

## Abstract

As DUNE and T2HK ramp up their efforts, it is a good time to examine what the oscillation probabilities actually are. I will develop a framework for neutrino oscillations in matter that leads to simple and precise expressions. These expressions are sufficiently accurate for current, planned, and proposed oscillation experiments. Further improvements to these expressions can be derived via either changing the basis or perturbation theory, or both. While other expressions exist on the market, I will show how these expressions are significantly more precise and as simple. I will explore how  $\Delta m_{ee}^2$  is modified in matter and how the previous techniques provide a simple and precise expression for this quantity as well. Finally, I will discuss recent results on understanding CP violation in the neutrino sector.

# Neutrino Oscillation Probabilities in Matter

Peter B. Denton

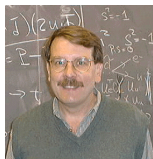
Caltech

May 20, 2019

**BROOKHAVEN**  
NATIONAL LABORATORY



# Analytic Oscillation Probability Collaborators



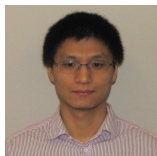
Stephen Parke



Hisakazu Minakata



Gabriela Barenboim



Xining Zhang



Christoph Ternes

1604.08167, 1806.01277, 1808.09453, 1902.00517,  
1902.07185

[github.com/PeterDenton/Nu-Pert](https://github.com/PeterDenton/Nu-Pert)

[github.com/PeterDenton/Nu-Pert-Compare](https://github.com/PeterDenton/Nu-Pert-Compare)

# Neutrino Oscillation Parameters Status

Six parameters:

1.  $\theta_{13} = (8.6 \pm 0.1)^\circ$
2.  $\theta_{12} = (33.8 \pm 0.8)^\circ$
3.  $\Delta m_{21}^2 = (7.4 \pm 0.2) \times 10^{-5} \text{ eV}^2$
4.  $\theta_{23} \sim 45^\circ$  (octant)
5.  $|\Delta m_{31}^2| = (2.52 \pm 0.03) \times 10^{-3} \text{ eV}^2$  (mass ordering)
6.  $\delta = ???$

NuFIT, 1811.05487

PMNS order allows for easy measurement of  $\theta_{13}$  and  $\theta_{12}$ .

$\theta_{23}$  and  $\delta_{\text{CP}}$  require full three-flavor description.

# Analytic Oscillation Probabilities in Matter

- ▶ Solar:  $P_{ee} \simeq \sin^2 \theta_\odot$

Approx: S. Mikheev, A. Smirnov, [Nuovo Cim. C9 \(1986\) 17-26](#)

Exact: S. Parke, [PRL 57 \(1986\) 2322](#)

- ▶ Long-baseline: All three flavors

Exact: H. Zaglauer, K. Schwarzer, [Z.Phys. C40 \(1988\) 273](#)

Approx: [PBD](#), H. Minakata, S. Parke, [1604.08167](#)

- ▶  $\nu_e$  disappearance (nu storm):

$$\widehat{\Delta m^2_{ee}} = \widehat{m^2_3} - (\widehat{m^2_1} + \widehat{m^2_2} - \Delta m^2_{21} c^2_{12})$$

[PBD](#), S. Parke, [1808.09453](#)

- ▶ Atmospheric?

# The Several Billion Dollar Question

What is  $P(\nu_\mu \rightarrow \nu_e)$ ?

$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = |\mathcal{A}_{\mu e}|^2 \quad \mathcal{A}_{\mu e} = \textcolor{red}{A}_{31} + e^{\pm i\Delta_{32}} \textcolor{blue}{A}_{21}$$

$$\textcolor{red}{A}_{31} = 2s_{13}c_{13}s_{23} \sin \Delta_{31}$$

$$\textcolor{blue}{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23}) \sin \Delta_{21}$$

$$\Delta_{ij} = \Delta m_{ij}^2 L / 4E$$

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$$\Delta_{ij} = \Delta m_{ij}^2 L / 4E$$

...in matter?

Now: NOvA, T2K, MINOS, ...

Upcoming: DUNE, T2HK, ...

Second maximum: T2HKK? ESSnuSB? ...

# Neutrino oscillations in vacuum: appearance

$$\delta = 0.0\pi$$

$$\Delta_{32} = 0.5\pi$$

NO



	$A_{31}$
	$A_{21}$
	$A_{\mu e}$

$$P(\nu_\mu \rightarrow \nu_e) = A_{\mu e} A_{\mu e}^*$$

Denton & Parke

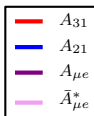
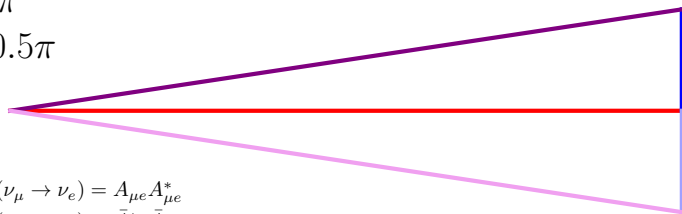


# Neutrino oscillations in vacuum: appearance

$$\delta = 0.0\pi$$

$$\Delta_{32} = 0.5\pi$$

NO



$$P(\nu_\mu \rightarrow \nu_e) = A_{\mu e} A_{\mu e}^*$$

$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \bar{A}_{\mu e}^* \bar{A}_{\mu e}$$

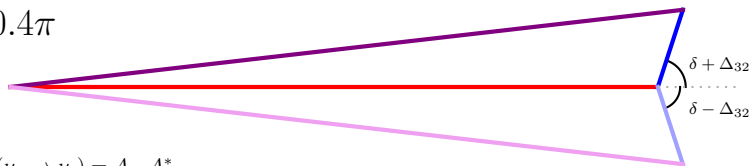
Denton & Parke

# Neutrino oscillations in vacuum: appearance

$$\delta = 0.0\pi$$

$$\Delta_{32} = 0.4\pi$$

NO



<span style="color: red;">—</span>	$A_{31}$
<span style="color: blue;">—</span>	$A_{21}$
<span style="color: purple;">—</span>	$A_{\mu e}$
<span style="color: magenta;">—</span>	$\bar{A}_{\mu e}^*$

$$P(\nu_\mu \rightarrow \nu_e) = A_{\mu e} A_{\mu e}^*$$

$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \bar{A}_{\mu e}^* \bar{A}_{\mu e}$$

Denton & Parke

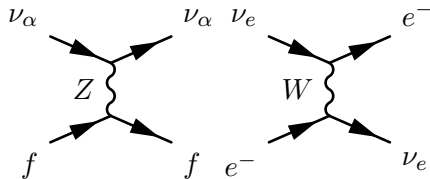
# Neutrino oscillations in vacuum: appearance

# Matter Effects Matter

Call Schrödinger equation's eigenvalues  $m_i^2$  and eigenvectors  $U_i$ .

$$\mathcal{A}(\nu_\alpha \rightarrow \nu_\beta) = \sum_i U_{\alpha i}^* U_{\beta i} e^{-im_i^2 L/2E} \quad P = |\mathcal{A}|^2$$

In **matter**  $\nu$ 's propagate in a **new basis** that depends on  $a \propto \rho E$ .



L. Wolfenstein, [PRD 17 \(1978\)](#)

Eigenvalues:  $m_i^2 \rightarrow \widehat{m_i^2}(a)$

Eigenvectors are given by  $\theta_{ij} \rightarrow \widehat{\theta_{ij}}(a) \quad \Leftarrow \quad \text{Unitarity}$

# Variable Matter Density

We assume  $\rho$  is constant. Is this okay?

If  $\rho$  varies only “slowly,” we can set  $\rho$  to the average:

$$\rho = \bar{\rho} = \frac{1}{L} \int_0^L \rho(x) dx$$

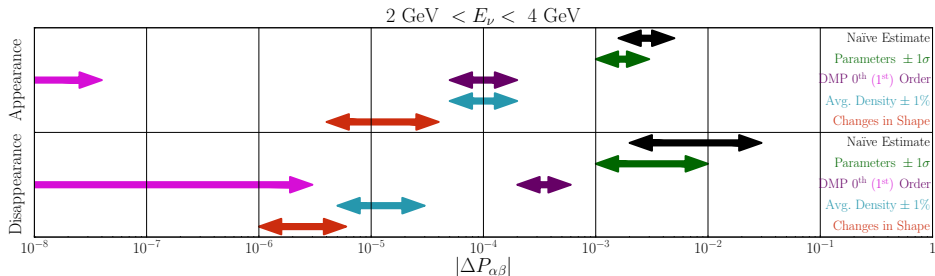
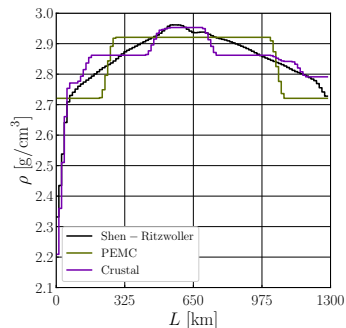
$\rho$  doesn't vary “too much” when

$$|\dot{\hat{\theta}}| \ll \left| \frac{\widehat{\Delta m^2}}{2E} \right|$$

True for DUNE?

# Variable Matter Density

This is a great approximation at DUNE: ✓!



K. Kelly, S. Parke, [1802.06784](https://arxiv.org/abs/1802.06784)

# Hamiltonian Dynamics

$$H = \frac{1}{2E} \left[ U \begin{pmatrix} 0 & & \\ & \Delta m_{21}^2 & \\ & & \Delta m_{31}^2 \end{pmatrix} U^\dagger + \begin{pmatrix} a & & \\ & 0 & \\ & & 0 \end{pmatrix} \right]$$

$$a = 2\sqrt{2}G_F N_e E$$

$$U = \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}$$

Find eigenvalues and eigenvectors:

$$H = \frac{1}{2E} \hat{U} \begin{pmatrix} 0 & & \\ & \widehat{\Delta m}_{21}^2 & \\ & & \widehat{\Delta m}_{31}^2 \end{pmatrix} \hat{U}^\dagger$$

Computationally works, but we can do better than a **black box**...

# Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation.

$$\widehat{m^2}_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}S - \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widehat{m^2}_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}S + \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widehat{m^2}_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2 - 3B}S$$

$$A = \Delta m_{21}^2 + \Delta m_{31}^2 + a$$

$$B = \Delta m_{21}^2 \Delta m_{31}^2 + a \left[ c_{13}^2 \Delta m_{31}^2 + (c_{12}^2 c_{13}^2 + s_{13}^2) \Delta m_{21}^2 \right]$$

$$C = a \Delta m_{21}^2 \Delta m_{31}^2 c_{12}^2 c_{13}^2$$

$$S = \cos \left\{ \frac{1}{3} \cos^{-1} \left[ \frac{2A^3 - 9AB + 27C}{2(A^2 - 3B)^{3/2}} \right] \right\}$$

H. Zaglauer, K. Schwarzer, [Z.Phys. C40 \(1988\) 273](#)

Traded one **black box** for another...



We're physicists so ...

# Perturbation theory

# Alternative Solutions

Perturbative expansion:

- ▶ Small matter potential:  $a/\Delta m^2$

Y. Li, Y. Wang, Z-z. Xing, [1605.00900](#)

- ▶  $s_{13}, s_{13}^2$

A. Cervera, et al., [hep-ph/0002108](#)

H. Minakata, [0910.5545](#)

K. Asano, H. Minakata, [1103.4387](#)

- ▶  $\Delta m_{21}^2/\Delta m_{31}^2 \sim 0.03$

J. Arafune, J. Sato, [hep-ph/9607437](#)

A. Cervera, et al., [hep-ph/0002108](#)

M. Freund, [hep-ph/0103300](#)

E. Akhmedov, et al., [hep-ph/0402175](#)

M. Blennow, A. Smirnov, [1306.2903](#)

H. Minakata, S. Parke, [1505.01826](#)

[PBD](#), H. Minakata, S. Parke, [1604.08167](#)

# Alternative Solutions: Example

$$P_0 = \sin^2 \theta_{23} \frac{\sin^2 2 \theta_{13}}{\hat{C}^2} \sin^2(\hat{\Delta} \hat{C}), \quad (36a)$$

$$P_{\sin \delta} = \frac{1}{2} \alpha \frac{\sin \delta \cos \theta_{13} \sin 2 \theta_{12} \sin 2 \theta_{13} \sin 2 \theta_{23}}{\hat{A} \hat{C} \cos \theta_{13}^2} \sin(\hat{C} \hat{\Delta})$$

$$\times \{\cos(\hat{C} \hat{\Delta}) - \cos((1 + \hat{A}) \hat{\Delta})\}, \quad (36b)$$

$$P_{\cos \delta} = \frac{1}{2} \alpha \frac{\cos \delta \cos \theta_{13} \sin 2 \theta_{12} \sin 2 \theta_{13} \sin 2 \theta_{23}}{\hat{A} \hat{C} \cos \theta_{13}^2} \sin(\hat{C} \hat{\Delta})$$

$$\times \{\sin((1 + \hat{A}) \hat{\Delta}) \mp \sin(\hat{C} \hat{\Delta})\}, \quad (36c)$$

$$P_1 = -\alpha \frac{1 - \hat{A} \cos 2 \theta_{13}}{\hat{C}^3} \sin^2 \theta_{12} \sin^2 2 \theta_{13} \sin^2 2 \theta_{23} \hat{\Delta}$$

$$\times \sin(2 \hat{\Delta} \hat{C}) + \alpha \frac{2 \hat{A}(-\hat{A} + \cos 2 \theta_{13})}{\hat{C}^4}$$

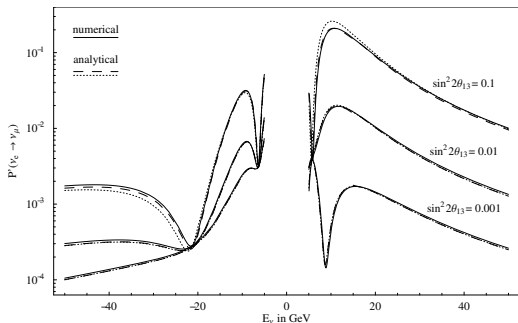
$$\times \sin^2 \theta_{12} \sin^2 2 \theta_{13} \sin^2 2 \theta_{23} \sin^2(\hat{\Delta} \hat{C}), \quad (36d)$$

$$P_2 = \alpha \frac{\mp 1 + \hat{C} \pm \hat{A} \cos 2 \theta_{13}}{2 \hat{C}^2 \hat{A} \cos^2 \theta_{13}} \cos \theta_{13} \sin 2 \theta_{12} \sin 2 \theta_{13}$$

$$\times \sin 2 \theta_{23} \sin^2(\hat{\Delta} \hat{C}), \quad (36e)$$

$$P_3 = \alpha^2 \frac{2 \hat{C} \cos^2 \theta_{23} \sin^2 2 \theta_{12}}{\hat{A}^2 \cos^2 \theta_{13} (\mp \hat{A} + \hat{C} \pm \cos 2 \theta_{13})}$$

$$\times \sin^2 \left( \frac{1}{2} (1 + \hat{A} \mp \hat{C}) \hat{\Delta} \right). \quad (36f)$$



M. Freund, [hep-ph/0103300](https://arxiv.org/abs/hep-ph/0103300)

# A Tale of Two Tools

Split the Hamiltonian into:

- ▶ Large, diagonal part ( $H_0$ )
- ▶ Small, *off-diagonal* part ( $H_1$ )
- ▶ Improves precision at zeroth order
- ▶ Naturally leads to using  $\Delta m_{ee}^2 \equiv c_{12}^2 \Delta m_{31}^2 + s_{12}^2 \Delta m_{32}^2$

H. Nunokawa, S. Parke, R. Zukanovich, [hep-ph/0503283](#)

## 1. Rotations:

- ▶ A two-flavor rotation only requires solving a quadratic
- ▶ Diagonalize away the big off-diagonal terms
- ▶ Follows the order of the PMNS matrix

## 2. Perturbative expansion:

- ▶ Smallness parameter is  $|\epsilon'| \leq 0.015$
- ▶ Correct eigenvalues ( $\widetilde{m_i^2}$ ) and eigenvectors ( $\widetilde{\theta_{ij}}$ )
- ▶ Eigenvalues already include 1<sup>st</sup> order corrections at 0<sup>th</sup> order
- ▶ Can improve the precision to arbitrary order

# “What is $\Delta m_{ee}^2$ ?”

$$\Delta m_{ee}^2 = c_{12}^2 \Delta m_{31}^2 + s_{12}^2 \Delta m_{32}^2 = \Delta m_{31}^2 - s_{12}^2 \Delta m_{21}^2$$

H. Nunokawa, S. Parke, R. Funchal, [hep-ph/0503283](#)

S. Parke, [1601.07464](#)

Additional expressions for  $\Delta m_{\mu\mu}^2, \Delta m_{\tau\tau}^2$

Useful definitions:

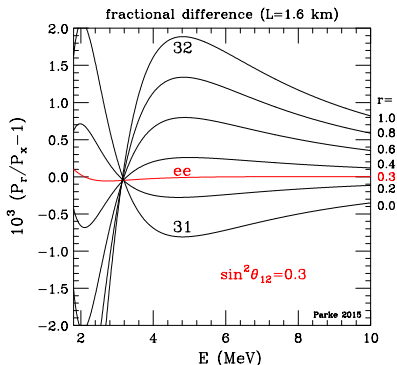
- ▶  $\nu_e$  weighted average of atmospheric splittings:

$$m_3^2 - \frac{|U_{e1}|^2 m_1^2 + |U_{e2}|^2 m_2^2}{|U_{e1}|^2 + |U_{e2}|^2}$$

- ▶ Measured by reactor experiments with smallest  $L/E$  error
- ▶ Simple form:

$$P_{ee} \approx 1 - \sin^2 2\theta_{13} \sin^2 \Delta_{ee}$$

$$\Delta_{ij} = \Delta m_{ij}^2 L / 4E$$



# Atmospheric Resonance



1.  $U_{23}(\theta_{23}, \delta)$  commutes with **matter potential**
  2. Largest off-diagonal term:  
 $s_{13}c_{13}\Delta m_{ee}^2$  in the 1-3 position
- 
- ▶ Eigenvalues still cross at the solar resonance:
    - ▶ No perturbation theory there
  - ▶ Smallness parameter:
    - ▶ After  $U_{23}$ :  $s_{13}c_{13} = 0.15$
    - ▶ After  $U_{13}$ :  $s_{12}c_{12} \frac{\Delta m_{21}^2}{\Delta m_{ee}^2} = 0.015$

Vacuum  
Rot.

Matter  
Rot.

MP15

# Solar Resonance



### 3. Largest off-diagonal term:

$s_{12}c_{12}c_{\tilde{\theta}_{13}-\theta_{13}}\Delta m_{21}^2$  in the 1-2 position

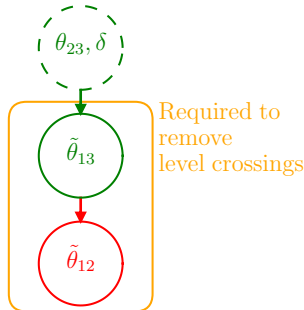
- ▶ Largest except for  $\nu$ 's above the atmospheric resonance
- ▶  $|\epsilon'| < 0.015$ , zero in vacuum
- ▶ Perturbation theory valid everywhere now
- ▶ Rotation order matches PMNS
- ▶ Take vacuum expressions, replace  $\theta_{13}$ ,  $\theta_{12}$ , and  $\Delta m_{ij}^2$
- ▶ Extremely precise  $|\Delta P/P| < 10^{-3}$

Vacuum  
Rot.

Matter  
Rot.

MP15  
DMP16

# Solar Resonance



### 3. Largest off-diagonal term:

$s_{12}c_{12}c_{\tilde{\theta}_{13}-\theta_{13}}\Delta m_{21}^2$  in the 1-2 position

- ▶ Largest except for  $\nu$ 's above the atmospheric resonance
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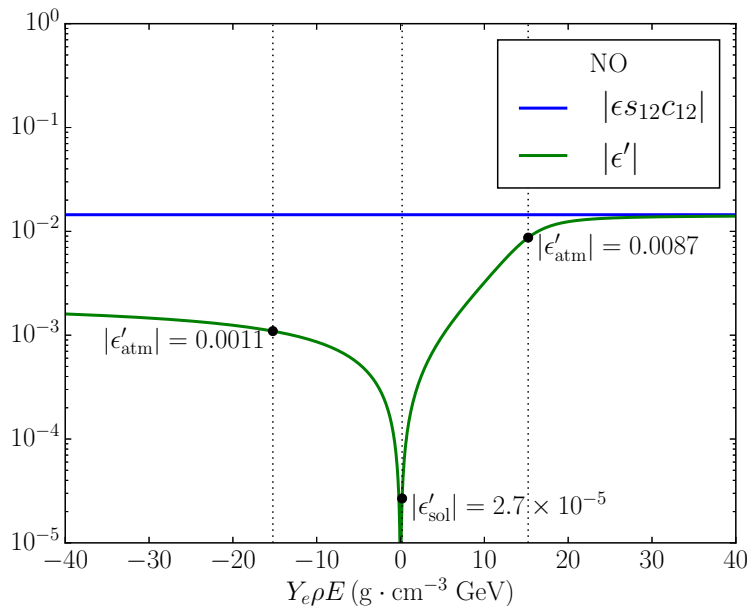
Vacuum  
Rot.

Matter  
Rot.

MP15  
DMP16



# Expansion Parameter



## Probability in Matter: DMP 0<sup>th</sup>

Matter expression  $\Rightarrow$  Vacuum expression

$$\tilde{P}_{\alpha\beta}(\Delta m_{21}^2, \Delta m_{31}^2, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\Delta \widetilde{m}_{21}^2, \Delta \widetilde{m}_{31}^2, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

# Probability in Matter: DMP 0<sup>th</sup>

Matter expression  $\Rightarrow$  Vacuum expression

$$\tilde{P}_{\alpha\beta}(\Delta m_{21}^2, \Delta m_{31}^2, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\widetilde{\Delta m}_{21}^2, \widetilde{\Delta m}_{31}^2, \tilde{\theta}_{13}, \tilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\tilde{\theta}_{13} = \frac{\Delta m_{ee}^2 \cos 2\theta_{13} - a}{\widetilde{\Delta m}_{ee}^2}$$

$$\widetilde{\Delta m}_{ee}^2 = \Delta m_{ee}^2 \sqrt{(\cos 2\theta_{13} - a/\Delta m_{ee}^2)^2 + \sin^2 2\theta_{13}}$$

# Probability in Matter: DMP 0<sup>th</sup>

Matter expression  $\Rightarrow$  Vacuum expression

$$\tilde{P}_{\alpha\beta}(\Delta m_{21}^2, \Delta m_{31}^2, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\Delta \widetilde{m}_{21}^2, \Delta \widetilde{m}_{31}^2, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m_{ee}^2 \cos 2\theta_{13} - a}{\Delta \widetilde{m}_{ee}^2}$$

$$\Delta \widetilde{m}_{ee}^2 = \Delta m_{ee}^2 \sqrt{(\cos 2\theta_{13} - a/\Delta m_{ee}^2)^2 + \sin^2 2\theta_{13}}$$

$$\cos 2\widetilde{\theta}_{12} = \frac{\Delta m_{21}^2 \cos 2\theta_{12} - a_{12}}{\Delta \widetilde{m}_{21}^2}, \quad a_{12} = (a + \Delta m_{ee}^2 - \Delta \widetilde{m}_{ee}^2)/2$$

$$\Delta \widetilde{m}_{21}^2 = \Delta m_{21}^2 \sqrt{(\cos 2\theta_{12} - a_{12}/\Delta m_{21}^2)^2 + \cos^2(\widetilde{\theta}_{13} - \theta_{13}) \sin^2 2\theta_{12}}$$

# Probability in Matter: DMP 0<sup>th</sup>

Matter expression  $\Rightarrow$  Vacuum expression

$$\tilde{P}_{\alpha\beta}(\Delta m_{21}^2, \Delta m_{31}^2, \theta_{13}, \theta_{12}, \theta_{23}, \delta) = P_{\alpha\beta}(\Delta \widetilde{m}_{21}^2, \Delta \widetilde{m}_{31}^2, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

Same expression, 4 new variables.

$$\cos 2\widetilde{\theta}_{13} = \frac{\Delta m_{ee}^2 \cos 2\theta_{13} - a}{\Delta \widetilde{m}_{ee}^2}$$

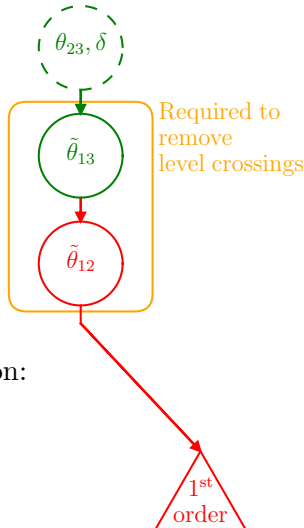
$$\Delta \widetilde{m}_{ee}^2 = \Delta m_{ee}^2 \sqrt{(\cos 2\theta_{13} - a/\Delta m_{ee}^2)^2 + \sin^2 2\theta_{13}}$$

$$\cos 2\widetilde{\theta}_{12} = \frac{\Delta m_{21}^2 \cos 2\theta_{12} - a_{12}}{\Delta \widetilde{m}_{21}^2}, \quad a_{12} = (a + \Delta m_{ee}^2 - \Delta \widetilde{m}_{ee}^2)/2$$

$$\Delta \widetilde{m}_{21}^2 = \Delta m_{21}^2 \sqrt{(\cos 2\theta_{12} - a_{12}/\Delta m_{21}^2)^2 + \cos^2(\widetilde{\theta}_{13} - \theta_{13}) \sin^2 2\theta_{12}}$$

$$\Delta \widetilde{m}_{31}^2 = \Delta m_{31}^2 + \frac{1}{4}a + \frac{1}{2}(\Delta \widetilde{m}_{21}^2 - \Delta m_{21}^2) + \frac{3}{4}(\Delta \widetilde{m}_{ee}^2 - \Delta m_{ee}^2)$$

# Improve with Perturbation



4.  $\gtrsim 2$  orders of magnitude of improvement in precision:  
 $|\Delta P/P| < 10^{-6}$

- ▶ Eigenvalues need no correction
- ▶ Compact form utilizes a  
 $\widetilde{m}^2_1 \leftrightarrow \widetilde{m}^2_2, \widetilde{\theta}_{12} \leftrightarrow \widetilde{\theta}_{12} \pm \pi/2$  symmetry

Vacuum  
Rot.

Matter  
Rot.

Pert.

MP15

DMP16

## $\widetilde{m}_{1,2}^2 - \widetilde{\theta}_{12}$ Symmetry

From the shape of  $U_{12}(\widetilde{\theta}_{12})$ , it is clear that the probabilities are invariant under a simultaneous interchange of

$$\widetilde{m}_1^2 \leftrightarrow \widetilde{m}_2^2, \quad \text{and} \quad \widetilde{\theta}_{12} \rightarrow \widetilde{\theta}_{12} \pm \frac{\pi}{2}.$$

Since only even powers of  $\widetilde{\theta}_{12}$  trig functions  $c_{12}^2, s_{12}^2, c_{12}s_{12}, \cos(2\widetilde{\theta}_{12}), \sin(2\widetilde{\theta}_{12})$  appear in the probabilities, the sign degeneracy is irrelevant.

More usefully, we can write that the probabilities are invariant under the simultaneous interchange of

$$\widetilde{m}_1^2 \leftrightarrow \widetilde{m}_2^2, \quad c_{12}^2 \leftrightarrow s_{12}^2, \quad \text{and} \quad c_{12}s_{12} \rightarrow -c_{12}s_{12}.$$

This interchange constrains the  $\sin^2 \Delta_{21}$  term, and the  $\sin^2 \Delta_{32}$  term easily follows from the  $\sin^2 \Delta_{31}$  term.

## General Form of the First Order Coefficients

$$P_{\alpha\beta} = \delta_{\alpha\beta} + 4C_{21}^{\alpha\beta} \sin^2 \Delta_{21} + 4C_{31}^{\alpha\beta} \sin^2 \Delta_{31} + 4C_{32}^{\alpha\beta} \sin^2 \Delta_{32} \\ + 8D^{\alpha\beta} \sin \Delta_{21} \sin \Delta_{31} \sin \Delta_{32}$$

Can reduce 8 expressions down to 3:

$$(C_{21}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left( \frac{F_1^{\alpha\beta}}{\widetilde{\Delta m_{31}^2}} + \frac{F_2^{\alpha\beta}}{\widetilde{\Delta m_{32}^2}} \right)$$

$$(C_{31}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left( \frac{F_1^{\alpha\beta} + G_1^{\alpha\beta}}{\widetilde{\Delta m_{31}^2}} - \frac{F_2^{\alpha\beta}}{\widetilde{\Delta m_{32}^2}} \right)$$

$$(C_{32}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left( -\frac{F_1^{\alpha\beta}}{\widetilde{\Delta m_{31}^2}} + \frac{F_2^{\alpha\beta} + G_2^{\alpha\beta}}{\widetilde{\Delta m_{32}^2}} \right)$$

$$(D^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left( \frac{K_1^{\alpha\beta}}{\widetilde{\Delta m_{31}^2}} - \frac{K_2^{\alpha\beta}}{\widetilde{\Delta m_{32}^2}} \right)$$

$$K_1^{\alpha\beta} = \mp s_{23} c_{23} \widetilde{c_{13}^2} s_{12}^2 (c_{13}^2 \widetilde{c_{12}^2} - s_{13}^2) s_\delta, \quad \alpha \neq \beta$$



# First Order Coefficients

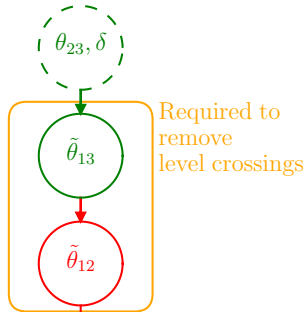
$\nu_\alpha \rightarrow \nu_\beta$	$F_1^{\alpha\beta}$
$\nu_e \rightarrow \nu_e$	$-2c_{13}^3 \widetilde{s_{13}} s_{12}^3 \widetilde{c_{12}}$
$\nu_\mu \rightarrow \nu_e$	$c_{13}^2 \widetilde{s_{12}} [s_{13} \widetilde{s_{12}} \widetilde{c_{12}} (c_{23}^2 + c_{213} s_{23}^2) - s_{23} c_{23} (s_{13}^2 \widetilde{s_{12}}^2 + c_{213}^2 \widetilde{c_{12}}^2) c_\delta]$
$\nu_\mu \rightarrow \nu_\mu$	$2c_{13} \widetilde{s_{12}} (s_{23}^2 \widetilde{s_{13}} \widetilde{c_{12}} + s_{23} c_{23} s_{12} \widetilde{c_\delta}) \times (c_{23}^2 \widetilde{c_{12}}^2 - 2s_{23} c_{23} s_{13} \widetilde{s_{12}} \widetilde{c_{12}} c_\delta + s_{23}^2 \widetilde{s_{13}}^2 \widetilde{s_{12}}^2)$

$\nu_\alpha \rightarrow \nu_\beta$	$G_1^{\alpha\beta}$
$\nu_e \rightarrow \nu_e$	$2s_{13} \widetilde{c_{13}} \widetilde{s_{12}} \widetilde{c_{12}} c_{213}$
$\nu_\mu \rightarrow \nu_e$	$-2s_{13} \widetilde{c_{13}} \widetilde{s_{12}} (s_{23}^2 c_{213} \widetilde{c_{12}} - s_{23} c_{23} s_{13} \widetilde{s_{12}} c_\delta)$
$\nu_\mu \rightarrow \nu_\mu$	$-2c_{13} \widetilde{s_{12}} (s_{23}^2 \widetilde{s_{13}} \widetilde{c_{12}} + s_{23} c_{23} s_{12} \widetilde{c_\delta}) \times (1 - 2c_{13}^2 s_{23}^2)$

Three channels gives them all with unitarity!

# Higher Orders

5.  $\gtrsim 2$  more orders of magnitude of improvement per order:  
 $|\Delta P/P| < 10^{-9}, \dots$



1<sup>st</sup>  
order

2<sup>nd</sup>  
order

$\vdots$

Vacuum  
Rot.

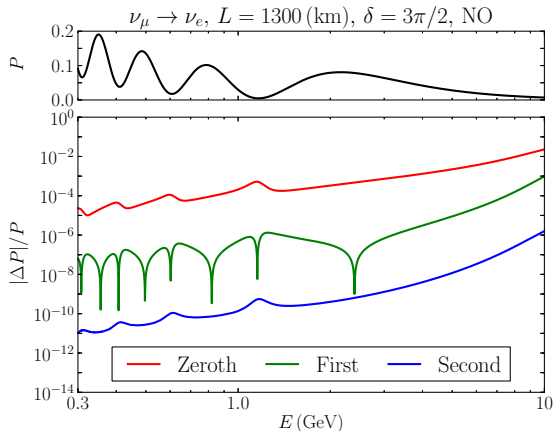
Matter  
Rot.

Pert.

MP15

DMP16

# Precision



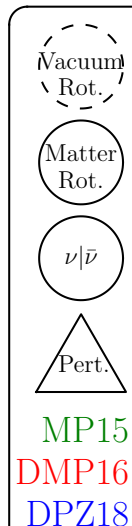
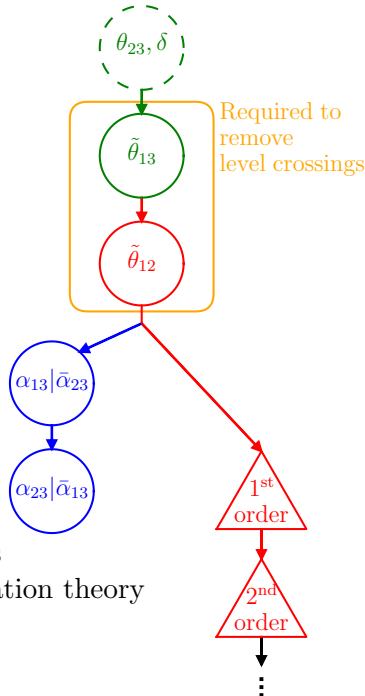
DUNE: NO, $\delta = 3\pi/2$		First min	First max
$P(\nu_\mu \rightarrow \nu_e)$		0.0047	0.081
$E$ (GeV)		1.2	2.2
$\frac{ \Delta P }{P}$	Zeroth	$5 \times 10^{-4}$	$4 \times 10^{-4}$
	First	$3 \times 10^{-7}$	$2 \times 10^{-7}$
	Second	$6 \times 10^{-10}$	$5 \times 10^{-10}$

# More Rotations

Instead continue to diagonalize large terms

4. 1-3 sector for  $\nu$ 's  
2-3 sector for  $\bar{\nu}$ 's
5. Then opposite

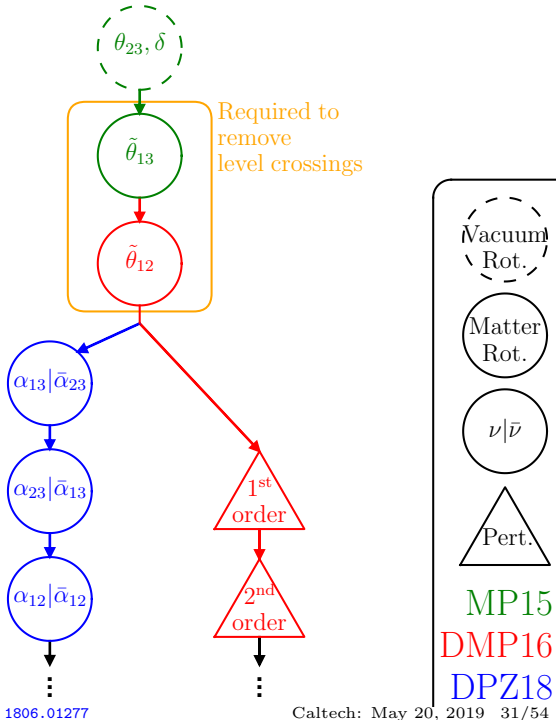
- 2 additional rotations  
≡ 1 order of perturbation theory



# Even More Rotations

6. 1-2 sector for either  $\nu/\bar{\nu}$

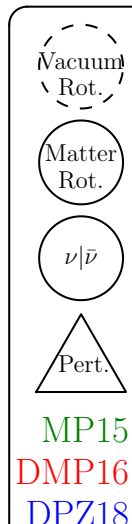
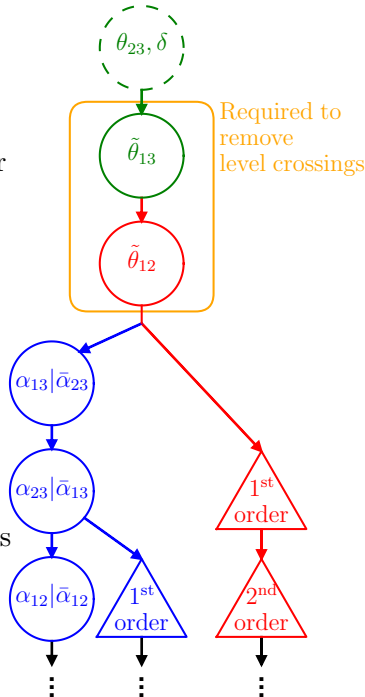
► 3 additional rotations  
 $\equiv$  2 orders of pert. th.



# More Options

## 6. Perturbation theory after 2 additional rotations

- ▶ 2 additional rotations + 1 order of pert. th.  $\equiv$  2 orders of pert. th.
- ▶ The precision of rotations follows Fibonacci sequence



# Rotation Precision Follows the Fibonacci Sequence

Given  $H = H_0 + H_1$ , perform successive rotations,

Rotation #	0	1	2	3	4	5	6	7 ...
Size of correction	1	1	2	3	5	8	13	21 ...

Necessary conditions:

1.  $H_1$  is Hermitian
2.  $H_1$  has no diagonal entries
3. At least one off-diagonal entry is zero

Each additional rotation is of uniform complexity,  
perturbation theory is of increasing complexity

# Verifying the CPV Term in Matter

The amount of CPV is

$$J \sin \Delta_{21} \sin \Delta_{31} \sin \Delta_{32}$$

where the Jarlskog is

$$J = 8c_{12}s_{12}c_{13}^2s_{13}c_{23}s_{23}s_{\delta}$$

C. Jarlskog, [PRL 55 \(1985\)](#)

The exact term in matter is known to be

$$\frac{\hat{J}}{J} = \frac{\Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2}{\widehat{\Delta m}_{21}^2 \widehat{\Delta m}_{31}^2 \widehat{\Delta m}_{32}^2}$$

V. Naumov, [Int. J. Mod. Phys. 1992](#)

P. Harrison, W. Scott, [hep-ph/9912435](#)

Our expression reproduces this order by order in  $\epsilon'$  for all channels.



# CPV in Matter

CPV in matter can be written sans  $\cos(\frac{1}{3}\cos^{-1}(\dots))$  term.

$$\frac{\widehat{J}}{J} = \frac{\Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2}{\widehat{\Delta m}_{21}^2 \widehat{\Delta m}_{31}^2 \widehat{\Delta m}_{32}^2}$$

$$\left(\widehat{\Delta m}_{21}^2 \widehat{\Delta m}_{31}^2 \widehat{\Delta m}_{32}^2\right)^2 = (A^2 - 4B)(B^2 - 4AC) + (2AB - 27C)C$$

$$A \equiv \sum_j \widehat{m}_{2j}^2 = \Delta m_{31}^2 + \Delta m_{21}^2 + a$$

$$B \equiv \sum_{j>k} \widehat{m}_{2j}^2 \widehat{m}_{2k}^2 = \Delta m_{31}^2 \Delta m_{21}^2 + a(\Delta m_{ee}^2 c_{13}^2 + \Delta m_{21}^2)$$

$$C \equiv \prod_j \widehat{m}_{2j}^2 = a \Delta m_{31}^2 \Delta m_{21}^2 c_{13}^2 c_{12}^2$$

This is the *only* oscillation quantity in matter that can be written exactly without  $\cos(\frac{1}{3}\cos^{-1}(\dots))$ !

# CPV in Matter

Thus  $\hat{J}^2$  is fourth order in matter potential:  
only two matter corrections are really needed.

# CPV in Matter

Thus  $\hat{J}^2$  is fourth order in matter potential:  
only two matter corrections are really needed.

CPV in matter can be approximated:

$$\frac{\hat{J}}{J} \approx \frac{1}{\mathcal{S}_{\text{atm}} \mathcal{S}_{\odot}}$$

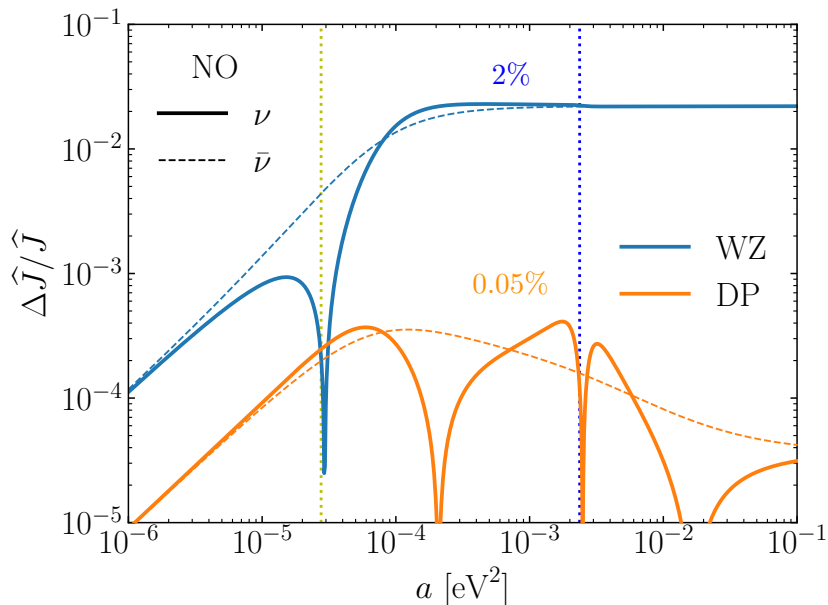
$$\mathcal{S}_{\text{atm}} = \sqrt{(\cos 2\theta_{13} - a/\Delta m_{ee}^2)^2 + \sin^2 2\theta_{13}}$$

$$\mathcal{S}_{\odot} = \sqrt{(\cos 2\theta_{12} - c_{13}^2 a/\Delta m_{21}^2)^2 + \sin^2 2\theta_{12}}$$

PBD, Parke, [1902.07185](#)

See also X. Wang, S. Zhou, [1901.10882](#)

# CPV In Matter Approximation Precision



# Is DMP the best?

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yes

# Other Expressions

We were not the first to examine this problem.

- Madrid: drop  $\frac{\Delta m_{21}^2}{\Delta m_{31}^2}$  and  $s_{13}$  terms;  $\sim |\text{sum of two amplitudes}|^2$

$$P_{\mu e} = 4s_{23}^2 s_{13}^2 c_{13}^2 \left( \frac{\Delta m_{31}^2}{b} \right)^2 \sin^2 \Delta_b + 4c_{23}^2 s_{12}^2 c_{12}^2 \left( \frac{\Delta m_{21}^2}{a} \right)^2 \sin^2 \Delta_a \\ + 8J_r \frac{\Delta m_{21}^2}{a} \frac{\Delta m_{31}^2}{b} \sin \Delta_a \sin \Delta_b \cos(\delta + \Delta_{31}) , \quad b = a - \Delta m_{31}^2$$

A. Cervera, et al., [hep-ph/0002108](#)

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A. Cervera, et al., [hep-ph/0002108](#)

E. Akhmedov, et al., [hep-ph/0402175](#)

A. Friedland, C. Lunardini, [hep-ph/0606101](#)

H. Nunokawa, S. Parke, J. Valle, [0710.0554](#)



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A. Friedland, C. Lunardini, [hep-ph/0606101](#)

H. Nunokawa, S. Parke, J. Valle, [0710.0554](#)

- AKT: from mass basis rotated 12 then 23 converted into 13
  - $\Delta m_{ee}^2$  appears all over the expressions

S. Agarwalla, Y. Kao, T. Takeuchi, [1302.6773](#)

# Other Expressions

- ▶ AM: Powers of  $s_{13}^2 \simeq \frac{\Delta m_{21}^2}{\Delta m_{31}^2}$  through the 5/2 order

K. Asano, H. Minakata, [1103.4387](#)

- ▶ Various other expressions

J. Arafune, M. Koike, J. Sato, [hep-ph/9703351](#)

M. Freund, [hep-ph/0103300](#)

E. Akhmedov, et al., [hep-ph/0402175](#)

Others...

Which is best?

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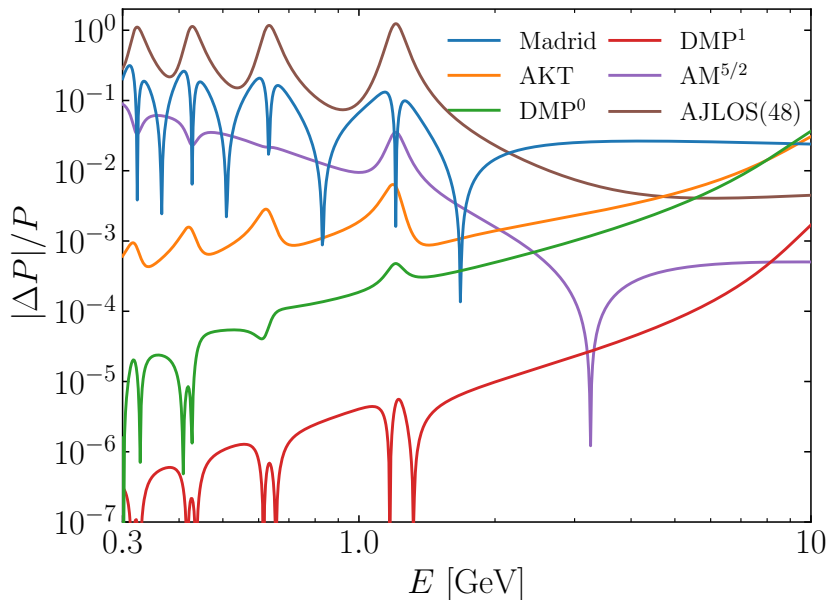
E. Akhmedov, et al., [hep-ph/0402175](#)

Others...

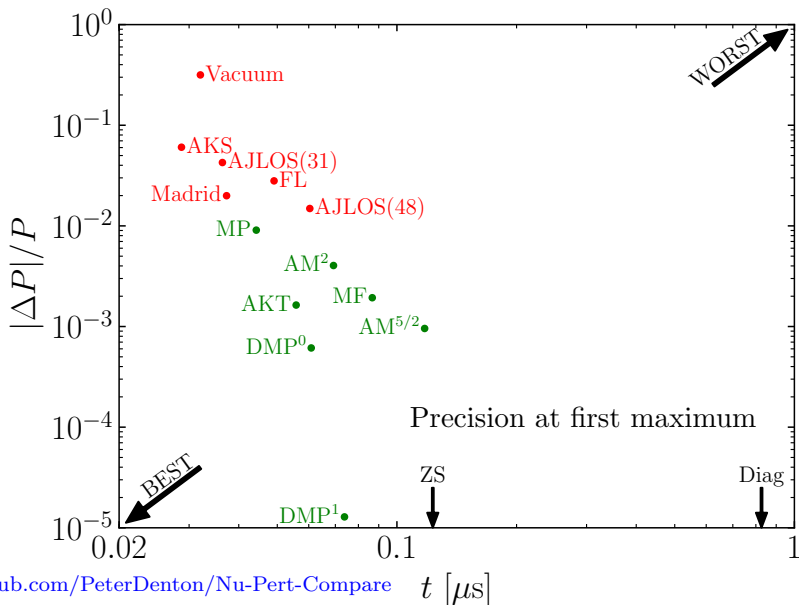
Which is best?

What does “best” mean?

# Comparative Precision ( $L = 1300$ km)



Speed  $\propto$  Simplicity



[github.com/PeterDenton/Nu-Pert-Compare](https://github.com/PeterDenton/Nu-Pert-Compare)

1902.00517

# Proper Expansions

Parameter  $x$  is an *expansion parameter* iff

$$\lim_{x \rightarrow 0} P_{\text{approx}}(x) = P_{\text{exact}}(x = 0)$$

	$\epsilon$	$s_{13}$	$a/\Delta m_{31}^2$	
Madrid(like)	✗	✗	✗	Cervera+, <a href="#">hep-ph/0002108</a>
AKT	✓	✓	✓	Agarwalla+, <a href="#">1302.6773</a>
MP	✓	✗	✗	Minakata, Parke, <a href="#">1505.01826</a>
<b>DMP</b>	✓	✓	✓	<b>PBD+</b> , <a href="#">1604.08167</a>
AKS	✗	✗	✗	Arafune+, <a href="#">hep-ph/9703351</a>
MF	✓	✗	✗	Freund, <a href="#">hep-ph/0103300</a>
AJLOS(48)	✓	✗	✗	Akhmedov+, <a href="#">hep-ph/0402175</a>
AM	✗	✗	✗	Asano, Minakata, <a href="#">1103.4387</a>

$$\epsilon \equiv \frac{\Delta m_{21}^2}{\Delta m_{ee}^2}$$

# Comparative Review

- ▶ Many expressions in the literature (12 considered)
- ▶ Most are not at the 1% level
- ▶ Most are not exact in vacuum
- ▶ Changing the basis to remove level crossings seems best
  - ▶ AKT, (MP), DMP
  - ▶  $\Delta m_{ee}^2$  naturally appears (regardless of the name)
- ▶ The order of rotations matters:
  - ▶ Constant 23 rotation, then in matter: 13, 12
- ▶ First order DMP corrections are quite simple

# The Effective $\Delta m_{ee}^2$ in Matter

$$\Delta m_{ee}^2 = c_{12}^2 \Delta m_{31}^2 + s_{12}^2 \Delta m_{32}^2$$

$\Delta m_{ee}^2$  is an important quantity for understanding oscillations:

- ▶ Optimal expression for MBL reactor experiments

H. Nunokawa, S. Parke, R. Zukanovich, [hep-ph/0503283](#)

S. Parke, [1601.07464](#)

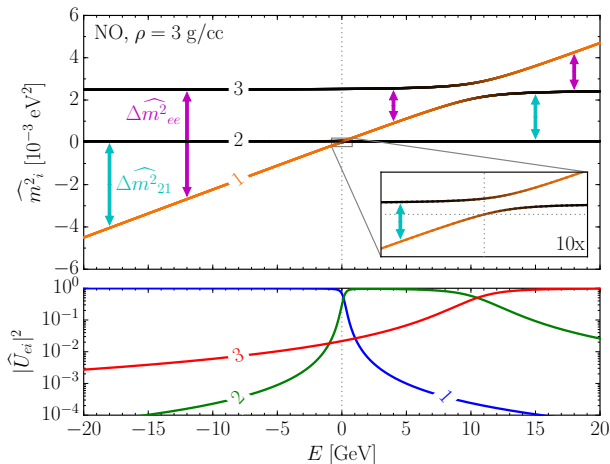
- ▶ Shows up naturally in DMP on long-baseline matter effect

How does  $\Delta m_{ee}^2$  evolve in **matter**?

Two-flavor approximation for  $\nu_e$  disappearance in **matter**?



# Asymptotic Evolution of $\widehat{\Delta m^2_{ee}}$



$$\widehat{\Delta m^2_{ee}} = \begin{cases} \widehat{m^2_3} - \widehat{m^2_1} & E \rightarrow -\infty \\ \widehat{m^2_3} - \widehat{m^2_2} & E \rightarrow +\infty \end{cases}$$

## Intermediate Evolution of $\widehat{\Delta m^2_{ee}}$

$$\widehat{\Delta m^2_{ee}} = \begin{cases} \widehat{m^2_3} - \widehat{m^2_1} & E \rightarrow -\infty \\ \widehat{m^2_3} - \widehat{m^2_2} & E \rightarrow +\infty \end{cases}$$

Since  $\widehat{m^2_2}(E \rightarrow -\infty) = \widehat{m^2_1}(E \rightarrow +\infty) = \text{constant}$ ,  
call  $m_0^2 \equiv \Delta m_{21}^2 c_{12}^2$

Now we can define

$$\Delta \widehat{m^2_{ee}} \equiv \widehat{m^2_3} - (\widehat{m^2_1} + \widehat{m^2_2} - m_0^2)$$

$$\Delta \widehat{m^2_{ee}} - \Delta m_{ee}^2 = (\widehat{m^2_3} - m_3^2) - (\widehat{m^2_1} - m_1^2) - (\widehat{m^2_2} - m_2^2)$$

Easy to see that  $\Delta \widehat{m^2_{ee}}(E=0) = \Delta m_{ee}^2$

Relationship to vacuum expression?

# Relationship to Vacuum Expression

In vacuum we can equivalently write:

$$\Delta m_{ee}^2 = \begin{cases} c_{12}^2 \Delta m_{31}^2 + s_{12}^2 \Delta m_{32}^2 \\ m_3^2 - (m_1^2 + m_2^2 - m_0^2) \end{cases}$$

Elevate everything to **matter** equivalent,  
except  $m_0^2$  which we know we want to be a constant.

$$\widehat{\Delta m}_{ee}^2 = \widehat{m}_3^2 - (\widehat{m}_1^2 + \widehat{m}_2^2 - m_0^2)$$

$$\Delta \widehat{m}_{EE}^2 = c_{12}^2 \Delta \widehat{m}_{31}^2 + s_{12}^2 \Delta \widehat{m}_{32}^2$$

The difference between these similar formulas:

$$\Delta_{Ee} = \widehat{m}_1^2 + c_{12}^2 \Delta \widehat{m}_{21}^2 - c_{12}^2 \Delta m_{21}^2$$

$$< 0.3\%$$

## A Third Option

Avoid  $\cos(\frac{1}{3} \cos^{-1} \dots)$ , use DMP:

$$\widetilde{\Delta m^2_{ee, \text{DMP}}} \equiv \Delta m^2_{ee} \sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

Which is best?

## What does *best* mean?

Using one  $\Delta m^2 \Rightarrow$  using a two-flavor picture:

$$P_{ee} \approx 1 - \sin^2 2\hat{\theta}_{13} \sin^2 \frac{\widehat{\Delta m^2}_{ee} L}{4E}$$

$\Rightarrow$  want the first minimum correct

Take exact expression at  $dP/dL = 0$  for a given  $E$ , then

$$\frac{\widehat{\Delta m^2}_{ee} L}{4E} = \frac{\pi}{2}$$

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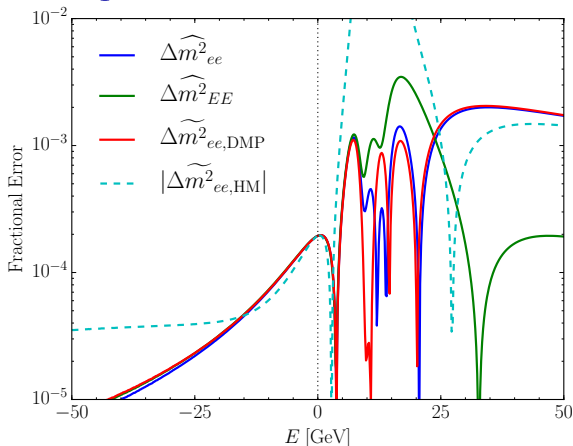
$$\frac{\widehat{\Delta m^2}_{ee} L}{4E} = \frac{\pi}{2}$$

Could do  $dP/dE = 0$  for a given  $L$ ,

H. Minakata, [1702.03332](#)

but  $L/E$  show up together except where  $a = a(E)$  appears,  
and both  $\widehat{\theta}_{13}$ ,  $\widehat{\Delta m^2}_{ee}$  are complicated functions of  $a$ .

# Comparison of Two-Flavor Precision



The  $\widehat{\Delta m^2_{ee}}$  expression also leads to a simple rewriting of the eigenvalues.

HM: H. Minakata, [1702.03332](#)  
21 term in probability not included

The winner is:  $\widehat{\Delta m^2_{ee}} \equiv \widehat{m^2_3} - (\widehat{m^2_1} + \widehat{m^2_2} - m_0^2)$  !

Precision is better than 0.06%

# Depth of Oscillations

The depth of the minimum is well-described by

$$\begin{aligned}\sin^2 2\widehat{\theta}_{13} &\approx \sin^2 2\theta_{13} \left( \frac{\Delta m_{ee}^2}{\widehat{\Delta m_{ee}^2}} \right)^2 \\ &\approx \frac{\sin^2 2\theta_{13}}{(\cos^2 2\theta_{13} - a/\Delta m_{ee}^2)^2 + \sin^2 2\theta_{13}}\end{aligned}$$

Using DMP



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Using DMP

The disappearance probability in matter is well described by

$$P_{ee} \approx 1 - \sin^2 2\theta_{13} \left( \frac{\Delta m_{ee}^2}{\widehat{\Delta m_{ee}^2}} \right)^2 \sin^2 \frac{\widehat{\Delta m_{ee}^2} L}{4E}$$

$$\begin{aligned}\widehat{\Delta m_{ee}^2} &\equiv \widehat{m_3^2} - (\widehat{m_1^2} + \widehat{m_2^2}) \\ &\quad - [m_3^2 - (m_1^2 + m_2^2)] + \Delta m_{ee}^2\end{aligned}$$

# New Physics

DUNE and T2HK will unprecedented capabilities to test the three-neutrino oscillation picture

Extend DMP to new physics progress report:

- ▶ Sterile
- ▶ NSI?
- ▶ Neutrino decay?
- ▶ Decoherence?

S. Parke, X. Zhang, [1905.01356](#)

# Key Points

- ▶ Long-baseline oscillations are fundamentally three-flavor
- ▶ Include 1<sup>st</sup> order corrections in 0<sup>th</sup> order eigenvalues ( $\Delta m_{ee}^2$ )
- ▶ Rotate **large terms first**  $\Rightarrow$  PMNS order, removes level crossings
- ▶ 0<sup>th</sup> order probabilities: **same structure as vacuum** probabilities
- ▶ 0<sup>th</sup> order: **accurate** enough for current & future experiments
- ▶ DMP is the most precise while just as simple
- ▶ Exact and approximate CPV in matter are **simpler** than expected
- ▶ Same tools can also describe  $\nu_e$  disappearance in matter:  $\widehat{\Delta m_{ee}^2}$

# Backups

# Neutrino Oscillations in Vacuum: Disappearance

It is easy to calculate the *exact* disappearance expression in vacuum:

$$P(\nu_\alpha \rightarrow \nu_\alpha) = 1 - 4 \sum_{i < j} |U_{\alpha i}|^2 |U_{\alpha j}|^2 \sin^2 \Delta_{ji}$$

For the electron case this expression is simple:

$$\begin{aligned} P(\nu_e \rightarrow \nu_e) = 1 & \\ & - 4c_{12}^2 s_{12}^2 c_{13}^4 \sin^2 \Delta_{21} \\ & - 4c_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{31} \\ & - 4s_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{32} \end{aligned}$$

$$\begin{aligned} \Delta_{ij} &= \frac{\Delta m_{ij}^2 L}{4E} \\ \Delta m_{ij}^2 &= m_i^2 - m_j^2 \end{aligned}$$

# A Simple Solution

For two-flavor oscillations:

$$P(\nu_\alpha \rightarrow \nu_\beta) = \sin^2 2\theta \sin^2 \left( \frac{\Delta m^2 L}{4E} \right)$$

- ▶ Solar:  $\theta_{21}$ ,  $\Delta m_{21}^2$
- ▶ Reactor:  $\theta_{13}$ ,  $\Delta m_{ee}^2$

# Our Methodology

- ▶ Start with  $\epsilon = \frac{\Delta m_{21}^2}{\Delta m_{ee}^2} = 0.03$
- ▶ Perform one fixed and two variable rotations:  $(\theta_{23}, \delta), \tilde{\theta}_{13}, \tilde{\theta}_{12}$
- ▶ Write the probabilities with simple  $L/E$  dependence:

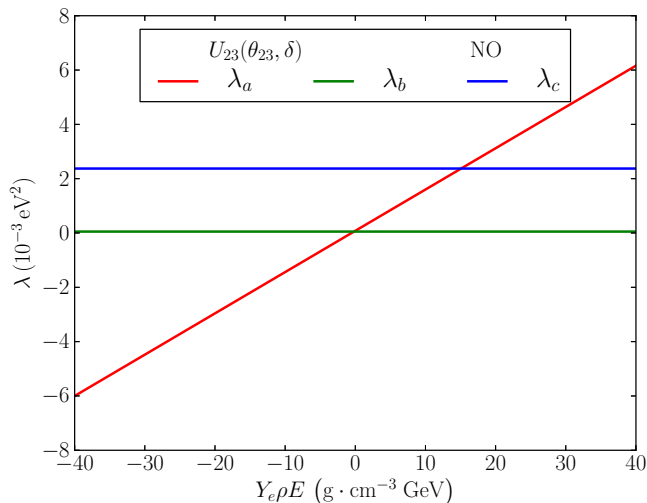
$$P(\nu_\alpha \rightarrow \nu_\beta) = \delta_{\alpha\beta} - \sum_{i < j} \Re [U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}] \sin^2 \Delta_{ij} \\ + 8\Im [U_{\alpha 1} U_{\beta 2}^* U_{\alpha 2}^* U_{\beta 1}] \sin \Delta_{32} \sin \Delta_{31} \sin \Delta_{21}$$

C. Jarlskog: [PRL 55 \(1985\)](#)

Nonvanishing Wronskian  $\Rightarrow$  fewest number of  $L/E$  functions

Clear that the **CPV** term is  $\mathcal{O}[(L/E)^3]$  not  $\mathcal{O}[(L/E)^1]$

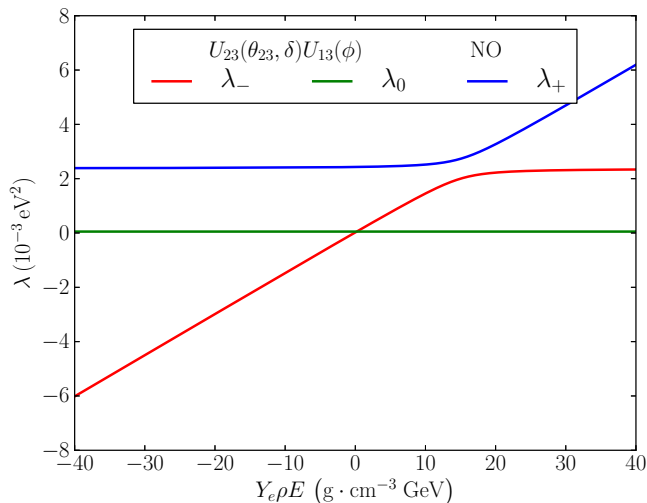
# Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m}_a^2 = a + (s_{13}^2 + \epsilon s_{12}^2) \Delta m_{ee}^2, \quad \widetilde{m}_b^2 = \epsilon c_{12}^2 \Delta m_{ee}^2, \quad \widetilde{m}_c^2 = (c_{13}^2 + \epsilon s_{12}^2) \Delta m_{ee}^2$$



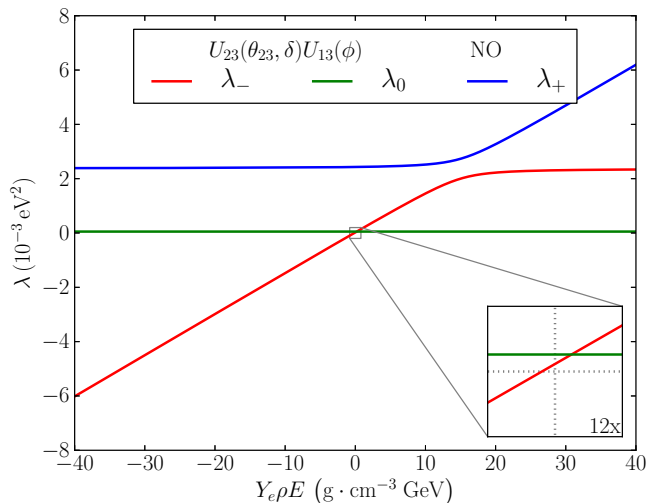
# Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m}_{\mp}^2 = \frac{1}{2} \left[ (\widetilde{m}_a^2 + \widetilde{m}_c^2) \mp \text{sgn}(\Delta m_{ee}^2) \sqrt{(\widetilde{m}_c^2 - \widetilde{m}_a^2)^2 + (2s_{13}c_{13}\Delta m_{ee}^2)^2} \right]$$

$$\widetilde{m}_0^2 = \widetilde{m}_b^2$$

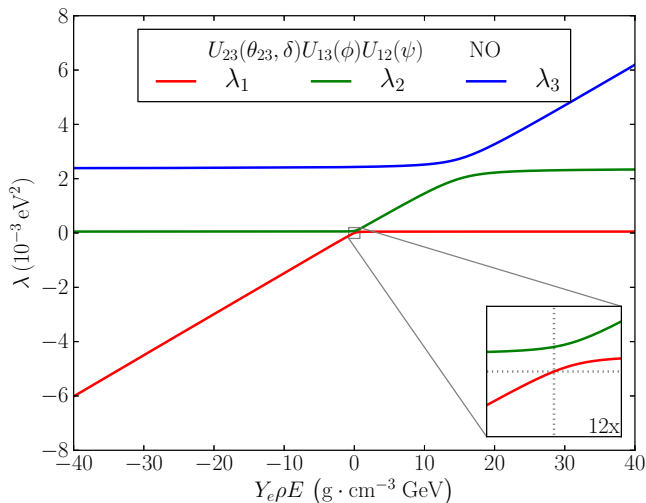
# Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m}_{\mp}^2 = \frac{1}{2} \left[ (\widetilde{m}_a^2 + \widetilde{m}_c^2) \mp \text{sgn}(\Delta m_{ee}^2) \sqrt{(\widetilde{m}_c^2 - \widetilde{m}_a^2)^2 + (2s_{13}c_{13}\Delta m_{ee}^2)^2} \right]$$

$$\widetilde{m}_0^2 = \widetilde{m}_b^2$$

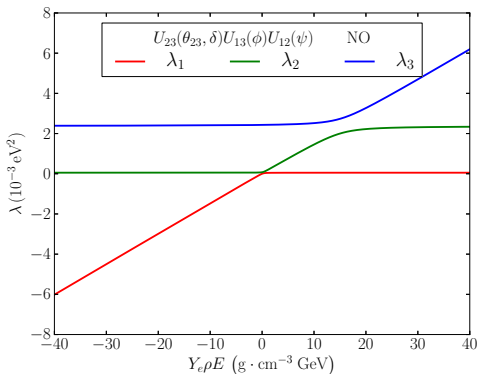
# Eigenvalues in Matter: Two Rotations are Needed



$$\widetilde{m}_{1,2}^2 = \frac{1}{2} \left[ (\widetilde{m}_0^2 + \widetilde{m}_-^2) \mp \sqrt{(\widetilde{m}_0^2 - \widetilde{m}_-^2)^2 + (2\epsilon c_{(\widetilde{\theta}_{13}-\theta_{13})} c_{12} s_{12} \Delta m_{ee}^2)^2} \right]$$

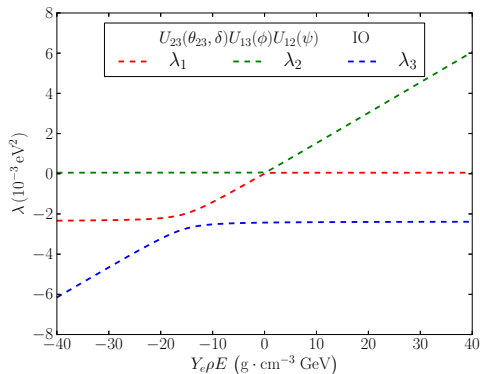
$$\widetilde{m}_3^2 = \widetilde{m}_+^2$$

# Eigenvalues in Matter: Mass Ordering



NO

$$\widetilde{m}^2_1 < \widetilde{m}^2_2 < \widetilde{m}^2_3$$



IO

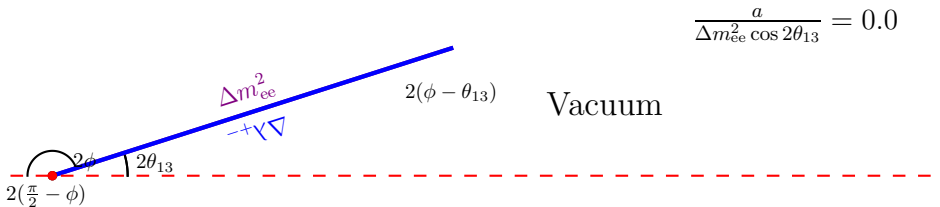
$$\widetilde{m}^2_3 < \widetilde{m}^2_1 < \widetilde{m}^2_2$$

# 1 + 2 Rotations

1. Perform a constant  $U_{23}(\theta_{23}, \delta)$  rotation
  - ▶  $U_{23}$  commutes with the matter potential
  - ▶ Resultant Hamiltonian is real
  - ▶ ‘Expansion parameter’ is  $c_{13}s_{13} = 0.15$  at this point
2. Diagonalize the diagonal and  $\mathcal{O}(\epsilon^0)$  off-diagonal terms with  $U_{13}(\tilde{\theta}_{13})$ 
  - ▶  $\tilde{\theta}_{13}(a=0) = \theta_{13}$
  - ▶ Expansion parameter is  $c_{12}s_{12} \frac{\Delta m_{21}^2}{\Delta m_{ee}^2} = 0.015$

H. Minakata, S. Parke, [1505.01826](#)
3. Diagonalize the terms non-zero in vacuum with  $U_{12}(\tilde{\theta}_{12})$ 
  - ▶  $\tilde{\theta}_{12}(a=0) = \theta_{12}$
  - ▶ Expansion parameter is now  $\epsilon' = c_{12}s_{12} s_{(\tilde{\theta}_{13}-\theta_{13})} \frac{\Delta m_{21}^2}{\Delta m_{ee}^2} < 0.015$
  - ▶  $\epsilon'(a=0) = 0$

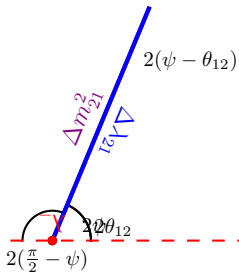
# 31 Triangle



Denton & Parke

# 31 Triangle

# 21 Triangle



$$\frac{a}{\Delta m_{21}^2 \frac{\cos 2\theta_{12}}{c_{13}^2}} = 0.0$$

Vacuum

Denton & Parke



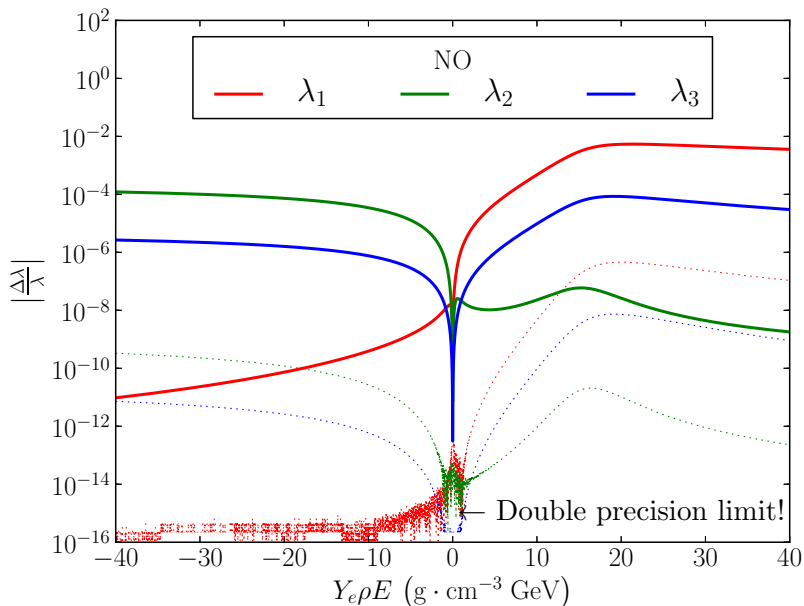
# 21 Triangle

# Exact Neutrino Oscillations in Matter: Mixing Angles

$$\begin{aligned}
 s_{12}^2 &= \frac{-\left[(\widehat{m}_2^2)^2 - \alpha \widehat{m}_2^2 + \beta\right] \Delta \widetilde{m}_{31}^2}{\left[(\widehat{m}_1^2)^2 - \alpha \widehat{m}_1^2 + \beta\right] \Delta \widetilde{m}_{32}^2 - \left[(\widehat{m}_2^2)^2 - \alpha \widehat{m}_2^2 + \beta\right] \Delta \widetilde{m}_{31}^2} \\
 s_{13}^2 &= \frac{(\widehat{m}_3^2)^2 - \alpha \widehat{m}_3^2 + \beta}{\Delta \widetilde{m}_{31}^2 \Delta \widetilde{m}_{32}^2} \\
 s_{23}^2 &= \frac{s_{23}^2 E^2 + c_{23}^2 F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2} \\
 e^{-i\delta} &= \frac{c_{23}^2 s_{23}^2 (e^{-i\delta} E^2 - e^{i\delta} F^2) + (c_{23}^2 - s_{23}^2) EF}{\sqrt{(s_{23}^2 E^2 + c_{23}^2 F^2 + 2EFc_{23}s_{23}c_\delta) (c_{23}^2 E^2 + s_{23}^2 F^2 - 2EFc_{23}s_{23}c_\delta)}} \\
 \alpha &= c_{13}^2 \Delta m_{31}^2 + (c_{12}^2 c_{13}^2 + s_{13}^2) \Delta m_{21}^2, \quad \beta = c_{12}^2 c_{13}^2 \Delta m_{21}^2 \Delta m_{31}^2 \\
 E &= c_{13}s_{13} \left[ \left( \widehat{m}_3^2 - \Delta m_{21}^2 \right) \Delta m_{31}^2 - s_{12}^2 \left( \widehat{m}_3^2 - \Delta m_{31}^2 \right) \Delta m_{21}^2 \right] \\
 F &= c_{12}s_{12}c_{13} \left( \widehat{m}_3^2 - \Delta m_{31}^2 \right) \Delta m_{21}^2
 \end{aligned}$$

H. Zaglauer, K. Schwarzer, [Z.Phys. C40 \(1988\) 273](#)

# Eigenvalues: Precision



## Hamiltonians

After a constant  $(\theta_{23}, \delta)$  rotation,  $2E\tilde{H} =$

$$\begin{pmatrix} \widetilde{m}_a^2 & & s_{13}c_{13}\Delta m_{ee}^2 \\ & \widetilde{m}_b^2 & \\ s_{13}c_{13}\Delta m_{ee}^2 & & \widetilde{m}_c^2 \end{pmatrix} + \epsilon s_{12}c_{12}\Delta m_{ee}^2 \begin{pmatrix} & c_{13} & \\ c_{13} & & -s_{13} \\ & -s_{13} & \end{pmatrix}$$

After a  $U_{13}(\tilde{\theta}_{13})$  rotation,  $2E\hat{H} =$

$$\begin{pmatrix} \widetilde{m}_-^2 & & \\ & \widetilde{m}_0^2 & \\ & & \widetilde{m}_+^2 \end{pmatrix} + \epsilon c_{12}s_{12}\Delta m_{ee}^2 \begin{pmatrix} & c_{(\tilde{\theta}_{13}-\theta_{13})} & \\ c_{(\tilde{\theta}_{13}-\theta_{13})} & & s_{(\tilde{\theta}_{13}-\theta_{13})} \\ & s_{(\tilde{\theta}_{13}-\theta_{13})} & \end{pmatrix}$$

After a  $U_{12}(\tilde{\theta}_{12})$  rotation,  $2E\check{H} =$

$$\begin{pmatrix} \widetilde{m}_1^2 & & \\ & \widetilde{m}_2^2 & \\ & & \widetilde{m}_3^2 \end{pmatrix} + \epsilon s_{(\tilde{\theta}_{13}-\theta_{13})} s_{12}c_{12}\Delta m_{ee}^2 \begin{pmatrix} & & -s_{\widetilde{12}} \\ & c_{\widetilde{12}} & \\ -s_{\widetilde{12}} & c_{\widetilde{12}} & \end{pmatrix}$$

# Perturbative Expansion

Hamiltonian:  $\check{H} = \check{H}_0 + \check{H}_1$

$$\check{H}_0 = \frac{1}{2E} \begin{pmatrix} \widetilde{m^2}_1 & & \\ & \widetilde{m^2}_2 & \\ & & \widetilde{m^2}_3 \end{pmatrix}, \quad \check{H}_1 = \epsilon' \frac{\Delta m_{ee}^2}{2E} \begin{pmatrix} & -s_{12} \widetilde{c_{12}} \\ -s_{12} \widetilde{c_{12}} & c_{12} \widetilde{c_{12}} \end{pmatrix}$$

Eigenvalues:  $\widetilde{m^2}_i^{\text{ex}} = \widetilde{m^2}_i + \widetilde{m^2}_i^{(1)} + \widetilde{m^2}_i^{(2)} + \dots$

$$\widetilde{m^2}_i^{(1)} = 2E(\check{H}_1)_{ii} = 0$$

$$\widetilde{m^2}_i^{(2)} = \sum_{k \neq i} \frac{[2E(\check{H}_1)_{ik}]^2}{\Delta \widetilde{m^2}_{ik}}$$

# Perturbative Expansion: Eigenvectors

Use vacuum expressions with  $U \rightarrow V$  where

$$V = \tilde{U}W$$

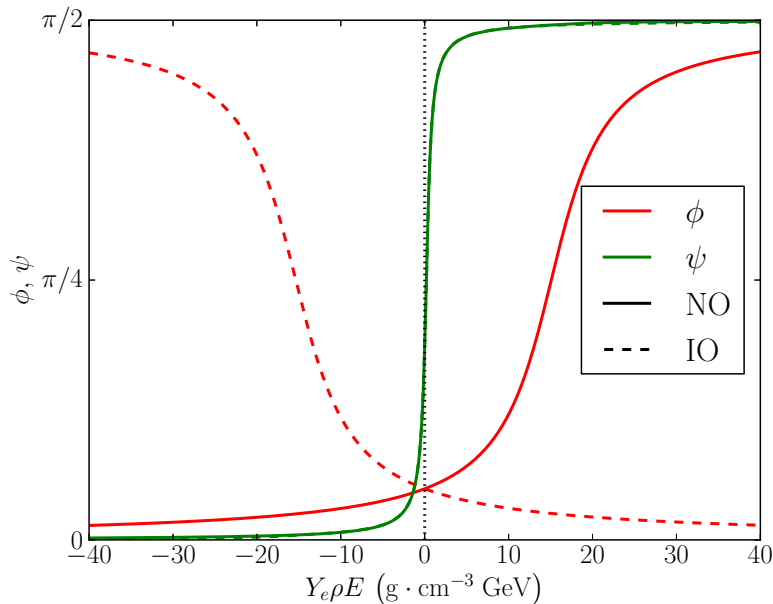
$\tilde{U}$  is  $U$  with  $\theta_{13} \rightarrow \tilde{\theta}_{13}$  and  $\theta_{12} \rightarrow \tilde{\theta}_{12}$ ,

$$W = W_0 + W_1 + W_2 + \dots \quad W_0 = \mathbb{1}$$

$$W_1 = \epsilon' \Delta m_{ee}^2 \begin{pmatrix} & -\frac{\widetilde{s_{12}}}{\widetilde{\Delta m^2_{31}}} \\ \frac{\widetilde{s_{12}}}{\widetilde{\Delta m^2_{31}}} & -\frac{\widetilde{c_{12}}}{\widetilde{\Delta m^2_{32}}} \end{pmatrix}$$

$$W_2 = -\epsilon'^2 \frac{(\Delta m_{ee}^2)^2}{2} \begin{pmatrix} \frac{\widetilde{s_{12}^2}}{(\widetilde{\Delta m^2_{31}})^2} & -\frac{\widetilde{s_{212}}}{\widetilde{\Delta m^2_{32}} \widetilde{\Delta m^2_{21}}} \\ \frac{\widetilde{s_{212}}}{\widetilde{\Delta m^2_{31}} \widetilde{\Delta m^2_{21}}} & \frac{\widetilde{c_{12}^2}}{(\widetilde{\Delta m^2_{32}})^2} \end{pmatrix} \left[ \frac{\widetilde{c_{12}^2}}{(\widetilde{\Delta m^2_{32}})^2} + \frac{\widetilde{s_{12}^2}}{(\widetilde{\Delta m^2_{31}})^2} \right]$$

# The Two Matter Angles



# Zeroth Order Coefficients

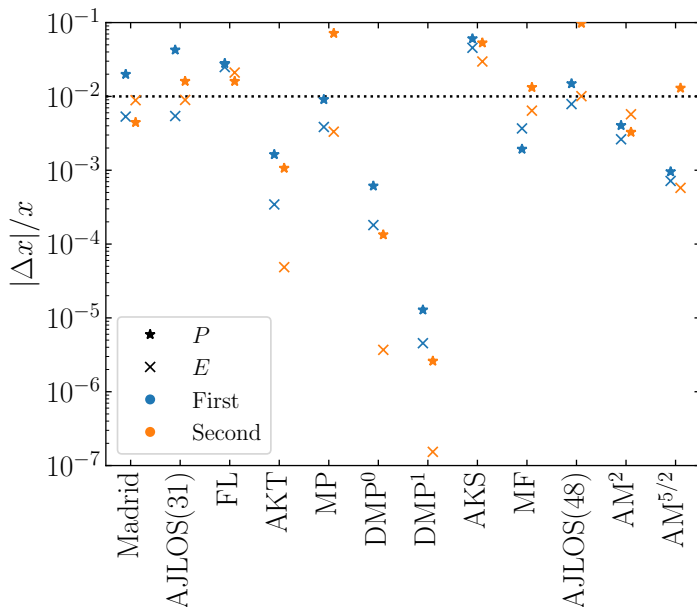
$$P_{\alpha\beta} = \delta_{\alpha\beta} + 4C_{21}^{\alpha\beta} \sin^2 \Delta_{21} + 4C_{31}^{\alpha\beta} \sin^2 \Delta_{31} + 4C_{32}^{\alpha\beta} \sin^2 \Delta_{32} \\ + 8D^{\alpha\beta} \sin \Delta_{21} \sin \Delta_{31} \sin \Delta_{32}$$

$\nu_\alpha \rightarrow \nu_\beta$	$(C_{21}^{\alpha\beta})^{(0)}$	
$\nu_e \rightarrow \nu_e$	$-c_{13}^2 \widetilde{s_{12}^2} \widetilde{c_{12}^2}$	
$\nu_\mu \rightarrow \nu_e$	$\widetilde{c_{13}^2} \widetilde{s_{12}^2} \widetilde{c_{12}^2} (c_{23}^2 - \widetilde{s_{13}^2} \widetilde{s_{23}^2}) + c_{212} \widetilde{J_r^m} c_\delta$	
$\nu_\mu \rightarrow \nu_\mu$	$-(c_{23}^2 \widetilde{c_{12}^2} + s_{23}^2 \widetilde{s_{13}^2} \widetilde{s_{12}^2}) (c_{23}^2 \widetilde{s_{12}^2} + s_{23}^2 \widetilde{s_{13}^2} \widetilde{c_{12}^2}) \\ - 2(c_{23}^2 - \widetilde{s_{13}^2} \widetilde{s_{23}^2}) c_{212} \widetilde{J_{rr}^m} c_\delta + (2 \widetilde{J_{rr}^m} c_\delta)^2$	
$\nu_\alpha \rightarrow \nu_\beta$	$(C_{31}^{\alpha\beta})^{(0)}$	$(D^{\alpha\beta})^{(0)}$
$\nu_e \rightarrow \nu_e$	$-c_{13}^2 \widetilde{s_{13}^2} \widetilde{c_{12}^2}$	0
$\nu_\mu \rightarrow \nu_e$	$\widetilde{s_{13}^2} \widetilde{c_{13}^2} \widetilde{c_{12}^2} \widetilde{s_{23}^2} + \widetilde{J_r^m} c_\delta$	$-\widetilde{J_r^m} s_\delta$
$\nu_\mu \rightarrow \nu_\mu$	$-\widetilde{c_{13}^2} \widetilde{s_{23}^2} (\widetilde{c_{23}^2} \widetilde{s_{12}^2} + s_{23}^2 \widetilde{s_{13}^2} \widetilde{c_{12}^2}) \\ - 2s_{23}^2 \widetilde{J_r^m} c_\delta$	0

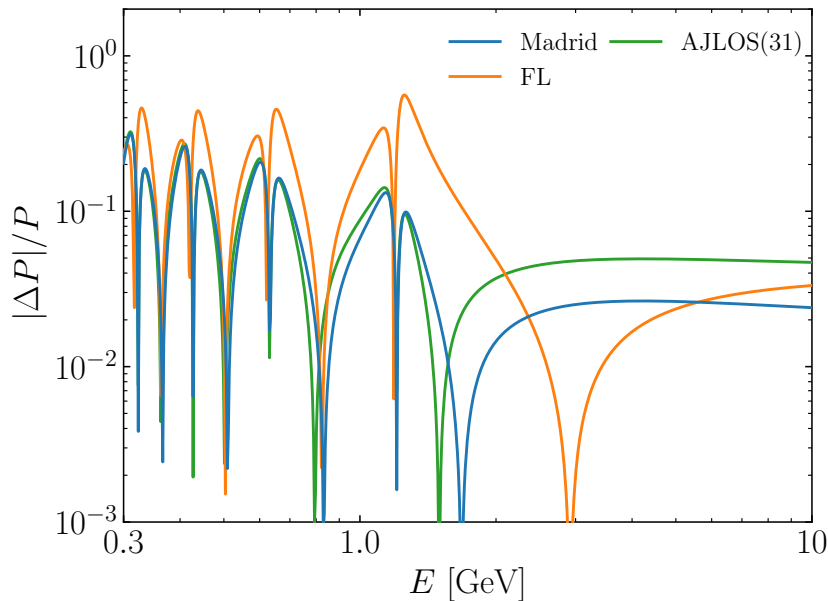
$$\widetilde{J_r^m} \equiv \widetilde{s_{12}} \widetilde{c_{12}} \widetilde{s_{13}} \widetilde{c_{13}^2} s_{23} c_{23}, \quad \widetilde{J_{rr}^m} \equiv \widetilde{J_r^m} / \widetilde{c_{13}^2}$$



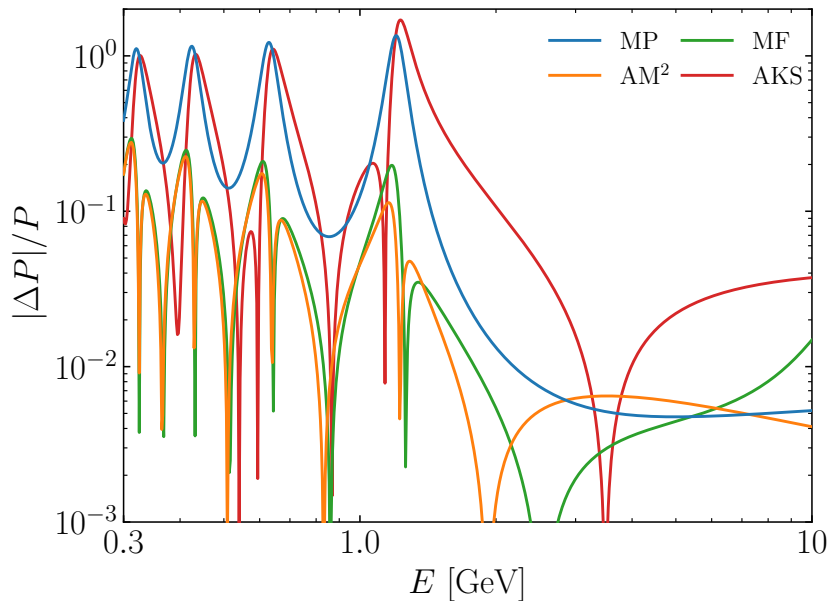
# Comparative Precision: At the Peaks



# Comparative Precision



# Comparative Precision



# Use DMP!

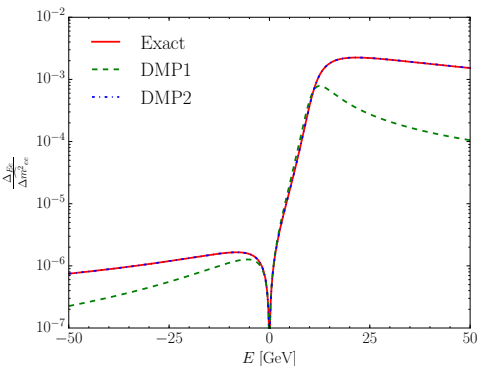
At zeroth order:  $\Delta_{Ee}^{(0)} = 0$

At first order, only correction is to  $\tilde{\theta}_{12}$ ,

PBD, S. Parke, X. Zhang, [1806.01277](#)

$$\Delta_{Ee}^{(1)} = t_{13}^2 s_{12}^2 c_{12}^2 \sin 2\theta_{13} a \frac{(\Delta m_{21}^2)^2}{\widetilde{\Delta m_{32}^2} \widetilde{\Delta m_{31}^2}}$$

At second order eigenvalues are also corrected,  $\Delta_{Ee}^{(2)} = \dots$



Error is quantified with DMP<sup>2</sup>:

- ▶ First order isn't enough, ...
- ▶ Second is
- ▶ Exact in vacuum

# Angles in Matter

Angles receive corrections at first order:

$$\widetilde{\theta}_{12}^{(1)} = \epsilon' \Delta m_{ee}^2 s_{12} \widetilde{c}_{12} \left( \frac{1}{\widetilde{\Delta m^2_{32}}} - \frac{1}{\widetilde{\Delta m^2_{31}}} \right)$$

$$\widetilde{\theta}_{13}^{(1)} = -\epsilon' \Delta m_{ee}^2 \frac{s_{13} \widetilde{c}_{13}}{c_{13}} \left( \frac{s_{12}^2}{\widetilde{\Delta m^2_{31}}} + \frac{c_{12}^2}{\widetilde{\Delta m^2_{32}}} \right)$$

$$\widetilde{\theta}_{23}^{(1)} = \epsilon' \Delta m_{ee}^2 \frac{c_\delta}{c_{13}} \left( \frac{s_{12}^2}{\widetilde{\Delta m^2_{31}}} + \frac{c_{12}^2}{\widetilde{\Delta m^2_{32}}} \right)$$

$$\widetilde{\delta}^{(1)} = -\epsilon' \Delta m_{ee}^2 \frac{2c_{223}s_\delta}{s_{223}c_{13}} \left( \frac{s_{12}^2}{\widetilde{\Delta m^2_{31}}} + \frac{c_{12}^2}{\widetilde{\Delta m^2_{32}}} \right)$$

Second order: see paper