = \frac{1}{2E}\begin{pmatrix} \widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3 \end{pmatrix}\,, \quad \check{H}_1 = \epsilon' \frac{\Delta m^2_{ee}}{2E}\begin{pmatrix} & & -s_{\widetilde{12}} \\ & & c_{\widetilde{12}} \\ -s_{\widetilde{12}} & c_{\widetilde{12}} & \end{pmatrix}$$

Eigenvalues: $\widetilde{m^2}_i^{\text{ex}} = \widetilde{m^2}_i + \widetilde{m^2}_i^{(1)} + \widetilde{m^2}_i^{(2)} + \ldots$

$$\widetilde{m^2}_i^{(1)} = 2E(\check{H}_1)_{ii} = 0$$

$$\widetilde{m^2}_i^{(2)} = \sum_{k \neq i} \frac{[2E(\check{H}_1)_{ik}]^2}{\Delta\widetilde{m^2}_{ik}}$$

Perturbative Expansion: Eigenvectors

Use vacuum expressions with $U \to V$ where

$$V = \widetilde{U} W$$

$\widetilde{U}$ is U with $\theta_{13} \to \widetilde{\theta}_{13}$ and $\theta_{12} \to \widetilde{\theta}_{12}$,

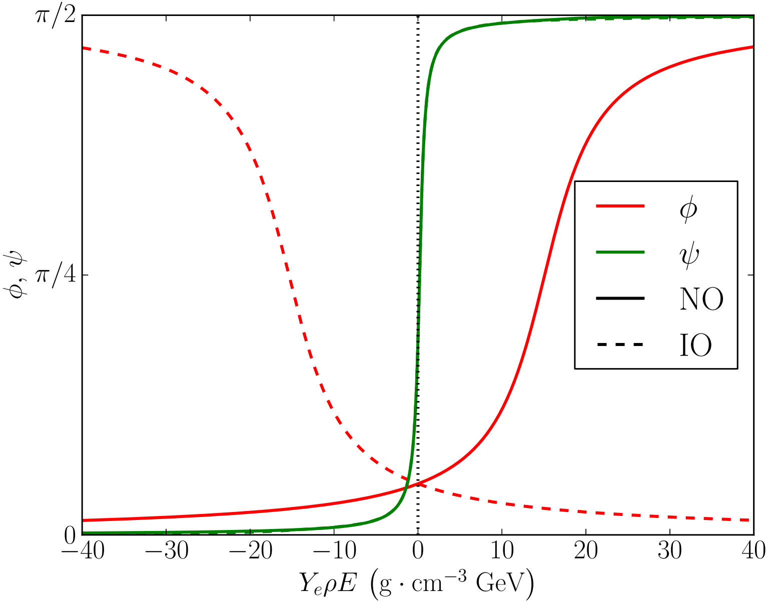
$W = W_0 + W_1 + W_2 + \ldots$

$$W_0 = \mathbb{1}$$

$$W_1 = \epsilon' \Delta m^2_{ee} \begin{pmatrix} & & -\frac{s_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} \\ & & \frac{c_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} \\ \frac{s_{\widetilde{12}}}{\widetilde{\Delta m^2}_{31}} & -\frac{c_{\widetilde{12}}}{\widetilde{\Delta m^2}_{32}} & \end{pmatrix}$$

$$W_2 = -\epsilon'^2 \frac{(\Delta m^2_{ee})^2}{2} \begin{pmatrix} \frac{s^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{31})^2} & -\frac{s_{2\widetilde{12}}}{\widetilde{\Delta m^2}_{32} \widetilde{\Delta m^2}_{21}} & \\ \frac{s_{2\widetilde{12}}}{\widetilde{\Delta m^2}_{31} \widetilde{\Delta m^2}_{21}} & \frac{c^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{32})^2} & \\ & & \left[\frac{c^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{32})^2} + \frac{s^2_{\widetilde{12}}}{(\widetilde{\Delta m^2}_{31})^2} \right] \end{pmatrix}$$

The Two Matter Angles



Zeroth Order Coefficients

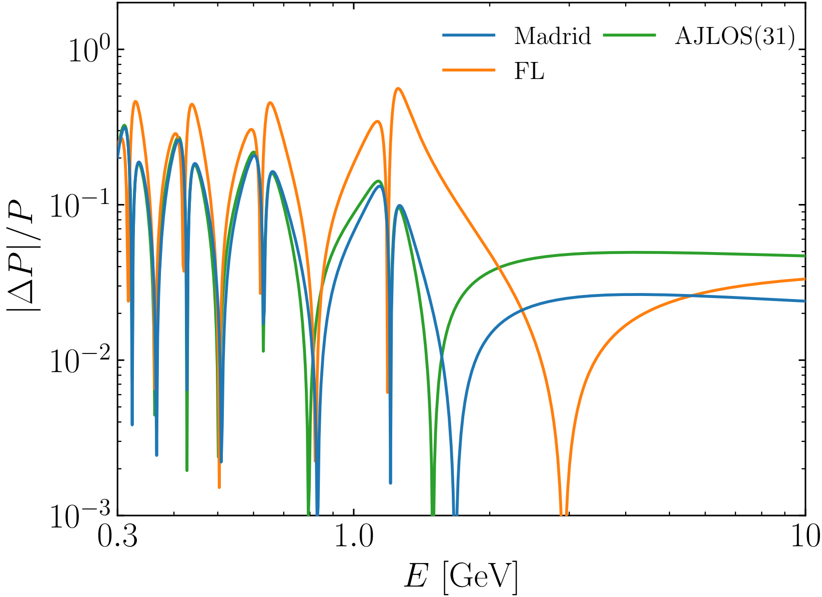
$$P_{\alpha\beta} = \delta_{\alpha\beta} + 4C_{21}^{\alpha\beta}\sin^2\Delta_{21} + 4C_{31}^{\alpha\beta}\sin^2\Delta_{31} + 4C_{32}^{\alpha\beta}\sin^2\Delta_{32} + 8D^{\alpha\beta}\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32}$$

$\nu_\alpha \to \nu_\beta$	$(C_{21}^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_{\widetilde{13}}^4 s_{\widetilde{12}}^2 c_{\widetilde{12}}^2$
$\nu_\mu \to \nu_e$	$c_{\widetilde{13}}^2 s_{\widetilde{12}}^2 c_{\widetilde{12}}^2 (c_{23}^2 - s_{\widetilde{13}}^2 s_{23}^2) + c_{2\widetilde{12}} J_r^m c_\delta$
$\nu_\mu \to \nu_\mu$	$-(c_{23}^2 c_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 s_{\widetilde{12}}^2)(c_{23}^2 s_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2)$ $-2(c_{23}^2 - s_{\widetilde{13}}^2 s_{23}^2) c_{2\widetilde{12}} J_{rr}^m c_\delta + (2J_{rr}^m c_\delta)^2$

$\nu_\alpha \to \nu_\beta$	$(C_{31}^{\alpha\beta})^{(0)}$	$(D^{\alpha\beta})^{(0)}$
$\nu_e \to \nu_e$	$-c_{\widetilde{13}}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2$	0
$\nu_\mu \to \nu_e$	$s_{\widetilde{13}}^2 c_{\widetilde{13}}^2 c_{\widetilde{12}}^2 s_{23}^2 + J_r^m c_\delta$	$-J_r^m s_\delta$
$\nu_\mu \to \nu_\mu$	$-c_{\widetilde{13}}^2 s_{23}^2 (c_{23}^2 s_{\widetilde{12}}^2 + s_{23}^2 s_{\widetilde{13}}^2 c_{\widetilde{12}}^2)$ $-2s_{23}^2 J_r^m c_\delta$	0

$$J_r^m \equiv s_{\widetilde{12}} c_{\widetilde{12}} s_{\widetilde{13}} c_{\widetilde{13}}^2 s_{23} c_{23}, \quad J_{rr}^m \equiv J_r^m / c_{\widetilde{13}}^2$$

Comparative Precision



Comparative Precision