

## Abstract

Expressing neutrino oscillation probabilities in matter can be done either approximately with very good precision, or exactly with complicated expressions. Exact solutions require solving for the eigenvalues and, in turn, the eigenvectors. While there is no shortcut for the exact expressions for the eigenvalues, given those the eigenvectors can be determined in a straightforward fashion. Finally, I will show that while CPV in matter appears to extremely complicated, it is much simpler than expected both exactly and after approximations.

# Recent Results in Neutrino Oscillation Theory

Peter B. Denton

CERN Neutrino Platform

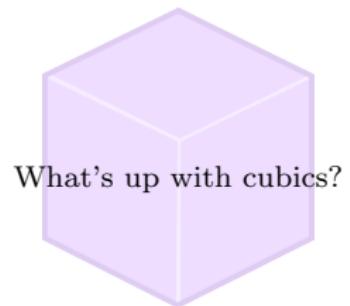
October 8, 2019

**BROOKHAVEN**  
NATIONAL LABORATORY



# Path

1. **Statement** of oscillation question
2. Get the eigen**values**
3. Get the eigen**vectors**
4. **CP** violation in matter



# **Statement** of oscillation question

# The Big Question

What is  $P(\nu_\mu \rightarrow \nu_e)$ ?

$$P({}^{\langle}\bar{\nu}_{\mu}{}^{\rangle} \rightarrow {}^{\langle}\bar{\nu}_e{}^{\rangle}) = |\mathcal{A}_{\mu e}|^2 \quad \mathcal{A}_{\mu e} = \mathcal{A}_{31} + e^{\pm i\Delta_{32}} \mathcal{A}_{21}$$

$$\mathcal{A}_{31} = 2s_{13}c_{13}s_{23} \sin \Delta_{31}$$

$$\mathcal{A}_{21} = 2s_{12}c_{13}(c_{12}c_{23}e^{i\delta} - s_{12}s_{13}s_{23}) \sin \Delta_{21}$$

$$\Delta_{ij} = \Delta m^2 {}_{ij} L / 4E$$

...in matter?

Now: NOvA, T2K, MINOS, ...

Upcoming: DUNE, T2HK, ...

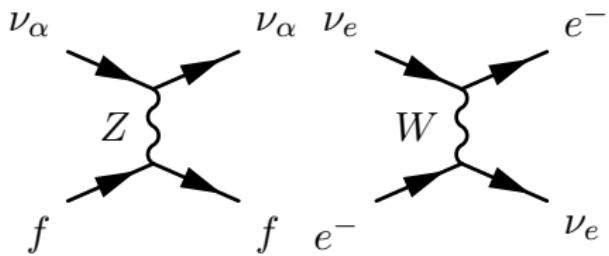
Second maximum: T2HKK? ESSnuSB? ...

# Matter Effects Matter

Call Schrödinger equation's eigenvalues  $m_i^2$  and eigenvectors  $U_i$ .

$$\mathcal{A}(\nu_\alpha \rightarrow \nu_\beta) = \sum_{i=1}^3 U_{\alpha i}^* U_{\beta i} e^{-im_i^2 L/2E} \quad P = |\mathcal{A}|^2$$

In matter  $\nu$ 's propagate in a new basis that depends on  $a \propto \rho E$ .



L. Wolfenstein, PRD 17 (1978)

Eigenvalues:  $m_i^2 \rightarrow \widehat{m^2}_i(a)$

Eigenvectors are given by  $\theta_{ij} \rightarrow \widehat{\theta}_{ij}(a)$        $\Leftarrow$       Unitarity

# Hamiltonian Dynamics

$$H = \frac{1}{2E} \left[ U \begin{pmatrix} 0 & & \\ & \Delta m_{21}^2 & \\ & & \Delta m_{31}^2 \end{pmatrix} U^\dagger + \begin{pmatrix} a & 0 & \\ & 0 & \\ & & 0 \end{pmatrix} \right]$$
$$a = 2\sqrt{2}G_F N_e E$$

$$U = \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}$$

Find eigenvalues and eigenvectors:

$$H = \frac{1}{2E} \widehat{U} \begin{pmatrix} 0 & & \\ & \widehat{\Delta m^2}_{21} & \\ & & \widehat{\Delta m^2}_{31} \end{pmatrix} \widehat{U}^\dagger$$

J. Kopp [physics/0610206](#)

Computationally works, but we can do better than a **black box**...  
Analytic expression?

Get the eigen**values**

# Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation: **eigenvalues**

G. Cardano *Ars Magna* 1545

V. Barger, et al. PRD 22 (1980) 2718

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

“Unfortunately, the algebra is rather impenetrable.”

V. Barger, et al.

Then write down **eigenvectors** (mixing angles)

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

K. Kimura, A. Takamura, H. Yokomakura [hep-ph/0205295](#)

PBD, S. Parke, X. Zhang [1907.02534](#)

# The Cubic

## Math history aside

Linear:  $ax + b = 0$



Quadratic:  $ax^2 + bx + c = 0$



Cubic:  $ax^3 + bx^2 + cx + d = 0$



Quartic:  $ax^4 + bx^3 + cx^2 + dx + e = 0$



Qunitic+:  $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$



Abel-Ruffini theorem, 1824

# Eigenvalues Analytically: The Exact Solution

The cubic solution (in neutrino terms)

$$\widehat{m^2}_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3BS} - \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widehat{m^2}_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2 - 3BS} + \frac{\sqrt{3}}{3}\sqrt{A^2 - 3B}\sqrt{1 - S^2}$$

$$\widehat{m^2}_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2 - 3BS}$$

$$A = \Delta m_{21}^2 + \Delta m_{31}^2 + a$$

$$B = \Delta m_{21}^2 \Delta m_{31}^2 + a [c_{13}^2 \Delta m_{31}^2 + (c_{12}^2 c_{13}^2 + s_{13}^2) \Delta m_{21}^2]$$

$$C = a \Delta m_{21}^2 \Delta m_{31}^2 c_{12}^2 c_{13}^2$$

$$S = \cos \left\{ \frac{1}{3} \cos^{-1} \left[ \frac{2A^3 - 9AB + 27C}{2(A^2 - 3B)^{3/2}} \right] \right\}$$

Get the eigen**vectors**

# Exact Neutrino Oscillations in Matter: Mixing Angles

$$s_{\widehat{12}}^2 = \frac{- \left[ (\widehat{m^2}_2)^2 - \alpha \widehat{m^2}_2 + \beta \right] \Delta \widehat{m^2}_{31}}{\left[ (\widehat{m^2}_1)^2 - \alpha \widehat{m^2}_1 + \beta \right] \Delta \widehat{m^2}_{32} - \left[ (\widehat{m^2}_2)^2 - \alpha \widehat{m^2}_2 + \beta \right] \Delta \widehat{m^2}_{31}}$$

$$s_{\widehat{13}}^2 = \frac{(\widehat{m^2}_3)^2 - \alpha \widehat{m^2}_3 + \beta}{\Delta \widehat{m^2}_{31} \Delta \widehat{m^2}_{32}}$$

$$s_{\widehat{23}}^2 = \frac{s_{23}^2 E^2 + c_{23}^2 F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\widehat{\delta}} = \frac{c_{23}s_{23} (e^{-i\delta}E^2 - e^{i\delta}F^2) + (c_{23}^2 - s_{23}^2) EF}{\sqrt{(s_{23}^2 E^2 + c_{23}^2 F^2 + 2EFc_{23}s_{23}c_\delta) (c_{23}^2 E^2 + s_{23}^2 F^2 - 2EFc_{23}s_{23}c_\delta)}}$$

$$\alpha = c_{13}^2 \Delta m_{31}^2 + (c_{12}^2 c_{13}^2 + s_{13}^2) \Delta m_{21}^2, \quad \beta = c_{12}^2 c_{13}^2 \Delta m_{21}^2 \Delta m_{31}^2$$

$$E = c_{13}s_{13} \left[ \left( \widehat{m^2}_3 - \Delta m_{21}^2 \right) \Delta m_{31}^2 - s_{12}^2 \left( \widehat{m^2}_3 - \Delta m_{31}^2 \right) \Delta m_{21}^2 \right]$$

$$F = c_{12}s_{12}c_{13} \left( \widehat{m^2}_3 - \Delta m_{31}^2 \right) \Delta m_{21}^2$$

H. Zaglauer, K. Schwarzer, [Z.Phys. C40 \(1988\) 273](#)



# Eigenvalues to Eigenvectors

KTY pushed calculating the eigenvectors from the eigenvalues.

K. Kimura, A. Takamura, H. Yokomakura [hep-ph/0205295](#)

Expressions for:

$$\hat{U}_{\alpha i} \hat{U}_{\beta i}^*$$

for  $\alpha \neq \beta$ .

Wanted to preserve phase information for  $\hat{\delta}$ .

# Eigenvalues: the Rosetta Stone

We realized:

$$|\widehat{U}_{\alpha i}|^2 = \frac{(\widehat{m^2}_i - \xi_\alpha)(\widehat{m^2}_i - \chi_\alpha)}{\Delta\widehat{m^2}_{ij}\Delta\widehat{m^2}_{ik}}$$

PBD, S. Parke, X. Zhang [1907.02534](#)

where  $\xi_\alpha$  and  $\chi_\alpha$  are the submatrix eigenvalues of  $H_\alpha$

$$H = \begin{pmatrix} H_{\alpha\alpha} & H_{\alpha\beta} & H_{\alpha\gamma} \\ H_{\beta\alpha} & H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\alpha} & H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix} \quad \rightarrow \quad H_\alpha = \begin{pmatrix} H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix}$$

e.g.

$$\xi_e + \chi_e = \Delta m_{21}^2 + \Delta m_{ee}^2 c_{13}^2$$

$$\xi_e \chi_e = \Delta m_{21}^2 [\Delta m_{ee}^2 c_{13}^2 c_{12}^2 + \Delta m_{21}^2 (s_{12}^2 c_{12}^2 - s_{13}^2 s_{12}^2 c_{12}^2)]$$

$$\Delta m_{ee}^2 = c_{12}^2 \Delta m_{31}^2 + s_{12}^2 \Delta m_{32}^2$$

H. Nunokawa, S. Parke, R. Z. Funchal, [hep-ph/0503283](#)

# Eigenvalues: the Rosetta Stone

$$s_{\widehat{13}}^2 = |\widehat{U}_{e3}|^2 = \frac{(\widehat{m^2}_3 - \xi_e)(\widehat{m^2}_3 - \chi_e)}{\Delta \widehat{m^2}_{31} \Delta \widehat{m^2}_{32}}$$

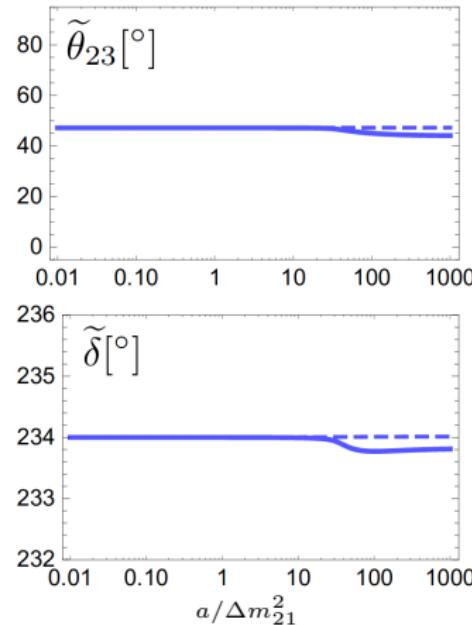
$$s_{\widehat{12}}^2 c_{\widehat{13}}^2 = |\widehat{U}_{e2}|^2 = -\frac{(\widehat{m^2}_2 - \xi_e)(\widehat{m^2}_2 - \chi_e)}{\Delta \widehat{m^2}_{32} \Delta \widehat{m^2}_{21}}$$

$$s_{\widehat{23}}^2 c_{\widehat{13}}^2 = |\widehat{U}_{\mu 3}|^2 = \frac{(\widehat{m^2}_3 - \xi_\mu)(\widehat{m^2}_3 - \chi_\mu)}{\Delta \widehat{m^2}_{31} \Delta \widehat{m^2}_{32}}$$

What about  $\widehat{\delta}$ ?

# CPV From Rosetta

$\hat{\delta}$  nearly constant, but have to get it right



Toshev identity:

$$\sin \hat{\delta} = \frac{\sin 2\theta_{23}}{\sin 2\hat{\theta}_{23}} \sin \delta$$

Z-z. Xing, S. Zhou  
Y-L. Zhou, [1802.00990](#)

S. Toshev [MPL A6 \(1991\) 455](#)

Get the sign of  $\cos \hat{\delta}$  from e.g.  $|\hat{U}_{\mu 1}|^2$ .

# EIGENVECTORS FROM EIGENVALUES

PETER B. DENTON, STEPHEN J. PARKE, TERENCE TAO, AND XINING ZHANG

ABSTRACT. We present a new method of succinctly determining eigenvectors from eigenvalues. Specifically, we relate the norm squared of the elements of eigenvectors to the eigenvalues and the submatrix eigenvalues.

$$|v_{i,j}|^2 = \frac{\prod_{k=1}^{n-1} (\lambda_i - \xi_{j,k})}{\prod_{k=1; k \neq i}^n (\lambda_i - \lambda_k)}$$

# Proofs

1. From previous result with  $n - 1$  subvectors using derivatives

L. Erdos, B. Schlein, H-T. Yau, [0711.1730](#)

T. Tao, V. Vu, [0906.0510](#)

2. Geometric formulation with exterior algebra
3. Using determinants and a Cauchy-Binet variant
4. Adjugate matrices

Can get off-diagonal elements, thus CP phase

5. Cramer's rule
6. Two other mathematicians provided other proofs
7. Another mathematician generalized it to all square matrices

[https://terrytao.wordpress.com/2019/08/13/  
eigenvectors-from-eigenvalues/](https://terrytao.wordpress.com/2019/08/13/eigenvectors-from-eigenvalues/)

# **CP** violation in matter

# The CPV Term in Matter

The amount of CPV is

$$P_{\alpha\beta} - \bar{P}_{\alpha\beta} = \pm 16J \sin \Delta_{21} \sin \Delta_{31} \sin \Delta_{32} \quad \alpha \neq \beta$$

where the Jarlskog is

$$J \equiv \Im[U_{\alpha i} U_{\beta j} U_{\alpha j}^* U_{\beta i}^*] \quad \alpha \neq \beta, i \neq j$$

$$J = c_{12} s_{12} c_{13}^2 s_{13} c_{23} s_{23} \sin \delta$$

C. Jarlskog, [PRL 55 \(1985\)](#)

The exact term in matter is known to be

$$\frac{\hat{J}}{J} = \frac{\Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2}{\Delta \hat{m}_{21}^2 \Delta \hat{m}_{31}^2 \Delta \hat{m}_{32}^2}$$

V. Naumov, [IJMP 1992](#)

P. Harrison, W. Scott, [hep-ph/9912435](#)

# CPV in Matter

CPV in matter can be written sans  $\cos(\frac{1}{3} \cos^{-1}(\dots))$  term.

$$\frac{\widehat{J}}{J} = \frac{\Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2}{\widehat{\Delta m^2}_{21} \widehat{\Delta m^2}_{31} \widehat{\Delta m^2}_{32}}$$

$$\left( \widehat{\Delta m^2}_{21} \widehat{\Delta m^2}_{31} \widehat{\Delta m^2}_{32} \right)^2 = (A^2 - 4B)(B^2 - 4AC) + (2AB - 27C)C$$

$$A \equiv \sum_j \widehat{m^2}_j = \Delta m_{31}^2 + \Delta m_{21}^2 + a$$

$$B \equiv \sum_{j>k} \widehat{m^2}_j \widehat{m^2}_k = \Delta m_{31}^2 \Delta m_{21}^2 + a(\Delta m_{ee}^2 c_{13}^2 + \Delta m_{21}^2)$$

$$C \equiv \prod_j \widehat{m^2}_j = a \Delta m_{31}^2 \Delta m_{21}^2 c_{13}^2 c_{12}^2$$

**This is the only oscillation quantity in matter that can be written exactly without  $\cos(\frac{1}{3} \cos^{-1}(\dots))$ !**

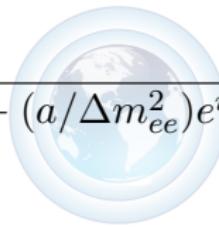
H. Yokomakura, K. Kimura, A. Takamura, [hep-ph/0009141](https://arxiv.org/abs/hep-ph/0009141)

# CPV Factorizes

Thus  $\hat{J}^{-2}$  is fourth order in matter potential:  
only two matter corrections are needed.

$$\frac{\hat{J}}{J} = \frac{1}{|1 - (a/\alpha_1)e^{i2\theta_1}| |1 - (a/\alpha_2)e^{i2\theta_2}|}$$

CPV in matter can be well approximated:

$$\frac{\hat{J}}{J} \approx \frac{1}{|1 - (a/\Delta m_{ee}^2)e^{i2\theta_{13}}| |1 - (\textcolor{orange}{c_{13}^2} a/\Delta m_{21}^2)e^{i2\theta_{12}}|}$$


PBD, Parke, [1902.07185](#)

See also X. Wang, S. Zhou, [1901.10882](#)

Precise at the < 0.04% level!

# Key Cancellation

Expect  $s_{13}^2$  or  $\Delta m_{21}^2/\Delta m_{ee}^2 \sim 2 - 3\%$  precision

The atmospheric term:

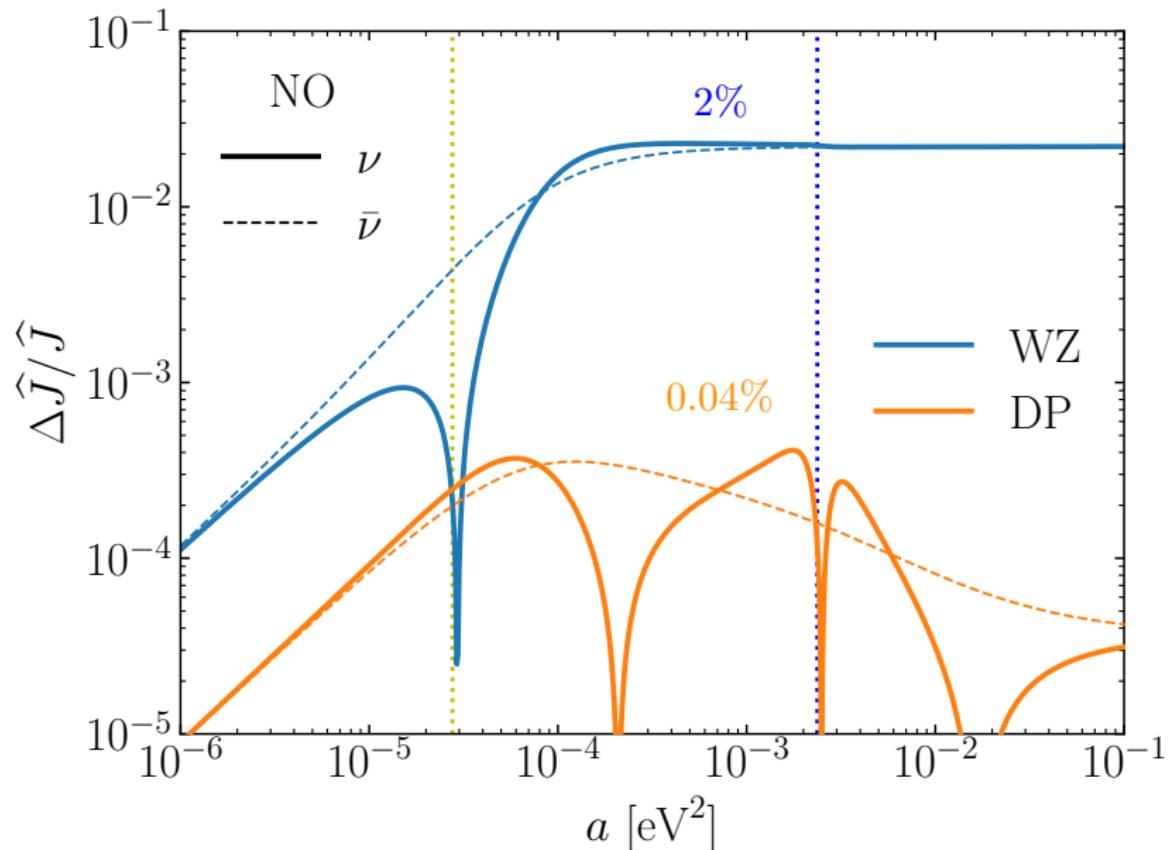
$$\begin{array}{ccc} \Delta m_{31}^2 & \Delta m_{ee}^2 & \Delta m_{32}^2 \\ \text{X} & \checkmark & \text{X} \end{array}$$

Solar correction:

$$\begin{array}{ccc} 1 & c_{13}^2 & \cos 2\theta_{13} \\ \text{X} & \checkmark & \text{X} \end{array}$$

$$\frac{\Delta \hat{J}}{\hat{J}} \sim \mathcal{O}\left(s_{13}^2 \frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right) + \mathcal{O}\left[\left(\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right)^2\right] \sim 0.04\%$$

# CPV In Matter Approximation Precision



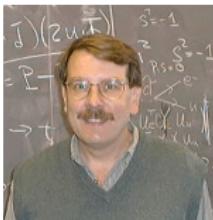
## Key Points

- ▶ Understanding probabilities in matter is key to current/future LBL
- ▶ Long-baseline oscillations are fundamentally three-flavor
- ▶ Eigenvectors follow from eigenvalues
- ▶ Exact and approximate CPV in matter are simpler than expected

# Questions!

# Backups

# Analytic Oscillation Probability Collaborators



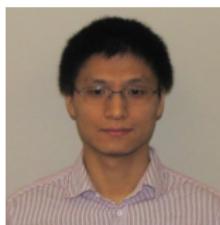
Stephen Parke



Hisakazu Minakata



Gabriela Barenboim



Xining Zhang



Christoph Ternes



Terry Tao

[1604.08167](#), [1806.01277](#), [1808.09453](#),  
[1902.00517](#), [1902.07185](#), [1907.02534](#)  
[1908.03795](#), [1909.02009](#)

[github.com/PeterDenton/Nu-Pert](https://github.com/PeterDenton/Nu-Pert)

[github.com/PeterDenton/Nu-Pert-Compare](https://github.com/PeterDenton/Nu-Pert-Compare)

# Analytic Oscillation Probabilities in Matter

- ☒ Solar:  $P_{ee} \simeq \sin^2 \theta_\odot$

Approx: S. Mikheev, A. Smirnov, [Nuovo Cim. C9 \(1986\) 17-26](#)

Exact: S. Parke, [PRL 57 \(1986\) 2322](#)

- ☒ Long-baseline: All three flavors

Exact: H. Zaglauer, K. Schwarzer, [Z.Phys. C40 \(1988\) 273](#)

Approx: PBD, H. Minakata, S. Parke, [1604.08167](#)

- ☒  $\nu_e$  disappearance (neutrino factory):

$$\Delta\widehat{m^2}_{ee} = \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - \Delta m^2_{21} c^2_{12})$$

PBD, S. Parke, [1808.09453](#)

- ☐ Atmospheric

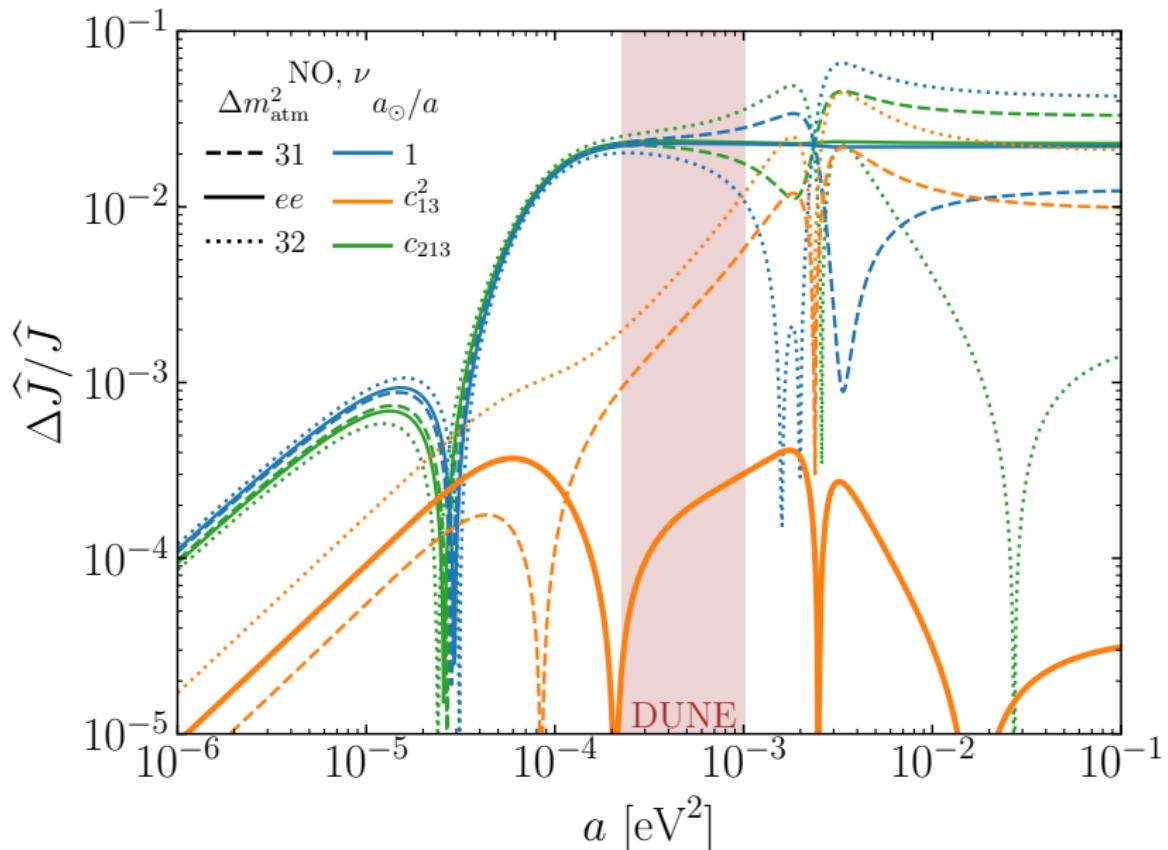
# Values and Vectors

Probability amplitude:

$$\mathcal{A}_{\alpha\beta} = \sum_i \hat{U}_{\alpha i}^* \hat{U}_{\beta i} e^{-im^2_i L/2E}$$

- ▶ Eigen**values** give the frequencies of the oscillations
  - Where should DUNE be?
- ▶ Eigen**vectors** give the amplitudes of the oscillations
  - How many events will DUNE see?

# Factorization Conditions



# The Cubic

## Math history aside

### 1. Ancients (20-16C BC)

Babylonians, Greeks, Chinese, Indians, Egyptians:  
thought about cubics, calculated cube roots

$$x^3 = a$$

### 2. Chinese Wang Xiaotong (7C AD):

numerically solved 25 general cubics

### 3. Persian Omar Khayyam (11C AD):

realized there are multiple solutions

# The Cubic

## Math history aside: The Italian Job (16C AD)

### 4. Scipione del **Ferro**:

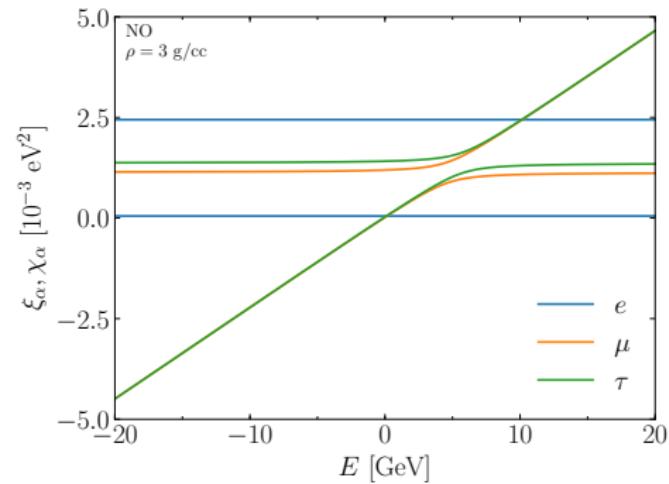
Secret solution, nearly all (didn't know negative numbers)

$$x^3 + mx = n$$

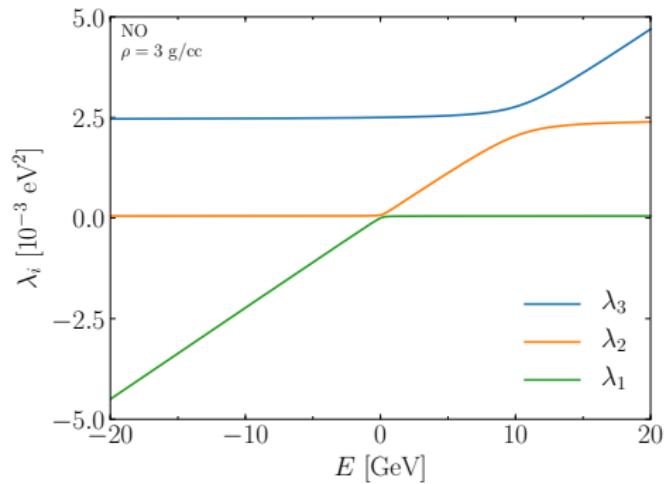
5. Antonio **Fiore**: Ferro's student, from just before his death
6. Niccol **Tartaglia**: Claimed a solution, was challenged by Fiore
7. Gerolamo **Cardano**: Gets Tartaglia's (winner) solution, promises to keep it secret. Later publishes Ferro's solution via Fiore
8. Tartaglia challenges Cardano who denies it. Cardano's student **Ferrari** accepted, Tartaglia lost along with prestige and income

Quartic was (nearly) solved around the same time by Ferrari,  
*before* the cubic solution was published

# Submatrix Eigenvalues



Submatrix Eigenvalues



Eigenvalues

# In General

Two flavor:

$$|\widehat{U}_{\alpha i}|^2 = \frac{\widehat{m^2}_i - \xi_\alpha}{\Delta \widehat{m^2}_{ij}}$$

leads to

$$\begin{aligned} \sin^2 \widehat{\theta} &= |\widehat{U}_{e2}|^2 = \frac{\widehat{m^2}_2 - \xi_e}{\widehat{m^2}_2 - \widehat{m^2}_1} \\ &= \frac{1}{2} \left( 1 - \frac{\Delta m^2 \cos 2\theta - a}{\sqrt{(\Delta m^2 \cos 2\theta - a)^2 + (\Delta m^2 \sin 2\theta)^2}} \right) \end{aligned}$$

Numerically checked for  $N = 4, 5$ .

True for all  $N$ ?

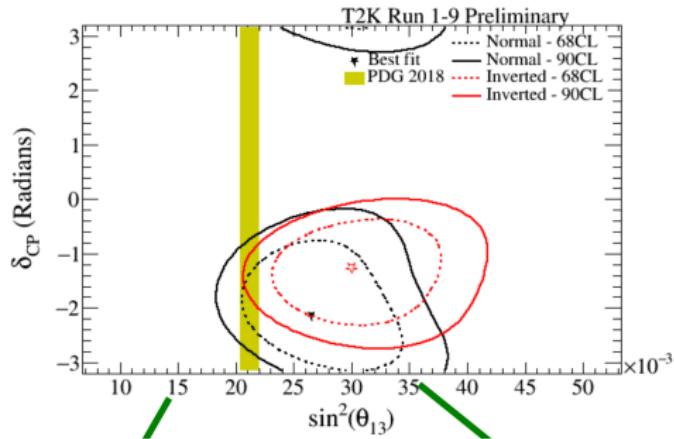
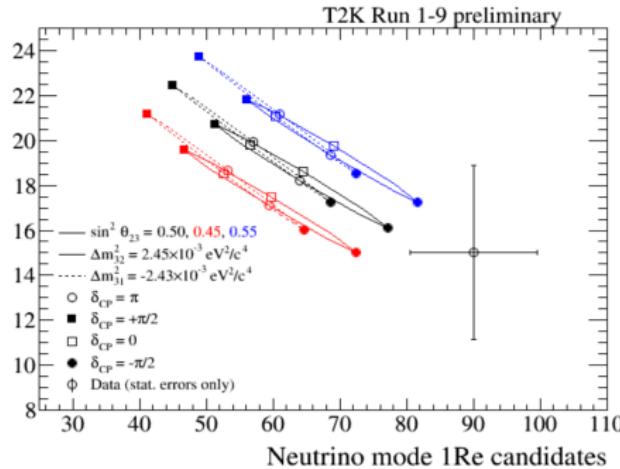
# A Cheery Firehose

1. Terry posted on the same question with a different answer
2. We emailed our result,  $< 2$  hours later:
  - ▶ “Very nice identity!”
  - ▶ New result
  - ▶ 3 distinct proofs
3. 6d later, we’ve sorted 1 proof, send a draft,  $< 1$  hr later:
  - ▶ Agrees to a paper
  - ▶ Adds a corollary
  - ▶ Adds several new observation
4. Barely processed that, another email  $< \frac{1}{2}$  day later
  - ▶ A more general proof
5. We sent confirmation that the  $v_i$  are normed
6.  $< 1$  day showed how that followed from proof 4

“He’s famously like a cheery firehose of mathematics  
Guess he’s power-washing you today”

# CPV Tension at T2K

Antineutrino mode 1Re candidates



$$J = c_{12}s_{12}c_{13}^2s_{13}c_{23}s_{23} \sin \delta$$

## CPV Factorizes Part II

- ▶ Option 1: Use NHS identity and  $\Delta\widehat{m^2}$ 's
- ▶ Option 2: Use the angles?

$$\frac{\widehat{J}}{J} = \frac{s_{12}\widehat{c_{12}}s_{13}\widehat{c_{13}^2}s_{23}\widehat{c_{23}}\sin\widehat{\delta}}{s_{12}c_{12}s_{13}\widehat{c_{13}^2}s_{23}c_{23}\sin\delta}$$

Toshev:  $\theta_{23}, \delta$ :

S. Toshev [MPL A6 \(1991\) 455](#)

$$\frac{\widehat{J}}{J} = \frac{s_{12}\widehat{c_{12}}s_{13}\widehat{c_{13}^2}}{s_{12}c_{12}s_{13}\widehat{c_{13}^2}}$$

How to split up?

From DMP:

[PBD](#), H. Minakata, S. Parke, [1604.08167](#)

$$\frac{1}{|1 - (a/\Delta m_{ee}^2)e^{i2\theta_{13}}|} \approx \frac{s_{13}\widehat{c_{13}}}{s_{13}c_{13}} \sim 0.4\%$$

Hopefully:

$$\frac{1}{|1 - (c_{13}^2 a / \Delta m_{21}^2) e^{i2\theta_{12}}|} \approx \frac{s_{12}\widehat{c_{12}}c_{13}}{s_{12}c_{12}c_{13}} \sim 0.4\%$$