PROBLEM & REVISION SHEETS: DISCRETE TIME FINANCE (MATH5320M)

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Abstract. You should carefully try to solve all the exercises each week as a preparation for the seminars where the solutions will be **discussed**. Full solutions to the problems will only be provided at the end of teaching, so you should make sure to write your own solutions based on your own work and the discussions during the seminars. Numerical solutions can be found at the end of each problem sheet. You can use these to verify if your solutions are correct.

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Problem Sheets

Problem Sheet 1

This sheet will be discussed in the seminar on Friday, 4th October 2024. The solutions will be discussed in the seminar 1 week later.

Problem 1.1. What is the price of an option with payoff $h(S_1) = S_1$?

Solution

The price of the option will be equal to S_0. One explanation is using replicating strategies. The obvious replicating strategy is $(x = S_0, \phi = 1)$. Therefore the option should also have price S_0.

Alternatively, one can use risk-neutral probabilities, namely $\tilde{p} = \frac{1+r-d}{u-d}$ and $1 - \tilde{p} = \frac{u-1-r}{u-d}$. To calculate the price we have:

$$E_{\tilde{\mathbb{P}}}\Big(\frac{h(S_1)}{1+r}\Big) = E_{\tilde{\mathbb{P}}}\Big(\frac{S_1}{1+r}\Big) = S_0,$$

by the properties of the risk-neutral measure.

Problem 1.2. Compute the hedging strategies (the replicating strategies) for a European Call and Put options with the strike price $K = 1$ and the maturity time $t = 1$ in the elementary market model with parameters: $r = \frac{1}{3}$, $S_0 = 1$, $u = 2$, $d = \frac{1}{2}$, $p = \frac{3}{4}$.

Solution

$$S_1 = \begin{cases} 2, & H \\ \frac{1}{2}, & T. \end{cases}$$

The call option has payoff

$$P^c = \begin{cases} 1, & H \\ 0, & T. \end{cases}$$

and the put option has payoff

$$P^p = \begin{cases} 0, & H \\ \frac{1}{2}, & T. \end{cases}$$

The replication strategy for the call is

$$(x - \phi S_0)(1 + r) + \phi S_1(H) = P^c(H)$$
$$(x - \phi S_0)(1 + r) + \phi S_1(T) = P^c(T)$$

i.e.

$$(x - \phi)\frac{4}{3} + \phi 2 = 1$$
$$(x - \phi)\frac{4}{3} + \phi\frac{1}{2} = 0$$

Solving this equation gives

$$\phi = \frac{1-0}{2-\frac{1}{2}} = \frac{2}{3}.$$

Substituting this back in gives $x = \frac{5}{12}$.

Doing the same for the put gives $\phi = -\frac{1}{3}$ and $x = \frac{1}{6}$.

Problem 1.3. Prove that the condition $d < 1 + r < u$ implies that there is no arbitrage in the elementary single period market model.

Solution Arbitrage is a strategy that

(1) starts with no money, so $x = 0$,
(2) has zero probability of losing money, so $V_1(x, \phi) \geq 0$,
(3) has positive probability of making money, so $V_1(x, \phi) > 0$ for at least one event.

Recall that

$$V_1(x, \phi) = \begin{cases} (x - \phi S_0)(1 + r) + \phi S_0 u, & H \\ (x - \phi S_0)(1 + r) + \phi S_0 d, & T \end{cases}.$$

Let $x = 0$, $\phi > 0$. It then follows that

$$V_1 = \begin{cases} S_0\phi(u - (1 + r)) > 0, & H \\ S_0\phi(d - (1 + r)) < 0, & T \end{cases},$$

which violates the second condition.

Problem 1.4. Consider the elementary single period market model with parameters: $r = \frac{1}{3}$, $S_0 = 1$, $u = 2$, $d = \frac{1}{2}$, $p = \frac{3}{4}$. Compute the price of a digital call option with strike price K which is characterized by the payoff function

$$h(S_1) = \begin{cases} 1, & \text{if } S_1 \geq K, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that the price depends on a real number K which is a parameter of the option.

Solution S_1 is as in the previous exercise. If $K > 2$ we have that the payoff of the digital option is always 0, so the replicating strategy is $(x, \phi) = (0, 0)$.

If $\frac{1}{2} < K \leq 2$, the payoff is 1 in H and 0 in T. This gives the system of equations

$$(x - \phi)\frac{4}{3} + \phi 2 = 1$$
$$(x - \phi)\frac{4}{3} + \frac{1}{2}\phi = 0$$

Subtracting the two gives

$$\phi = \frac{1}{2 - \frac{1}{2}} = \frac{2}{3}$$

and so $x = \frac{5}{12}$.

If $K \leq \frac{1}{2}$, the payoff is always 1. Calculating as above, $\phi = 0$ and $x = \frac{3}{4}$.

Remark: You will be able to solve the following 3 problems after our second lecture!

Problem 1.5. Give a proof of the Put-Call Parity.

Solution The formula:

$$C - P = S_0 - \frac{1}{1+r}K.$$

Consider two portfolios.

- Portfolio 1: 1 call option + $K\frac{1}{1+r}$ in cash
  At time $t = 1$ this portfolio has has value

$$(S_1 - K)^+ + K = \begin{cases} S_1 - K + K, & S_1 > K, \\ 0 + K, & S_1 \leq K. \end{cases}$$

- Portfolio 2: 1 put option + 1 share of stock
  At time $t = 1$ this portfolio has has value

$$(K - S_1)^+ + S_1 = \begin{cases} S_1, & S_1 > K \\ K - S_1 + S_1, & S_1 \leq K \end{cases}$$

The two portfolios have equal payoffs at $t = 1$, so to avoid arbitrage they need to have the same value at $t = 0$.

Problem 1.6. Under the assumption that the elementary single period market model is arbitrage free show that there exists a random variable Z such that the price of an European option with the payoff $h(S_1)$ can be computed by the formula

$$x = \mathbb{E}\left(Zh(S_1)\right).$$

Solution Assuming no arbitrage gives us that there exists a risk-neutral measure. We want to show that there exists a Z s.t.

$$x = \mathbb{E}(Zh(S_1))$$

and

$$x = \mathbb{E}_{\tilde{\mathbb{P}}}(\frac{h(S_1)}{1+r})$$

The second equation can be expressed as

$$x = \frac{1}{1+r}(\tilde{p}h(S_1(H)) + (1-\tilde{p})h(S_1(T))$$

while the first equation gives

$$x = pZ(H)h(S_1(H)) + (1-p)Z(T)h(S_1(T))$$

Combining the two gives that if we set

$$Z(H) = \frac{\tilde{p}}{p}\frac{1}{1+r}$$

and

$$Z(T) = \frac{1-\tilde{p}}{1-p}\frac{1}{1+r},$$

Z satisfies the requirement. Such a Z is called the Radon-Nikodym derivative or the State-Price density.

Problem 1.7. Consider the elementary single period market model with parameters $r = \frac{1}{3}$, $S_0 = 1$, $u = 2$, $d = \frac{1}{2}$, $p = \frac{3}{4}$. Find all probability measures $\mathbb{Q}$ that assign positive values to both states H and T and satisfy

$$S_0 = \mathbb{E}_{\mathbb{Q}} \frac{S_1}{1+r},$$

where

$$\mathbb{E}_{\mathbb{Q}} \frac{S_1}{1+r} = \frac{S_1(H)}{1+r}\mathbb{Q}(H) + \frac{S_1(T)}{1+r}\mathbb{Q}(T).$$

Solution $\mathbb{Q}$ is a probability measure so $\mathbb{Q}(H) + \mathbb{Q}(t) = 1$. We also want $\mathbb{Q}(H)$ and $\mathbb{Q}(T)$ to be strictly positive.

Since $\mathbb{Q}$ satisfies $S_0 = \mathbb{E}_{\mathbb{Q}} \frac{S_1}{1+r}$ by assumption, this gives that

$$1 = \frac{1}{1+r}(\mathbb{Q}(H)u + \mathbb{Q}(T)d).$$

Combined with $\mathbb{Q}(H) + \mathbb{Q}(t) = 1$ this gives

$$\mathbb{Q}(T) = \frac{u-1-r}{u-d}$$

and

$$\mathbb{Q}(H) = \frac{1+r-d}{u-d}.$$

In our example this gives $\mathbb{Q}(H) = \frac{5}{9} > 0$ and $\mathbb{Q}(T) = \frac{4}{9} > 0$, so this is the unique solution to the problem.

Numerical solutions

P1.2: $(x, \phi) = (\frac{5}{12}, \frac{2}{3})$ and $(x, \phi) = (\frac{1}{6}, -\frac{1}{3})$. **P1.4:** For $K > 2$, $(x, \phi) = (0, 0)$. For $K > \frac{1}{2}$, $(x, \phi) = (\frac{5}{12}, \frac{2}{3})$. For $K \leq \frac{1}{2}$, $(x, \phi) = (\frac{3}{4}, 0)$. **P1.7:** $\mathbb{Q}(H) = \frac{5}{9}$ and $\mathbb{Q}(T) = \frac{4}{9}$.

Problem Sheet 2

This sheet will be discussed in the seminar on Friday, 11th October 2024. The solutions will be discussed in the seminar 1 week later.

Problem 2.1. Verify the following two equations:

$$\hat{V}_1(x,\phi) = \frac{V_1(x,\phi)}{1+r}, \qquad \hat{V}_1(x,\phi) = \hat{V}_0(x,\phi) + \hat{G}(x,\phi).$$

Solution We calculate starting from the right-hand side.

$$\begin{aligned}\frac{V_1(x,\phi)}{1+r} &= \frac{(x - \sum_{i=1}^n \phi^i S_0^i)(1+r) + \sum_{i=1}^n S_1^i \phi^i}{1+r} \\ &= (x - \sum_{i=1}^n \phi^i S_0^i) + \sum_{i=1}^n \phi^i \frac{S_1^i}{1+r} \\ &= x - \sum_{i=1}^n \phi^i S_0^i + \sum_{i=1}^n \phi^i \hat{S}_1^i = \hat{V}_1(x,\phi).\end{aligned}$$

For the second equality, we calculate

$$\begin{aligned}\hat{V}_1(x,\phi) &= x - \sum_{i=1}^n \phi^i S_0^i + \sum_{i=1}^n \phi^i \hat{S}_1^i \\ &= x + \sum_{i=1}^n \phi^i (\hat{S}_1^i - S_0^i) = x + \sum_{i=1}^n \phi^i \Delta \hat{S}^i \\ &= x + \hat{G}(x,\phi) = V_0(x,\phi) + \hat{G}(x,\phi) = \hat{V}_0(x,\phi) + \hat{G}(x,\phi).\end{aligned}$$

Problem 2.2. Prove Proposition 1.2.6. (This is merely a reformulation of the definition of arbitrage; don't be afraid!)

Solution We first verify that the first of the two conditions is equivalent to the definition. Since $V_0(x,\phi) = \hat{V}_0(x,\phi)$, the first condition holds by definition. The second condition of arbitrage also holds since $1+r > 0$ by assumption, so that when $V_1(x,\phi) \geq 0$, $\hat{V}_1(x,\phi) = V_1(x,\phi)/(1+r) \geq 0$ as well and similarly in reverse. The final condition is equally satisfied for the same ω due to $(1+r) > 0$.

The equivalence of the two conditions from Proposition 1.16. can be proven similarly since:

$$\hat{G}(x,\phi) = \hat{V}_1(0,\phi)$$

Remark: You will be able to solve the following 3 problems after our third lecture!

Problem 2.3. Consider a model where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $r = \frac{1}{9}$, $S_0 = 5$ and $S_1(\omega)$ is given by the table:

	ω_1	ω_2	ω_3
S_1^1	$\frac{20}{3}$	$\frac{40}{9}$	$\frac{30}{9}$

Compute all risk neutral measures and all attainable contingent claims for this model. Is the model arbitrage free?

Solution

$$\begin{tikzcd}[column sep=large] & S_1(\omega_1) = \frac{20}{3} \\ S_0 = 5 \arrow[ur] \arrow[r] \arrow[dr] & S_1(\omega_2) = \frac{40}{9} \\ & S_1(\omega_3) = \frac{30}{9} \end{tikzcd}$$

$$B_0 \longrightarrow B_1 = \frac{10}{9} B_0$$

A risk-neutral probability measure should satisfy

$$\sum_{i=1}^{3} \mathbb{Q}(\omega_i) = 1$$

and

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{S_1}{1+r}\right) = S_0$$

or equivalently

$$\begin{aligned} q_1 + q_2 + q_3 &= 1 \\ \frac{1}{1+r}(q_1 S_1(\omega_1) + q_2 S_1(\omega_2) + q_3 S_1(\omega_3)) &= S_0 \\ q_1 \geq 0, q_2 \geq 0, q_3 &\geq 0 \end{aligned}$$

This gives us

$$\begin{aligned} q_1 + q_2 + q_3 &= 1 \\ \frac{9}{10}\left(q_1 \frac{20}{3} + q_2 \frac{40}{9} + q_3 \frac{30}{9}\right) &= 5 \end{aligned}$$

We substitute $q_3 = 1 - q_2 - q_1$ into the second equation to get

$$6q_1 + 4q_2 + 3(1 - q_1 - q_2) = 5$$

i.e.

$$\begin{aligned} q_3 &= 1 - q_1 - q_2 \\ 3q_1 + q_2 &= 2 \end{aligned}$$

so $q_3 = 2q_1 - 1$ and $q_2 = 2 - q_1$.

If we write $q_1 = \lambda$, this gives

$$\mathbb{Q} = \begin{pmatrix} \lambda \\ 2 - 3\lambda \\ 2\lambda - 1 \end{pmatrix}$$

where the first line gives $\lambda > 0$, the second that $\lambda < \frac{2}{3}$ and the third $\lambda > \frac{1}{2}$, therefore $\lambda \in (\frac{1}{2}, \frac{2}{3})$

We therefore found an infinite number of risk neutral measures. The *fundamental theorem of asset pricing* tells us that the existence of a risk neutral measure guarantees no arbitrage.

We now focus on finding all attainable contingent claims. Consider for example an option with payoff

	ω_1	ω_2	ω_3
S_1^1	$\frac{20}{9}$	$\frac{10}{9}$	$\frac{5}{9}$

Let us find its price using the risk neutral measures:

$$\begin{aligned} x &= \mathbb{E}_{\mathbb{Q}}(\frac{X}{1+r}) \\ &= 2\lambda + (2-3\lambda) + (2\lambda - 1)\frac{1}{2} \\ &= \frac{3}{2}. \end{aligned}$$

Note that the price does not depend on λ, i.e. it does not depend on the choice of the risk neutral measure. This makes the claim attainable (i.e. there exists a replicating strategy).

In general, a claim is attainable if

$$\begin{aligned} (x - \phi S_0)(1+r) + \phi S_1(\omega_1) &= X(\omega_1) \\ (x - \phi S_0)(1+r) + \phi S_1(\omega_2) &= X(\omega_2) \\ (x - \phi S_0)(1+r) + \phi S_1(\omega_3) &= X(\omega_3) \end{aligned}$$

This can be rewritten as

$$\phi \begin{pmatrix} \Delta\hat{S}(\omega_1) \\ \Delta\hat{S}(\omega_2) \\ \Delta\hat{S}(\omega_3) \end{pmatrix} + x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{1+r} \begin{pmatrix} X(\omega_1) \\ X(\omega_2) \\ X(\omega_3) \end{pmatrix},$$

with $x, \phi \in \mathbb{R}$.

The space of attainable claims is therefore

$$\begin{aligned} T &:= \{x \in \mathbb{R}^3 : \phi \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{1+r} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \phi, x \in \mathbb{R}\} \\ &= \mathrm{Span}(\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) \end{aligned}$$

Problem 2.4. Compute $\mathbb{W}$ and $\mathbb{W}^{\perp}$ for the model in Problem 2.3.

Solution We know from the previous question that

$$\begin{pmatrix} \Delta\hat{S}(\omega_1) \\ \Delta\hat{S}(\omega_2) \\ \Delta\hat{S}(\omega_3) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

By definition

$$\mathbb{W} = \{x \in \mathbb{R}^3 | x = \hat{G}(x, \phi), \phi \in \mathbb{R}\}$$

and

$$\hat{G}(x,\phi) = \phi\Delta\hat{S} = \phi\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

Therefore

$$\mathbb{W} = \{x \in \mathbb{R}^3 | x = \phi\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \phi \in \mathbb{R}\}.$$

$$\begin{aligned}\mathbb{W}^{\perp} &= \{Y \in \mathbb{R}^3 | \langle Y, X\rangle = 0, X \in \mathbb{W}\} \\ &= \{Y \in \mathbb{R}^3 | Y_1 - Y_2 - 2Y_3 = 0\}\end{aligned}$$

Recall that $\mathbb{W}^{\perp} \cap \mathcal{P}^{+}$ is the set of all risk-neutral measures, so that any triple from $\mathbb{W}^{\perp}$ that also represents a strictly positive probability measure is a risk-neutral measure.

Problem 2.5. Consider a market model with two stocks, a money market account and four possible states of the world: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. The risk free interest rate is equal to $r = 20\%$. Prices of both stocks at time $t = 0$ are equal to 1. The prices at time $t = 1$ are given by the following table:

	ω_1	ω_2	ω_3	ω_4
S_1^1	4	1	3	1
S_1^2	1.2	0.2	1.2	2

- Find the set of risk neutral probability measures.
- Find the set of attainable contingent claims for the model.

Solution Let the risk neutral measure be $\mathbb{Q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$. We look for all $\mathbb{Q}$ that satisfy

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(\Delta\hat{S}^1) &= 0 \\ \mathbb{E}_{\mathbb{Q}}(\Delta\hat{S}^2) &= 0\end{aligned}$$

This is equivalent to the three equations

$$\begin{aligned}4q_1 + q_2 + 3q_3 + q_4 &= 1.2 \\ 1.2q_1 + 0.2q_2 + 1.2q_3 + 2q_4 &= 1.2 \\ q_1 + q_2 + q_3 + q_4 &= 1\end{aligned}$$

If we write this as a matrix and diagonalise it we get

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 3 & 1 & 1.2 \\ 1.2 & 0.2 & 1.2 & 2 & 1.2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & -3 & -2.8 \\ 0 & -1 & 0 & 0.8 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} & 1 & \frac{2.8}{3} \\ 0 & 0 & \frac{1}{3} & 1.8 & \frac{2.8}{5} \end{array}\right)$$

From the last row we get that $q_3 = 2.8 - 5.4q_4$. Row 2 gives $q_2 = 0.8q_4$ and Row 1 gives $q_1 = -1.8 + 2.6q_4$, so

$$\mathbb{Q} = \begin{pmatrix} -1.8 + 3.6\lambda \\ 0.8\lambda \\ 2.8 - 5.4\lambda \\ \lambda \end{pmatrix}$$

Since all q_i need to be positive, this gives that $\lambda \in (0.5, 0.5185)$.

We next look for all attainable claims. By repeating the steps from the previous question we get

$$T := \mathrm{Span}(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{28}{12} \\ -\frac{2}{12} \\ \frac{18}{12} \\ -\frac{2}{12} \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{10}{12} \\ 0 \\ \frac{8}{12} \end{pmatrix})$$

Numerical solutions

P2.3: $\mathbb{Q} = (\lambda, 2-3\lambda, 2\lambda-1)^T$, $\lambda \in (\frac{1}{2}, \frac{2}{3})$. $T = \mathrm{Span}\left((1,-1,-2)^T, (1,1,1)^T\right)$.

P2.4: $\mathbb{W} = \mathrm{Span}\left((1,-1,-2)^T\right)$, $\mathbb{W}^{\perp} = \{Y \in \mathbb{R} | Y_1 - Y_2 - 2Y_3 =)\}$. **P2.5:** $\mathbb{Q} = (-1.8 + 3.6\lambda, 0.8\lambda, 2.8 - 5.4\lambda, \lambda)^T$, $\lambda \in (0.5, 0.5185)$.

$$T = \mathrm{Span}\left((1,1,1,1)^T, \left(\frac{28}{12}, -\frac{2}{12}, \frac{18}{12}, -\frac{2}{12}\right)^T, \left(0, -\frac{10}{12}, 0, \frac{8}{12}\right)^T\right).$$

Problem Sheet 3

This sheet will be discussed in the seminar on Friday, 18th October 2024. The solutions will be discussed in the seminar 1 week later.

Problem 3.1. Prove that if $\tilde{\mathbb{P}}$ is a risk neutral measure and (x, ϕ) is a trading strategy in the general single period market model then

$$\mathbb{E}_{\tilde{\mathbb{P}}}\left(\frac{V_1(x,\phi)}{1+r}\right) = x.$$

Solution We calculate directly

$$\begin{aligned}
\mathbb{E}_{\tilde{\mathbb{P}}}(\frac{V_1(x,\phi)}{1+r}) &= \mathbb{E}_{\tilde{\mathbb{P}}}(\frac{(x-\sum_{i=1}^n \phi^i S_0^i)(1+r)+\sum_{i=1}^n \phi^i S_1^i}{1+r}) \\
&= \mathbb{E}_{\tilde{\mathbb{P}}}(x-\sum_{i=1}^n \phi^i S_0^i + \sum_{i=1}^n \phi^i \frac{S_1^i}{1+r}) \\
&= \mathbb{E}_{\tilde{\mathbb{P}}}(x+\sum_{i=1}^n \phi^i (\frac{S_1^i}{1+r} - S_0^i)) \\
&= \mathbb{E}_{\tilde{\mathbb{P}}}(x+\sum_{i=1}^n \phi^i \Delta\hat{S}^i) = x + \sum_{i=1}^n \phi^i \mathbb{E}_{\tilde{\mathbb{P}}}(\Delta\hat{S}^i) \\
&= x,
\end{aligned}$$

where in the last step we used the definition of a risk neutral measure.

Problem 3.2. Consider a one-period market model consisting of a **bond** and a **stock** with the following prices.

- At time 0:

$$B_0^1 = 1 \quad and \quad S_0^2 = 4$$

- At time 1:

	ω_1	ω_2	ω_3
B_1^1	1	1	1
S_1^2	6	4	3

(a) Explain briefly why the interest rate is equal to 0%.
(b) Compute the set of risk neutral measures for this model.
(c) Is this market arbitrage-free? Explain your answer.
(d) Is this model complete? Give an explanation.
(e) *ABC Bank* decides to introduce an investment product that protects investors from adverse stock performance. This product has a payoff:

	ω_1	ω_2	ω_3
S_1^3	0	1	1.5

Show that it is an attainable claim.

Solution

(a) $B_1 = (1+r)B_0$ if and only if $r = 0$.

(b) We are solving the system of two equations with 3 unknowns:

$$q_1 + q_2 + q_3 = 1$$
$$6q_1 + 4q_2 + 3q_3 = 4.$$

Substituting q_3 from the first equation into the second, we get

$$q_3 = 1 - q_1 - q_2$$
$$q_2 = 1 - 3q_1,$$

and substituting q_2 from the second equation into the first yields

$$\mathbb{Q} = \begin{pmatrix} \lambda \\ 1 - 3\lambda \\ 2\lambda \end{pmatrix}.$$

Here, $\lambda > 0$ due to the first entry and $\lambda < 1/3$ due to the second entry, so $\lambda \in (0, 1/3)$

(c) Yes, the fundamental theorem of asset pricing states that if at least one risk-neutral measure exists, then there is no arbitrage.

(d) The market is incomplete since more than one risk-neutral measure exists.

(e) We have two approaches:

Approach 1: A claim is attainable if a replicating strategy exists. We solve the system

$$(x - \phi 4) + \phi 6 = 0$$
$$(x - 4\phi) + \phi 4 = 1$$
$$(x - 4\phi) + \phi 3 = 1.5$$

Subtracting the 2nd equality from the first one gives

$$2\phi = -1 \Leftrightarrow \phi = -\frac{1}{2} \Leftrightarrow x = 1.$$

Checking the 3rd equality does not yield a contradiction, so the claim is attaninable.

Approach 2: We calculate

$$\mathbb{E}_{\mathbb{Q}}(\frac{S_1^3}{1+r}) = 0\lambda + (1 - 3\lambda) + 2\lambda\frac{3}{2} = 1.$$

Since the price does not depend on λ, the claim is attainable.

Problem 3.3. Consider a model where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $r = \frac{1}{9}$, $S_0 = 5$ and $S_1(\omega)$ is given by the table:

	ω_1	ω_2	ω_3
S_1^1	$\frac{20}{3}$	$\frac{40}{9}$	$\frac{30}{9}$

This is the model from Problem 2.3. In this model consider a call option with the strike price $K = \frac{50}{9}$, i.e. $X = (S_1 - K)^+$. Compute the set of all prices of this option that comply with the no arbitrage principle.

Solution The payout of X is

	ω_1	ω_2	ω_3
X	$\frac{10}{9}$	0	0

Recall that

$$\mathbb{Q} = \begin{pmatrix} \lambda \\ 2-3\lambda \\ 2\lambda - 1 \end{pmatrix},$$

with $\lambda \in (\frac{1}{2}, \frac{2}{3})$. We next calculate

$$\mathbb{E}_{\mathbb{Q}}(\frac{X}{1+r}) = \lambda \frac{10}{9}\frac{9}{10} = \lambda \in (\frac{1}{2}, \frac{2}{3}).$$

Any price in the above interval does not introduce arbitrage. The claim is not attainable since the price depends on λ.

Problem 3.4. Consider an arbitrage free general single period market model. Given a contingent claim X characterize the set of all its prices that comply with the no arbitrage principle. Hint: use the set $\mathbb{M}$ of risk neutral measures.

Solution

$$x = \mathbb{E}_{\mathbb{Q}}(\frac{X}{1+r}) = \frac{1}{1+r}\langle X, \mathbb{Q}\rangle = \frac{1}{1+r}(q_1 X_1 + \cdots + q_k X_k),$$

with $\mathbb{Q} \in \mathbb{W}^{\perp} \cap \mathcal{P}^{+}$. Here (as in the lectures):

$$\mathbb{W} = \{X \in \mathbb{R}^k : X = \hat{G}(x, \phi)\}$$
$$\mathbb{W}^{\perp} = \{Y \in R^k : \langle X, Y\rangle = 0, X \in \mathbb{W}\}$$
$$\mathcal{P}^{+} = \{Z \in \mathbb{R}^k : \sum_{i=1}^{k} Z_i = 1, Z_i > 0\, \forall i\}$$

Numerical solutions
P3.2: (b) $\mathbb{Q} = (\lambda, 1-3\lambda, 2\lambda)$ with $\lambda \in (0, 1/3)$. **P3.3:** $\mathbb{E}_{\mathbb{Q}}(\frac{X}{1+r}) = \lambda \in (\frac{1}{2}, \frac{2}{3})$.

Problem Sheet 4

This sheet will be discussed in the seminar on Friday, 25th October 2024. The solutions will be discussed in the seminar 1 week later.
Remark: You will be able to solve the following 2 problems after our fifth lecture! You can already attempt Problem 4.2 (b) and think about (c).

Problem 4.1. In the context of a multiperiod market model with $t = 0, \dots, T$ show that if there exists a trading strategy $\phi = (\phi_t)_{0 \le t \le T}$ and $t^* \le T$ such that

(1) $V_0(\phi) = 0$
(2) $V_{t^*}(\phi) \ge 0$
(3) $\mathbb{E}(V_{t^*}(\phi)) > 0$

then there exists an arbitrage. Hint: construct an arbitrage from the given trading strategy.

Solution We require

(1) $V_0(\phi) = 0$,
(2) $V_T(\phi) \ge 0$,
(3) $\mathbb{E}(V_T(\phi)) > 0$.

To build the arbitrage, do the following:

- Follow the arbitrage strategy ϕ from $t = 0$ to $t = t^*$.
- At $t = t^*$, cash in the arbitrage opportunity and invest it all in the riskless asset (i.e. the money market account). Therefore, for $t \ge t^*$,

$$\phi_t = \begin{pmatrix} V_{t^*}/B_{t^*} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

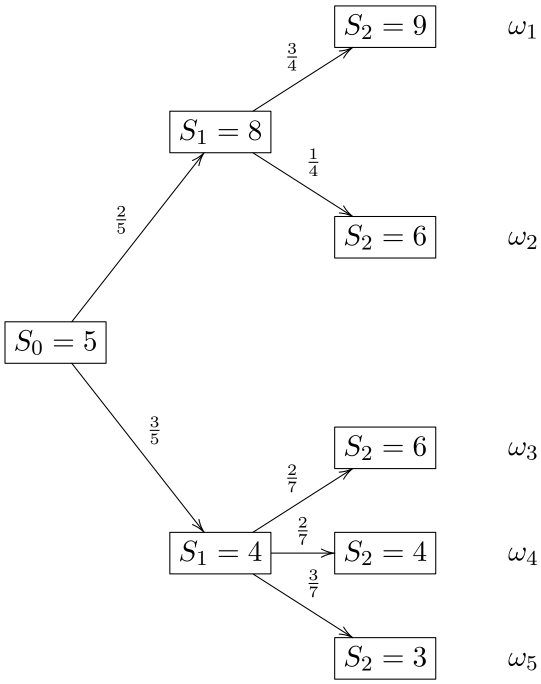
By definition of the arbitrage strategy, there exists an $\omega \in \Omega$ such that $V_{t^*}(\phi)(\omega) > 0$ and therefore ϕ_t above is not identically equal 0 for at least one ω.

At time T, the value of ϕ is equal to

$$V_T(\phi) = \frac{V_{t^*}}{B_{t^*}} B_{t^*}(1+r)^{T-t^*} = V_{t^*}(1+r)^{T-t^*} \ge 0$$

and $V_T(\phi) > 0$ on at least one ω by the above.

Problem 4.2. Consider a market model with $T = 2$, i.e. $t = 0, 1, 2$, and riskless interest rate $r = \frac{1}{4}$, and one risky asset whose prices are described by the following diagram:



(a) Compute the probability measure $\mathbb{P}$ for the model using the probability values on the arrows.
(b) Find a filtration generated by the asset prices.
(c) Check if the following trading strategies are adapted:

	$t=0$					$t=1$				
	ω_1	ω_2	ω_3	ω_4	ω_5	ω_1	ω_2	ω_3	ω_4	ω_5
ϕ^0	$\frac{12}{5}$	$\frac{12}{5}$	$\frac{17}{3}$	$\frac{17}{3}$	$\frac{17}{3}$	$\frac{12}{5}$	$\frac{12}{5}$	$\frac{17}{3}$	$\frac{17}{3}$	$\frac{17}{3}$
ϕ^1	-17	-17	-8	-8	-8	-17	-17	-8	-8	-8
ϕ^0	2	2	2	2	2	2	2	2	3	3
ϕ^1	-1	-1	-1	-1	-1	$-\frac{5}{3}$	$-\frac{5}{3}$	2	2	2
ϕ^0	2	2	2	2	2	2	2	3	3	3
ϕ^1	-1	-1	-1	-1	-1	$-\frac{5}{3}$	$-\frac{5}{3}$	2	2	2

(d) Correct ϕ_1^0 (i.e. ϕ^0 at $t = 1$) in such a way that the strategy given by the table below if self-financing:

	$t=0$					$t=1$				
	ω_1	ω_2	ω_3	ω_4	ω_5	ω_1	ω_2	ω_3	ω_4	ω_5
ϕ^0	2	2	2	2	2	2	2	3	3	3
ϕ^1	-1	-1	-1	-1	-1	$-\frac{5}{2}$	$-\frac{5}{2}$	2	2	2

(e) Compute the initial capital needed to set up the above (fixed) trading strategy.

(f) Compute the value process for the trading strategy obtained in (d), i.e. for the self-financing version!

Solution

(a)

$$\begin{aligned}\mathbb{P}(\omega_1) &= \mathbb{P}(S_1 = 8 \,\&\, S_2 = 9)\\ &= \mathbb{P}(\omega_1|\{\omega_1,\omega_2\})\mathbb{P}(\{\omega_1,\omega_2\}) + \mathbb{P}(\omega_1|\{\omega_3,\omega_4,\omega_5\})\mathbb{P}(\{\omega_3,\omega_4,\omega_5\})\\ &= \frac{3}{4}\frac{2}{5} + 0\frac{3}{5} = \frac{3}{10}\\ \mathbb{P}(\omega_2) &= \frac{2}{5}\frac{1}{4} = \frac{1}{10}\\ \mathbb{P}(\omega_3) &= \frac{3}{5}\frac{2}{7} = \frac{6}{35}\\ \mathbb{P}(\omega_4) &= \frac{3}{5}\frac{2}{7} = \frac{6}{35}\\ \mathbb{P}(\omega_5) &= \frac{3}{5}\frac{3}{7} = \frac{9}{35}\end{aligned}$$

We also calculate for later use

$$B_0 = 1 \to B_1 = \frac{5}{4} \to B_2 = \frac{25}{16}$$

(b) At $t = 0$, $S_0 = 5$ is constant in all Ω. Consequently, $A_0 = \{\Omega\}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

At $t = 1$, $S_1 = \begin{cases} 8 & \text{for } \{\omega_1,\omega_2\}\\ 4 & \text{for } \{\omega_3,\omega_4,\omega_5\}\end{cases}$. The resulting partition of Ω is $A_1^1 = \{\omega_1,\omega_2\}$ and $A_1^2 = \{\omega_3,\omega_4,\omega_5\}$. The sigma algebra generated by S_1 is therefore $\mathcal{G}_1 = \{\emptyset, \{\omega_1,\omega_2\}, \{\omega_3,\omega_4,\omega_5\}, \Omega\}$. Since $\mathcal{F}_0 \subset \mathcal{G}_1$, we have that $\mathcal{G}_1 = \mathcal{F}_1$.

At $t = 2$, $S_2 = \begin{cases} 9 & \text{for } \omega_1\\ 6 & \text{for } \{\omega_2,\omega_3\}\\ 4 & \text{for } \omega_4\\ 3 & \text{for } \omega_5\end{cases}$. The partition for $\mathcal{G}_2$ is therefore $\{\omega_1\}$, $\{\omega_2,\omega_3\}$, $\{\omega_4\}$ and $\{\omega_5\}$. To get the filtration at $t = 2$, we need to combine $\mathcal{F}_1$ and $\mathcal{G}_2$. The only partition that can reproduce all partition sets of $\mathcal{F}_1$ and $\mathcal{G}_2$ is

$$A_2^1 = \{\omega_1\}, A_2^2 = \{\omega_2\}, A_2^3 = \{\omega_3\}, A_2^4 = \{\omega_4\}, A_2^5 = \{\omega_5\}$$

and so $\mathcal{F}_2 = \mathcal{P}(\Omega)$.

(c) The first strategy is not adapted since ϕ^1 is not constant on Ω at $t = 0$. The second strategy is not adapted since ϕ^0 is not constant on the sets $\{\omega_1,\omega_2\}$ and $\{\omega_3,\omega_4,\omega_5\}$ at $t = 1$. The third strategy is adapted, since it is constant on all partition sets of $\mathcal{F}_0$ and $\mathcal{F}_1$.

(d) We will modify the values of ϕ^0 at time $t = 1$ to ensure the strategy is self financing. We have that

$$V_0(\phi) = \phi_0^0 B_0 + \phi_0^1 S_0 = 2 \cdot 1 - 1 \cdot 5 = -3.$$

In order to be self financing, we require that $V_1(\phi)^- = V_1(\phi)^+$, i.e.

$$\phi_0^0 B_1 + \phi_0^1 S_1 = \phi_1^0 B_1 + \phi_1^1 S_1.$$

On $\{\omega_1, \omega_2\}$, this translates to

$$\phi_0^0(1+r) + \phi_0^1 S_1(\omega_1, \omega_2) = \phi_1^0(\omega_1, \omega_2)(1+r) + \phi_1^1(\omega_1, \omega_2) S_1(\omega_1, \omega_2)$$
$$2\frac{5}{4} - 1 \cdot 8 = \phi_1^0(\omega_1, \omega_2)\frac{5}{4} - \frac{5}{2}8,$$

so $\phi_1^0(\omega_1, \omega_2) = \frac{58}{5}$. A similar calculation on $\{\omega_3, \omega_4, \omega_5\}$ gives $\phi_1^0(\omega_3, \omega_4, \omega_5) = -\frac{38}{5}$.

(e) We already calculated that

$$V_0(\phi) = \phi_0^0 B_0 + \phi_0^1 S_0 = 2 \cdot 1 - 1 \cdot 5 = -3.$$

(f) We have that

$$V_1(\phi) = \begin{cases} 2\frac{5}{4} - 1 \cdot 8 = -\frac{11}{2} & \text{for } \{\omega_1, \omega_2\} \\ 2\frac{5}{4} - 1 \cdot 4 = -\frac{3}{2} & \text{for } \{\omega_3, \omega_4, \omega_5\} \end{cases}$$

and

$$V_2(\phi) = \phi_1^0 B_2 + \phi_1^1 S_2 = \begin{cases} \frac{58}{5}\left(\frac{5}{4}\right)^2 - \frac{5}{2}9 = -4.375 & \text{for } \omega_1 \\ \frac{58}{5}\left(\frac{5}{4}\right)^2 - \frac{5}{2}6 = 3.125 & \text{for } \omega_2 \\ -\frac{38}{5}\left(\frac{5}{4}\right)^2 + 2 \cdot 6 = 0.125 & \text{for } \omega_3 \\ -\frac{38}{5}\left(\frac{5}{4}\right)^2 + 2 \cdot 4 = -3.875 & \text{for } \omega_4 \\ -\frac{38}{5}\left(\frac{5}{4}\right)^2 + 2 \cdot 3 = -5.875 & \text{for } \omega_5 \end{cases}.$$

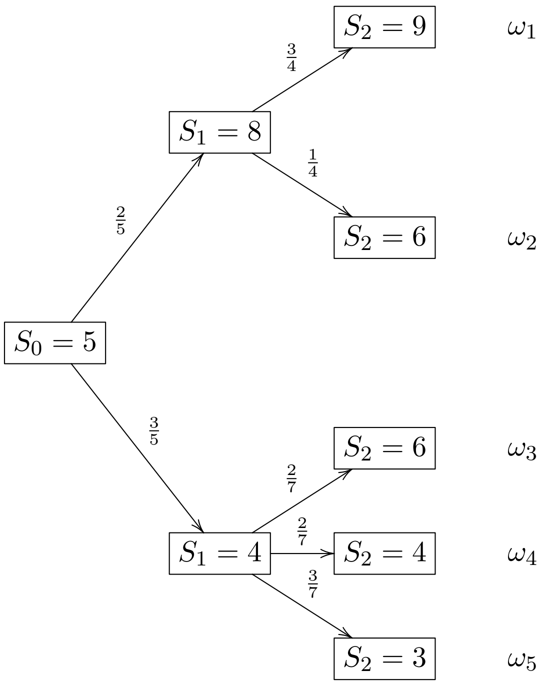
Numerical solutions

P4.2: (a) $\mathbb{P}(\omega_1) = \frac{3}{10}$, $\mathbb{P}(\omega_2) = \frac{1}{10}$, $\mathbb{P}(\omega_3) = \frac{6}{35}$, $\mathbb{P}(\omega_4) = \frac{6}{35}$ and $\mathbb{P}(\omega_5) = \frac{9}{35}$. (b) $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{G}_1 = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5\}, \Omega\}$. $\mathcal{F}_2 = \mathcal{P}(\Omega)$. (c) Not adapted, not adapted, adapted. (d) $\phi_1^0(\omega_1, \omega_2) = \frac{58}{5}$ and $\phi_1^0(\omega_3, \omega_4, \omega_5) = -\frac{38}{5}$. (e) $V_0(\phi) = -3$ (f) $V_1(\phi)(\omega_1, \omega_2) = -\frac{11}{2}$, $V_1(\phi)(\omega_3, \omega_4, \omega_5) = -\frac{3}{2}$, $V_2(\phi)(\omega_1) = -4.375$, $V_2(\phi)(\omega_2) = -3.125$, $V_2(\phi)(\omega_3) = 0.125$, $V_2(\phi)(\omega_4) = -3.875$ and $V_2(\phi)(\omega_5) = -5.875$

Problem Sheet 5

This sheet will be discussed in the seminar on Friday, 1st November 2024. The solutions will be discussed in the seminar 1 week later.

Problem 5.1. Consider a market model with a money market account with one-period interest rate equal to 20% and a risky asset whose prices are given by the following diagram:



(a) What are the dynamics of the money market account, i.e. find values for B_0, B_1 and B_2.
(b) Recall that a multiperiod contingent claim is a random variable X representing the payoff at $t = 2$. Find the payoffs, i.e. the random variables X, for the following options:
- a call option with the strike price K equal to 5,
- a put option with the strike price equal to 5,
- a binary put option with the strike price equal to 4.2.

(c) Try to calculate a hedging strategy for the binary put option with strike price equal to 4.2.
(d) Is this market model complete?

Solution

(a)

$$B_0 = 1, \quad B_1 = (1 + r)B_0 = 1.2, \quad B_2 = (1 + r)^2 B_0 = 1.44.$$

(b) Note first that

$$X^C = (S_2 - 5)^+,$$
$$X^P = (5 - S_2)^+,$$
$$X^{BP} = \begin{cases} 1, & S_2 \leq 4.2 \\ 0, & S_2 > 4.2 \end{cases}.$$

We have therefore that the payoffs are as follows:

Ω	X^C	X^P	X^{BP}
ω_1	4	0	0
ω_2	1	0	0
ω_3	1	0	0
ω_4	0	1	1
ω_5	0	2	1

(c) In the lower branch of the tree we are solving the system of equations

$$\phi_0^1 1.2^2 + \phi_1^1 6 = 0$$
$$\phi_0^1 1.2^2 + \phi_1^1 4 = 1$$
$$\phi_0^1 1.2^2 + \phi_1^1 3 = 1.$$

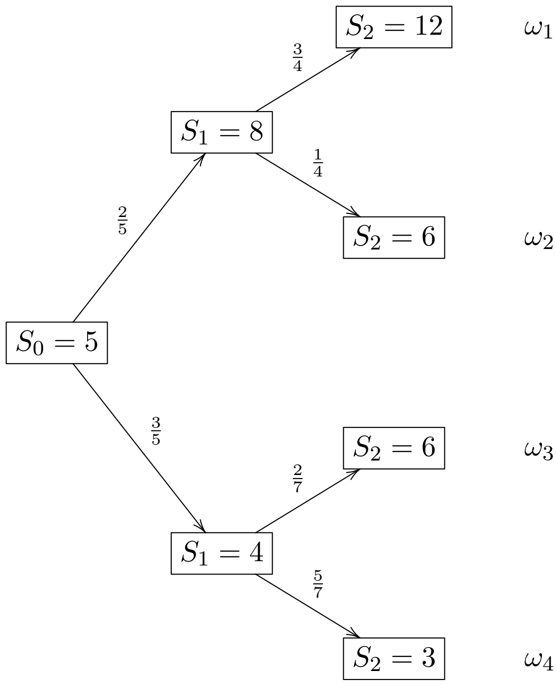
The second and third equations yields $\phi_1^1 = 0$ and therefore $\phi_1^0 = \frac{1}{1.2^2}$. Checking whether this solution satisfies the first equation we get

$$1 + 0 = 0,$$

which is a contradiction. It follows that no replicating strategy exists.

(d) No, the market is not complete. We know this since we have an unattainable claim X^{BP} (we verified this in the previous task) and therefore the market cannot be complete (by definition of completeness).

Problem 5.2. Consider a market model with $T = 2$, i.e. $t = 0, 1, 2$, and riskless interest rate $r = \frac{1}{4}$, and one risky asset whose prices are described by the following diagram:



(Do not use decimal approximations in calculations.)

(a) Compute a hedge for a European Put option with the strike price equal to $K = 5$.
(b) What is the price one has to pay for this option at the moment $t = 0$?
(c) What would be a fair price for this option to be paid at $t = 1$? Is it a deterministic value or a random variable?
(d) Show that the hedging strategy you calculated is self-financing.

Solution

(a) We first calculate the payoff of the put option: The replicating strategy at

Ω	X^P
ω_1	0
ω_2	0
ω_3	0
ω_4	2

the node $S_1 = 8$ can be quickly calculated to satisfy $(\phi_1^0, \phi_1^1) = (0, 0)$. This also gives us that the value at this node is $x_1 = 0$.

At the node $S_1 = 4$ we need to solve the system

$$\phi_1^0(1+r)^2 + \phi_1^1 6 = 0,$$
$$\phi_1^0(1+r)^2 + \phi_1^1 3 = 2.$$

Solving this system we get that $\phi_1^1 = \frac{0-2}{6-3} = -\frac{2}{3}$ and consequently $\phi_1^0 = 4 \cdot \frac{16}{25}$. The value at this node is therefore $x_1 = \phi_1^0(1+r) + \phi_1^1 4 = \frac{8}{15}$.
Remark 1: Note that the replicating strategy at this node requires us to hold a negative amount of the underlying stock and a positive amount of the risk free asset. This is because the payoff of the put option (at this node) increases if the stock price falls, so we are required to hold a negative amount of the stock to hedge this fall. For a call option, this would be reversed - the payoff of a call option tends to increase if the underlying stock increases in value, so a hedging strategy will therefore have to hold a positive amount of the stock so that the two move in the same direction.
Remark 2: The above remark holds for holding long positions of the option. When shorting, the reverse becomes true. For example, a hedging strategy for a short position in a European put option will require a positive amount of the stock, since a fall in the stock price increases the value of the put, and since we are shorting the put our losses would therefore increase. Put simply, a short in a put loses more money if the stock value drops, so to hedge against it we need to hold a positive amount of stock so that our hedge moves in the same direction as the stock.

We now proceed to calculate the replicating strategy at the node $S_0 = 5$, using the values we have calculated at the previous two nodes. We are therefore solving the system of equations

$$\phi_0^0(1+r) + \phi_0^1 8 = 0,$$
$$\phi_0^0(1+r) + \phi_0^1 4 = \frac{8}{15}.$$

Calculating as before, we get $\phi_0^1 = \frac{-\frac{8}{15}}{8-4} = -\frac{2}{15}$ and $\phi_0^0 = \frac{64}{75}$. Calculating the value of this strategy at time $t = 0$ we get

$$x_0 = \phi_0^0 + \phi_0^1 5 = \frac{14}{75} \sim 0.187.$$

Note that the comment and argument from Remark 1 applies here as well.

(b) The price of the option at $t = 0$ is $x_0 = 0.187$ as calculated above.

(c) The price at $t = 1$ is a random variable, depending on which node we are in. The price is therefore

$$x_1 = \begin{cases} 0, & \{\omega_1, \omega_2\} \\ \frac{8}{15}, & \{\omega_3, \omega_4\} \end{cases}.$$

(d) We need to check that at each node, no money is entering or leaving the system: At node $S_1 = 8$ we have

$$\phi_0^0(1+r) + \phi_0^1 8 = 0,$$

which matches the value of x_1 at this node. At the node $S_1 = 4$ we have

$$\phi_0^0(1+r) + \phi_0^1 4 = \frac{8}{15},$$

which matches the value x_1 at this node.

Numerical solutions

P5.1: (a) $B_0 = 1$, $B_1 = (1+r)B_0 = 1.2$, $B_2 = (1+r)^2 B_0 = 1.44$. (b) $X^C = (4,1,1,0,0)^T$, $X^P = (0,0,0,1,2)^T$, $X^{BP} = (0,0,0,1,1)^T$. **P5.2:** (b) $x_0 \sim 0.187$. (c) $x_1 = \begin{cases} 0, & \{\omega_1,\omega_2\} \\ \frac{8}{15}, & \{\omega_3,\omega_4\} \end{cases}$.

Problem Sheet 6

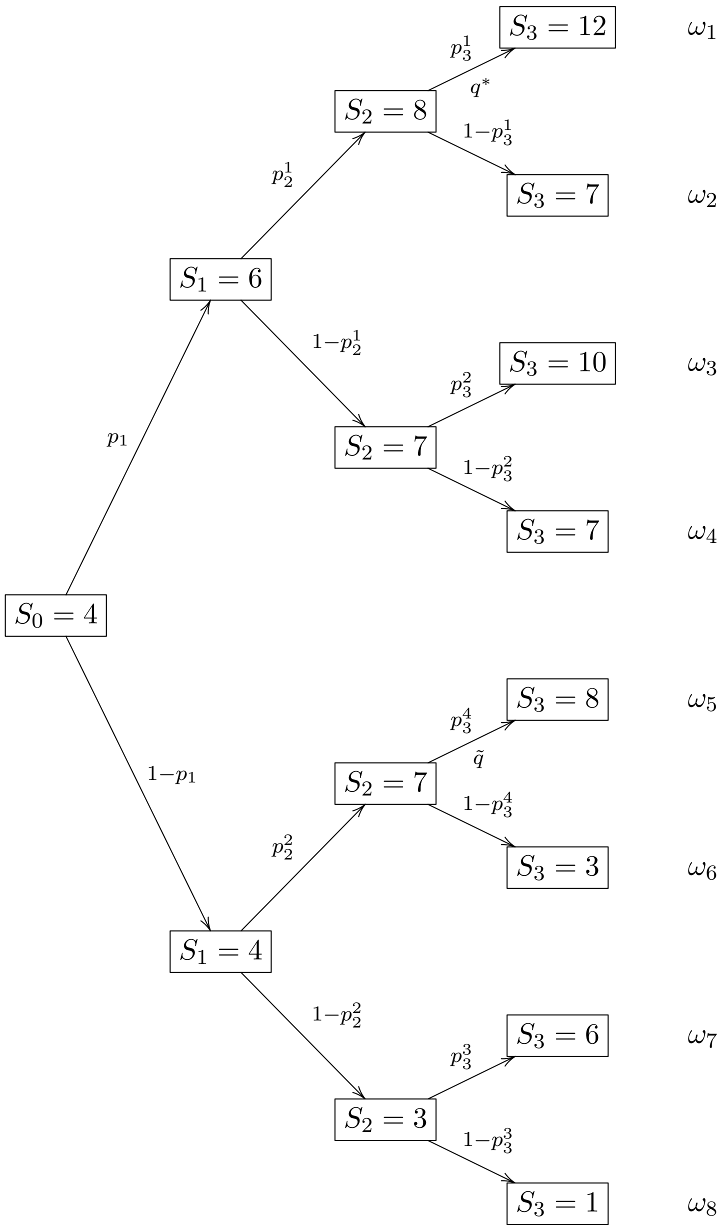
This sheet will be discussed in the seminar on Friday, 8th November 2024. The solutions will be discussed in the seminar 1 week later.

Problem 6.1. Consider a three period market model with the riskless interest rate equal to 25% (in every period) and one risky asset whose dynamics are given by the table (together with the physical measure $\mathbb{P}$):

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$\mathbb{P}$	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125
S_0	4	4	4	4	4	4	4	4
S_1	6	6	6	6	4	4	4	4
S_2	8	8	7	7	7	7	3	3
S_3	12	7	10	7	8	3	6	1

(a) Find the filtration generated by the asset prices.
(b) Compute the set of risk neutral measures.
(c) Decide whether the market model is arbitrage free.
(d) Construct a tree representing the dynamics of the risky asset. Remember the probabilities on the arrows.

Solution We start by drawing the tree in order to help us visualise the process.



(a) At $t = 0$ we have

$$\mathcal{F}_0 = \{\emptyset, \Omega\}.$$

At $t = 1$, the filtration is

$$\mathcal{F}_1 = \{\emptyset, \Omega, \underbrace{\{\omega_1, \omega_2, \omega_3, \omega_4\}}_{A_1^1}, \underbrace{\{\omega_5, \omega_6, \omega_7, \omega_8\}}_{A_1^2}, \}.$$

At $t = 2$, the σ-algebra is generated by the set

$$\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5, \omega_6\}, \{\omega_7, \omega_8\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\},$$

which has the partition set

$$A_2^1 = \{\omega_1, \omega_2\}, A_2^2 = \{\omega_3, \omega_4\}, A_2^3 = \{\omega_5, \omega_6\}, A_2^4 = \{\omega_7, \omega_8\}.$$

At $t = 3$, the σ-algebra is generated by the set

$$\{\{\omega_1\}, \{\omega_2, \omega_4\}, \{\omega_3\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8\}\},$$

for which the partition is

$$\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\},$$

so that $\mathcal{F}_3 = \mathcal{P}(\Omega)$.

(b) The risk neutral measures are

$$\mathbb{Q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_8 \end{pmatrix}$$

satisfying:

$$q_1 + \cdots + q_8 = 1. \tag{6.1}$$

Furthermore, they need to satisfy

$$\mathbb{E}_{\mathbb{Q}}(\frac{S_1}{1+r}|\mathcal{F}_0) = S_0,$$

i.e.

$$\frac{1}{1.25}((q_1 + q_2 + q_3 + q_4)6 + (q_5 + q_6 + q_7 + q_8)4) = 4. \tag{6.2}$$

Next, it needs to hold that

$$\mathbb{E}_{\mathbb{Q}}(\frac{S_2}{1+r}|\mathcal{F}_1) = S_1,$$

i.e. on A_1^1

$$\frac{1}{1.25}\left(\frac{q_1 + q_2}{q_1 + q_2 + q_3 + q_4}8 + \frac{q_3 + q_4}{q_1 + q_2 + q_3 + q_4}7\right) = 6 \tag{6.3}$$

and on A_1^2

$$\frac{1}{1.25}\left(\frac{q_1 + q_2}{q_1 + q_2 + q_3 + q_4}7 + \frac{q_3 + q_4}{q_1 + q_2 + q_3 + q_4}3\right) = 4. \tag{6.4}$$

From

$$\mathbb{E}_{\mathbb{Q}}(\frac{S_3}{1+r}|\mathcal{F}_2) = S_2$$

we get the 4 equations

$$\frac{1}{1.25}\left(\frac{q_1}{q_1 + q_2}12 + \frac{q_2}{q_1 + q_2}7\right) = 8 \tag{6.5}$$

$$\frac{1}{1.25}\left(\frac{q_3}{q_3+q_4}10+\frac{q_3}{q_3+q_4}7\right)=7 \tag{6.6}$$

$$\frac{1}{1.25}\left(\frac{q_5}{q_5+q_6}8+\frac{q_6}{q_5+q_6}3\right)=7 \tag{6.7}$$

$$\frac{1}{1.25}\left(\frac{q_7}{q_7+q_8}6+\frac{q_8}{q_7+q_8}1\right)=3 \tag{6.8}$$

If we tried to solve this system of 8 equations with 8 unknowns, we would obtain no solutions, telling us there is no risk neutral measure.

A quicker way to convince ourselves that that is the case, is trying to find the risk-neutral probabilities (recall that these are the conditional probabilities) on the arrows in the tree. We get at the very top branch $S_2 = 8$ that the risk neutral (conditional) probability has to satisfy

$$q^* = \frac{1+r-d^*}{u^*-d^*} = \frac{1.25-\frac{7}{8}}{\frac{12}{8}-\frac{7}{8}} = \frac{3}{5},$$

which would be ok. If we however consider the node $S_2 = 7$ in the lower half of the tree (i.e. on $\{\omega_5, \omega_6\}$), we get

$$\tilde{q} = \frac{1.25-\frac{3}{7}}{\frac{8}{7}-\frac{3}{7}} = \frac{23}{20} > 1$$

which is not a (conditional) probability, so a risk neutral measure cannot exists at least on this branch of the tree and therefore not for the whole tree.

(c) The market is not arbitrage free due to the Fundamental theorem of asset pricing, since no risk-neutral measure exists.

(d) We calculate the probabilities on the arrows. Starting with the two top most states of the tree, we have the two equations

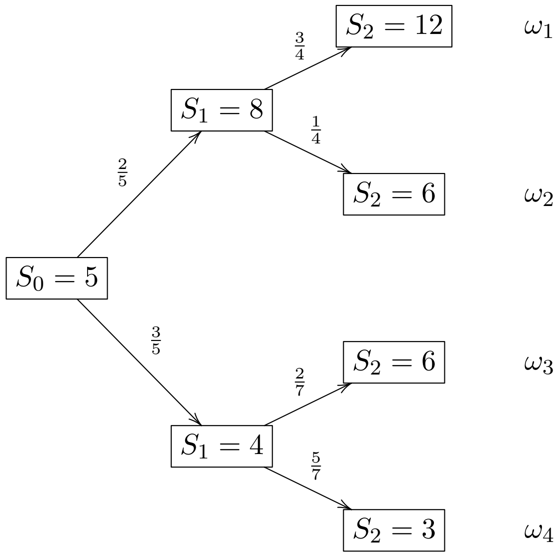
$$\mathbb{P}(\omega_1) = p_1 p_2^1 p_3^1 = 0.5^3,$$
$$\mathbb{P}(\omega_2) = p_1 p_2^1 (1-p_3^1) = 0.5^3.$$

Dividing them we obtain

$$\frac{p_3^1}{1-p_3^1} = 1 \Leftrightarrow p_3^1 = 0.5.$$

A similar calculation across all branches gives us that all probabilities are equal to 0.5.

Problem 6.2. Fill a table with asset prices at $t = 0, 1, 2$ (as in the above exercise) for the risky asset whose prices are described by the following diagram:



Solution We calculate from the arrows

$$\mathbb{P}(\omega_1) = \frac{2}{5} \cdot \frac{3}{4} = 0.3,$$

$$\mathbb{P}(\omega_2) = \frac{2}{5} \cdot \frac{1}{4} = 0.1,$$

$$\mathbb{P}(\omega_3) = \frac{3}{5} \cdot \frac{2}{7} = \frac{6}{35},$$

and

$$\mathbb{P}(\omega_4) = \frac{3}{5} \cdot \frac{5}{7} = \frac{15}{35}.$$

The resulting table is

	ω_1	ω_2	ω_3	ω_4
$\mathbb{P}$	0.3	0.1	$\frac{6}{35}$	$\frac{15}{35}$
S_0	5	5	5	5
S_1	8	8	4	4
S_2	12	6	6	3

Problem 6.3. Show that if ϕ is an adapted and self-financing trading strategy and $\mathbb{Q}$ is a risk neutral measure, then

$$\hat{V}_s(\phi) = \mathbb{E}_{\mathbb{Q}}(\hat{V}_t(\phi)|\mathcal{F}_s), \qquad 0 \le s \le t \le T.$$

In particular,

$$V_t(\phi) = B_t\, \mathbb{E}_{\mathbb{Q}}\left(\frac{V_T(\phi)}{B_T} \middle| \mathcal{F}_t \right), \qquad t = 0, 1, \ldots, T.$$

Solution We start with the right hand side and calculate

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(\hat{V}_t(\phi)|\mathcal{F}_s) &= \mathbb{E}_{\mathbb{Q}}(\hat{V}_s(\phi) + \underbrace{\hat{G}_t^s(\phi)}_{\text{discounted gains from } s \text{ to } t} |\mathcal{F}_s) && \text{(Proposition 2.1.} \\
&= \hat{V}_s(\phi) + \mathbb{E}_{\mathbb{Q}}(\hat{G}_t^s(\phi)|\mathcal{F}_s) && (\hat{V}_s(\phi) \text{ is measurable w.r.t.} \\
&= \hat{V}_s(\phi) + \mathbb{E}_{\mathbb{Q}}(\sum_{i=1}^{n}\sum_{j=s}^{t-1} \phi_j^i \Delta \hat{S}_{j+1}^i |\mathcal{F}_s) \\
&= \hat{V}_s(\phi) + \sum_{i=1}^{n}\sum_{j=s}^{t-1} \mathbb{E}_{\mathbb{Q}}(\phi_j^i \Delta \hat{S}_{j+1}^i |\mathcal{F}_s) \\
&= \hat{V}_s(\phi) + \sum_{i=1}^{n}\sum_{j=s}^{t-1} \mathbb{E}_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}(\phi_j^i \Delta \hat{S}_{j+1}^i |\mathcal{F}_j)|\mathcal{F}_s) \\
&= \hat{V}_s(\phi) + \sum_{i=1}^{n}\sum_{j=s}^{t-1} \mathbb{E}_{\mathbb{Q}}(\phi_j^i \underbrace{\mathbb{E}_{\mathbb{Q}}(\Delta \hat{S}_{j+1}^i |\mathcal{F}_j)}_{=0 \text{ by definition of r.n.m.}} |\mathcal{F}_s) \\
&= \hat{V}_s(\phi).
\end{aligned}
$$

This gives us that

$$
\begin{aligned}
\hat{V}_s(\phi) &= \mathbb{E}_{\mathbb{Q}}(\hat{V}_t(\phi)|\mathcal{F}_s) \\
\frac{V_s(\phi)}{B_s} &= \mathbb{E}_{\mathbb{Q}}(\frac{V_t(\phi)}{B_t}|\mathcal{F}_s) \\
V_s(\phi) &= B_s \mathbb{E}_{\mathbb{Q}}(\frac{V_t(\phi)}{B_t}|\mathcal{F}_s)
\end{aligned}
$$

If we change $s \to t$ and $t \to T$, this gives us the desired second formula.

Numerical solutions P6.1: (a) $\mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\}$, $\mathcal{F}_2 = \mathcal{P}(\Omega)$. (b) The set is empty. (c) All conditional probabilities are equal 0.5.

Problem Sheet 7

This sheet will be discussed in the seminar on Friday, 15th November 2024. The solutions will be discussed in the seminar 1 week later.

Problem 7.1. Show that the following Utility Functions satisfy the properties $1-3$ from Definition 3.1.1 in the Notes. Calculate the coefficients of risk aversion (absolute and relative).

(a) $u(x) = 5\ln(x)$.
(b) $u(x) = x^{1/3}$ (it is an example of the power utility function).

Solution

(a) We consider $u(x) = 5\ln(x)$ with $x > 0$. The first derivative is $u'(x) = \frac{5}{x} > 0$, still for $x > 0$, so condition 2 is satisfied. The second derivative is $u''(x) = -\frac{5}{x^2} < 0$ for $x > 0$ and so the 3rd condition is satisfied.
Furthermore,
$$\lim_{x\downarrow 0} u'(x) = \lim_{x\downarrow 0} \frac{5}{x} = \infty$$
and
$$\lim_{x\uparrow\infty} u'(x) = \lim_{x\uparrow\infty} \frac{5}{x} = 0,$$
so condition 1 holds. We next calculate the two coefficients of risk aversion:
$$\alpha(x) = -\frac{u''(x)}{u'(x)} = -\frac{-\frac{5}{x^2}}{\frac{5}{x}} = \frac{1}{x},$$
which shows decreasing absolute risk aversion (DARA).
$$r(x) = -\frac{xu''(x)}{u'(x)} = 1,$$
which shows a constant relative risk aversion (CRRA).

(b) We repeat the above for $u(x) = x^{1/3}$. This gives us:
$$u'(x) = \frac{1}{3}x^{-2/3} > 0, \quad x > 0$$
and therefore u is increasing.
$$u''(x) = -\frac{2}{9}x^{-5/3} < 0, \quad x > 0$$
and thus u is concave.
$$\lim_{x\downarrow 0} u'(x) = \lim_{x\downarrow 0} \frac{1}{3x^{2/3}} = \infty$$
and
$$\lim_{x\uparrow\infty} u'(x) = \lim_{x\uparrow\infty} \frac{1}{3x^{2/3}} = 0$$
and consequently condition 1 holds. Finally,
$$\alpha(x) = -\frac{-\frac{2}{9}x^{-5/3}}{\frac{1}{3}x^{-2/3}} = \frac{2}{3}x^{-1}$$

and
$$r(x) = \frac{2}{3},$$
so we again have DARA and CRRA.

Problem 7.2. Derive a formula for the arbitrage free price at time $t = 0$ for the European put option with payoff $X = (K - S_T)^+$ in the binomial market model with parameters $u = d^{-1}$,r, T.

Solution The price of the option at $t = 0$ will be given by
$$V_0 = \mathbb{E}_{\mathbb{Q}}(\frac{1}{(1+r)^T}X) = \frac{1}{(1+r)^T}\sum_{k=0}^{T}\binom{T}{k}q^k(1-q)^{T-k}\max\{0, K - S_0u^kd^{T-k}\}.$$
The payoff is non-zero when $K - S_0u^kd^{T-k} > 0$, which is equivalent to $k < \frac{\log(\frac{K}{S_0d^T})}{\log(\frac{u}{d})}$. Let $\hat{k}$ be the largest integer for which this inequality holds. We can then calculate further that
$$\begin{aligned} V_0 &= \frac{1}{(1+r)^T}\sum_{k=0}^{\hat{k}}\binom{T}{k}q^k(1-q)^{T-k}(K - S_0u^kd^{T-k}) \\ &= \frac{K}{(1+r)^T}\sum_{k=0}^{\hat{k}}\binom{T}{k}q^k(1-q)^{T-k} - \frac{S_0}{(1+r)^T}\sum_{k=0}^{\hat{k}}\binom{T}{k}q^k(1-q)^{T-k}u^kd^{T-k} \\ &= \frac{K}{(1+r)^T}\sum_{k=0}^{\hat{k}}\binom{T}{k}q^k(1-q)^{T-k} - S_0\sum_{k=0}^{\hat{k}}\binom{T}{k}\left(\frac{qu}{1+r}\right)^k\left(\frac{(1-q)d}{1+r}\right)^{T-k} \end{aligned}$$

Problem 7.3. Answer the following questions as precisely as possible.

(a) In the two-state single period model with $\Omega = \{H, T\}$, $S_1(H) = uS_0$, $S_1(T) = S_0d$, where S_0 is the initial stock value and $d < 1 + r < u$, show that
$$\tilde{P}(H) = \frac{1+r-d}{u-d}, \quad \tilde{P}(T) = \frac{u-1-r}{u-d}$$
is a risk-neutral probability measure.

(b) When is a contingent claim called attainable? In the general single-period market model, are all contingent claims attainable when the market is arbitrage-free?

(c) In a single period model with three states of the world ($\Omega = \{\omega_1, \omega_2, \omega_3\}$), one stock and a money market account available, we are given the set $\mathbb{M}$ of risk neutral measures:
$$\mathbb{M} = \{\begin{pmatrix} \lambda \\ 1 - \frac{3}{2}\lambda \\ \frac{1}{2}\lambda \end{pmatrix}, \ \lambda \in (0, \frac{2}{3})\}.$$
Give an example of an attainable contingent claim in this market.

(d) Write down the filtration generated by a binomial tree with 2 time periods ($t = 0, 1, 2$) and $u = 2, d = 1/2, S_0 = 10$.

Solution

(a) We first check that $\tilde{\mathbb{P}}(H) + \tilde{\mathbb{P}}(T) = 1$, which can quickly be verified to hold. We also have that $\tilde{\mathbb{P}}(H) > 0$ and $\tilde{\mathbb{P}}(T) > 0$ since $d < 1 + r < u$. It remains to show that this probability measure is risk neutral. We calculate

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{P}}}(\frac{S_1}{1+r}) &= \frac{1}{1+r}(\tilde{\mathbb{P}}(H)S_1(H) + \tilde{\mathbb{P}}(T)S_1(T)) \\ &= \frac{1}{1+r}(\frac{1+r-d}{u-d}S_0 u + \frac{u-1-r}{u-d}S_0 d) \\ &= \frac{S_0}{1+r}(\frac{u-ru-du+du-d-rd}{u-d}) \\ &= \frac{S_0}{1+r}\frac{(u-d)(1+r)}{u-d} \\ &= S_0.\end{aligned}$$

Note: This makes the discounted stock price a *martingale* (a stochastic process that in expectation maintains its value).

(b) A contingent claim is attainable if and only if there exists a replicating strategy. A market being arbitrage-free does not necessarily mean complete. Therefore, in a market that is arbitrage-free but not complete, not every contingent claim is attainable.

(c) This market is incomplete because we have more than 1 risk-neutral measure. We want to find a contingent claim that is attainable. This means that a replicating strategy exists for this claim, which is equivalent to the price of the claim being the same for all risk-neutral measures, i.e. the price must not depend on λ. If we calculate the price of a general claim, it holds that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(\frac{X}{1+r}) &= \frac{1}{1+r}(\lambda X(\omega_1) + (1 - \frac{3}{2}\lambda)X(\omega_2) + \frac{1}{2}\lambda X(\omega_3)) \\ &= \lambda\frac{1}{1+r}(X(\omega_1) - \frac{3}{2}X(\omega_2) + \frac{1}{2}X(\omega_3)) + \frac{X(\omega_2)}{1+r}.\end{aligned}$$

Therefore, if $X(\omega_1) - \frac{3}{2}X(\omega_2) + \frac{1}{2}X(\omega_3) = 0$, the price does not depend on λ and the claim is therefore attainable.

(d) As we've already seen before, the filtrations are

$$\begin{aligned}\mathcal{F}_0 &= \{\emptyset, \Omega\}, \\ \mathcal{F}_1 &= \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}, \\ \mathcal{F}_2 &= \mathcal{P}(\Omega).\end{aligned}$$

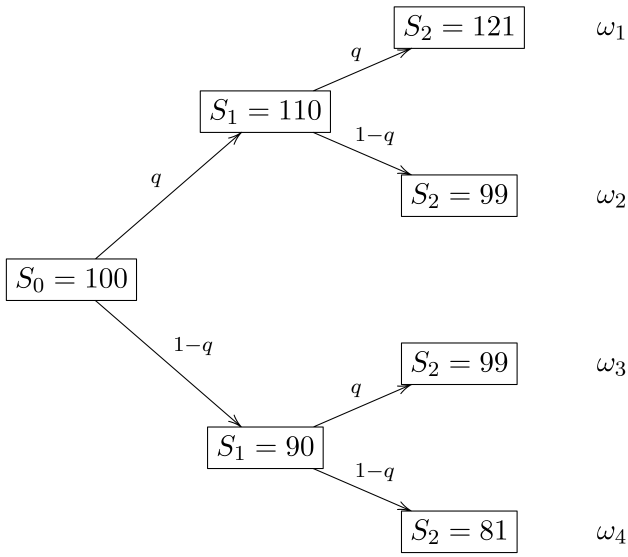
Problem 7.4. Assume that a stock with an initial price of $S_0 = 100$ follows the binomial model with $u = 1.1$, $d = 0.9$ over the next two periods. There is a **European Call option** with payoff $X = (S_2 - K)^+$ at time $t = 2$ on this stock. The interest rate for one period is $r = 0.02$ and interest is **compounded continuously**.

(a) Find the value at time 0 of this option as a function of the strike price K.

(b) Find a hedging strategy for the above call option with the strike $K = 100$. Justify why this strategy is adapted to the filtration generated by S_t. (*Hint: Part (2) can be solved independently of (1)*)

Solution

(a) The price evolution of S can be seen in the following tree:



We therefore get different payoffs depending on whether

- $K \geq 121$: $X = 0$ for all $\omega \in \Omega$,
- $K < 81$: $X = \begin{cases} 121 - K, & \omega_1 \\ 99 - K, & \omega_2 \\ 99 - K, & \omega_3 \\ 81 - K, & \omega_4 \end{cases}$,
- $81 \leq K < 99$: $X = \begin{cases} 121 - K, & \omega_1 \\ 99 - K, & \omega_2 \\ 99 - K, & \omega_3 \\ 0, & \omega_4 \end{cases}$ and
- $99 \leq K < 121$: $X = \begin{cases} 121 - K, & \omega_1 \\ 0, & \omega_2 \\ 0, & \omega_3 \\ 0, & \omega_4 \end{cases}$

The evolution of the money market is

$$B_0 = 1 \to B_1 = e^{0.02} \to B_2 = e^{2 \cdot 0.002}.$$

We can calculate $q = \frac{e^{0.02} - d}{u - d} = 0.6010067$. We then get for

- $K \geq 121$: $X_0 = 0$,
- $K < 81$: $X_0 = e^{-2 \cdot 0.02}(q^2(121 - K) + 2q(1 - q)(99 - K) + (1 - q)^2(81 - K))$,

- $81 \le K < 99$: $X_0 = e^{-2 \cdot 0.02}(q^2(121-K) + 2q(1-q)(99-K))$ and
- $99 \le K < 121$: $X_0 = e^{-2 \cdot 0.02} q^2(121-K)$

(b) For $K = 100$ the payoff is $X = \begin{cases} 21, & \omega_1 \\ 0, & \omega_2 \\ 0, & \omega_3 \\ 0, & \omega_4 \end{cases}$. The replicating strategy will trivially hold 0 assets at the node $S_1 = 90$ and the corresponding value of the replicating strategy is also $x_B = 0$. At the node $S_1 = 110$ we calculate

$$\phi_A^0 B_2 + \phi_A^1 S_2(\omega_1) = X(\omega_1)$$
$$\phi_A^0 B_2 + \phi_A^1 S_2(\omega_2) = X(\omega_2)$$

which is equivalent to

$$\phi_A^0 e^{0.04} + \phi_A^1 121 = 21$$
$$\phi_A^0 e^{0.04} + \phi_A^1 99 = 0$$

This yields $\phi_A^1 = \frac{21}{22}$ and $\phi_A^0 = -90.7946$. The value of the replicating strategy at this node is $x_A = \phi_A^0 e^{0.02} + \phi_A^1 110 = 12.37$

At the node $S_0 = 100$, we get using the above values the system

$$\phi_C^0 B_1 + \phi_C^1 S_1(\omega_1, \omega_2) = x_A$$
$$\phi_C^0 B_1 + \phi_C^1 S_1(\omega_3, \omega_4) = x_B$$

which is equivalent to

$$\phi_C^0 e^{0.02} + \phi_C^1 110 = 12.37$$
$$\phi_C^0 e^{0.02} + \phi_C^1 90 = 0$$

which gives $\phi_C^1 = 0.6186$ and $\phi_C^0 = -54.56$. The value $x_C = \phi_C^0 1 + \phi_C^1 100 = 7.2870$.

The filtration is

$$\mathcal{F}_0 = \{\emptyset, \Omega\},$$
$$\mathcal{F}_1 = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\},$$
$$\mathcal{F}_2 = \mathcal{P}(\Omega).$$

Our trading strategy satisfies the following: $\phi_0^0 = \phi_C^0 = -54.56$ is constant on all of Ω. Likewise $\phi_0^1 = \phi_C^1 = 0.6186$ is constant on all of Ω.

$\phi_1^0 = \begin{cases} \phi_A^0 = -90.97, & \omega_1, \omega_2 \\ \phi_B^0 = 0, & \omega_3, \omega_4 \end{cases}$ is constant on the partition sets of $\mathcal{F}_1$, as is $\phi_1^1 = \begin{cases} \phi_A^1 = \frac{21}{22}, & \omega_1, \omega_2 \\ \phi_B^1 = 0, & \omega_3, \omega_4 \end{cases}$. The strategy is therefore constant on each partition set of $\mathcal{F}_t, t = 0, 1$, so it is adapted.

Numerical solutions P7.1: (a) $\alpha(x) = \frac{1}{x}$, $r(x) = 1$. (b) $\alpha(x) = \frac{2}{3}x^{-1}$, $r(x) = \frac{2}{3}$.

Problem Sheet 8

This sheet will be discussed in the seminar on Friday, 22nd November 2024. The solutions will be discussed in the seminar 1 week later.

Problem 8.1. Consider a (risk averse) utility function $u : \mathbb{R}_+ \to \mathbb{R}$. Show that the inverse function $I : \mathbb{R}_+ \to \mathbb{R}_+$ of the derivative $u' : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly decreasing and satisfies

$$\lim_{x\to 0} I(x) = \infty, \qquad \lim_{x\to\infty} I(x) = 0.$$

Hint: You may assume that u is twice differentiable.

Solution Recall that $I(x) := [u'(x)]^{-1}$, so $u'(I(x)) = x$ and $I(u'(x)) = x$. This in particular means that

$$(u'(I(x)))' = x',$$

which is equivalent to

$$u''(I(x))I'(x) = 1.$$

Expressing $I'(x)$ from the above we have

$$I'(x) = \frac{1}{u''(I(x))} < 0,$$

where the inequality follows since $u''(x) < 0$.

Furthermore,

$$\lim_{x\downarrow 0} I(x) = \lim_{x\uparrow\infty} I(u'(x)) = \lim_{x\uparrow\infty} x = \infty,$$

and similarly

$$\lim_{x\uparrow\infty} I(x) = \lim_{x\downarrow 0} I(u'(x)) = \lim_{x\downarrow 0} x = 0.$$

Problem 8.2. Compute the function I for the following utility functions:

$$u(x) = \log(x), \qquad u(x) = \frac{1}{\gamma}x^\gamma, \quad \gamma \in (0,1).$$

For each of these functions calculate the coefficients of absolute and relative risk aversion and draw graphs of u, u' and I.

Solution We begin with $u(x) = \log(x)$. We have then $u'(x) = \frac{1}{x}$ and $u''(x) = -\frac{1}{x^2}$. Setting $u'(I(x)) = x$, this is equivalent to $\frac{1}{I(x)} = x$, i.e. $I(x) = \frac{1}{x}$. The coefficient of absolute risk aversion is then

$$\alpha(x) = -\frac{u''(x)}{u'(x)} = \frac{1}{x} \quad \text{(DARA)}$$

and the coefficient of relative risk aversion is

$$r(x) = 1 \quad \text{(CRRA)}.$$

The plots of u and u' (which is the same for I) are in Figure 1.

We next consider $u(x) = \frac{1}{\gamma}x^\gamma$, with $\gamma \in (0,1)$. We get $u'(x) = x^{\gamma-1}$ and $u''(x) = (\gamma-1)x^{\gamma-2} < 0$, where the latter holds since $\gamma < 1$. To find

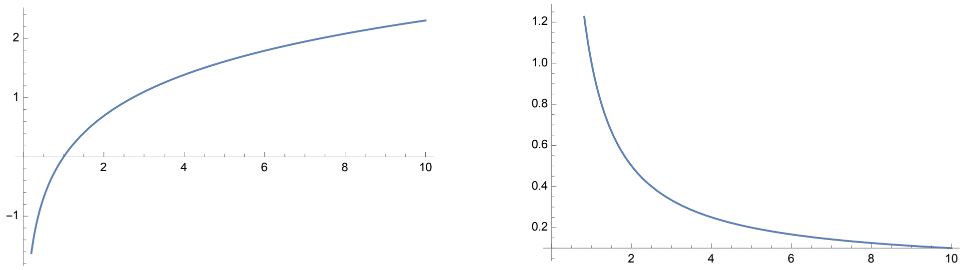


FIGURE 1. The plots for $u(x) = \ln(x)$.

the inverse, we want $u'(I(x)) = x$, i.e. $(I(x))^{\gamma-1} = x$. This gives that $I(x) = x^{\frac{1}{\gamma-1}}$. Consequently, we have

$$\alpha(x) = -\frac{(\gamma-1)x^{\gamma-2}}{x^{\gamma-1}} = -(\gamma-1)x^{-1} = \frac{1-\gamma}{x} > 0 \quad \text{(DARA)}$$

and

$$r(x) = 1-\gamma \quad \text{(CRRA)}.$$

The plots (for $\gamma = \frac{1}{2}$) are in Figure 2.

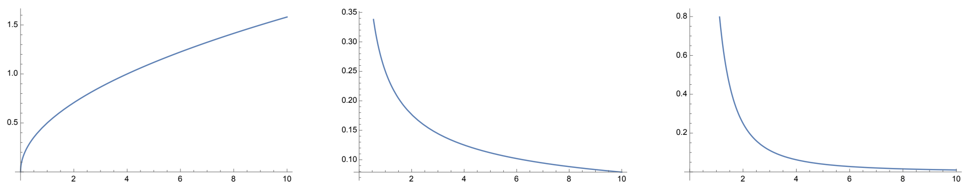


FIGURE 2. The plots for $u(x) = \frac{1}{\gamma}x^{\gamma}$ with $\gamma = \frac{1}{2}$.

Problem 8.3. Consider a general single period market model with $n = 1$, $k = 2$, $r = \frac{1}{9}$, $S_0 = 5$, $S_1(\omega_1) = \frac{20}{3}$, $S_1(\omega_2) = \frac{40}{9}$ and $\mathbb{P}(\omega_1) = \frac{3}{5}$. Solve *Step 1* of the optimal portfolio problem for the following utility function:

$$u(x) = \frac{1}{\gamma}x^{\gamma}, \quad \gamma \in (0,1).$$

Remember that the solution depends on the initial investment x, i.e. x should be a parameter of the solution.

Proceed now to *Step 2*: Find an optimal trading strategy for $u(x) = 3x^{1/3}$ and the initial investment equal to 5.

Solution

Step 1: We want to find $W(\omega_1)$ and $W(\omega_2)$ that maximise $\mathbb{E}(u(W))$, limiting ourselves to only attainable W. We know from the lecture that the method of Lagrange multipliers gives an equation for λ

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{1}{1+r} I\left(\lambda \frac{L}{1+r}\right)\right) = x$$

which we can then use to set

$$W(\omega_i) = I(\lambda \frac{L(\omega_1)}{1+r}).$$

In order to determine that $L(\omega_i)$ is, we first need to know $\mathbb{P}$ and $\mathbb{Q}$. $\mathbb{P}$ is given in the exercise text, so it remains to find the risk neutral measure. Since our model is a single period two state model, we can use the (by now) well known formula

$$q = \frac{1+r-d}{u-d} = \frac{\frac{10}{9} - \frac{40/0}{5}}{\frac{20/3}{5} - \frac{40/0}{5}} = \frac{1}{2}.$$

We therefore have that $\mathbb{Q}(\omega_1) = q = \frac{1}{2}$ and $\mathbb{Q}(\omega_2) = 1 - q = \frac{1}{2}$. The state price density L is therefore given by

$$L(\omega_1) = \frac{\mathbb{Q}(\omega_1)}{\mathbb{P}(\omega_1)} = \frac{\frac{1}{2}}{\frac{3}{5}} = \frac{5}{6},$$
$$L(\omega_2) = \frac{\mathbb{Q}(\omega_2)}{\mathbb{P}(\omega_2)} = \frac{\frac{1}{2}}{\frac{2}{5}} = \frac{5}{4}.$$

To determine λ, we need $I(x)$. Using the previous exercise, we know $I(x) = x^{\frac{1}{\gamma-1}}$. This gives that λ has to solve the equation

$$\frac{1}{1+r}\left[q(\lambda\frac{L(\omega_1)}{1+r})^{\frac{1}{\gamma-1}} + (1-q)(\lambda\frac{L(\omega_2)}{1+r})^{\frac{1}{\gamma-1}}\right] = x.$$

Expressing λ, we get

$$\lambda^{\frac{1}{\gamma-1}} = \frac{(1+r)x}{q(\frac{L(\omega_1)}{1+r})^{\frac{1}{\gamma-1}} + (1-q)(\frac{L(\omega_2)}{1+r})^{\frac{1}{\gamma-1}}}$$

(Note that although we have not plugged the numbers in yet, all of the above values are known).

We are now ready to calculate what W is:

$$\begin{aligned} W(\omega_1) &= I(\lambda\frac{L(\omega_1)}{1+r}) \\ &= \lambda^{\frac{1}{\gamma-1}}\left(\frac{L(\omega_1)}{1+r}\right)^{\frac{1}{\gamma-1}} \\ &= \frac{(1+r)x\left(\frac{L(\omega_1)}{1+r}\right)^{\frac{1}{\gamma-1}}}{q(\frac{L(\omega_1)}{1+r})^{\frac{1}{\gamma-1}} + (1-q)(\frac{L(\omega_2)}{1+r})^{\frac{1}{\gamma-1}}} \end{aligned}$$

Using now $\gamma = \frac{1}{3}$, $u(x) = 3x^{\frac{1}{3}}$ and $x = 5$, this gives

$$W(\omega_1) = 7.194773.$$

A similar calculation gives $W(\omega_2) = 3.91634$.

Step 2: We are now ready to find a replicating strategy for W. The system we are solving is

$$(x - S_0\phi)(1+r) + \phi S_1 = W,$$

which translates to the two equations

$$(5 - 5\phi)\frac{10}{9} + \phi\frac{20}{3} = 7.194773,$$
$$(5 - 5\phi)\frac{10}{9} + \phi\frac{40}{9} = 3.91634.$$

This system has the solution $\phi = 1.4753$.

(Note that we had two equations, but only one unknown. The reason we knew the system would be solvable is because we are working in a complete market and we constructed W to be attainable with a starting investment of $x = 5$).

Numerical solutions P8.2: For $\log(x)$, $a(x) = \frac{1}{x}$ and $r(x) = 1$. For $\frac{1}{\gamma}x^{\gamma}$, $a(x) = \frac{1-\gamma}{x}$ and $r(x) = 1 - \gamma$. **P8.3:** For $x = 5$,$W = (7.194773, 3.91634)$, $(x, \phi) = (5, 1.4753)$.

Problem Sheet 9

This sheet will be discussed in the seminar on Friday, 29th November 2024. The solutions will be discussed in the seminar 1 week later.
Remark: You will be able to solve the following Problem 9.2 (f) after the Lecture on December 3rd.

Problem 9.1. Consider the following single period market model with state space $\Omega = \{\omega_1, \omega_2, \omega_3\}$. There are two stocks available. The initial stock prices are $S_0^1 = 10$ and $S_0^2 = 6$. The stock prices evolve according to the following table:

	ω_1	ω_2	ω_3
S_1^1	5	10	14
S_1^2	10	12	4

The probabilities of the states under the subjective measure $\mathbb{P}$ are assumed to be $\mathbb{P}(\omega_1) = \frac{3}{5}$ and $\mathbb{P}(\omega_2) = \mathbb{P}(\omega_3) = \frac{1}{5}$.

(a) Compute the expectation, the variance and the covariance matrix of the returns R^1 and R^2 of the stocks under the subjective probability measure. Hint: you need to compute the returns of the stocks first using the prices given.
(b) Assume that we can invest 1 unit of money to a portfolio consisting of F_1 units of stock 1 and F_2 units of stock 2. Calculate F_1, F_2 so that the portfolio has minimum variance. What is the expected return of this portfolio?

Solution

(a) We first calculate the corresponding returns:

	ω_1	ω_2	ω_3
R^1	-0.5	0	0.4
R^2	2/3	1	−1/3
$\mathbb{P}$	3/5	1/5	1/5

The expected values are then

$$\mathbb{E}(R^1) = \frac{3}{5}(-0.5) + \frac{1}{5}0 + \frac{1}{5}0.4 = -0.22$$

and

$$\mathbb{E}(R^2) = \cdots = \frac{8}{15}.$$

The second moments are

$$\mathbb{E}((R^1)^2) = \frac{3}{5}(-0.5)^2 + \frac{1}{5}0^2 + \frac{1}{5}0.4^2 = 0.182$$

and

$$\mathbb{E}((R^2)^2) = \cdots = 0.488.$$

The variances are therefore

$$\mathrm{Var}(R^1) = \mathbb{E}((R^1)^2) - (\mathbb{E}(R^1))^2 = 0.1336$$

and
$$\mathrm{Var}(R^2) = 0.2043.$$
The covariance is
$$\begin{aligned}\mathrm{Cov}(R^1, R^2) &= \mathbb{E}(R^1R^2) - \mathbb{E}(R^1)\mathbb{E}(R^2)\\ &= \frac{3}{5}(-0.5)\frac{2}{3} + \frac{1}{5}0\cdot 1 + \frac{1}{5}0.4(-\frac{1}{3}) - (-0.22)\frac{8}{15}\\ &= -0.1094.\end{aligned}$$

(b) We are solving the problem
$$\min_{w_1,w_2} \mathrm{Var}(R) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12},$$
$$\text{s.t. } w_1 + w_2 = 1.$$
We solve this by finding the minimum of the Lagrange function
$$L(w_1, w_2, \lambda) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} - \lambda(w_1 + w_2 - 1),$$
which can (in this case) be simplified by substituting $w_2 = 1 - w_1$, which simplifies to finding the minimum of
$$L(w_1) = w_1^2\sigma_1^2 + (1 - w_1)^2\sigma_2^2 + 2w_1(1 - w_1)\sigma_{12}.$$
Taking the derivative, we get
$$\frac{\partial L}{\partial w_1} = 2w_1\sigma_1^2 - 2(1 - w_1)\sigma_2^2 + 2(1 - w_1)\sigma_{12} - 2w_1\sigma_{12} = 0,$$
which solving for w_1 with the values of σ_1, σ_2 and σ_{12} substituted in gives
$$w_1 = 0.5637, w_2 = 0.4363.$$
The return of this a portfolio with these weights is
$$\mathbb{E}(R) = w_1\mathbb{E}(R^1) + w_2\mathbb{E}(R^2) = 0.1087.$$
To determine how many shares we buy, note that the investor invests the proportion w_1 of the 1 unit of money that is being invested (i.e. w_1 units of money) into the first share, which at the time of purchase costs 10 units of money. The investor therefore buys $w_1/10$ shares of the first stock. Similarly, the investor buys $w_2/6$ shares of the second stock.

Problem 9.2. Consider a general single period market model with two risky assets and a money market account with interest rate $r = 0.1$. The state space consists of 3 states: $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Prices of risky assets at $t = 0$ are $S_0^1 = 4$, $S_0^2 = 10$. Time $t = 1$ prices are given by the following table:

	ω_1	ω_2	ω_3
S_1^1	3	5	7
S_1^2	15	20	10

The probabilities of the states under the physical (subjective) measure $\mathbb{P}$ are assumed to be $\mathbb{P}(\omega_1) = \frac{1}{6}$ and $\mathbb{P}(\omega_2) = \frac{1}{3}$ and $\mathbb{P}(\omega_3) = \frac{1}{2}$.

(a) Compute the returns R^1, R^2 of the stocks.

(b) Compute the expectation, the variance and the covariance matrix of the returns of the stocks.
(c) Solve the Mean-Variance portfolio problem: find an optimal portfolio with the expected return equal to 50%.
(d) Using Lemma 3.4.2, find an optimal portfolio with the expected return equal to 100%.
(e) Find the Market Portfolio. What is the expected return of the market portfolio?
(f) Find the price for the call option $(S_1^2 - 10)^+$ determined by the CAPM principle.

Solution

(a) The returns are

	ω_1	ω_2	ω_3
R^1	$-1/4$	$1/4$	$3/4$
R^2	$1/2$	1	0

(b) We calculate like we did in Problem 9.1 and get

$$\mathbb{E}(R^1) = \frac{5}{12}, \quad \mathbb{E}(R^2) = \frac{5}{12}, \quad \mathbb{E}((R^1)^2) = 0.3125, \quad \mathbb{E}((R^2)^2) = 0.375,$$

which gives $\mathrm{Var}(R^1) = 0.1389$ and $\mathrm{Var}(R^2) = 0.2014$. The covariance is $\mathrm{Cov}(R^1, R^2) = -0.1111$.

(c) We are searching for the proportional weights w_0, w_1 and w_2 satisfying

$$\begin{aligned} &\min \mathrm{Var}(R) \\ &\text{s.t. } \mathbb{E}(R) = \rho = 0.5 \\ &\qquad w_0 + w_1 + w_2 = 1. \end{aligned}$$

The Lagrangian is then

$$\mathcal{L}(w_1, w_2, \lambda) = \frac{1}{2}(w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}) + \lambda((1-w_1-w_2)r + w_1\bar{R}^1 + w_2\bar{R}^2 - 0.5).$$

This leads to the equations

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial w_1} &= w_1\sigma_1^2 + w_2\sigma_{12} + \lambda\bar{R}^1 - \lambda r = 0 \\ \frac{\partial \mathcal{L}}{\partial w_2} &= w_2\sigma_2^2 + w_1\sigma_{12} + \lambda\bar{R}^2 - \lambda r = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= (1 - w_1 - w_2)r + w_1\bar{R}^1 + w_2\bar{R}^2 - 0.5 = 0 \end{aligned}$$

Solving the system gives $w_1 = 0.702$ and $w_2 = 0.562$, which therefore also gives $w_0 = 1 - w_1 - w_2 = -0.27$.

(d) We need $\mathbb{E}(\tilde{R}) = \tilde{r} = 1$. We have from (c) a portfolio $\hat{R}$ with expected return $\rho = 0.5$. Using the formula from the previous exercise, we get

$$\gamma = \frac{\tilde{r} - \rho}{r - \rho} = \frac{1 - 0.5}{0.1 - 0.5} = -\frac{0.5}{0.4} = -1.25.$$

The weights in the new portfolio are therefore

$$\tilde{w}_1 = (1-\gamma)w_1 = 2.25w_1 = 1.58$$

and

$$\tilde{w}_2 = (1-\gamma)w_2 = 2.25w_2 = 1.264.$$

$\tilde{w}_0$ can now be calculated either by adding γ and $(1-\gamma)w_0$ together, or simply using that $\tilde{w}_0 = 1 - \tilde{w}_1 - \tilde{w}_2$, either of which gives $\tilde{w}_0 = -1.894$.

(e) To find the market portfolio, we can start with the portfolio from (c) and find γ so that the resulting portfolio has $\tilde{w}_0 = 0$. This is equivalent to requiring that $(1-\gamma)(w_1 + w_2) = 1$, i.e.

$$(1-\gamma)1.264 = 1,$$

so that the suitable $\gamma = 0.209$. The resulting market portfolio weights are then

$$w_1^M = (1-\gamma)w_1 = (1-0.209)0.702 = 0.555$$

and

$$w_2^M = (1-\gamma)w_2 = (1-0.209)0.562 = 0.445.$$

The expected return of the market portfolio is

$$\mathbb{E}(R^M) = w_1^M \bar{R}^1 + w_2^M \bar{R}^2 = \frac{5}{12}$$

(f) We begin by calculating all of the necessary values

	ω_1	ω_2	ω_3
R^M	0.08375	0.58375	0.41625
X	5	10	0

Here, $R^M(\omega_1)$ was for example calculated as

$$R^M(\omega_1) = w_1^M R^1(\omega_1) + w_2^M R^2(\omega_1) = 0.555(-\frac{1}{4}) + 0.445(\frac{1}{2}) = 0.08375.$$

To calculate $\mathrm{Var}(R^M)$ we can calculate the first and second moment of R^M as in Problem 9.1(a), or calculate directly

$$\mathrm{Var}(R^M) = (w_1^M)^2\sigma_1^2 + (w_2^M)^2\sigma_2^2 + 2w_1^M w_2^M \sigma_{12} = 0.0231.$$

The CAPM then tells us

$$\mathbb{E}(\underbrace{R}_{\frac{X-x}{x}}) - r = \frac{\mathrm{Cov}(R, R^M)}{\mathrm{Var}(R^M)}(\mathbb{E}(R^M) - r)$$

so

$$x = \frac{\mathbb{E}(X) - \frac{\mathrm{Cov}(X,R^M)}{\mathrm{Var}(R^M)}(\mathbb{E}(R^M) - r)}{1+r}.$$

To calculate the above, we still require $\mathrm{Cov}(X, R^M)$, which we calculate next. We have

$$\mathbb{E}(XR^M) = \frac{1}{6}\cdot 5\cdot 0.08375 + \frac{1}{3}\cdot 10\cdot 0.5838 + \frac{1}{2}\cdot 0\cdot 0.41625 = 2.015792$$

and
$$\mathbb{E}(X) = \frac{1}{6} \cdot 5 + \frac{1}{3} \cdot 10 + \frac{1}{2} \cdot 0 = \frac{25}{6}.$$
Consequently, the covariance is
$$\mathrm{Cov}(X, R^M) = 2.015792 - \frac{25}{6}\frac{5}{12} = 0.27968.$$
The resulting β is then
$$\beta = \frac{\mathrm{Cov}(X, R^M)}{\mathrm{Var}(R^M)} = 12.10736$$
and we can finally calculate the price
$$x = \frac{\mathbb{E}(X) - \beta(\bar{R}^M - r)}{1 + r} = \frac{\frac{25}{6} - 12.10736(\frac{5}{12} - 0.1)}{1.1} = 0.30243,$$
which is the CAPM price of the call option. With this price, the expected return of the option is
$$\mathbb{E}(R^X) = \mathbb{E}(\frac{X - x}{x}) = \frac{\mathbb{E}(X) - x}{x} = \frac{\frac{25}{6} - 0.30243}{0.30243} = 12.77729.$$

Numerical solutions P9.1: (a) $\bar{R}^1 = -0.22$, $\bar{R}^2 = 8/15$, $\mathrm{Var}(R^1) = 0.1336$, $\mathrm{Var}(R^2) = 0.2043$, $\mathrm{Cov}(R^1, R^2) = -1094$. (b) $F_1 = 0.0564$, $F_2 = 0.0727$. **P9.2:** (b) $\mathbb{E}(R^1) = \frac{5}{12}$, $\mathbb{E}(R^2) = \frac{5}{12}$, $\mathrm{Var}(R^1) = 0.1389$, $\mathrm{Var}(R^2) = 0.2014$, $\mathrm{Cov}(R^1, R^2) = -0.1111$. (c) $w_0 = -0.27$, $w_1 = 0.702$, $w_2 = 0.562$. (d) $\tilde{w}_0 = -1.894$, $\tilde{w}_1 = 1.58$, $\tilde{w}_2 = 1.264$. (e) $w_1^M = 0.555$, $w_2^M = 0.445$, $\mathbb{E}(R^M) = 5/12$. (f) $x = 0.30243$.

Revision Sheets

Revision Sheet 1 (Pricing)

Problem 1.1. (a) The price of a European Put option with strike $K = 50$ in a single period market with risk-free interest rate $r = 0.1$ per period is equal to $P = 1.2$. The current price of the underlying stock is $S_0 = 60$. Find the price of a European Call option on the same stock with the same strike and maturity as the Put above

(b) Do you agree or disagree with the following statement? Justify your answer. *"In an arbitrage-free but incomplete market model, there might be contingent claims that have a unique risk-neutral price".*

(c) Define the two Arrow-Pratt coefficients of risk aversion and calculate their values for the following utility function:

$$u(x) = \sqrt{x}.$$

(d) Consider a two-period market model with one stock and one risk-free bond. The risk-free interest rate is equal to $r = 0.2$ per period and the stock's price dynamics are given in the following table:

	ω_1	ω_2	ω_3	ω_4
S_0	100	100	100	100
S_1	110	110	90	90
S_2	121	99	99	81

Show that this model is **not** arbitrage free.

Solution

(a) From the Put-Call parity, we have that

$$\text{Price of Call} - \text{Price of Put} = S_0 - \frac{1}{1+r}K.$$

In our case, this results in

$$\begin{aligned}\text{Price of Call} &= \text{Price of Put} + S_0 - \frac{1}{1+r}K \\ &= 1.2 + 60 - \frac{1}{1.1} \cdot 50 \\ &\simeq 15.75.\end{aligned}$$

(b) I agree. These are the attainable contingent claims.

(c) The Coefficient of Absolute Risk Aversion is

$$a(x) = -\frac{u''(x)}{u'(x)}.$$

The Coefficient of Relative Risk Aversion is

$$r(x) = -\frac{xu''(x)}{u'(x)}.$$

For $u(x) = \sqrt{x}$, we have

$$u'(x) = (x^{1/2})' = \frac{1}{2}x^{-1/2}$$

and

$$u''(x) = -\frac{1}{4}x^{-3/2}.$$

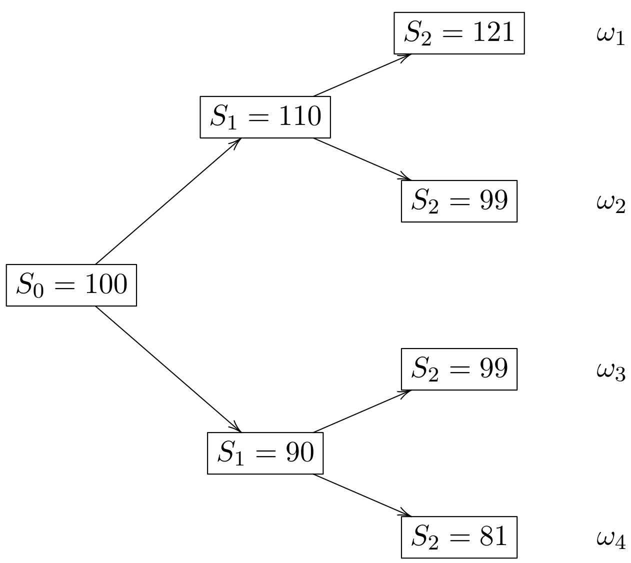
Consequently,

$$a(x) = -\frac{-\frac{1}{4}x^{-3/2}}{\frac{1}{2}x^{-1/2}} = \frac{1}{2x} \quad \text{(Decreasing Absolute Risk Aversion)}$$

and

$$r(x) = \frac{1}{2} \quad \text{(Constant Relative Risk Aversion).}$$

(d) We can draw the tree for this market model:



We can quickly see that at each step the stock price goes up by a factor $u = 1.1$ or down by a factor $d = 0.9$. Therefore, each individual one-period conditional tree has arbitrage because $u = 1.1 < 1.2 = 1 + r$.

Problem 1.2. Consider the single period coin-toss market model ($\Omega = \{H, T\} = (\omega_1, \omega_2)$) with one stock and one bank account. The known parameters are $S_0 = 10, S_1(H) = 12$ and $r = 0.1$.

(a) What is the range of possible values $S_1(T)$ such that there is no arbitrage in the model?
(b) Consider the case $S_1(T) = 9$ for this and the remaining parts of Problem 1.2. Give the risk neutral measure $\mathbb{P}$, that is, $\mathbb{P}(H)$ and $\mathbb{P}(T)$.
(c) What is the solution of the hedging problem of the European Put option with strike price $K = 11$? What should be the price of the option?

(d) Draw the sets $\mathcal{P}^+, \mathbb{A}, \mathbb{W}$ and $\mathbb{W}^\perp$ in the (T, H) plane, that is, the set of probability measures, the location of the arbitrage strategies, the location of the value processes with zero initial investment and its perpendicular subspace, respectively.

(e) State the fundamental theorem of asset pricing for a general discrete time market model. Explain the statement of the theorem using the sets $\mathcal{P}^+, \mathbb{A}, \mathbb{W}$ and $\mathbb{W}^\perp$.

(f) Explain the statement of the theorem for the above single period coin toss model (remember that $S_1(T) = 9$).

Solution

$$\begin{tikzcd}[column sep=small]
 & \boxed{S_1(H) = 12} \\
\boxed{S_0 = 10} \arrow[ur] \arrow[dr] & \\
 & \boxed{S_1(T) = ?}
\end{tikzcd}$$

(a) To ensure there is no arbitrage in the market, we must have $d < 1 + r$, that is

$$\frac{S_1(T)}{S_0} < 1 + r \Leftrightarrow \frac{S_1(T)}{10} < 1.1 \Leftrightarrow S_1(T) < 11.$$

Since the stock price should always remain positive, we have $0 < S_1(T) < 11$.

(b) We have $S_1(T) = 9$. The risk-neutral measure should satisfy

$$\tilde{p}\frac{S_1(H)}{1+r} + (1-\tilde{p})\frac{S_1(T)}{1+r} = S_0.$$

Plugging in the values, this is

$$\tilde{p} \cdot 12 + (1 - \tilde{p}) \cdot 9 = 11 \Leftrightarrow 3\tilde{p} = 2 \Leftrightarrow \tilde{p} = \frac{2}{3}.$$

Therefore $\mathbb{P}(H) = \frac{2}{3}$ and $\mathbb{P}(T) = \frac{1}{3}$. Alternatively, we can calculate $u = \frac{12}{10}$ and $d = \frac{9}{10}$ and

$$\tilde{p} = \frac{1 + r - d}{u - d} = \frac{1.1 - 0.9}{1.2 - 0.9} = \frac{2}{3}$$

and the rest is then as above.

(c) Since the payoff of the European Put option is $X^P = (K - S_1)^+$, we are dealing with

$$\begin{tikzcd}[column sep=small]
 & \boxed{X^P(H) = 0} \\
\boxed{x^P = ?} \arrow[ur] \arrow[dr] & \\
 & \boxed{X^P(T) = 2}
\end{tikzcd}$$

To find the hedging/replicating strategy, we must solve the equation

$$(x - \phi S_0)(1 + r) + \phi S_1 = X,$$

which in our case is equivalent to the system

$$(x - \phi \cdot 10) \cdot 1.1 + 12 \cdot \phi = 0,$$
$$(x - \phi \cdot 10) \cdot 1.1 + 9 \cdot \phi = 2.$$

Solving for ϕ gives

$$\phi = \frac{0 - 2}{12 - 9} = -\frac{2}{3}.$$

Plugging this into the first equation then gives

$$(x + \frac{2}{3} \cdot 10) \cdot 1.1 - 12 \cdot \frac{2}{3} = 0 \Leftrightarrow x = 0.606.$$

The hedging strategy therefore starts with an initial investment $x = 0.606$ and "buying" $-\frac{2}{3}$ of a share of the stock (so in fact, short-selling), as well as investing $0.606 + \frac{2}{3} \cdot 10 \simeq 7.2$ into the money market.

The price of the option should coincide with the price of the replicating strategy, so $x^P = 0.606$.

(d) We have

$$\Delta\hat{S}_1 = \begin{cases} \frac{12}{1.1} - 10 = 0.9091, & H \\ \frac{9}{1.1} - 10 = -1.8182, & T \end{cases}.$$

The sets we want to draw are

$$\mathbb{W} = \{X \in \mathbb{R}^2 | X = \hat{G}(x, \phi) = \phi \Delta\hat{S}_1\},$$
$$\mathbb{W}^\perp = \{Y \in \mathbb{R}^2 | \langle X, Y \rangle = 0, X \in \mathbb{W}\},$$
$$\mathbb{A} = \{X \in \mathbb{R}^2 | X \geq 0, X \neq 0\},$$

and

$$\mathcal{P}^+ = \{X \in \mathbb{R}^2 | X(H) + X(T) = 1, X(H), X(T) > 0\}.$$

Figure 4 shows the corresponding sets on the plane $\mathbb{R}^2$.

(e) The fundamental theorem of asset pricing states that a model is arbitrage-free if and only if there exists at least one risk-neutral measure.

The set of risk-neutral measures is given by $\mathcal{P}^+ \cap \mathbb{W}^\perp$. The set of arbitrage strategies is given by $\mathbb{W} \cap \mathbb{A}$. Therefore, the previous statement is equivalent to the following statement.

The market model is arbitrage-free if and only if $\mathcal{P}^+ \cap \mathbb{W}^\perp$ is non-empty. This statement is equivalent to $\mathbb{W} \cap \mathbb{A}$ being empty.

(f) In this model, we have calculated the risk-neutral measure as $\mathbb{P}(H) = \frac{2}{3}$ and $\mathbb{P}(T) = \frac{1}{3}$. As we can see in Figure 4, the $\Delta\hat{S}_1$ values have different signs. Therefore, the set $\mathbb{W}$ is a line in the 2nd and 4th quadrant. Its orthogonal complement is the perpendicular line that crosses $\mathcal{P}^+$ in the point $(\frac{1}{3}, \frac{2}{3})$.

Problem 1.3. Consider a one-period market model consisting of a **bond** and a **stock** with the following prices.

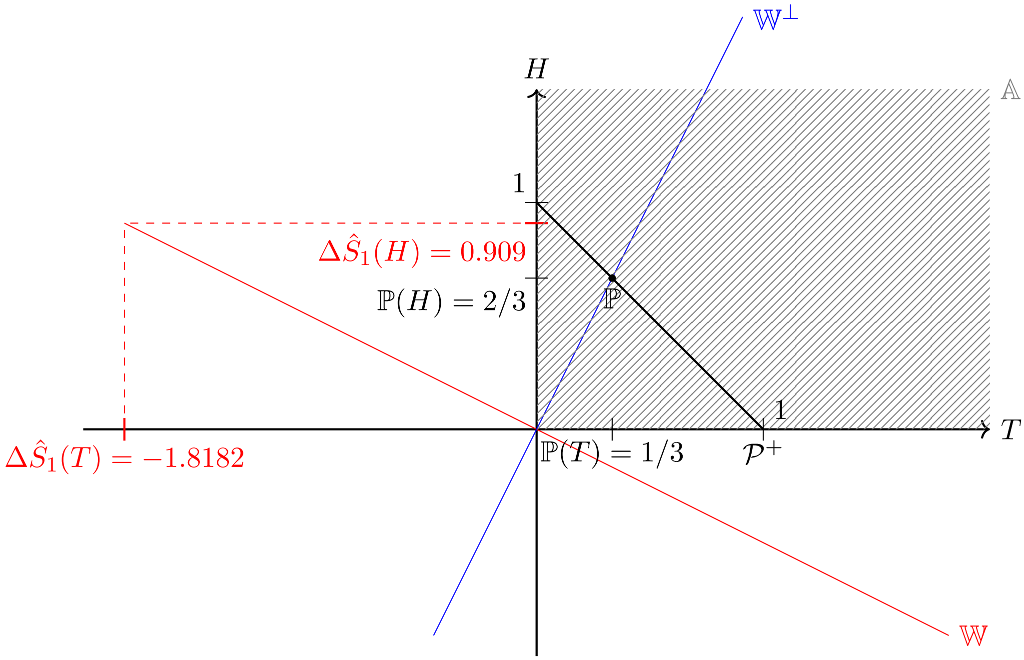


FIGURE 3. The sets $\mathbb{W}$ (red line), $\mathbb{W}^\perp$ (blue line), $\mathbb{A}$ (grey wavy area) and $\mathcal{P}^+$ (diagonal line in black). The intersection of $\mathcal{P}^+$ with $\mathbb{W}^\perp$ is the sought after risk neutral measure (within the quadrant $\mathbb{A}^+$).

- At time 0:

$$B_0 = 1 \quad \textit{and} \quad S_0^1 = 3$$

- At time 1:

	ω_1	ω_2	ω_3
B_1	1.1	1.1	1.1
S_1^1	6	4	2

(a) Calculate the interest rate r.
(b) Compute the set of risk neutral measures for this model.
(c) Is this market arbitrage-free? Why?
(d) Is this market complete? Why?
(e) *ABC Bank* decides to introduce a new product with payoff

	ω_1	ω_2	ω_3
S_1^2	0	2	0

It is decided that the this product will be sold at a price of S_0^2=1/2. Check that the market is now complete and arbitrage-free.

Solution

(a) $r = \dfrac{B_1}{B_0} - 1 = \dfrac{1.1}{1} - 1 = 0.1.$

(b) We want $\mathbb{E}_{\mathbb{Q}}(\frac{S_1}{1+r}) = S_0$. This is equivalent to the system

$$\begin{aligned} 6q_1 + 4q_2 + 2q_3 &= 3 \cdot 1.1, \\ q_1 + q_2 + q_3 &= 1. \\ &\Updownarrow \\ 6(1 - q_2 - q_3) + 4q_2 + 2q_3 &= 3.3, \\ q_1 &= 1 - q_2 - q_3. \\ &\Updownarrow \\ 2q_2 + 4q_3 &= 2.7, \\ q_1 &= 1 - q_2 - q_3. \\ &\Updownarrow \\ q_2 &= 1.35 - 2q_3, \\ q_1 &= q_3 - 0.35. \end{aligned}$$

Therefore, $\mathbb{Q} = \begin{pmatrix} \lambda - 0.35 \\ 1.35 - 2\lambda \\ \lambda \end{pmatrix}$. Since this needs to be a risk-neutral measure, it must hold that $\lambda > 0$ (since $q_3 > 0$), $\lambda < 0.675$ (since $q_2 > 0$, i.e. $1.35 - 2\lambda > 0$) and $\lambda > 0.35$ (since $q_1 > 0$). Combined, this gives $\lambda \in (0.35, 0.675)$. Different parameterisations of course give equivalent solutions.

(c) Yes, from the fundamental theorem of asset pricing, this market is arbitrage-free since there exists at least one risk-neutral measure.

(d) No, this is an incomplete market as there is more than 1 risk-neutral measure.

(e) Let us find the new risk neutral measure when the asset is included. We are therefore solving the system

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\frac{S_1^1}{1+r}) &= S_0^1 \\ \mathbb{E}_{\mathbb{Q}}(\frac{S_1^2}{1+r}) &= S_0^2 \\ \sum_{i=1}^{3} \mathbb{Q}(\omega_i) &= 1 \\ &\Updownarrow \\ 6q_1 + 4q_2 + 2q_3 &= 3.3 \\ 2q_2 &= 0.5 \cdot 1.1 \\ q_1 + q_2 + q_3 &= 1. \end{aligned}$$

This system has a *unique* solution $\mathbb{Q} = \begin{pmatrix} 0.1875 \\ 0.275 \\ 0.5375 \end{pmatrix}$. The market is therefore now arbitrage-free and complete.

Revision Sheet 2 (Investment)

Problem 2.1. The correlation ρ between assets A and B is 0.1. The other data is given in the following table:

Asset	$\bar{R}$	σ
A	10%	15%
B	18%	30%

(a) Find the proportions α of A and $(1-\alpha)$ of B that define a portfolio of A and B having minimum standard deviation.
(b) What is the value of this minimum standard deviation?
(c) What is the expected return of this portfolio?

Solution

(a) We first calculate the covariance as

$$\sigma_{AB} = \rho\sigma_A\sigma_B = 0.0045.$$

We are therefore looking for the minimiser of

$$\mathcal{L}(\alpha) = \frac{1}{2}(\alpha^2\sigma_A^2 + (1-\alpha)^2\sigma_B^2 + 2\alpha(1-\alpha)\sigma_{AB}).$$

Note: We could have set the proportions of the two assets independently and added the condition that their sum needs to be equal to 1. This would lead to the same solution once we substitute the condition into the remaining equations.

Taking the partial derivative and setting it to zero we have

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \alpha\sigma_A^2 - (1-\alpha)\sigma_B^2 + \sigma_{AB} - 2\alpha\sigma_{AB} = 0,$$

which is equivalent to

$$\alpha = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}} = 0.826.$$

(b) The standard deviation is equal to

$$SD = \sqrt{\alpha^2\sigma_A^2 + (1-\alpha)^2\sigma_B^2 + 2\alpha(1-\alpha)\sigma_{AB}} \stackrel{\alpha=0.826}{=} 0.139.$$

(c) The expected return is

$$\bar{R} = \alpha\bar{R}^A + (1-\alpha)\bar{R}^B \simeq 0.114.$$

Problem 2.2. Two stocks are available. The corresponding expected rates of return are $\bar{R}^1$ and $\bar{R}^2$. The corresponding variances and covariance are σ_1^2, σ_2^2 and σ_{12}. What percentages of the total investment should be invested in each of the two stocks to minimise the total variance of return of the resulting portfolio? What is the mean rate of return of this portfolio?

Solution Let α be the proportion of wealth invested in the 1st asset. This problem is the same as the previous, with the actual values left in their full generality.

We are therefore solving

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \alpha\sigma_1^2 - (1-\alpha)\sigma_2^2 + \sigma_{12} - 2\alpha\sigma_{12} = 0,$$

which gives the solution

$$\alpha = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}.$$

Consequently,

$$1 - \alpha = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}.$$

The expected return is therefore

$$\bar{R} = \alpha \bar{R}^1 + (1-\alpha)\bar{R}^2 = \frac{(\sigma_2^2 - \sigma_{12})\bar{R}^1 + (\sigma_1^2 - \sigma_{12})\bar{R}^2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

Problem 2.3. Suppose there are n assets which are uncorrelated. You may invest in any one, or in any combination of them. The mean rate of return $\bar{R}$ is the same for each asset, but the variances are different. The return on asset i has a variance σ_i^2 for $i = 1, \dots, n$.

(a) Show the situation on a mean-variance diagram. Describe the efficient set.
(b) Find the minimum-variance point. Express your result in terms of

$$\bar{\sigma}^2 = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)^{-1}.$$

Solution

(a) The minimum-variance diagram is depicted in Figure 4. The efficient set is the red line in the diagram.

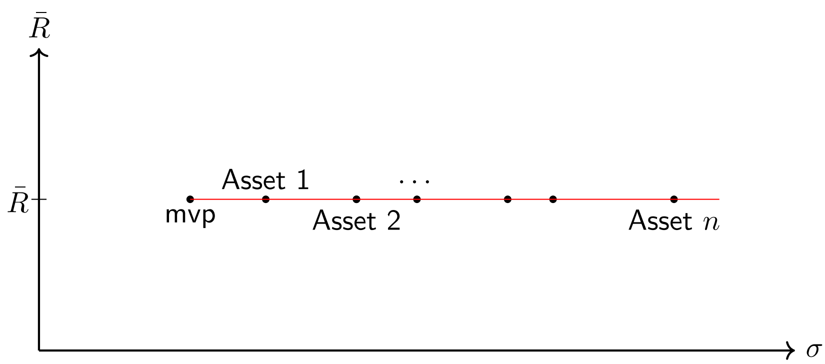


FIGURE 4. The mean-variance diagram, with *mvp* being the minimum variance portfolio and the red line depicting the efficient set. The assets must not necessarily be ordered in an increasing order and are presented here as such for clarity.

(b) Since the assets are uncorrelated, we are solving the problem

$$\min \sum_{i=1}^{n} w_i^2 \sigma_i^2$$
$$\text{s.t.} \sum_{i=1}^{n} w_i = 1.$$

The corresponding Lagrangian is

$$\mathcal{L}(w_i, \lambda) = \frac{1}{2}(\sum_{i=1}^{n} w_i^2 \sigma_i^2) - \lambda(\sum_{i=1}^{n} w_i - 1).$$

Takin partial derivatives, we obtain the system

$$\frac{\partial \mathcal{L}}{\partial w_i} = w_i \sigma_i^2 - \lambda = 0 \Leftrightarrow w_i = \frac{\lambda}{\sigma_i^2} \qquad i = 1 \dots n$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Leftrightarrow \sum_{i=1}^{n} w_i = 1.$$

Substituting the first n equations into the last one, we have that it is equivalent to

$$\sum_{i=1}^{n} \frac{\lambda}{\sigma_i^2} = 1 \Leftrightarrow \lambda = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\right)^{-1} =: \bar{\sigma}^2.$$

Therefore, the proportions of asset i in the minimum variance portfolio is equal to $w_i = \frac{\bar{\sigma}^2}{\sigma_i^2}$.

Problem 2.4. There are just three assets with rates of return R_1, R_2 and R_3, respectively. The covariance matrix and the expected rates of return are

$$V = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.1 & 0.2 & 0.1 \\ 0 & 0.1 & 0.2 \end{bmatrix} \quad \bar{R} = \begin{bmatrix} 0.4 \\ 0.8 \\ 0.8 \end{bmatrix}$$

(a) Find the minimum-variance portfolio (Hint: By symmetry $w_1 = w_3$).
(b) Find another efficient portfolio by setting $\lambda_1 = 1$, $\lambda_2 = 0$.

Solution

(a) Using the hint that $w_1 = w_3$, the resulting Lagrangian is

$$\begin{aligned}\mathcal{L}(w_1, w_2, \mu) &= \frac{1}{2}(2w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} + 2w_2w_1\sigma_{23}) - \mu(2w_1 + w_2 - 1) \\ &= \frac{1}{2}(0.4w_1^2 + 0.2w_2^2 + 0.4w_1w_2) - \lambda_2(2w_1 + w_2 - 1).\end{aligned}$$

Note: The full Lagrangian would also depend on λ_1 as the factor in front of the condition that we want our portfolio to have a given expected return. Since we are looking for the minimum variance portfolio without requiring a fixed return, the condition is not present, i.e. $\lambda_1 = 0$.

Taking the partial derivatives, we obtain

$$\frac{\partial \mathcal{L}}{\partial w_1} = 0.4w_1 + 0.2w_2 - 2\mu = 0$$
$$\frac{\partial \mathcal{L}}{\partial w_2} = 0.2w_1 + 0.2w_2 - \mu = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = 2w_1 + w_2 - 1 = 0.$$

The solution to this system is $w_1 = w_3 = \frac{1}{2}$ and $w_2 = 0$.

(b) The Lagrangian with $\lambda_2 = 0, \lambda_1 = 1$ is

$$\mathcal{L} = \frac{1}{2}(w_1^2\sigma_1^2 + w_2^2\sigma^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{12} + 2w_2w_3\sigma_{23}) - (w_1\bar{R}_1 + w2\bar{R}_2 + w_3\bar{R}_3 - \bar{r}),$$

where $\bar{r}$ is some (unknown and irrelevant) desired return. Taking partial derivatives we obtain

$$\frac{\partial \mathcal{L}}{\partial w_1} = 0.2w_1 + 0.1w_2 - \bar{R}_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial w_2} = 0.2w_2 + 0.1w_1 + 0.1w_3 - \bar{R}_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial w_3} = 0.2w_3 + 0.1w_2 - \bar{R}_3 = 0$$

The solution to the above system is $w_1 = 1$, $w_2 = 2$ and $w_3 = 3$. These values do not add up to 1 (since we set $\mu = 0$ and therefore did not require this), so we need to renormalise. The combined weights are $w_1 + w_2 + w_3 = 6$, so the renormalised weights are

$$\tilde{w}_1 = \frac{1}{6}, \quad \tilde{w}_2 = \frac{2}{6}, \quad \tilde{w}_3 = \frac{3}{6}.$$

Problem 2.5. Assume a single-period market model with 2 stocks, 1 bond and 3 possible states of the world with subjective probabilities $\mathbb{P}(\omega_1) = 1/2$ and $\mathbb{P}(\omega_2) = \mathbb{P}(\omega_3) = 1/4$. At time $t = 0$, the prices of the stocks are $S_0^1 = 5$ and $S_0^2 = 4$. The risk free interest rate is $r_f = 0.2$. The prices of the stocks at time $t = 1$ are given in the following table:

	ω_1	ω_2	ω_3
S_1^1	3	6	9
S_1^2	5	6	4

(a) Solve the Markowitz optimisation problem for the portfolio with expected return $\bar{r} = 0.3$.
(b) Using the optimal portfolio found in the previous subquestion, find an optimal portfolio with expected return $\tilde{r} = 0.6$.
(c) Find the market portfolio and its expected return.

Solution We first calculate the returns, mean returns and variances.

	ω_1	ω_2	ω_3
R_1	-0.4	0.2	0.8
R_2	0.25	0.5	0

The expected returns, second moments, variances and covariances are therefore

$$\begin{aligned}
\bar{R}^1 &= 0.05 \\
\bar{R}^2 &= 0.25 \\
\mathbb{E}(R_1^2) &= 0.25 \\
\mathbb{E}(R_2^2) &= 0.09375 \\
\sigma_1^2 &= 0.2475 \\
\sigma_2^2 &= 0.03125 \\
\sigma_{12} &= \mathbb{E}(R_1 R_2) - \mathbb{E}(R_1)\mathbb{E}(R_2) = -0.0375.
\end{aligned}$$

(a) The Lagrangian is

$$\mathcal{L}(w_1, w_2, \lambda) = \frac{1}{2}(w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}) - \lambda(w_1\bar{R}_1 + w_2\bar{R}_2 + (1 - w_1 - w_2)r_f - \bar{r}).$$

Taking partial derivatives, we obtain the system of equations

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial w_1} &= w_1\sigma_1^2 + w_2\sigma_{12} - \lambda(\bar{R}_1 - r_f) = 0 \\
\frac{\partial \mathcal{L}}{\partial w_2} &= w_2\sigma_2^2 + w_1\sigma_{12} - \lambda(\bar{R}_2 - r_f) = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= w_1\bar{R}_1 + w_2\bar{R}_2 + (1 - w_1 - w_2)r_f - \bar{r} = 0.
\end{aligned}$$

Solving this system we obtain $w_1 \simeq -0.37$, $w_2 \simeq 0.89$ and $w_0 \simeq 0.48$.

(b) The previous solution was for an expected return $\bar{r} = 0.3$. We now need an optimal portfolio for the expected return $\tilde{r} = 0.6$. We have

$$\gamma = \frac{\tilde{r} - \bar{r}}{r_f - \bar{r}} = \frac{0.6 - 0.3}{0.2 - 0.3} = -3$$

and $1 - \gamma = 4$. The new weights will therefore be

$$\begin{aligned}
\tilde{w}_1 &= (1 - \gamma)w_1 = -1.48 \\
\tilde{w}_2 &= (1 - \gamma)w_2 = 3.56 \\
\tilde{w}_0 &= 1 - \tilde{w}_1 - \tilde{w}_2 = -1.08.
\end{aligned}$$

(c) In the market portfolio it must hold that $w_1^M + w_2^M = 1$, i,e. $w_0^M = 0$. Therefore, we are looking for $(1 - \gamma)$ such that

$$(1 - \gamma)(w_1 + w_2) = 1 \Leftrightarrow (1 - \gamma) = \frac{1}{0.89 - 0.37} = 1.92.$$

Therefore,

$$\begin{aligned}
w_1^M &= -0.36 \cdot 1.92 = -0.71 \\
w_2^M &= 0.89 \cdot 1.92 = 1.71.
\end{aligned}$$

Problem 2.6. Solve Question 3 from Problem Sheet 8, for the logarithmic utility function (i.e. $u(x) = \ln(x)$).

Solution From Question 3, we have $r = \frac{1}{9}$, $S_0 = 5$, $S_1(\omega_1) = \frac{20}{3}$, $S_1(\omega_2) = \frac{40}{9}$, $\mathbb{P}(\omega_1) = \frac{3}{5}$ and $\mathbb{P}(\omega_2) = \frac{2}{5}$.

We first obtain $u'(x) = \frac{1}{x}$ and therefore

$$u'(I(x)) = x \Leftrightarrow \frac{1}{I(x)} = x \Leftrightarrow I(x) = \frac{1}{x}.$$

Since $u = \frac{\frac{20}{3}}{5} = \frac{4}{3} > \frac{10}{9} = 1 + r > \frac{8}{9} = \frac{\frac{40}{9}}{5} = d$, the market is arbitrage free and complete, and we can use the two-step approach.

Step 1: Find $\mathbb{Q}$: $q = \frac{1+r-d}{u-d} = \frac{1}{2} = \mathbb{Q}(\omega_1)$ and so $\mathbb{Q}(\omega_2) = \frac{1}{2}$.

The State-Price density is therefore

$$L(\omega_1) = \frac{\mathbb{Q}(\omega_1)}{\mathbb{P}(\omega_1)} = \frac{5}{6}$$
$$L(\omega_2) = \frac{\mathbb{Q}(\omega_2)}{\mathbb{P}(\omega_2)} = \frac{5}{4}.$$

We must now solve the system

$$W(\omega_i) = I(\lambda \frac{L(\omega_i)}{1+r}) \qquad i = 1, 2$$
$$\mathbb{E}_{\mathbb{Q}}(\frac{1}{1+r} I(\lambda \frac{L}{1+r})) = x.$$

We start by solving the second equation with respect to λ:

$$\mathbb{E}_{\mathbb{Q}}(\frac{1}{1+r} I(\lambda \frac{L}{1+r})) = x$$
$$\Updownarrow$$
$$\mathbb{E}_{\mathbb{Q}}(\frac{1}{\lambda L}) = x$$
$$\Updownarrow$$
$$\frac{1}{\lambda}\left(\mathbb{Q}(\omega_1)\frac{\mathbb{P}(\omega_1)}{\mathbb{Q}(\omega_1)} + \mathbb{Q}(\omega_2)\frac{\mathbb{P}(\omega_2)}{\mathbb{Q}(\omega_2)}\right) = x$$
$$\Updownarrow$$
$$\frac{1}{\lambda} = x$$
$$\Updownarrow$$
$$\lambda = \frac{1}{x}.$$

Substituting λ into the first equation, we get:

$$W(\omega_i) = \frac{(1+r)\cdot x}{L(\omega_i)}, \quad i = 1, 2.$$

Step 2: We are solving the problem for the initial investment $x = 5$. The optimal attainable wealths are therefore

$$W(\omega_1) = \frac{\frac{10}{9}\cdot 5}{\frac{5}{6}} = \frac{60}{9},$$
$$W(\omega_2) = \frac{\frac{10}{9}\cdot 5}{\frac{5}{4}} = \frac{40}{9}.$$

We now find the replicating strategy for the above wealths, so we are solving the system

$$(x - \phi S_0)(1+r) + \phi S_1(\omega_1) = W(\omega_1)$$
$$(x - \phi S_0)(1+r) + \phi S_1(\omega_2) = W(\omega_2).$$

This yields

$$\phi = \frac{\frac{60-40}{9}}{\frac{60-40}{9}} = 1.$$

The optimal strategy is therefore $(x, \phi) = (5, 1)$, or put into words, for the starting investment of 5 units of money, the optimal strategy is to buy one share of the stock and to invest $5 - 1 \cdot 5 = 0$ into the money account.