Maximal Sizes of Weak $(2, 1)$-Sum-Free Sets in Finite Abelian Groups

Peter E. Francis

Gettysburg College

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Ordinary Sumsets

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We will write hA for the h-fold sumset of A, which consists of sums of exactly h terms of A:

$$hA = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \mathbb{N}_0, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

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Let $A = \{1, 2, 3\} \subset \mathbb{Z}_6$.Then,

$$\begin{aligned} 2A &= \{1+1,\ \ 1+2,\ \ 1+3,\ \ 2+2,\ \ 2+3,\ \ 3+3\} \\ &= \{0, 2, 3, 4, 5\}. \end{aligned}$$

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Let $A = \{1, 2, 3\} \subset \mathbb{Z}_6$. Then,

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Example 5

$\mu\hat{}(\mathbb{Z}_4, \{2, 1\}) = 2.$

$\lvert A \rvert$	$A \subseteq \mathbb{Z}_4$	$2\hat{}A$	$2\hat{}A \cap A$
3	$\{0, 1, 2\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$
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3	$\{0, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
3	$\{1, 2, 3\}$	$\{1, 3, 4\}$	$\{1, 3\}$
2	$\{1, 3\}$	$\{0, 2\}$	$\emptyset$

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Observation

For any G,

$$\mu\hat{}(G, \{2,1\}) \geq \mu(G, \{2,1\}).$$

Solving $\mu(\mathbb{Z}_n, \{2, 1\})$

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Theorem G.4 (Diananda and Yap)

For all positive integers n, we have

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Define:

$$v_1(x, 3) = \begin{cases} \left(1 + \frac{1}{p}\right) \frac{x}{3} & \text{if } x \text{ has prime divisors congruent to 2 mod 3,} \\ & \text{and } p \text{ is the smallest such divisor;} \\ \left\lfloor \frac{x}{3} \right\rfloor & \text{otherwise.} \end{cases}$$

Solving $\mu(G, \{2, 1\})$

Theorem G.18 (Green and Ruzsa)

Let κ be the exponent of G. Then

$$\mu(G, \{2, 1\}) = \mu(\mathbb{Z}_\kappa, \{2, 1\}) \cdot \frac{n}{\kappa} = v_1(\kappa, 3) \cdot \frac{n}{\kappa}.$$

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Example 6

Let $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{24}$. Then the exponent of G is $\kappa = 24$, so

$$\mu(G, \{2, 1\}) = v_1(24, 3) \cdot 8 = 12 \cdot 8 = 96.$$

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Theorem G.67 (Zannier)

For all positive integers we have

$$\mu\hat{}(\mathbb{Z}_n, \{2,1\}) = \begin{cases} \left(1+\frac{1}{p}\right)\frac{n}{3} & \text{if } n \text{ has prime divisors cong. to } 2(3) \\ & \text{and } p \text{ is the smallest such divisor;} \\ \left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

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Proposition 11, P

For any G with $|G| = n \equiv 0 \bmod 2$,

$$\mu\hat{}(G, \{2,1\}) = \frac{n}{2}.$$

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Theorem 14, P

For all positive $\kappa \equiv 1 \bmod 6$,

$$\mu\hat{}(\mathbb{Z}_\kappa^2, \{2,1\}) \geq \frac{\kappa - 1}{3} \cdot \kappa + 1.$$

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- **type III**: all divisors of $|G|$ are congruent to 1 mod 3

Type I

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If G is a group of type I, then

$$\mu\hat{}(G, \{2, 1\}) = \mu(G, \{2, 1\}).$$

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If G is a group of type II, then

$$\hat{\mu}(G, \{2, 1\}) \leq \mu(G, \{2, 1\}) + 1.$$

Corollary 11

If $w \equiv 1 \bmod 2$ has no prime divisors congruent to 2 mod 3, then

$$\hat{\mu}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2, 1\}) = \mu(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2, 1\}) + 1 = 3w + 1.$$

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$$D = \left\{\pm 1, \pm 3, \ldots, \pm \frac{\kappa - 4}{3}\right\} \subset \mathbb{Z}_\kappa, \text{ and}$$

$$A = \left\{\left(0, \ldots, 0, \frac{\kappa + 2}{3}\right)\right\} \cup (G_1 \times D) \subset G.$$

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Theorem 13

For every group G of type III,

$$\hat{\mu}(G, \{2, 1\}) \geq \mu(G, \{2, 1\}) + 1.$$

A Sketch of the Proof of Proposition 12

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Then define

$$D = \left\{\pm 1, \pm 3, \ldots, \pm \frac{25-4}{3}\right\} = \{-7, -5, -3, -1, 1, 3, 5, 7\} \subset \mathbb{Z}_{25}$$

and

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We will show that A is weakly $(2,1)$-sum free in G.

Since

$$D = \{-7, -5, -3, -1, 1, 3, 5, 7\} \equiv \{18, 20, 22, 24, 1, 3, 5, 7\}$$

is an arithmetic progression, we can quickly write

$$\begin{aligned} 2D &= \left\{0, \pm 2, \pm 4, \ldots, \pm 2 \cdot \frac{25-4}{3}\right\} \\ &= \{-14, -12, -10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10, 12, 14\} \\ &= \{11, 13, 15, 17, 19, 21, 23, 0, 2, 4, 6, 8, 10, 12, 14\}. \end{aligned}$$

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We can see that D is $(2,1)$-sum free, and we can partition $\mathbb{Z}_{25}$ as

$$\mathbb{Z}_{25} = D \cup \{9\} \cup 2D \cup \{16\}.$$

Now, if we take any (not necessarily distinct)

$$a_1, a_2 \in A \setminus \{(0, \ldots, 0, 9)\},$$

we know the last coordinates of a_1 and a_2 are in D, so the last coordinate of $a_1 + a_2$ is in $2D$, not in $D \cup \{9\}$;

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$$D + 9 = \{2, 4, 6, 8, 10, 12, 14, 16\} \subset (2D \cup \{16\}),$$

which is disjoint from D, so we are done. $\square$

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- A more general categorization on the groups G of type II for which
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is not known and would be valuable to find. It is curious that the value of $\hat{\mu}(G, \{2,1\})$ did not depend on the exponent of the group G for type I but seems to for type II.

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- It is also still open to prove or disprove that
$$\mu\hat{}(G, \{2, 1\}) = \mu(G, \{2, 1\}) + 1$$
for every group G of type III. This task presents to be very challenging.

Thank you!

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Questions?