

Abstract

We further develop and extend a recent perturbative framework for neutrino oscillations in uniform matter density so that the resulting oscillation probabilities are accurate for the complete matter potential versus baseline divided by neutrino energy plane. This extension also gives the exact oscillation probabilities in vacuum for all values of baseline divided by neutrino energy. The expansion parameter used is related to the ratio of the solar to the atmospheric Δm^2 scales but with a unique choice of the atmospheric Δm^2 such that certain first-order effects are taken into account in the zeroth-order Hamiltonian. Using a mixing matrix formulation, this framework has the exceptional feature that the neutrino oscillation probability in matter has the same structure as in vacuum, to all orders in the expansion parameter. It also contains all orders in the matter potential and $\sin \theta_{13}$. It facilitates immediate physical interpretation of the analytic results, and makes the expressions for the neutrino oscillation probabilities extremely compact and very accurate even at zeroth order in our perturbative expansion. The first and second order results are also given which improve the precision by approximately two or more orders of magnitude per perturbative order.

Analytic and compact perturbative expressions for neutrino oscillations in matter

Peter B. Denton

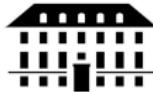
Campinas State University

October 19, 2017

Work done with S. Parke and H. Minakata.

[1604.08167 JHEP 1606 \(2016\) 051](https://arxiv.org/abs/1604.08167)

github.com/PeterDenton/Nu-Pert



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Neutrino oscillations in vacuum: disappearance

It is easy to calculate the *exact* disappearance expression in vacuum:

$$P(\nu_\alpha \rightarrow \nu_\alpha) = 1 - 4 \sum_{i < j} |U_{\alpha i}|^2 |U_{\alpha j}|^2 \sin^2 \Delta_{ji}.$$

For the electron case this expression is simple:

$$\begin{aligned} P(\nu_e \rightarrow \nu_e) = 1 \\ &\quad - 4c_{12}^2 s_{12}^2 c_{13}^4 \sin^2 \Delta_{21} \\ &\quad - 4c_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{31} \\ &\quad - 4s_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{32}. \end{aligned}$$

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$
$$\Delta m_{ij}^2 = m_i^2 - m_j^2$$

Neutrino oscillations in matter

The electron *flavor* interacts differently, modifying the Hamiltonian.

$$H^m = \frac{1}{2E} \left[U \begin{pmatrix} 0 & & \\ & \Delta m_{21}^2 & \\ & & \Delta m_{31}^2 \end{pmatrix} U^\dagger + \begin{pmatrix} a & & \\ & & \end{pmatrix} \right].$$

We could write

$$H^m = \frac{1}{2E} U^m \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} U^{m\dagger},$$

where the λ_i are the masses squared in matter,
and $\theta_{12}^m, \theta_{13}^m, \theta_{23}^m$, and δ^m and the angles and phase in matter.

$$\begin{aligned} U &\equiv U_{23}(\theta_{23}, \delta) U_{13}(\theta_{13}) U_{12}(\theta_{12}), \\ a &= 2EV_{\text{CC}} = 2\sqrt{2}G_F N_e E, \\ \Delta m_{ji}^2 &= m_j^2 - m_i^2. \end{aligned}$$

Exact neutrino oscillations in matter: eigenvalues

This has been done.

H. Zaglauer, K. Schwarzer, Z. Phys. C Particles and Fields, 40 (1988)

$$\lambda_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}S - \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\lambda_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3B}S + \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\lambda_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2 - 3B}S$$

$$A = \Delta m_{21}^2 + \Delta m_{31}^2 + a$$

$$B = \Delta m_{21}^2 \Delta m_{31}^2 + a [c_{13}^2 \Delta m_{31}^2 + (c_{12}^2 c_{13}^2 + s_{13}^2) \Delta m_{21}^2]$$

$$S = \cos \left\{ \frac{1}{3} \cos^{-1} \left[\frac{2A^3 - 9AB + 27C}{2(A^2 - 3B)^{3/2}} \right] \right\}$$

$$C = a \Delta m_{21}^2 \Delta m_{31}^2 c_{12}^2 c_{13}^2$$

Alternative solutions

- ▶ Numerical methods:

- ▶ Good for experiments.
- ▶ Important understanding can be missed (magic baseline).

P. Huber, W. Winter, [hep-ph/0301257](#)

A. Smirnov, [hep-ph/0610198](#)

- ▶ Perturbative expansion:

- ▶ Small matter potential: $a/\Delta m^2$.

Y. Li, Y. Wang, Z-z. Xing, [1605.00900](#)

- ▶ s_{13}, s_{13}^2 .

A. Cervera, et. al., [hep-ph/0002108](#)

H. Minakata, [0910.5545](#)

K. Asano, H. Minakata, [1103.4387](#)

- ▶ $\Delta m_{21}^2/\Delta m_{31}^2 \sim 0.03$.

J. Arafune, J. Sato, [hep-ph/9607437](#)

A. Cervera, et. al., [hep-ph/0002108](#)

M. Freund, [hep-ph/0103300](#)

E. Akhmedov, et. al., [hep-ph/0402175](#)

H. Minakata, S. Parke, [1505.01826](#)

PBD, H. Minakata, S. Parke, [1604.08167](#)

Alternative solutions: example

$$P_0 = \sin^2 \theta_{23} \frac{\sin^2 2 \theta_{13}}{\hat{C}^2} \sin^2(\hat{A}\hat{C}), \quad (36a)$$

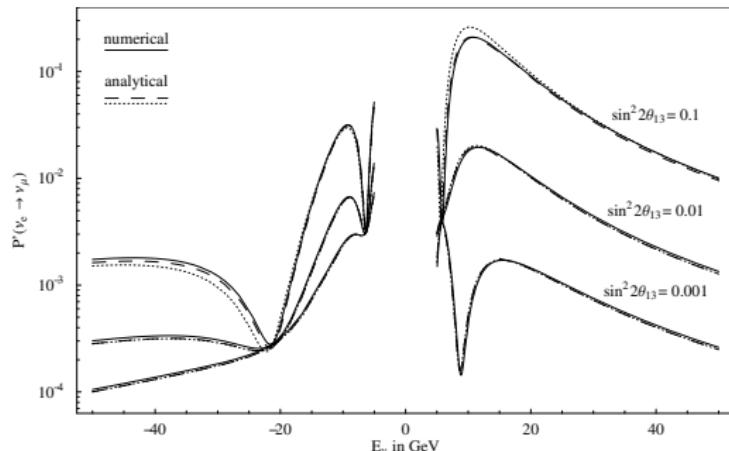
$$\begin{aligned} P_{\sin \delta} &= \frac{1}{2} \alpha \frac{\sin \delta \cos \theta_{13} \sin 2 \theta_{12} \sin 2 \theta_{13} \sin 2 \theta_{23}}{\hat{A}\hat{C} \cos \theta_{13}^2} \sin(\hat{C}\hat{\Delta}) \\ &\times \{\cos(\hat{C}\hat{\Delta}) - \cos((1+\hat{A})\hat{\Delta})\}, \end{aligned} \quad (36b)$$

$$\begin{aligned} P_{\cos \delta} &= \frac{1}{2} \alpha \frac{\cos \delta \cos \theta_{13} \sin 2 \theta_{12} \sin 2 \theta_{13} \sin 2 \theta_{23}}{\hat{A}\hat{C} \cos \theta_{13}^2} \sin(\hat{C}\hat{\Delta}) \\ &\times \{\sin((1+\hat{A})\hat{\Delta}) \mp \sin(\hat{C}\hat{\Delta})\}, \end{aligned} \quad (36c)$$

$$\begin{aligned} P_1 &= -\alpha \frac{1 - \hat{A} \cos 2 \theta_{13}}{\hat{C}^3} \sin^2 \theta_{12} \sin^2 2 \theta_{13} \sin^2 \theta_{23} \Delta \\ &\times \sin(2\hat{A}\hat{C}) + \alpha \frac{2\hat{A}(-\hat{A} + \cos 2 \theta_{13})}{\hat{C}^4} \\ &\times \sin^2 \theta_{12} \sin^2 2 \theta_{13} \sin^2 \theta_{23} \sin^2(\hat{A}\hat{C}), \end{aligned} \quad (36d)$$

$$\begin{aligned} P_2 &= \alpha \frac{\mp 1 + \hat{C} \pm \hat{A} \cos 2 \theta_{13}}{2\hat{C}^2\hat{A} \cos^2 \theta_{13}} \cos \theta_{13} \sin 2 \theta_{12} \sin 2 \theta_{13} \\ &\times \sin 2 \theta_{23} \sin^2(\hat{A}\hat{C}), \end{aligned} \quad (36e)$$

$$\begin{aligned} P_3 &= \alpha^2 \frac{2\hat{C} \cos^2 \theta_{23} \sin^2 \theta_{12}}{\hat{A}^2 \cos^2 \theta_{13} (\mp \hat{A} + \hat{C} \pm \cos 2 \theta_{13})} \\ &\times \sin^2 \left(\frac{1}{2}(1 + \hat{A} \mp \hat{C})\Delta \right), \end{aligned} \quad (36f)$$



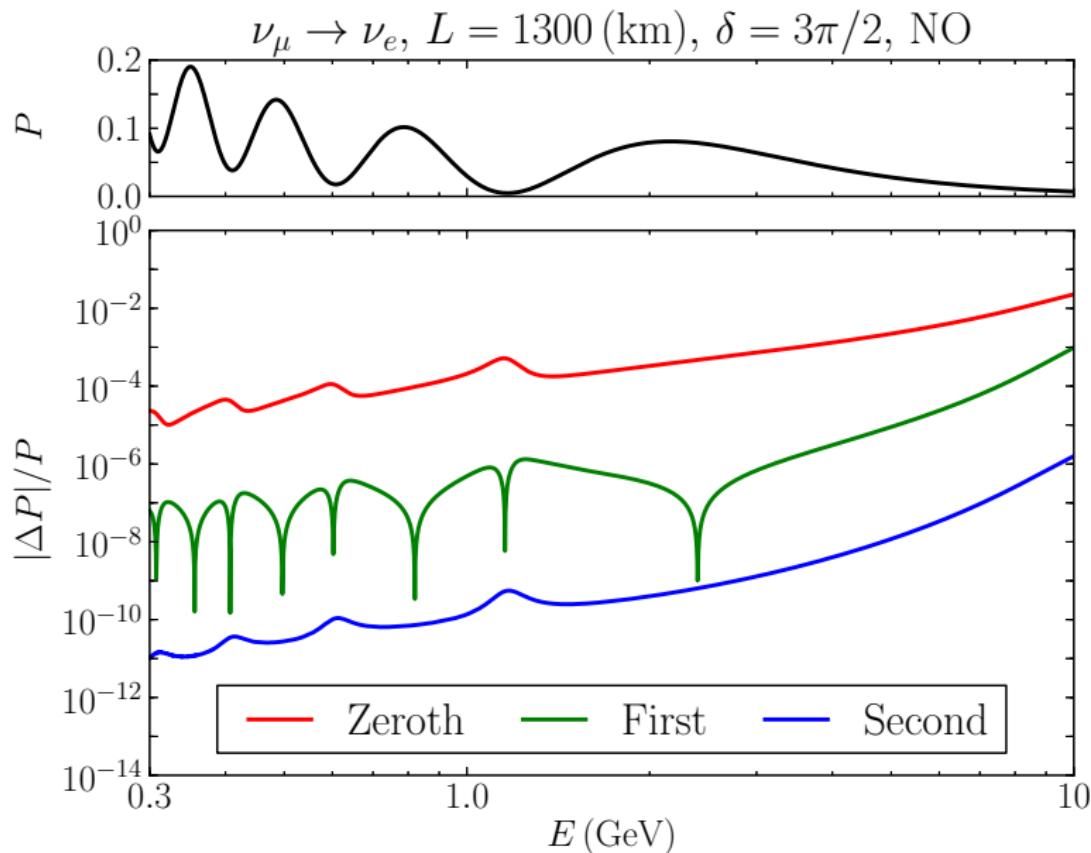
M. Freund, [hep-ph/0103300](https://arxiv.org/abs/hep-ph/0103300)

A perturbative description of neutrino oscillations in matter

Several goals:

1. Use $\epsilon \simeq \Delta m_{21}^2 / \Delta m_{31}^2 = 0.03$ as an expansion parameter.
2. Minimize the number of additional ‘matter’ angles/phases (2).
3. Extremely accurate for all channels, L/E , orderings, CP phases, . . .
4. Exact in the vacuum limit.
5. Reproduce the known CPV term in matter.

Precision order-by-order



Our methodology

- ▶ Start with $\epsilon = \frac{\Delta m_{21}^2}{\Delta m_{ee}^2} = 0.03$.
- ▶ Perform one fixed and two variable rotations: $(\theta_{23}, \delta), \phi, \psi$.
- ▶ Write the probabilities with simple L/E dependence:

$$P(\nu_\alpha \rightarrow \nu_\beta) = \delta_{\alpha\beta} - \sum_{i < j} \Re [U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}] \sin^2 \Delta_{ij}$$
$$+ 8 \Im [U_{\alpha 1} U_{\beta 2}^* U_{\alpha 2}^* U_{\beta 1}] \sin \Delta_{32} \sin \Delta_{31} \sin \Delta_{21}$$

C. Jarlskog: [PRL 55 \(1985\)](#)

Nonvanishing Wronskian \Rightarrow fewest number of L/E functions.

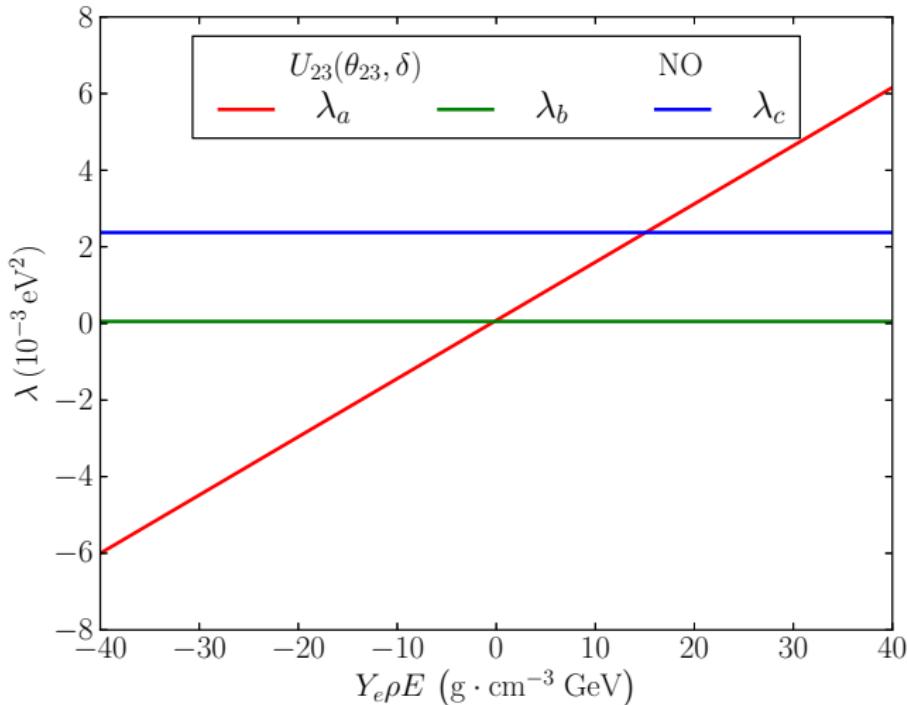
Clear that the **CPV** term is $\mathcal{O}[(L/E)^3]$ not $\mathcal{O}[(L/E)^1]$.

$$\Delta m_{ee}^2 = \Delta m_{31}^2 - s_{12}^2 \Delta m_{21}^2$$

H. Nunokawa, S. Parke, R. Zukanovich Funchal, [hep-ph/0503283](#)

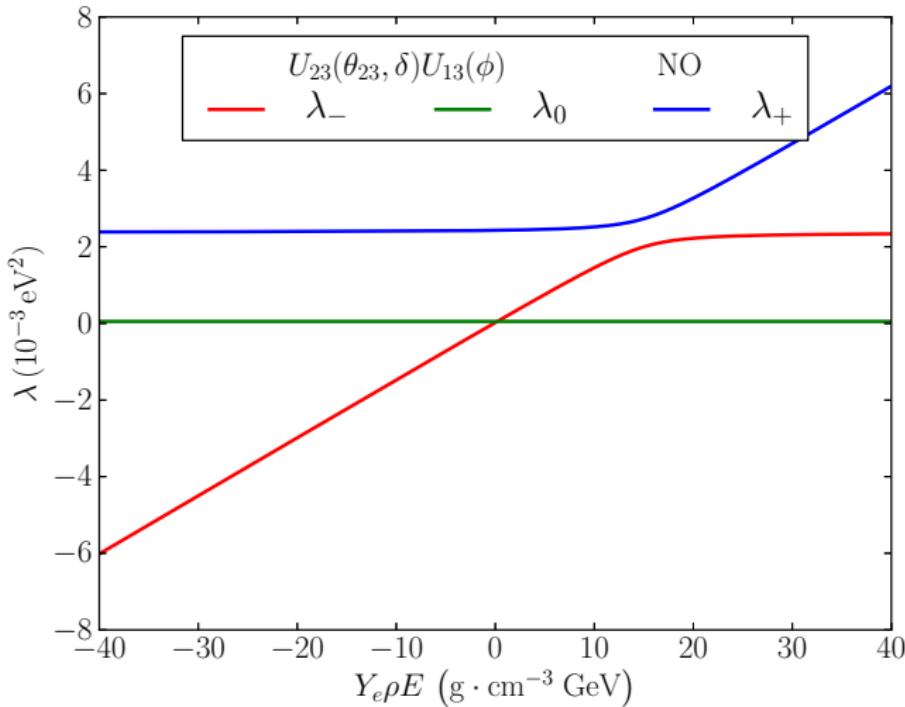
$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$

Eigenvalues in matter: two rotations are needed



$$\lambda_a = a + (s_{13}^2 + \epsilon s_{12}^2) \Delta m_{ee}^2, \quad \lambda_b = \epsilon c_{12}^2 \Delta m_{ee}^2, \quad \lambda_c = (c_{13}^2 + \epsilon s_{12}^2) \Delta m_{ee}^2.$$

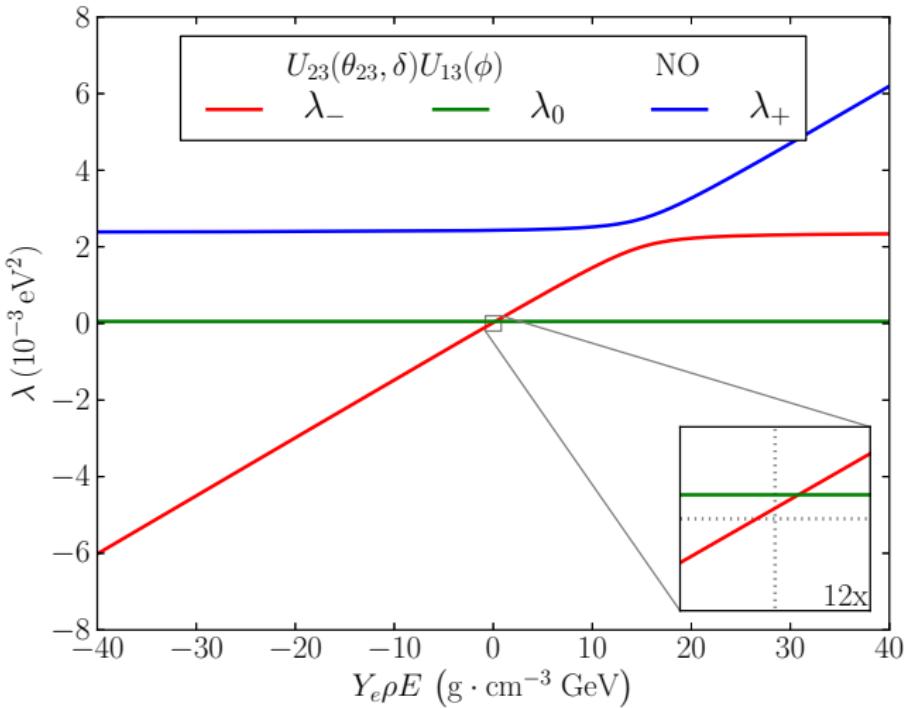
Eigenvalues in matter: two rotations are needed



$$\lambda_{\mp} = \frac{1}{2} \left[(\lambda_a + \lambda_c) \mp \text{sign}(\Delta m_{ee}^2) \sqrt{(\lambda_c - \lambda_a)^2 + 4(s_{13}c_{13}\Delta m_{ee}^2)^2} \right],$$

$$\lambda_0 = \lambda_b.$$

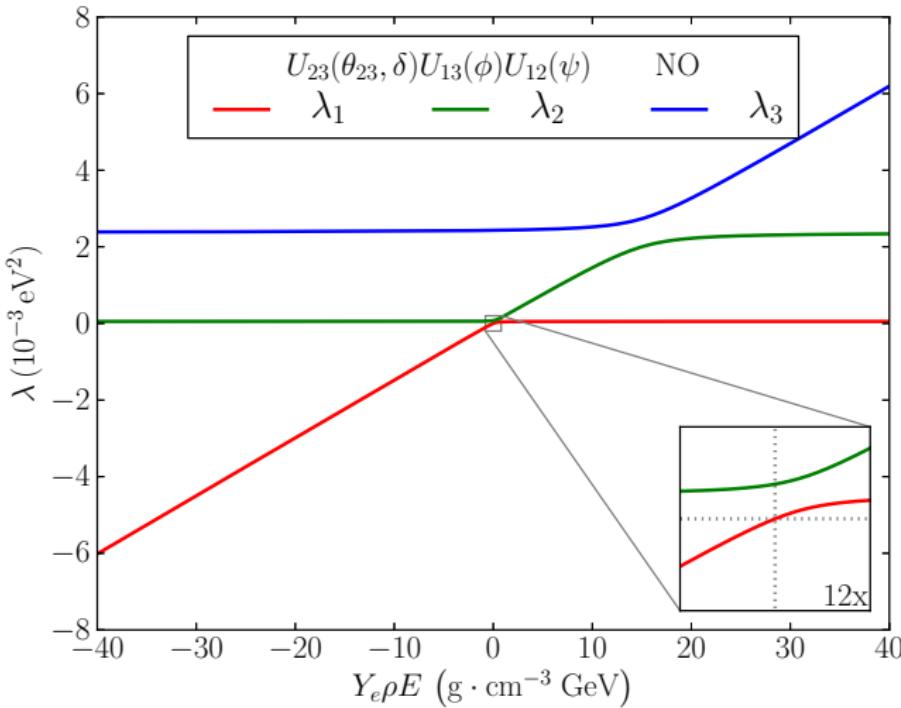
Eigenvalues in matter: two rotations are needed



$$\lambda_{\mp} = \frac{1}{2} \left[(\lambda_a + \lambda_c) \mp \text{sign}(\Delta m_{ee}^2) \sqrt{(\lambda_c - \lambda_a)^2 + 4(s_{13}c_{13}\Delta m_{ee}^2)^2} \right],$$

$$\lambda_0 = \lambda_b.$$

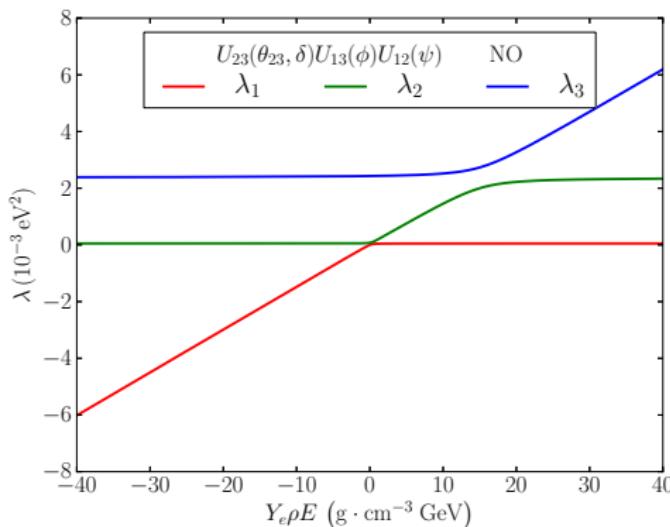
Eigenvalues in matter: two rotations are needed



$$\lambda_{1,2} = \frac{1}{2} \left[(\lambda_0 + \lambda_-) \pm \sqrt{(\lambda_0 - \lambda_-)^2 + 4(\epsilon c_{(\phi-\theta_{13})} c_{12} s_{12} \Delta m_{ee}^2)^2} \right],$$

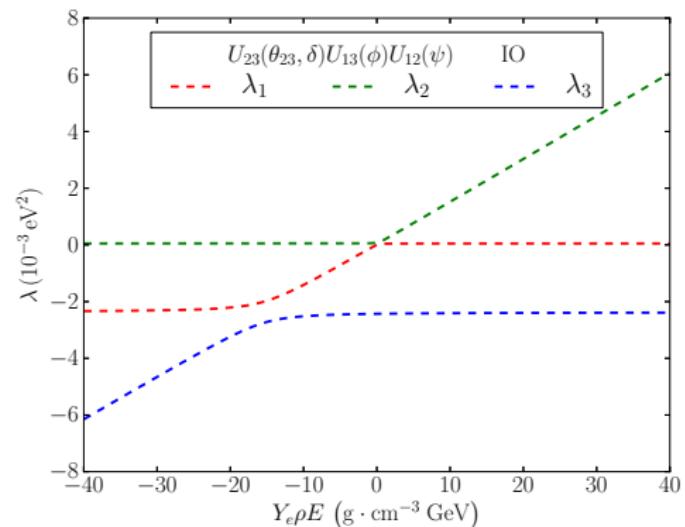
$$\lambda_3 = \lambda_+.$$

Eigenvalues in matter: mass ordering



NO

$$\lambda_1 < \lambda_2 < \lambda_3$$



IO

$$\lambda_3 < \lambda_1 < \lambda_2$$

$1 + 2$ rotations

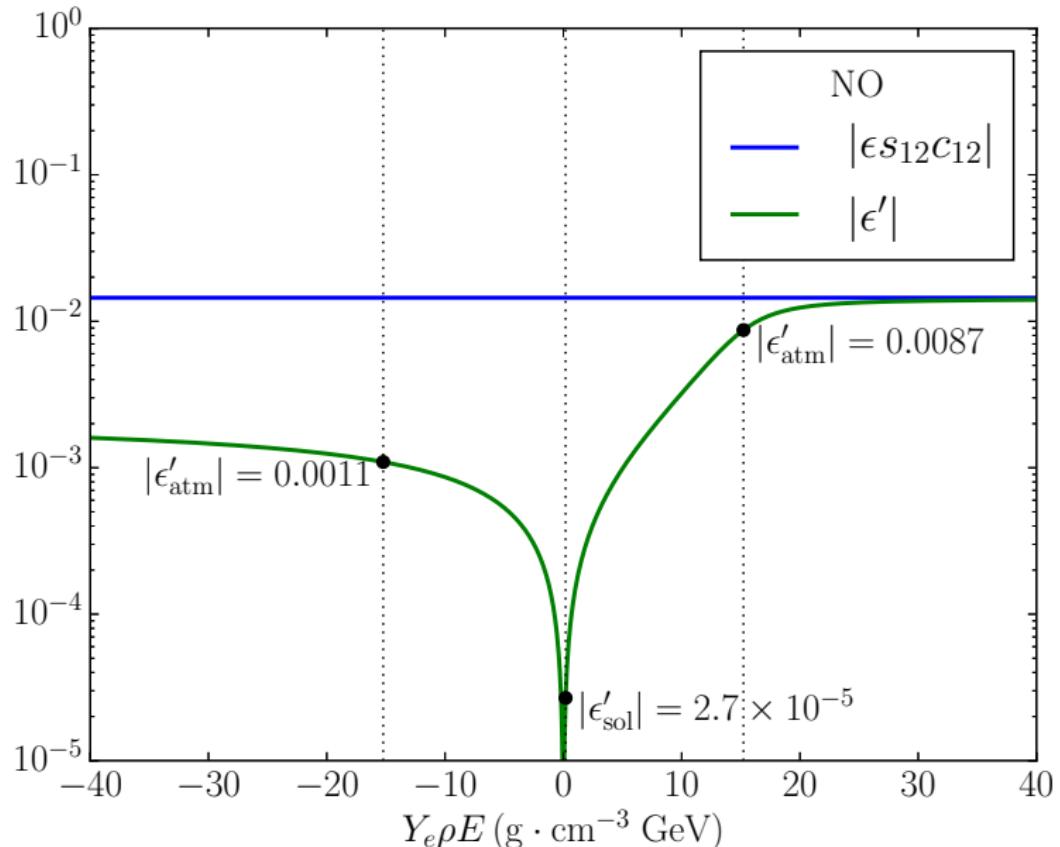
1. Perform a constant $U_{23}(\theta_{23}, \delta)$ rotation.
 - ▶ U_{23} commutes with the matter potential.
 - ▶ Resultant Hamiltonian is real.
 - ▶ ‘Expansion parameter’ is $c_{13}s_{13} = 0.15$ at this point.
2. Diagonalize the diagonal and $\mathcal{O}(\epsilon^0)$ off-diagonal terms with $U_{13}(\phi)$.
 - ▶ $\phi(a = 0) = \theta_{13}$.
 - ▶ Expansion parameter is $c_{12}s_{12}\frac{\Delta m_{21}^2}{\Delta m_{ee}^2} = 0.015$.

H. Minakata, S. Parke, [1505.01826](#)

3. Diagonalize the terms non-zero in vacuum with $U_{12}(\psi)$.

- ▶ $\psi(a = 0) = \theta_{12}$.
- ▶ Expansion parameter is now $\epsilon' = c_{12}s_{12} \cancel{s}_{(\phi - \theta_{13})} \frac{\Delta m_{21}^2}{\Delta m_{ee}^2} < 0.015$.
- ▶ $\epsilon'(a = 0) = 0$.

Expansion parameter



Angles in matter

The $\theta_{13} \rightarrow \phi$ angle in matter:

$$c_{2\phi} = \frac{\lambda_c - \lambda_a}{\lambda_+ - \lambda_-}, \quad s_{2\phi} = s_{2\theta_{13}} \frac{\Delta m_{ee}^2}{\lambda_+ - \lambda_-}.$$

The $\theta_{12} \rightarrow \psi$ angle matter:

$$c_{2\psi} = \frac{\lambda_0 - \lambda_-}{\lambda_2 - \lambda_1}, \quad s_{2\psi} = s_{2\theta_{12}} \epsilon c_{(\phi - \theta_{13})} \frac{\Delta m_{ee}^2}{\lambda_2 - \lambda_1}.$$

Simplicity at zeroth order

Simple L/E dependence:

$$P = \delta^{\alpha\beta} + 4C_{21}^{\alpha\beta} \sin^2 \Delta_{21} + 4C_{31}^{\alpha\beta} \sin^2 \Delta_{31} + 4C_{32}^{\alpha\beta} \sin^2 \Delta_{32} \\ + 8D^{\alpha\beta} \sin \Delta_{21} \sin \Delta_{31} \sin \Delta_{32}.$$

Zeroth order is equivalent to the vacuum probability with

$$\theta_{13} \rightarrow \phi, \quad \theta_{12} \rightarrow \psi,$$

$$\Delta m_{ij}^2 \rightarrow \Delta \lambda_{ij}.$$

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$

CPV term

It is known that the exact CPV term in matter is

$$P \supset \pm 8 \sin \delta c_{12} s_{12} c_{13}^2 s_{13} c_{23} s_{23} \prod_{i>j} \frac{\Delta m_{ij}^2}{\Delta \lambda_{ij}} \sin \Delta_{32}^m \sin \Delta_{31}^m \sin \Delta_{21}^m.$$

V. Naumov, Int. J. Mod. Phys. 1992

P. Harrison, W. Scott, [hep-ph/9912435](https://arxiv.org/abs/hep-ph/9912435)

Our expression reproduces this order by order in ϵ' .

Precision

At the first oscillation minimum and maximum:

DUNE: NO, $\delta = 3\pi/2$		First min	First max
$P(\nu_\mu \rightarrow \nu_e)$		0.0047	0.081
E (GeV)		1.2	2.2
$\frac{ \Delta P }{P}$	Zeroth	5×10^{-4}	4×10^{-4}
	First	3×10^{-7}	2×10^{-7}
	Second	6×10^{-10}	5×10^{-10}

Conclusions

- ▶ Matter effects provide a rich environment to test the neutrino sector.
- ▶ Analytic expression for neutrino oscillations with precision and clarity.
- ▶ Clear that exactly two matter rotations is the correct number.
- ▶ Our perturbative parameter is < 0.015 and is 0 in vacuum.
- ▶ Form emphasizes the L/E dependence of oscillations at all orders.
- ▶ Zeroth order expression matches the form of the vacuum expression.
- ▶ Zeroth order is sufficient for current and planned experiments.
- ▶ Code is available on github at github.com/PeterDenton/Nu-Pert.

Backups



Takaaki Kajita
Super K



Arthur McDonald
SNO

"for the discovery of neutrino oscillations,
which shows that neutrinos have mass"

Neutrino oscillations in vacuum

Neutrino oscillations in matter are described by this Hamiltonian written in the flavor basis:

$$H = \frac{1}{2E} U \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix} U^\dagger,$$

where the unitary mixing matrix is parameterized by

$$\begin{aligned} U &= \begin{pmatrix} 1 & & \\ c_{23} & s_{23}e^{i\delta} & \\ -s_{23}e^{-i\delta} & c_{23} & \end{pmatrix} \begin{pmatrix} c_{13} & & \\ & 1 & \\ -s_{13} & c_{13} & \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13} \\ -c_{23}s_{12} - e^{i\delta}c_{12}s_{13}s_{23} & c_{12}c_{23} - e^{i\delta}s_{12}s_{13}s_{23} & e^{i\delta}c_{13}s_{23} \\ e^{-i\delta}s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{23}s_{12}s_{13} - e^{-i\delta}c_{12}s_{23} & c_{13}c_{23} \end{pmatrix}. \end{aligned}$$

Neutrino properties

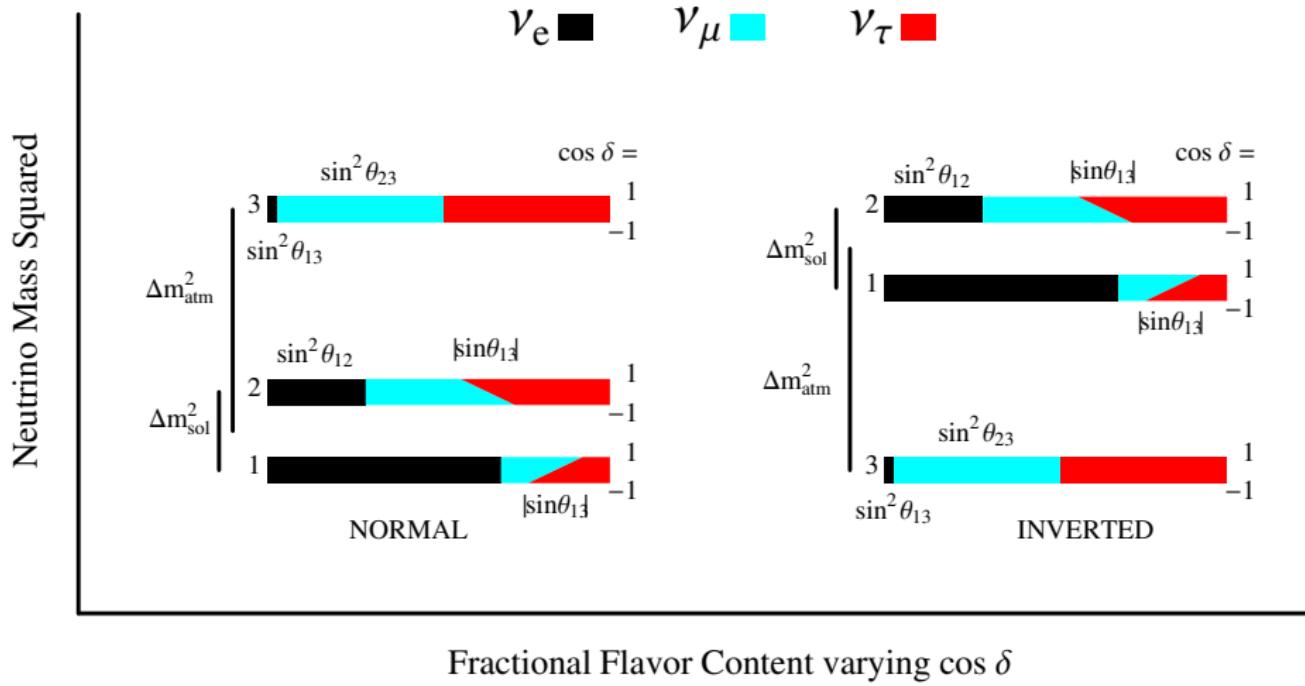
Current status:

- ▶ $\sin^2 \theta_{13} \sim 0.02.$
- ▶ $\Delta m_{21}^2 = 7.6 \times 10^{-5} \text{ eV}^2.$
- ▶ $\sin^2 \theta_{12} \sim 0.3.$
- ▶ $|\Delta m_{31}^2| \approx |\Delta m_{32}^2| \sim 2.4 \times 10^{-3} \text{ eV}^2.$
- ▶ $\sin^2 \theta_{23} \sim 0.5.$
- ▶ $\frac{|\Delta m_{31}^2|}{\Delta m_{21}^2} \sim 30.$
- ▶ $\delta \sim ???$

Unknowns:

- ▶ δ : CP phase.
- ▶ $\sin^2 \theta_{23} < 0.5$ or $\sin^2 \theta_{23} > 0.5$: octant (dominant flavor ν_3).
- ▶ $\Delta m_{31}^2 > 0$ or $\Delta m_{31}^2 < 0$: mass ordering.
- ▶ m_1^2 : absolute mass scale.
- ▶ α_1, α_2 : Majorana phases.
- ▶ Anomalies: Short baseline, reactor, gallium, ...
- ▶ New physics: NSI, magnetic moment, decay, steriles, ...

Neutrino properties



O. Mena, S. Parke, [hep-ph/0312131](https://arxiv.org/abs/hep-ph/0312131)

Neutrino oscillations in vacuum: appearance

For the $\nu_\mu \rightarrow \nu_e$ appearance channel,

$$A_{\mu e} = A_{31} + A_{21} e^{i(\delta + \Delta_{32})},$$

where

$$A_{31} = 2|U_{\mu 3}^*| U_{e 3} \sin \Delta_{31} = 2 s_{23} c_{13} s_{13} \sin \Delta_{31},$$

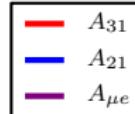
$$A_{21} = 2 U_{\mu 2}^* U_{e 2} \sin \Delta_{21} \approx 2 [c_{12} c_{23} s_{12} c_{13} + \mathcal{O}(s_{13})] \sin \Delta_{21}.$$

Neutrino oscillations in vacuum: appearance

$$\delta = 0.0\pi$$

$$\Delta_{32} = 0.5\pi$$

NO



$$P(\nu_\mu \rightarrow \nu_e) = A_{\mu e} A_{\mu e}^*$$

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Neutrino oscillations in vacuum: appearance

$$\delta = 0.0\pi$$

$$\Delta_{32} = 0.5\pi$$

NO

- A_{31}
- A_{21}
- $A_{\mu e}$
- $\bar{A}_{\mu e}^*$

$$P(\nu_\mu \rightarrow \nu_e) = A_{\mu e} A_{\mu e}^*$$
$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \bar{A}_{\mu e}^* \bar{A}_{\mu e}$$

Denton & Parke

Neutrino oscillations in vacuum: appearance

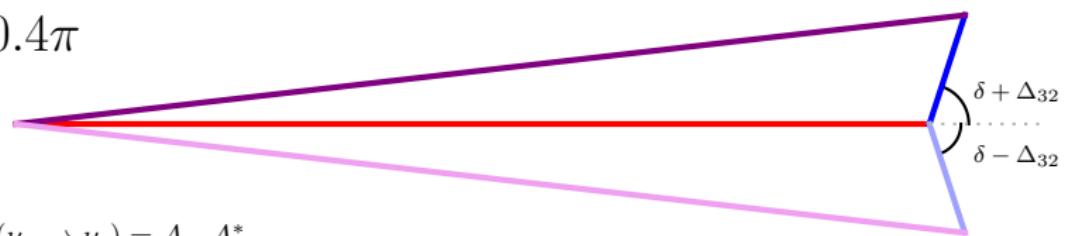
$$\delta = 0.0\pi$$

$$\Delta_{32} = 0.4\pi$$

NO

- A_{31}
- A_{21}
- $A_{\mu e}$
- $\bar{A}_{\mu e}^*$

$$P(\nu_\mu \rightarrow \nu_e) = A_{\mu e} A_{\mu e}^*$$
$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \bar{A}_{\mu e}^* \bar{A}_{\mu e}$$



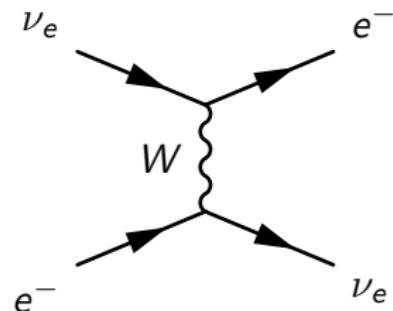
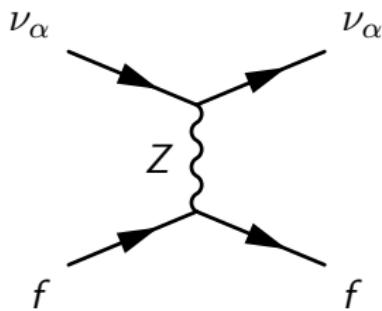
Denton & Parke

Neutrino oscillations in vacuum: appearance

Matter effects

Oscillation probabilities are different in matter.

L. Wolfenstein, PRD 17 (1978)



$$V_{NC} = \mp \sqrt{2} G_F \frac{1}{2} N_n$$

$$V_{CC} = \pm \sqrt{2} G_F N_e$$

Resolved the solar neutrino problem (MSW effect).

S. Mikheyev, A. Smirnov, Sov. J. Nucl. Phys. 42 (1985)

$N_{n(e)}$ is the neutron (electron) number density,
upper (lower) sign is for (anti) neutrinos.

Identity matrix

Oscillation probabilities are invariant under

$$H \rightarrow H + x\mathbb{1},$$

for some $x \in \mathbb{R}$.

So oscillation experiments cannot tell us about the absolute mass scale or V_{NC} .

Matter effect at long baseline experiments

$$\delta = 0.0\pi$$

$$\Delta_{32} = 0.5\pi$$

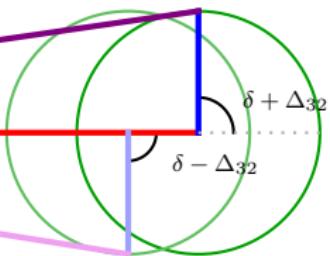
NO

T2K

- A_{31}^m
- A_{21}^m
- $A_{\mu e}^m$
- $\bar{A}_{\mu e}^{m*}$

$$P(\nu_\mu \rightarrow \nu_e) = A_{\mu e} A_{\mu e}^*$$

$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \bar{A}_{\mu e}^* \bar{A}_{\mu e}$$



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Matter effect at long baseline experiments

$$\delta = 0.0\pi$$

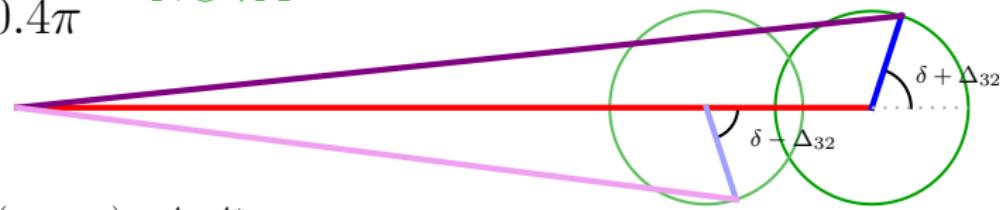
$$\Delta_{32} = 0.4\pi$$

NO

NOVA

- A_{31}^m
- A_{21}^m
- $A_{\mu e}^m$
- $\bar{A}_{\mu e}^{m*}$

$$P(\nu_\mu \rightarrow \nu_e) = A_{\mu e} A_{\mu e}^*$$
$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \bar{A}_{\mu e}^* \bar{A}_{\mu e}$$



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Matter effect at long baseline experiments

$$\delta = 0.0\pi$$

$$\Delta_{32} = 0.4\pi$$

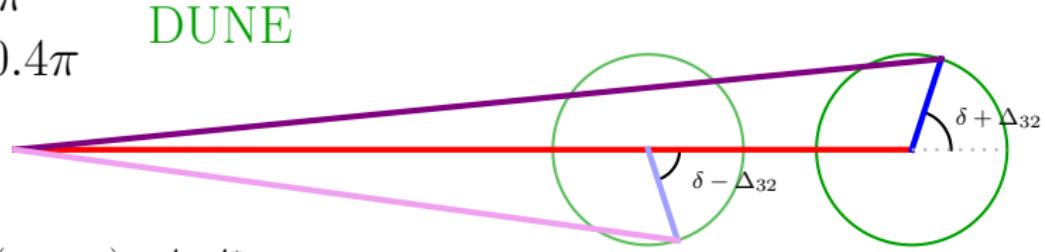
NO

- A_{31}^m
- A_{21}^m
- $A_{\mu e}^m$
- $\bar{A}_{\mu e}^{m*}$

DUNE

$$P(\nu_\mu \rightarrow \nu_e) = A_{\mu e} A_{\mu e}^*$$

$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \bar{A}_{\mu e}^* \bar{A}_{\mu e}$$



Denton & Parke

$$\text{NO: } P > \bar{P} \forall \delta$$

$$\text{IO: } P < \bar{P} \forall \delta$$

Exact neutrino oscillations in matter: mixing angles

$$s_{12}^{m2} = \frac{-(\lambda_2^2 - \alpha\lambda_2 + \beta)\Delta\lambda_{31}}{(\lambda_1^2 - \alpha\lambda_1 + \beta)\Delta\lambda_{32} - (\lambda_2^2 - \alpha\lambda_2 + \beta)\Delta\lambda_{31}}$$

$$s_{13}^{m2} = \frac{\lambda_3^2 - \alpha\lambda_3 + \beta}{\Delta\lambda_{31}\Delta\lambda_{32}}$$

$$s_{23}^{m2} = \frac{s_{23}^2 E^2 + c_{23}^2 F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

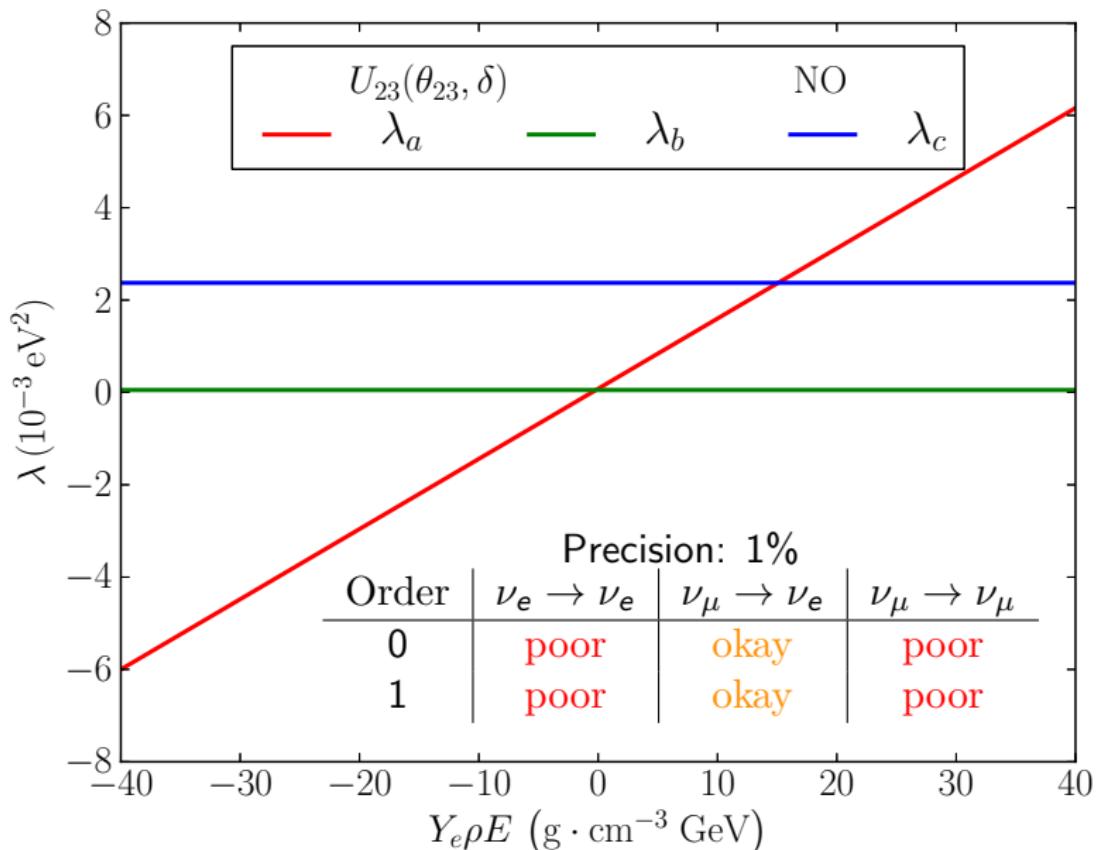
$$e^{-i\delta^m} = \frac{c_{23}^2 s_{23}^2 (e^{-i\delta} E^2 - e^{i\delta} F^2) + (c_{23}^2 - s_{23}^2) EF}{\sqrt{(s_{23}^2 E^2 + c_{23}^2 F^2 + 2EFc_{23}s_{23}c_\delta)(c_{23}^2 E^2 + s_{23}^2 F^2 - 2EFc_{23}s_{23}c_\delta)}}$$

$$\alpha = c_{13}^2 \Delta m_{31}^2 + (c_{12}^2 c_{13}^2 + s_{13}^2) \Delta m_{21}^2, \quad \beta = c_{12}^2 c_{13}^2 \Delta m_{21}^2 \Delta m_{31}^2$$

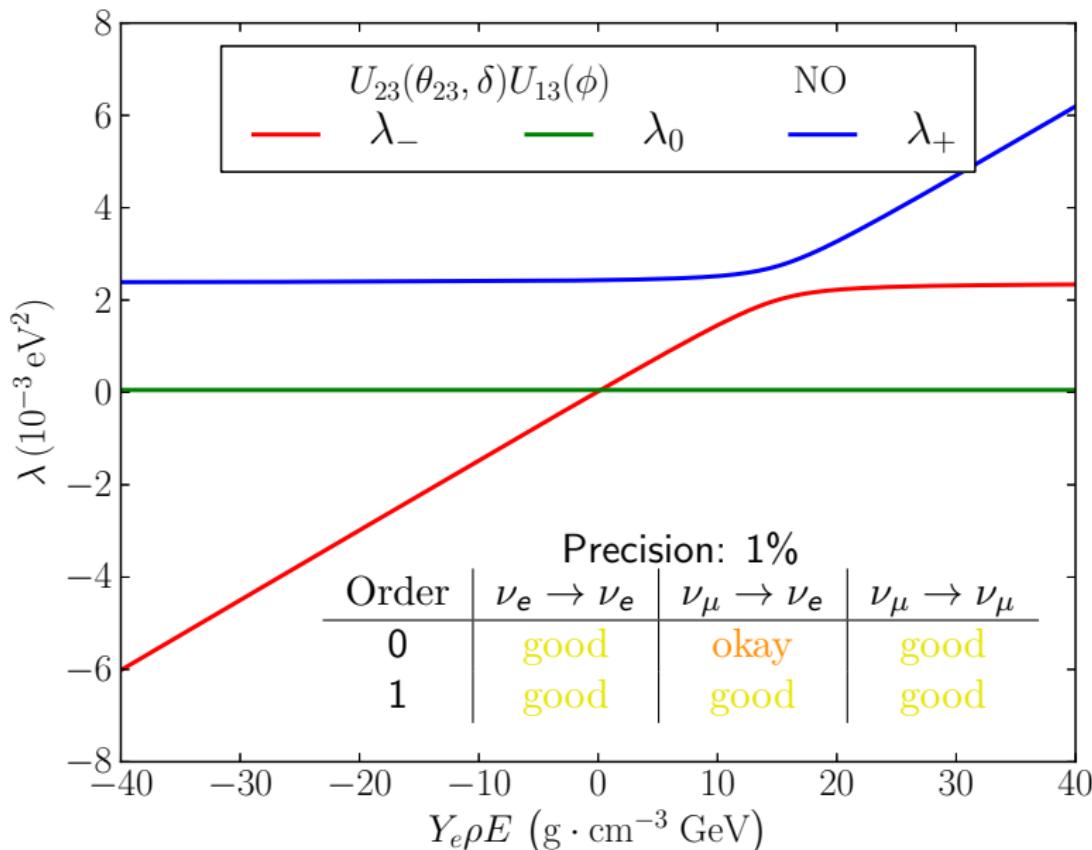
$$E = c_{13}s_{13} [(\lambda_3 - \Delta m_{21}^2) \Delta m_{31}^2 - s_{12}^2 (\lambda_3 - \Delta m_{31}^2) \Delta m_{21}^2]$$

$$F = c_{12}s_{12}c_{13} (\lambda_3 - \Delta m_{31}^2) \Delta m_{21}^2$$

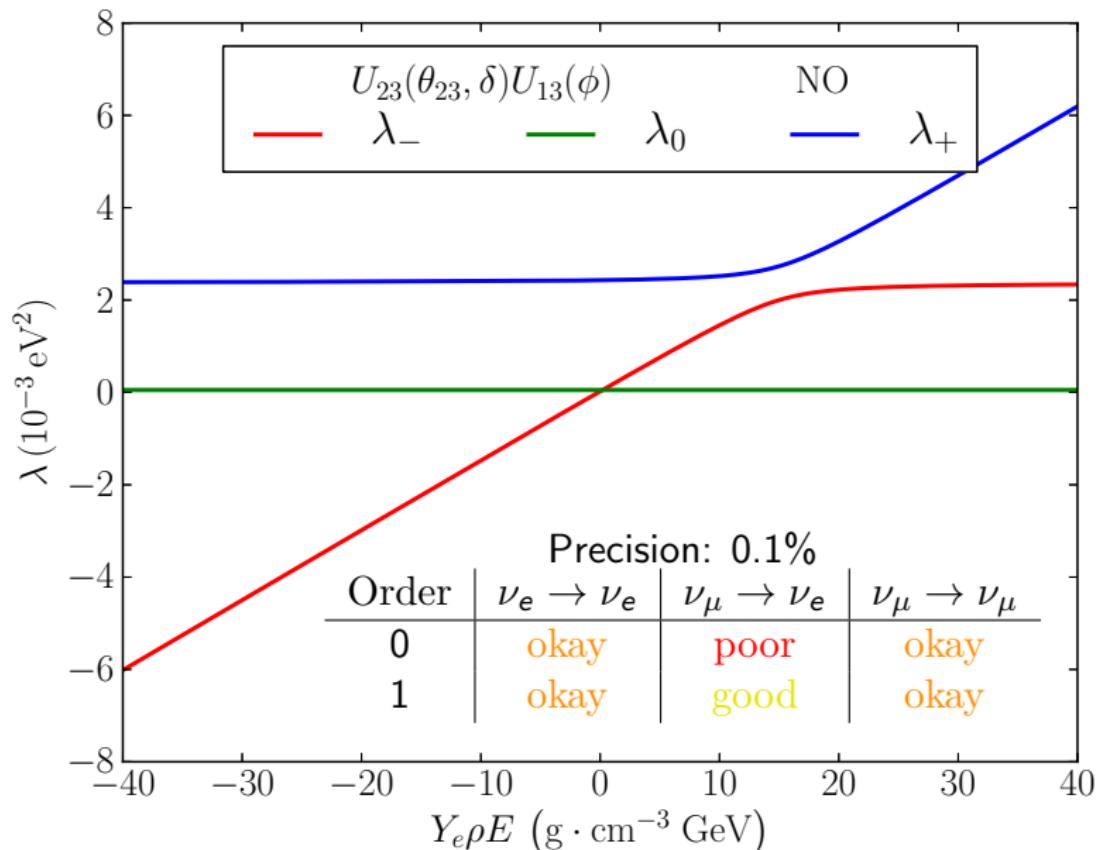
Eigenvalues in matter: two rotations are needed



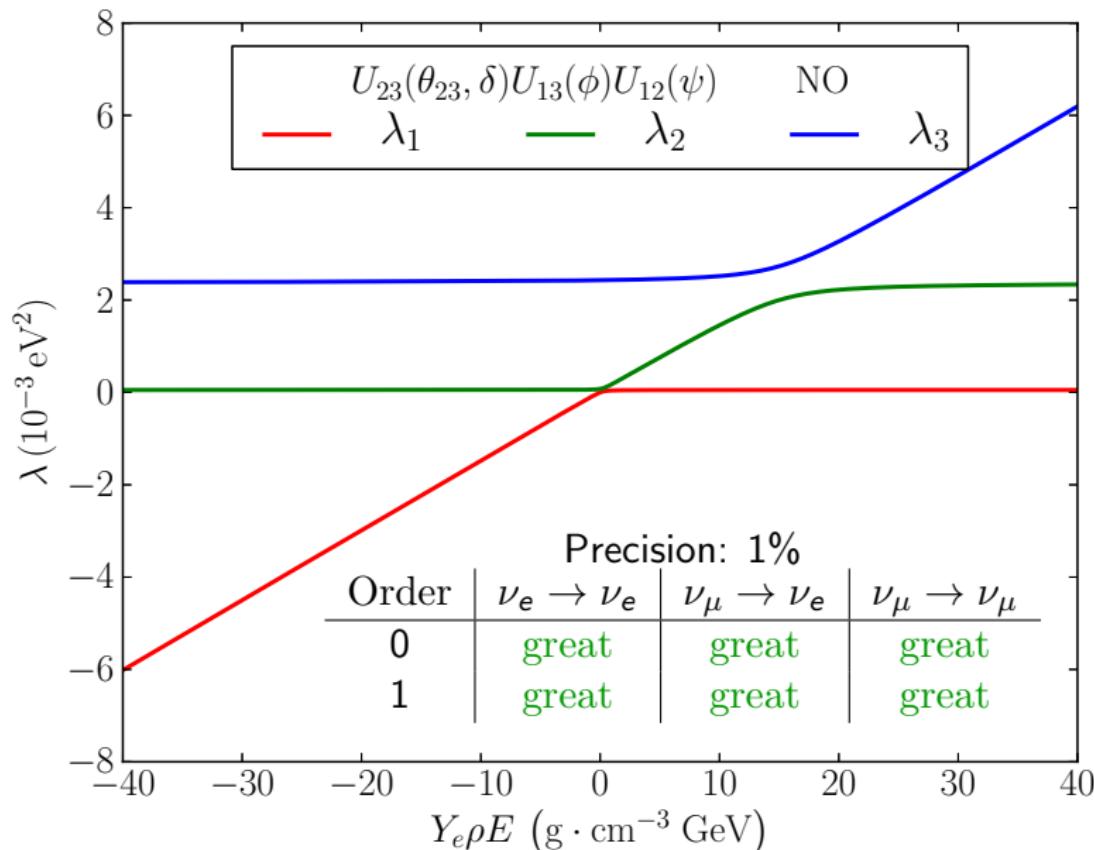
Eigenvalues in matter: two rotations are needed



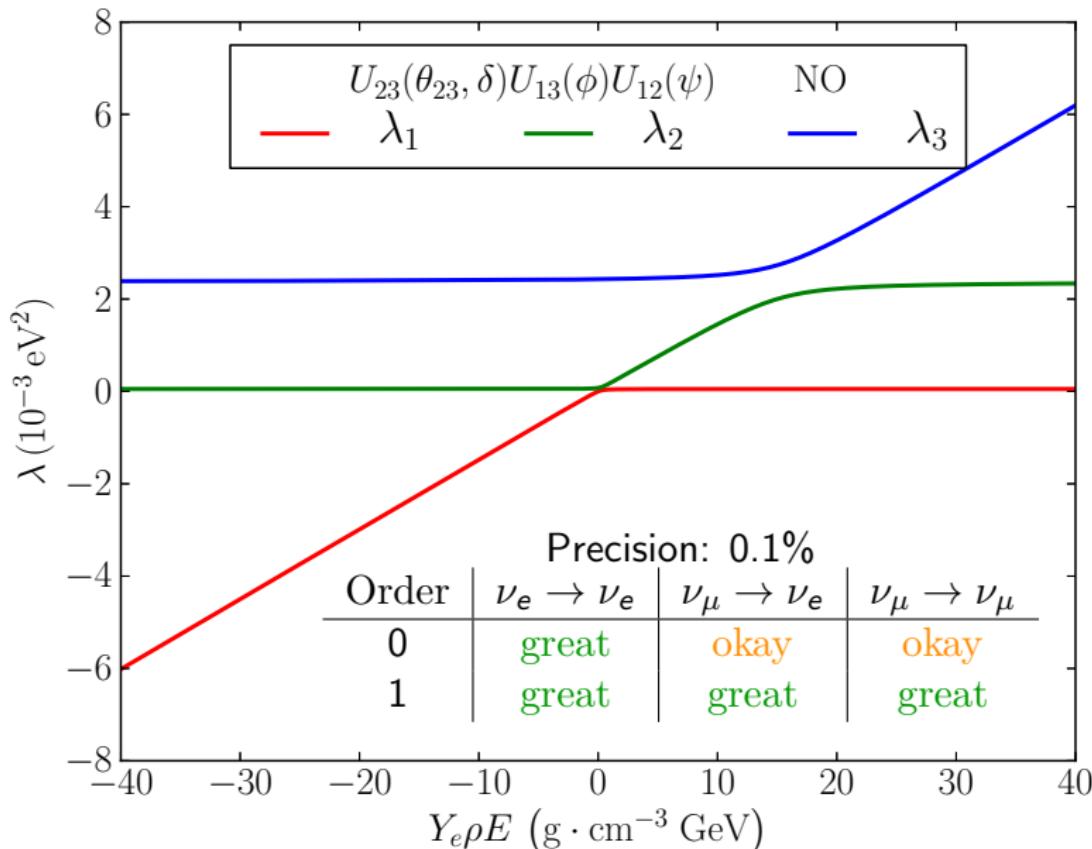
Eigenvalues in matter: two rotations are needed



Eigenvalues in matter: two rotations are needed



Eigenvalues in matter: two rotations are needed



Eigenvalues

Tilde basis after $U_{23}(\theta_{23}, \delta)$:

$$\lambda_a = a + (s_{13}^2 + \epsilon s_{12}^2) \Delta m_{ee}^2 ,$$

$$\lambda_b = \epsilon c_{12}^2 \Delta m_{ee}^2 ,$$

$$\lambda_c = (c_{13}^2 + \epsilon s_{12}^2) \Delta m_{ee}^2 .$$

Hat basis after $U_{13}(\phi)$:

$$\lambda_{\mp} = \frac{1}{2} \left[(\lambda_a + \lambda_c) \mp \text{sign}(\Delta m_{ee}^2) \sqrt{(\lambda_c - \lambda_a)^2 + 4(s_{13} c_{13} \Delta m_{ee}^2)^2} \right] ,$$

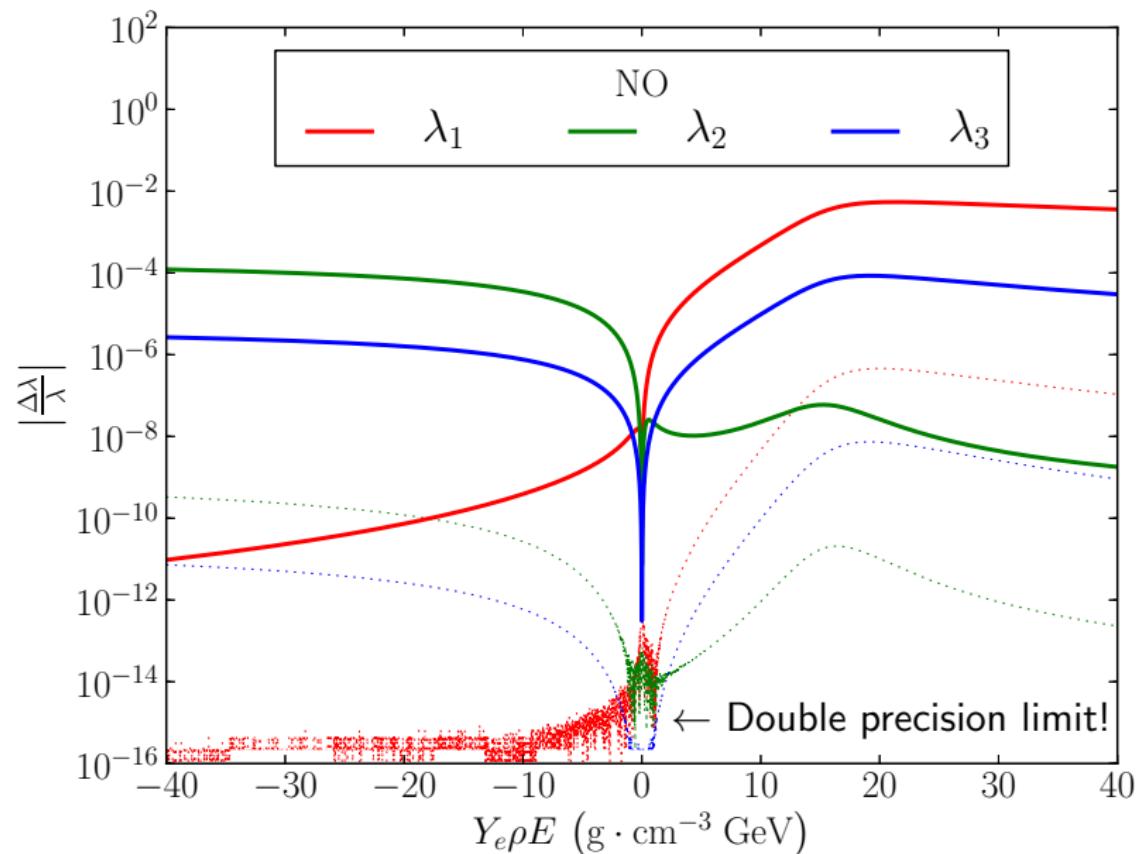
$$\lambda_0 = \lambda_b .$$

Check basis after $U_{12}(\psi)$:

$$\lambda_{1,2} = \frac{1}{2} \left[(\lambda_0 + \lambda_-) \mp \sqrt{(\lambda_0 - \lambda_-)^2 + 4(\epsilon c_{(\phi-\theta_{13})} c_{12} s_{12} \Delta m_{ee}^2)^2} \right] ,$$

$$\lambda_3 = \lambda_+ .$$

Eigenvalues: precision



Hamiltonians

After a constant (θ_{23}, δ) rotation, $2E\tilde{H} =$

$$\begin{pmatrix} \lambda_a & s_{13}c_{13}\Delta m_{ee}^2 \\ s_{13}c_{13}\Delta m_{ee}^2 & \lambda_b \\ & \lambda_c \end{pmatrix} + \epsilon s_{12}c_{12}\Delta m_{ee}^2 \begin{pmatrix} c_{13} & c_{13} \\ c_{13} & -s_{13} \\ -s_{13} & -s_{13} \end{pmatrix}.$$

After a $U_{13}(\phi)$ rotation, $2E\hat{H} =$

$$\begin{pmatrix} \lambda_- & & \\ & \lambda_0 & \\ & & \lambda_+ \end{pmatrix} + \epsilon c_{12}s_{12}\Delta m_{ee}^2 \begin{pmatrix} c_{(\phi-\theta_{13})} & c_{(\phi-\theta_{13})} \\ c_{(\phi-\theta_{13})} & s_{(\phi-\theta_{13})} \\ s_{(\phi-\theta_{13})} & s_{(\phi-\theta_{13})} \end{pmatrix}.$$

After a $U_{12}(\psi)$ rotation, $2E\check{H} =$

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} + \epsilon \textcolor{red}{s_{(\phi-\theta_{13})}} s_{12}c_{12}\Delta m_{ee}^2 \begin{pmatrix} -s_\psi & \\ -s_\psi & c_\psi \\ c_\psi & \end{pmatrix}.$$

Perturbative expansion

Hamiltonian: $\check{H} = \check{H}_0 + \check{H}_1$.

$$\check{H}_0 = \frac{1}{2E} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}, \quad \check{H}_1 = \epsilon' \frac{\Delta m_{ee}^2}{2E} \begin{pmatrix} & -s_\psi & \\ & c_\psi & \\ -s_\psi & c_\psi & \end{pmatrix}.$$

Eigenvalues: $\lambda_i^{\text{ex}} = \lambda_i + \lambda_i^{(1)} + \lambda_i^{(2)} + \dots$

$$\lambda_i^{(1)} = 2E(\check{H}_1)_{ii} = 0,$$

$$\lambda_i^{(2)} = \sum_{k \neq i} \frac{[2E(\check{H}_1)_{ik}]^2}{\Delta\lambda_{ik}}.$$

Perturbative expansion: eigenvectors

Use vacuum expressions with $U \rightarrow V$ where

$$V = U^m W,$$

U^m is U with $\theta_{13} \rightarrow \phi$ and $\theta_{12} \rightarrow \psi$,

$$W = W_0 + W_1 + W_2 + \dots$$

$$W_0 = \mathbb{1},$$

$$W_1 = \epsilon' \Delta m_{ee}^2 \begin{pmatrix} & -\frac{s_\psi}{\Delta \lambda_{31}} \\ & \frac{c_\psi}{\Delta \lambda_{32}} \\ \frac{s_\psi}{\Delta \lambda_{31}} & -\frac{c_\psi}{\Delta \lambda_{32}} \end{pmatrix},$$

$$W_2 = -\epsilon'^2 \frac{(\Delta m_{ee}^2)^2}{2} \begin{pmatrix} \frac{s_\psi^2}{(\Delta \lambda_{31})^2} & -\frac{s_{2\psi}}{\Delta \lambda_{32} \Delta \lambda_{21}} \\ \frac{s_{2\psi}}{\Delta \lambda_{31} \Delta \lambda_{21}} & \frac{c_\psi^2}{(\Delta \lambda_{32})^2} \\ & \left[\frac{c_\psi^2}{(\Delta \lambda_{32})^2} + \frac{s_\psi^2}{(\Delta \lambda_{31})^2} \right] \end{pmatrix}.$$

$\lambda_{1,2} - \psi$ interchange

From the shape of $U_{12}(\psi)$, it is clear that the probabilities are invariant under a simultaneous interchange of

$$\lambda_1 \leftrightarrow \lambda_2, \quad \text{and} \quad \psi \rightarrow \psi \pm \frac{\pi}{2}.$$

Since only even powers of ψ trig functions ($c_\psi^2, s_\psi^2, c_\psi s_\psi, c_{2\psi}, s_{2\psi}$) appear in the probabilities, the sign degeneracy is irrelevant.

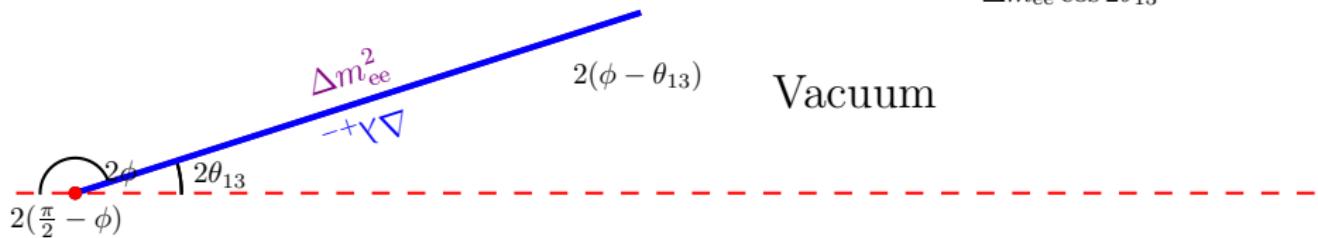
More usefully, we can write that the probabilities are invariant under the simultaneous interchange of

$$\lambda_1 \leftrightarrow \lambda_2, \quad c_\psi^2 \leftrightarrow s_\psi^2, \quad \text{and} \quad c_\psi s_\psi \rightarrow -c_\psi s_\psi.$$

This interchange constrains C_{21} ,
and C_{32} is then easily calculated from C_{31} .

31 Triangle

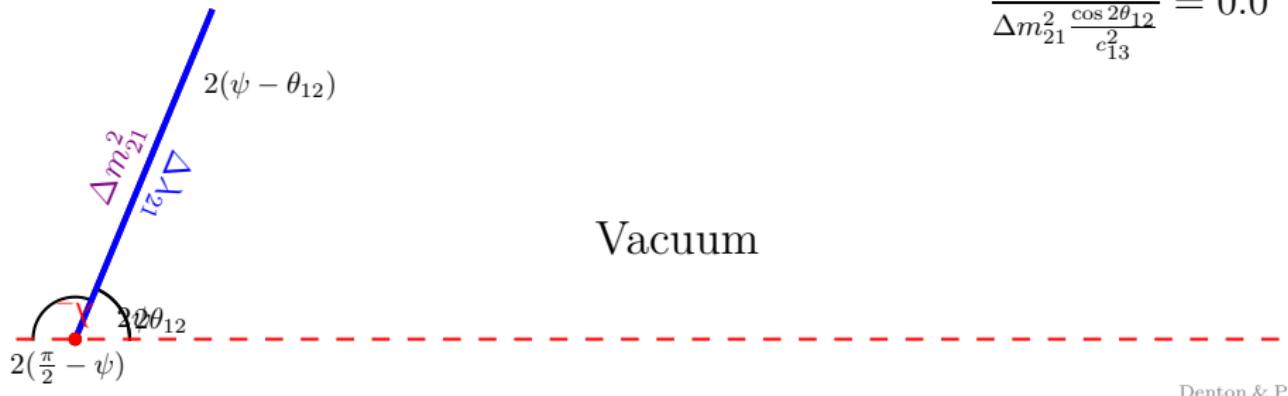
$$\frac{a}{\Delta m_{ee}^2 \cos 2\theta_{13}} = 0.0$$



Denton & Parke

31 Triangle

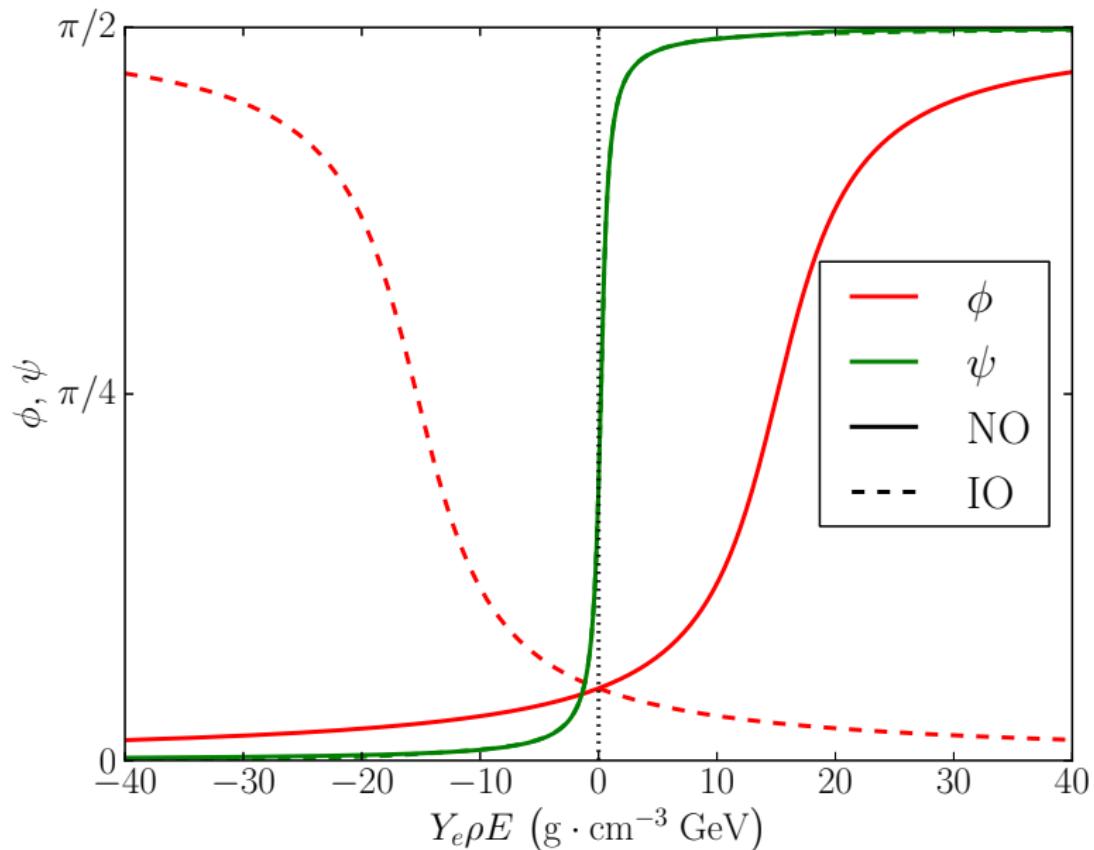
21 Triangle



Denton & Parke

21 Triangle

The two angles in matter



Zeroth order coefficients

$\nu_\alpha \rightarrow \nu_\beta$	$(C_{21}^{\alpha\beta})^{(0)}$
$\nu_e \rightarrow \nu_e$	$-c_\phi^4 s_\psi^2 c_\psi^2$
$\nu_\mu \rightarrow \nu_e$	$c_\phi^2 s_\psi^2 c_\psi^2 (c_{23}^2 - s_\phi^2 s_{23}^2) + c_{2\psi} J_r^m \cos \delta$
$\nu_\mu \rightarrow \nu_\mu$	$-(c_{23}^2 c_\psi^2 + s_{23}^2 s_\phi^2 s_\psi^2)(c_{23}^2 s_\psi^2 + s_{23}^2 s_\phi^2 c_\psi^2)$ $-2(c_{23}^2 - s_\phi^2 s_{23}^2)c_{2\psi} J_{rr}^m \cos \delta + (2J_{rr}^m \cos \delta)^2$

$\nu_\alpha \rightarrow \nu_\beta$	$(C_{31}^{\alpha\beta})^{(0)}$	$(D^{\alpha\beta})^{(0)}$
$\nu_e \rightarrow \nu_e$	$-c_\phi^2 s_\phi^2 c_\psi^2$	0
$\nu_\mu \rightarrow \nu_e$	$s_\phi^2 c_\phi^2 c_\psi^2 s_{23}^2 + J_r^m \cos \delta$	$-J_r^m \sin \delta$
$\nu_\mu \rightarrow \nu_\mu$	$-c_\phi^2 s_{23}^2 (c_{23}^2 s_\psi^2 + s_{23}^2 s_\phi^2 c_\psi^2)$ $-2s_{23}^2 J_r^m \cos \delta$	0

$$J_r^m \equiv s_\psi c_\psi s_\phi c_\phi^2 s_{23} c_{23}$$

$$J_{rr}^m \equiv J_r^m / c_\phi^2$$

General form of the first order coefficients

Can reduce 8 expressions down to 3:

$$(C_{21}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(\frac{F_1^{\alpha\beta}}{\Delta\lambda_{31}} + \frac{F_2^{\alpha\beta}}{\Delta\lambda_{32}} \right),$$

$$(C_{31}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(\frac{F_1^{\alpha\beta} + G_1^{\alpha\beta}}{\Delta\lambda_{31}} - \frac{F_2^{\alpha\beta}}{\Delta\lambda_{32}} \right),$$

$$(C_{32}^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(-\frac{F_1^{\alpha\beta}}{\Delta\lambda_{31}} + \frac{F_2^{\alpha\beta} + G_2^{\alpha\beta}}{\Delta\lambda_{32}} \right),$$

$$(D^{\alpha\beta})^{(1)} = \epsilon' \Delta m_{ee}^2 \left(\frac{K_1^{\alpha\beta}}{\Delta\lambda_{31}} - \frac{K_2^{\alpha\beta}}{\Delta\lambda_{32}} \right).$$

$$K_1^{\alpha\beta} = \begin{cases} 0 & \alpha = \beta \\ \mp s_{23} c_{23} c_\phi s_\psi^2 (c_\phi^2 c_\psi^2 - s_\phi^2) \sin \delta & \alpha \neq \beta \end{cases},$$

where the minus sign is for $\nu_\mu \rightarrow \nu_e$.

First order coefficients

$\nu_\alpha \rightarrow \nu_\beta$	$F_1^{\alpha\beta}$
$\nu_e \rightarrow \nu_e$	$-2c_\phi^3 s_\phi s_\psi^3 c_\psi$
$\nu_\mu \rightarrow \nu_e$	$c_\phi s_\psi^2 [s_\phi s_\psi c_\psi (c_{23}^2 + c_{2\phi} s_{23}^2) - s_{23} c_{23} (s_\phi^2 s_\psi^2 + c_{2\phi} c_\psi^2) \cos \delta]$
$\nu_\mu \rightarrow \nu_\mu$	$2c_\phi s_\psi (s_{23}^2 s_\phi c_\psi + s_{23} c_{23} s_\psi \cos \delta) \times (c_{23}^2 c_\psi^2 - 2s_{23} c_{23} s_\phi s_\psi c_\psi \cos \delta + s_{23}^2 s_\phi^2 s_\psi^2)$

$\nu_\alpha \rightarrow \nu_\beta$	$G_1^{\alpha\beta}$
$\nu_e \rightarrow \nu_e$	$2s_\phi c_\phi s_\psi c_\psi c_{2\phi}$
$\nu_\mu \rightarrow \nu_e$	$-2s_\phi c_\phi s_\psi (s_{23}^2 c_{2\phi} c_\psi - s_{23} c_{23} s_\phi s_\psi \cos \delta)$
$\nu_\mu \rightarrow \nu_\mu$	$-2c_\phi s_\psi (s_{23}^2 s_\phi c_\psi + s_{23} c_{23} s_\psi \cos \delta) \times (1 - 2c_\phi^2 s_{23}^2)$

Note on the PMNS matrix

The PMNS matrix typically has the CP phase associated with θ_{13} , while our unitary matrix has δ_{CP} associated with θ_{23} . The standard form is related to our form by multiplying the third row by $e^{i\delta}$ and the third column by $e^{-i\delta}$.

Our formulation is useful because after the U_{23} rotation the resulting Hamiltonian of the “tilde” basis is real.

B. Pontecorvo, Sov. Phys. JETP 7 1958

Z. Maki, M. Nakagawa, S. Sakata, Prog. Theor. Phys. 28 1962

A constant rotation in the 2 – 3 plane

Change basis by rotating $H^m \rightarrow \tilde{H}^m$ by $U_{23}(\theta_{23}, \delta)$

- ▶ U_{23} commutes with the matter potential,
- ▶ \tilde{H}^m is now entirely real.

$$\lambda_a = a + (s_{13}^2 + \epsilon s_{12}^2) \Delta m_{ee}^2$$

$$\lambda_b = \epsilon c_{12}^2 \Delta m_{ee}^2$$

$$\lambda_c = (c_{13}^2 + \epsilon s_{12}^2) \Delta m_{ee}^2$$

$$\Delta m_{ee}^2 = \Delta m_{31}^2 - s_{12}^2 \Delta m_{21}^2$$

$$\epsilon = \frac{\Delta m_{21}^2}{\Delta m_{ee}^2} \approx 0.03$$

H. Nunokawa, S. Parke, R. Zukanovich Funchal, [hep-ph/0503283](#)

A rotation in the 1 – 3 plane

Change basis again by rotating $\tilde{H}^m \rightarrow \hat{H}^m$ by $U_{13}(\phi)$

- ▶ $\phi(a = 0) = \theta_{13}$.

$$\lambda_{\mp} = \frac{1}{2} \left[(\lambda_a + \lambda_c) \mp \text{sign}(\Delta m_{ee}^2) \sqrt{(\lambda_c - \lambda_a)^2 + 4(s_{13}c_{13}\Delta m_{ee}^2)^2} \right]$$

$$\lambda_0 = \lambda_b$$

A rotation in the 1 – 2 plane

Change basis again by rotating $\hat{H}^m \rightarrow \check{H}^m$ by $U_{12}(\psi)$

- ▶ $\psi(a = 0) = \theta_{12}$,
- ▶ We have now created our own $U^m(\theta_{23}, \delta, \phi, \psi)$.

$$\lambda_{1,2} = \frac{1}{2} \left[(\lambda_0 + \lambda_-) \mp \sqrt{(\lambda_0 - \lambda_-)^2 + 4(\epsilon c_{(\phi-\theta_{13})} c_{12} s_{12} \Delta m_{ee}^2)^2} \right]$$
$$\lambda_3 = \lambda_+$$

Neutrino oscillation probabilities

$$\mathcal{A}(\nu_\alpha \rightarrow \nu_\beta) = \sum_{i=1}^3 U_{\alpha i}^* U_{\beta i} e^{-i \frac{m_i^2 L}{2E}}$$

$$\begin{aligned} P(\nu_\alpha \rightarrow \nu_\beta) &= |\mathcal{A}(\nu_\alpha \rightarrow \nu_\beta)|^2 \\ &= \delta_{\alpha\beta} - \sum_{i < j} \Re [U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}] \sin^2 \Delta_{ij} \\ &\quad + 8 \Im [U_{\alpha 1} U_{\beta 2}^* U_{\alpha 2}^* U_{\beta 1}] \sin \Delta_{32} \sin \Delta_{31} \sin \Delta_{21} \end{aligned}$$

C. Jarlskog, PRL 1985

Nonvanishing Wronskian \Rightarrow fewest number of L/E functions.

Clear that the CPV term is $\mathcal{O}[(L/E)^3]$ not $\mathcal{O}[(L/E)^1]$.

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E}$$

Variable Matter Density

This work assumed ρ is constant.

If ρ doesn't vary too much, we can set ρ to the average

$$\rho = \bar{\rho} = \frac{1}{L} \int_0^L \rho(x) dx .$$

ρ doesn't vary "too much" when

$$|\dot{\theta}^m| \ll \left| \frac{\Delta\lambda}{2E} \right| .$$